MASTER'S THESIS - EQUIDISTRIBUTION IN FAMILIES OF ABELIAN VARIETIES AND UNIFORMITY

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Abstract. In this master's thesis, we study in detail the article [14]. We show how to construct an equidistribution measure with respect to a generic sequence on a non-degenerate subvariety of an abelian variety, and how to apply this construction to prove a uniform version of the Bogomolov Conjecture, which can in turn be applied to obtain a uniform version of Faltings's Theorem. For this, we define concepts and use techniques from Arakelov geometry, such as Chern forms, Hermitian line bundles and arithmetic intersection numbers.

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Introduction

In this work, we will study the paper [14], which gives *equidistribution results in families of abelian varieties*. In this introduction, we will explain what we mean by this sentence and, most importantly, why we care about it. We start from the very beginning.

0.1. Diophantine Geometry

Determining the integer solutions of an equation is a very old mathematical problem: Diophantus, in the third century, already studied this kind of problem. Due to him, we name a polynomial equation with integer coefficients a *Diophantine equation*. But there are registers of the apparition of Diophantine equations even before him.

However, it took much longer for the subject to fully develop. It was in the nineteenth century, with the important contribution of Gauss, Dirichlet, Dedekind, Hilbert, and others, that *Algebraic Number Theory* was founded. Thereafter, our knowledge of the subject grew exponentially, and we were able to solve some very difficult problems, like *Fermat's Last Theorem*, which was proposed in 1637 and only fully solved in 1994.

Nevertheless, the real dream of number theorists was to find an algorithm to determine if any given Diophantine equation has a solution. This dream was frustrated by *Matiyasevich's Theorem*, which proved there is no such algorithm. So we have to content ourselves with working on particular types of equations or estimating the number of solutions. That is where *Diophantine geometry* enters.

Using techniques of both algebraic geometry and number theory, we can prove that the number of solutions of a Diophantine equation has much to do with some algebraic-geometric invariant of the equation - or better saying, of the curve it determines - called the *genus*. We can conclude, for example, that if the genus of the curve is at least 1, then the number of its integer solutions is finite (*Siegel's Theorem*), and if the genus of the curve is at least 2, then even the number of its rational solutions is finite (*Faltings's Theorem*). We can also conclude that if the genus of the curve is 1, then the number of rational points is, in some sense, a finitely generated abelian group (*Mordell-Weil Theorem*).

0.2. Rational Points

In order to apply the theory of schemes for counting rational points, we first need to know what a *rational point* means in this context. Here we recall the definition and discuss why this corresponds to what we expect to have:

Definition. Let X and T be S-schemes. A **T-point of X (over S)** is an S-morphism $T \to X$. Then the set of T-points of X (over S) is by definition $\mathrm{Hom}_S(T, X)$, which we also denote $X_S(T)$, or simply $X(T)$ if S is clear from context.

If $T = \operatorname{Spec} A$ is affine, we simply say **A-point of** X for a $\operatorname{Spec} A$-point of X, and denote the set of A-points of X by $X(A)$. Similar notational simplifications are performed if S is affine.

In the particular case in which $T = \operatorname{Spec} K$ is the spectrum of a field, we also call a K-point a **K-rational point**.

Now, let us test this definition. Let K be a field. We would expect the set of K-rational points of A_K^n to be K^n. In fact, the definition above would not be good if this were not true. This is indeed true, but we have to be a little careful, as we have to consider $\mathbb{A}_K^n$ and $\operatorname{Spec} K$ *as K-schemes* for this to work. By definition:

$$\begin{aligned} \mathbb{A}_K^n(K) = \operatorname{Hom}_K(\operatorname{Spec} K, \mathbb{A}_K^n) &= \operatorname{Hom}_K(\operatorname{Spec} K, \operatorname{Spec} K[x_1, \dots, x_n]) \\ &= \operatorname{Hom}_K(K[x_1, \dots, x_n], K) \cong K^n, \end{aligned}$$

since choosing a K-morphism from $K[x_1, \dots, x_n]$ to K is the same as assigning one element of K to each x_i, with no restrictions. Notice that the fact that we are considering morphisms of K-schemes is fundamental; otherwise, things could go wrong. For example, if we want to compute $\mathbb{A}_{\mathbb{Q}(i)}^n(\mathbb{Q}(i))$ as $\mathbb{Q}$-schemes, we obtain "extra points" due to the automorphism $i \mapsto -i$. By testing (being careful with the base scheme) other examples of the concept of T-points we see that this definition behaves as one would expect. For example, we see that the K-rational points of $\mathbb{P}_K^n$ correspond to $(K^{n+1} \setminus \{0\})/K^\times$.

If we want to consider K-rational points over the affine space and we are varying the field K, we can simply study the K-rational points of $\mathbb{A}_{\mathbb{Q}}^n$ as $\mathbb{Q}$-schemes (or $\mathbb{Z}$-schemes), instead of changing the "affine space scheme" and the base scheme as well. Indeed, by the same reasoning as above, $\mathbb{A}_{\mathbb{Q}}^n(K)$ corresponds to the group of ring homomorphisms $\mathbb{Q}[x_1, \dots, x_n] \to K$, which is clearly isomorphic to K^n.

More generally, if X and T are S-schemes and $X' := X \times_S T$, it easily follows from the universal property of the product that we have $X'_T(T) \cong X_S(T)$. Due to this correspondence, no confusion arises, for example, if we refer to the "K-rational points of the projective plane", since the sets $\mathbb{P}_K^n(K)$ (considering morphisms of K-schemes) and $\mathbb{P}_{\mathbb{Q}}^n(K)$ (considering morphisms of schemes) can be identified.

0.3. Heights

All the above results are proved with the help of *height functions*. They generalize the idea in elementary number theory of solving problems by limiting the absolute value of the solutions. A bit more precisely, let K be a number field. We want to define for every smooth projective variety V/K some function $h\colon V(K) \to \mathbb{R}$ that "measures the size" of the points of $V(K)$. In order to reproduce the arguments of elementary number theory, we would like to put some finiteness condition on h: for any constant C, the points of $V(K)$ with height bounded by C should be in a finite number.

Here we will just state the main results we will need about heights and refer the reader to [10, Part B] for a more complete study of these functions. We start by defining the height for projective spaces. We denote by $\Sigma(K)$ the set of places of K, and $\Sigma_\infty(K)$ the set of its Archimedean places.

Definition. Let $x = [x_0 : x_1 : \cdots : x_r] \in \mathbb{P}^r_K(K)$. The **height** of x (relative to K) is defined by the formula:

$$h_K(x) \coloneqq \log \prod_{\nu \in \Sigma(K)} \max\{\|x_0\|_\nu, \ldots, \|x_r\|_\nu\},$$

where $\|\cdot\|_\nu$ denotes the normalized absolute value in the completion K_ν.

The **absolute height** on $\mathbb{P}^r_{\mathbb{Q}}(\overline{\mathbb{Q}})$ is the function $h\colon \mathbb{P}^r_{\mathbb{Q}}(\overline{\mathbb{Q}}) \to \mathbb{R}_+$ given by $h(x) \coloneqq h_K(x)/[K : \mathbb{Q}]$, where K is any number field with $x \in \mathbb{P}^r_{\mathbb{Q}}(K)$.

It is not so difficult to see that the absolute height h is a well-defined function and satisfies some good finiteness properties. The heights behave well with respect to some classical maps in algebraic geometry:

Proposition 0.1. *Let $S_{r_1,r_2}\colon \mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \to \mathbb{P}^{r_1 r_2 + r_1 + r_2}$ be the Segre embedding. Consider the projections $\mathrm{pr}_i\colon \mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \to \mathbb{P}^{r_i}$, and let $\mathcal{O}(1,1)$ be the line bundle on $\mathbb{P}^{r_1} \times \mathbb{P}^{r_2}$ given by $\mathrm{pr}_1^* \mathcal{O}(1) \otimes \mathrm{pr}_2^* \mathcal{O}(1)$. Let also $\nu_d\colon \mathbb{P}^r \to \mathbb{P}^{\binom{r+d}{r}-1}$ be the Veronese embedding of degree d. Then:*

(a) $S^*_{r_1,r_2}\mathcal{O}(1) \sim \mathcal{O}(1,1)$.
(b) $h(S_{r_1,r_2}(x,y)) = h(x) + h(y)$ for all $x \in \mathbb{P}^{r_1}(\overline{\mathbb{Q}})$ and $y \in \mathbb{P}^{r_2}(\overline{\mathbb{Q}})$.
(c) $h(\nu_d(x)) = d \cdot h(x)$, for all $x \in \mathbb{P}^r(\overline{\mathbb{Q}})$.

(See [10, Proposition B.2.4]). The fact that we can extend the concept of heights for all smooth projective varieties over K is a very powerful result:

Proposition 0.2 (Weil's Height Machine)**.** *Let K be a number field. For every smooth projective variety V/K and every line bundle L on V, there exists a function $h_{V,L}\colon V(\overline{K}) \to \mathbb{R}$, such that we have the following properties:*

(a) (Normalization) Let H be a hyperplane line bundle on $\mathbb{P}^r_K$, and let h be the absolute height on $\mathbb{P}^r_K$ defined above. Then $h_{\mathbb{P}^r_K,H}(x) = h(x) + O(1)$, for all $x \in \mathbb{P}^n(\overline{K})$.
(b) (Functoriality) Let $\varphi\colon V \to W$ be a morphism and let L be a line bundle on W. Then $h_{V,\varphi^* L}(x) = h_{W,L}(\varphi(x)) + O(1)$ for all $x \in V(\overline{K})$.
(c) (Additivity) Let L, M be line bundles on V. Then $h_{V,L\otimes M}(x) = h_{V,L}(x) + h_{V,M}(x) + O(1)$ for all $x \in V(\overline{K})$.
(d) (Linear Equivalence) Let L, M be line bundles on V with $L \sim M$. Then $h_{V,L} = h_{V,M} + O(1)$, for all $x \in V(\overline{K})$.
(e) (Positivity) Let L be a line bundle on V with base locus B, and assume $B \neq V$. Then $h_{V,L}(x) \geq O(1)$ for all $x \in (V \setminus B)(\overline{K})$.
(f) (Finiteness) Let L be an ample line bundle on V. Then for every finite extension K'/K and every constant C, the set $\{x \in V(K')\colon h_{V,L}(x) \leq C\}$ is finite.
(g) (Uniqueness) The height functions $h_{V,L}$ are determined, up to $O(1)$, by normalization (a), functoriality (b) just for embeddings $\varphi\colon V \hookrightarrow \mathbb{P}^r_K$, and additivity (c).

The $O(1)$ depends on varieties, line bundles, and morphisms, but does not depend on the points.

(See [10, Theorems B.3.2 and B.3.6]). Even though the existence of these functions is a strong result on its own, with an additional hypothesis on the line bundle we can strengthen it even more by eliminating the "up to $O(1)$":

Proposition 0.3. *(Néron, Tate) Let V/K be a smooth variety defined over a number field, let L be a line bundle on V, and let $\varphi\colon V \to V$ be a morphism. Suppose that $\varphi^* L \sim L^{\otimes t}$ for some integer $t > 1$, and let $h_{V,L}\colon V(\overline{K}) \to \mathbb{R}$ be "the" height (uniquely determined up to $O(1)$) given by Weil's Height Machine. Then there is a unique function, called the **canonical height** (or **Néron-Tate height**) on V relative to φ and L:*

$$\hat{h}_{V,\varphi,L}\colon V(\overline{K}) \to \mathbb{R},$$

with the following two properties:

(i) $\hat{h}_{V,\varphi,L}(x) = h_{V,L}(x) + O(1)$ *for all* $x \in V(\overline{K})$.
(ii) $\hat{h}_{V,\varphi,L}(\varphi(x)) = t \cdot \hat{h}_{V,\varphi,L}(x)$ *for all* $x \in V(\overline{K})$.

The canonical height depends only on the equivalence class of L. Furthermore, it can be computed as the limit:

$$\hat{h}_{V,\varphi,L}(x) = \lim_{j\to\infty} \frac{1}{t^j} h_{V,L}(\varphi^j(x)),$$

where $\varphi^j = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{j \text{ times}}$.

(See [10, Theorem B.4.1]). Suppose further that L is ample. By the finiteness property of Weil heights (Proposition 0.2, item (f)) the number of $x \in V(K)$ with negative height is finite. Hence, by translation, we can associate this line bundle with a non-negative Weil height. This shows that the Néron-Tate height defined by the limit above is always non-negative, as it does not depend on the choice of a particular Weil height.

Besides, the points of canonical height zero are exactly the points $x \in \overline{K}$ for which the orbit of x by φ, $\{\varphi^j(x)\colon j \in \mathbb{N}\}$, is finite. Indeed, let $x \in V(\overline{K})$, and choose a finite extension K'/K such that[1] $x \in V(K')$. If the orbit of x by φ is finite, then so is its direct image for some height $h_{V,L}$. In particular, this set is bounded. This proves that $t^{-j} \cdot h_{V,L}(\varphi^j(x)) \to 0$, hence $\widehat{h}_{V,\varphi,L} = 0$. On the other hand, if $\widehat{h}_{V,\varphi,L}(x) = 0$, then for any $j \geq 1$:

$$h_{V,L}(\varphi^j(x)) = \widehat{h}_{V,\varphi,L}(\varphi^j(x)) + O(1) = t^j \cdot \widehat{h}_{V,\varphi,L}(x) + O(1) = O(1),$$

where we used Proposition 0.2, item (g), and property (ii) of the Néron-Tate height. Notice that this $O(1)$ is independent of j, hence all the $\varphi^j(x)$ are points in $V(K')$ with height limited above by some constant. This proves that the orbit of x by φ is finite, by Proposition 0.2, item (f).

[1]Notice that if we suppose only that the line bundle L is defined over a finite extension of K (not necessarily over K), by making K' bigger we can assume that L is defined over K', and the rest of the argument works the same.

0.4. Abelian Varieties

Abelian varieties are very well-behaved mathematical objects. Complex abelian varieties are exactly the complex torus that can be given a *polarization*, that is, which have a holomorphic line bundle L with a positive definite Chern form. We refer the reader to the book [16] for a complete study of complex abelian varieties, or to [10, Chapter A.5] for a shorter read.

We can generalize the notion of abelian variety for any base scheme:

Definition. An **abelian variety** over a scheme S is a connected projective algebraic group V over S.

It is not clear from the definition, but the projectivity of an abelian variety forces it to be a commutative algebraic group. So for every scheme T over S, we can denote the operation in $V(T)$ additively. In particular, we can define for every $n \in \mathbb{N}$ a multiplication map $[n]: V(T) \to V(T)$ given by $x \mapsto \underbrace{x + x + \cdots + x}_{n \text{ times}}$. By the Yoneda Lemma, this induces a morphism $[n]: V \to V$.

We denote the set of n-torsion points (the kernel of $[n]$) by $V[n]$. If V is an abelian variety of dimension g over an algebraically closed field whose characteristic does not divide n, then we have $V[n] \cong (\mathbb{Z}/n\,\mathbb{Z})^{2g}$. Besides, it can be shown that $[n]$ is surjective under these conditions. So $[n]: V \to V$ is an *isogeny*, that is, a morphism between abelian varieties which is surjective and has finite kernel. Notice that all of these results are trivial for complex tori.

It is also well-known how $[n]$ relates to pullbacks of line bundles:

Proposition 0.4. *(Mumford's formula) Let L be a line bundle on an abelian variety V, and let $[n]: V \to V$ be the multiplication-by-n map. Then:*

$$[n]^*L \sim L^{\otimes \frac{n^2+n}{2}} \otimes ([-1]^*L)^{\otimes \frac{n^2-n}{2}}.$$

In particular, if L is symmetric, that is, if $[-1]^*L \sim L$, then we have $[n]^*L \sim L^{\otimes n^2}$, and if L is antisymmetric, that is, if $[-1]^*L \sim L^{\otimes(-1)}$, then we have $[n]^*L \sim L^{\otimes n}$.

(See [10, Corollary A.7.2.5]). We can use the result above to associate a canonical height for every line bundle L on V. For every (anti)symmetric line bundle L and every integer n, by Mumford's formula and by Proposition 0.3 we can construct a height $\widehat{h}_{V,[n],L}$. This turns out to be independent of n (see for example [10, Theorem B.5.1 and Theorem B.5.5]). Hence we have a canonical height $\widehat{h}_{V,L}$ that only depends on L.

In the case of a general line bundle L, we can write

$$L^{\otimes 2} = (L \otimes [-1]^*L) \otimes (L \otimes ([-1]^*L)^{\otimes(-1)}).$$

This is the product of a symmetric and an antisymmetric line bundle. Thus, associating to L the function given by the average of the canonical heights of the line bundles $L \otimes [-1]^*L$ and $L \otimes ([-1]^*L)^{\otimes(-1)}$, we obtain the canonical Néron-Tate height $\widehat{h}_{V,L}$ associated to L.

Given an ample symmetric line bundle L over V (or over a base change of V by some finite extension of K), the points $x \in A(\overline{K})$ in which the canonical height $\widehat{h}_{V,L}$ is zero are exactly the torsion points of $A(\overline{K})$. Indeed, choose a finite extension K'/K so that $x \in A(K')$ and L is defined over K'. If x is torsion, there exists an integer $n \geq 2$ such that $[n](x) = 0$. So it is clear that the orbit of x by $[n]$ is finite, and as we already saw this implies $\widehat{h}_{V,L}(x) = \widehat{h}_{V,[n],L}(x) = 0$. On the other hand, if $\widehat{h}_{V,L}(x) = 0$, then by the same argument of the previous subsection, the set $\{[2^j](x) \colon j \in \mathbb{N}\}$ is finite. Hence there are integers $j_1 > j_2 \geq 0$ such that $[2^{j_1}](x) = [2^{j_2}](x)$. But this implies $[2^{j_1} - 2^{j_2}](x) = 0$, proving that x is torsion.

One of the most basic and important results in the theory of abelian varieties over number fields is the so-called *Mordell-Weil Theorem*:

Theorem 0.5 (Mordell-Weil Theorem)**.** *Let A be an abelian variety over a number field K. Then the group $A(K)$ of K-rational points of A is a finitely generated abelian group.*

The proof of this result is a great example of applying the theory of canonical heights, together with some deep number-theoretical facts and an elementary combinatorial argument using lattices.

For more information about abelian varieties, see [10], chapter A.7 for a short introduction, chapter B.5 for the study of canonical heights on these objects and part C for a proof of the Mordell-Weil theorem. For an extensive study of abelian varieties, see [20] or [7].

Another reason why abelian varieties are so important is that every smooth projective curve C of genus $g \geq 1$ can be embedded in an abelian variety $\mathrm{Jac}(C)$, called the *Jacobian* of C. This means that by considering the Jacobian of a curve and using results on abelian varieties, we can also get information about the curve. See [10], chapter A.6 and chapter A.8 for an introduction to Jacobians over $\mathbb{C}$ and over arbitrary fields, respectively. For a deeper study, see [7, Chapter 14].

0.5. Some (already proven) Conjectures

When embedding a curve C into its Jacobian J, we can ask which points of C correspond to torsion points on J. The following result, proven by Raynaud, states that if the genus of the curve is at least two then the number of such points is finite:

Theorem 0.6. *(Manin-Mumford Conjecture) Let K be a number field, and let C be a curve over K of genus $g \geq 2$. Let J be its Jacobian. Then, for any embedding $C \hookrightarrow J$, the set $C(\overline{K}) \cap J(\overline{K})_{\mathrm{tors}}$ is finite.*

The original proof of this result is given in [23]. On the other hand, Faltings's theorem, previously known as *Mordell conjecture*, states:

Theorem 0.7. *(Faltings's Theorem/Mordell Conjecture) Let K be a number field, and let C be a curve over K of genus $g \geq 2$. Then the set $C(K)$ is finite.*

(See [10, Part E] for a proof).

Definition. An abelian group Γ has **finite rank** if it contains a free finitely generated subgroup $\Gamma_0 \subseteq \Gamma$ such that for every $x \in \Gamma$ there exists an integer $n \geq 1$ such that $nx \in \Gamma_0$.

It is possible to prove that both groups $J(\overline{K})_{\mathrm{tors}}$ and $J(K)$ have finite rank (indeed, $J(\overline{K})_{\mathrm{tors}}$ has finite rank as it is torsion, and $J(K)$ has finite rank as it is finitely generated by the Mordell-Weil Theorem). So both results happen to be consequences of a stronger result called the *Mordell-Lang conjecture*:

Theorem 0.8 (Mordell-Lang Conjecture for curves)**.** *Let K be a number field, and let C be a curve over K of genus $g \geq 2$. Let J be its Jacobian. Let Γ be a subgroup of finite rank of $J(\overline{K})$. Then, for any embedding $C \hookrightarrow J$, the set $C(\overline{K}) \cap \Gamma$ is finite.*

(See [22]). The Manin-Mumford conjecture has a generalization called the *Bogomolov conjecture*:

Theorem 0.9. *(Bogomolov Conjecture) Let K be a number field, and let C be a curve over K of genus $g \geq 2$. Let J be its Jacobian, and let $\widehat{h}$ be the Néron-Tate height on J associated to an ample symmetric line bundle. Then, for any embedding $C \hookrightarrow J$, there exists an $\varepsilon > 0$ such that the set*

$$\{x \in C(\overline{K}) \colon \widehat{h}(x) < \varepsilon\}$$

is finite.

This was proved by Ullmo, Zhang, and Szpiro. See for example [26] or [25]. This is indeed a generalization of the Manin-Mumford conjecture since torsion points are exactly the points of height zero.

We will state these conjectures in greater generality later.

0.6. Equidistribution

The proof of the Bogomolov conjecture uses techniques of *equidistribution.* The idea is to take a sequence of points in a measure space that is so general that we can integrate functions by analyzing their values on the orbits of these points. Formally, let K be a number field and let X be a proper variety over K. Consider an Archimedean place $\nu \in \Sigma_\infty(K)$ and its corresponding inclusion $K \hookrightarrow \mathbb{C}_\nu$. Given a closed point $x \in X$, we define its **orbit** to be $O_\nu(x) \coloneqq (x \times_K \mathbb{C}_\nu)^{\mathrm{an}} \subseteq X^{\mathrm{an}}_{\mathbb{C}_\nu}$. We denote its cardinality by $\#O_\nu(x)$. As a set, we have:

$$O_\nu(x) = (\mathrm{Spec}(K(x) \otimes_K \mathbb{C}_\nu))(\mathbb{C}_\nu) = \mathrm{Hom}_{\mathbb{C}_\nu}(K(x) \otimes_K \mathbb{C}_\nu, \mathbb{C}_\nu) = \mathrm{Hom}_K(K(x), \mathbb{C}_\nu).$$

By Hilbert's Nullstellensatz, the residue field $K(x)$ of x is a finite extension of K. Hence, as $K(x)/K$ is a separable extension, the cardinality of the set $\mathrm{Hom}_K(K(x), \mathbb{C}_\nu)$ is the finite number $[K(x) : K]$. So $O_\nu(x)$ consists of a finite number of points, and we have $\#O_\nu(x) = [K(x) : K]$.

Given a probability measure μ_ν on $X^{\mathrm{an}}_{\mathbb{C}_\nu}$, a sequence of points (x_i) in X is said to be **equidistributed**, or to **satisfy the equidistribution property**, if for every compactly supported continuous function $f \colon X^{\mathrm{an}}_{\mathbb{C}_\nu} \to \mathbb{R}$ we have the equality:

$$\int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} f \, d\mu_\nu = \lim_{i \to \infty} \frac{1}{\#O_\nu(x_i)} \sum_{x \in O_\nu(x_i)} f(x).$$

0.7. Abelian Schemes

The paper [14] works with several abelian varieties at the same time. This is formalized in the notion of an *abelian scheme.* Given a base scheme S, we want a scheme $A \to S$ that "represents" a family of abelian varieties. We formalize this by imposing that, for every $s \in S$, the fiber $A|_s$ must be an abelian variety. So we have an abelian variety for each point of S.

Definition. An **abelian scheme** A over S is a smooth group scheme over S whose fibers are abelian varieties. Equivalently, it is a proper smooth group scheme over S whose fibers are geometrically connected.

It can be proven that A is a commutative group scheme over S. This gives us for each $n \in \mathbb{Z}$ a multiplication-by-n morphism $[n]\colon A \to A$ that restricts to multiplication-by-n in each fiber. Besides, if S is normal, then A is projective over S. We refer the reader to [24] and to [19, Chapter 6].

Now that we defined abelian schemes, we can understand better the main idea of [14]: given an abelian scheme A, we obtain equidistribution results for subvarieties $X \subseteq A$ which are *non-degenerate*. More precisely, we will prove that a *generic* sequence on X whose *heights* go to zero satisfies the equidistribution property for some choice of μ.

In the same way that the Bogomolov conjecture follows from classical equidistribution results over abelian varieties, from this equidistribution version over abelian schemes we obtain a version of the Bogomolov conjecture that gives us information about a whole family of abelian varieties simultaneously. This is what we call the *uniform Bogomolov conjecture*. As we would expect, this implies a *uniform Manin-Mumford conjecture*. We will go through the details in the next section.

1. Main Results of the Paper

Let S be a smooth, irreducible algebraic variety over a number field $K, \pi : A \to S$ an abelian scheme over S, $\iota : A \hookrightarrow \mathbb{P}^N_K$ an immersion. Moreover, let $n \geq 2$ be a fixed integer. For each irreducible, projective K-subvariety $X \subseteq A$ and every integer $k \geq 0$, we write $\overline{X}_k$ for the Zariski-closure of $\iota([n^k](X))$ in $\mathbb{P}^N_K$. We consider furthermore the graph $\Gamma_k \subset A \times A$ of $\left[n^k\right]\big|_X : X \to \left[n^k\right](X)$ and the Zariski closure $\overline{Y}_k$ of $(\iota \times \iota)(\Gamma_k)$ in $\mathbb{P}^N_K \times \mathbb{P}^N_K$. Notice that the projection on the first coordinate $\mathbb{P}^N_K \times \mathbb{P}^N_K \to \mathbb{P}^N_K$ induces a surjective, birational map $\phi_1 : \overline{Y}_k \to \overline{X}_0$. Similarly, the projection on the second coordinate induces a surjective map $\phi_2 : \overline{Y}_k \to \overline{X}_k$. We denote by $\mathscr{X}$ and $\mathscr{Y}_k$ the Zariski closures of X and $\overline{Y}_k$ in $\mathbb{P}^N_{\mathcal{O}_K}$. Notice that $\mathscr{X}$ is a flat, integral, and projective $\mathcal{O}_K$-scheme that is a *model* for X, that is, $X = \mathscr{X} \times_S K$. Besides, we call d the dimension of X. Observe that $\overline{X}_k$ and $\overline{Y}_k$ also have dimension d, and $\mathscr{X}$ has relative dimension d.

Furthermore, the notation pr_i will be universally used to denote the projection of a product to its i-th factor. Its meaning shall not cause confusion, since it will be clear in each context. We will also use the following notation: consider projections $\mathrm{pr}_i\colon \mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \to \mathbb{P}^{r_i}$. We write $\mathcal{O}(1,1)$ for the line bundle $\mathrm{pr}_1^* \mathcal{O}(1) \otimes \mathrm{pr}_2^* \mathcal{O}(1)$ on $\mathbb{P}^{r_1} \times \mathbb{P}^{r_2}$.

We will use this setup and these notations throughout the paper unless explicitly stated.

1.1. Equidistribution in Families of Abelian Varieties

In order to get equidistribution results, we have to take a sequence of points of X that is "generic" enough. We formalize this notion as follows:

Definition. Let $X \subseteq A$ be a subvariety. A sequence (x_i) of closed points of X is called X**-generic** if none of its subsequences is contained in a proper algebraic subvariety of X. If X is clear by the context, we call this sequence **generic** for short.

The next thing we have to do is to define a height on A. As A is an abelian scheme, for any $s \in S$ the fiber $A|_s$ is an abelian variety, which is stable by $[n]$. So we get a canonical height $\hat{h}_{A|_s,\iota^*\mathcal{O}(1)|_s}$ by setting, for each $x \in A|_s$:

$$\hat{h}_{A|_s,\iota^*\mathcal{O}(1)|_s}(x) \coloneqq \lim_{k\to\infty}\left(\frac{h_{A|_s,\iota^*\mathcal{O}(1)|_s}([n^k]x)}{n^{2k}}\right).$$

As $\iota^*\mathcal{O}(1)|_s$ is ample, this Néron-Tate height is non-negative.

Our goal is to generalize the notion of height to irreducible subvarieties of A. Let $\mathcal{X}$ be a flat, integral, and projective $\mathcal{O}_K$-scheme, and consider a $\mathscr{C}^\infty$-Hermitian line bundle[2] $\overline{\mathscr{L}}$ on $\mathcal{X}$. Then $\overline{\mathscr{L}}_K$ is a Hermitian line bundle on $\mathcal{X}_K$. Suppose $\overline{\mathscr{L}}_K$ is ample. Given an irreducible subvariety $Y \subseteq \mathcal{X}_K$ of dimension d', its Zariski closure $\mathcal{Y}$ in $\mathcal{X}$ is a flat, irreducible $\mathcal{O}_K$-scheme of relative dimension d' (see for example [18, Proposition 4.3.9 and Corollary 4.3.14]). Using the arithmetic intersection numbers and the definition of the degree of a subvariety[3], we can define the height:

$$h_{\overline{\mathscr{L}}}(Y) \coloneqq \frac{(\overline{\mathscr{L}}|_{\mathcal{Y}})^{d'+1}}{[K:\mathbb{Q}](d'+1)\deg_{\mathscr{L}_K}(Y)}.$$

Taking $\mathcal{X} = \mathbb{P}^N_{\mathcal{O}_K}$ we can define heights for subvarieties of $Y \subseteq \mathbb{P}^N_K$ with respect to $\overline{\mathcal{O}}(1)$ [4]. In particular, we can define $h_{\overline{\mathcal{O}}(1)}(y)$ for every closed point $y \in \mathbb{P}^N_K$, and the corresponding function $h_{\overline{\mathcal{O}}(1)}\colon \mathbb{P}^N_K(K) \to \mathbb{R}$ is a height in the usual sense.

We can also take $\mathcal{X} = \mathscr{X}$. In this case, given a subvariety $Y \subseteq X$, we can define its height with respect to a Hermitian line bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$. In the case of a closed point $x \in X$, we have $d' = 0$ and $\deg_{\mathscr{L}_K}(x) = [K(x) : K]$ (see Proposition B.1, item (a)). Hence we obtain

$$h_{\overline{\mathscr{L}}}(x) = \frac{\overline{\mathscr{L}}|_x}{[K(x):\mathbb{Q}]}.$$

The height of a subvariety is related to the heights of its points by the following inequalities:

Proposition 1.1. *(Zhang's Successive Minima) Let $Y \subseteq \mathcal{X}_K$ be a subvariety of dimension d', and let $\overline{\mathscr{L}}$ be a $\mathscr{C}^\infty$-Hermitian line bundle on Y. For $i = 1, \dots, d'$, define numbers:*

$$e_i \coloneqq \sup_{\substack{Z\subseteq Y \, subvariety \\ \operatorname{codim} Z = i}} \inf_{y\in Y\setminus Z} h_{\overline{\mathscr{L}}}(y).$$

Then $e_1 \geq e_2 \geq \cdots \geq e_{d'}$, and we have:

$$e_1 \geq h_{\overline{\mathscr{L}}}(Y) \geq \frac{e_1 + \cdots + e_{d'}}{d'}.$$

[2]See Appendix A.5.

[3]See appendix B.2.

[4]See appendix A.5 for the definition of $\overline{\mathcal{O}}(1)$.

(See [29, Theorem 1.10]). In particular, if $Y \subseteq \mathbb{P}^N_K$, we have $h_{\overline{\mathcal{O}}(1)}(Y) \geq 0$. Indeed, since for every $y \in Y$ we have $h_{\overline{\mathcal{O}}(1)}(y) = h(y) \geq 0$, we will also have $e_i \geq 0$ for each $1 \leq i \leq d'$, and hence $h_{\overline{\mathcal{O}}(1)}(Y) \geq 0$ by the proposition above.

Guided by the construction of the Néron-Tate height, we form the limit:

$$\hat{h}_\iota(X) := \limsup_{k\to\infty} \left(\frac{h_{\overline{\mathcal{O}}(1)}(\overline{X}_k)}{n^{2k}} \right) \in [0, \infty].$$

By the previous remark, this coincides with the Néron-Tate height for points $x \in A$, hence in this case the limit superior above is really a limit. Besides, for future use, notice the following easy consequence of the definition of $\hat{h}_\iota$:

Proposition 1.2. *Let $x \in A$ be a closed point, and let $k \geq 0$ be an integer. Then $\hat{h}_\iota(x) = n^{-2k}\hat{h}_\iota([n^k](x))$.*

Proof. We have:

$$\hat{h}_\iota([n^k](x)) = \lim_{j\to\infty} \left(\frac{h_{\overline{\mathcal{O}}(1)}([n^j][n^k](x))}{n^{2j}} \right) = n^{2k} \lim_{j\to\infty} \left(\frac{h_{\overline{\mathcal{O}}(1)}[n^{j+k}](x)}{n^{2(j+k)}} \right) = n^{2k}\hat{h}_\iota(x),$$

proving the statement. □

Writing $\overline{x}$ for the generic point of the closure of $x \in \mathscr{X}$, we have

$$h_{\overline{\mathscr{L}}}(x) = \frac{1}{[K(x) : \mathbb{Q}]} \left(\log \# \left(\mathscr{L}|_{\overline{x}} \,/\, (s|_{\overline{x}}) \right) - \sum_{\nu \in \Sigma_\infty(K)} \sum_{y \in O_\nu(x)} \delta_\nu \log \|s(y)\|_\nu \right) \tag{1}$$

for any non-zero rational section s of $\mathscr{L}_K$ such that $x \notin |\operatorname{div}(s)|$ (see [15, Section 2.7]).

We will also need the technical, but very important, concept of a non-degenerate variety. Fix an Archimedean place $\nu \in \Sigma_\infty(K)$ and a point $s \in S^{\mathrm{an}}_{\mathbb{C}_\nu}$. By [6, Proposition B.2] and the paragraph right after, there exist a connected open neighborhood $U \subseteq S^{\mathrm{an}}_{\mathbb{C}_\nu}$ of s and a real-analytic isomorphism

$$a \colon \pi^{-1}(U) \to (\mathbb{R} \,/\, \mathbb{Z})^{2\dim(A/S)} \times U$$

that restricts to a group homomorphism in the fiber over each point of U. Besides, the induced map

$$b = \mathrm{pr}_1 \circ a \colon \pi^{-1}(U) \to (\mathbb{R} \,/\, \mathbb{Z})^{2\dim(A/S)}$$

is unique up to post-composition with an automorphism of $(\mathbb{R} \,/\, \mathbb{Z})^{2\dim(A/S)}$. A map b like this is called a **Betti map**.

Definition. For an irreducible subvariety $X \subseteq A$ and a point $x \in (X^{\mathrm{sm}})^{\mathrm{an}}_{\mathbb{C}_\nu} \cap \pi^{-1}(U)$, we define $\mathrm{rank}_{\mathrm{Betti}}(X, x)$, the **Betti rank of X at** x, as the $\mathbb{R}$-dimension of

$$db(\mathrm{T}_{\mathbb{R},x}\, X_{\mathbb{C}_\nu^{\mathrm{an}}}) \subseteq \mathrm{T}_{b(x)}(\mathbb{R} \,/\, \mathbb{Z})^{2\dim(A/S)} = \mathbb{R}^{2\dim(A/S)}\,.$$

$X \subseteq A$ is called **non-degenerate** if there exists a point $x_0 \in (X^{\mathrm{sm}})^{\mathrm{an}}_{\mathbb{C}_\nu} \cap \pi^{-1}(U)$ such that $\mathrm{rank}_{\mathrm{Betti}}(X, x_0) = 2\dim X$. These definitions depend neither on the choice of U nor a.

We are mainly interested in the following result:

Relative equidistribution conjecture (REC). Let $X \subseteq A$ be a non-degenerate irreducible subvariety. For each place $\nu \in \Sigma_\infty(K)$, there exists a measure μ_ν on $X^{\mathrm{an}}_{\mathbb{C}_\nu}$ with the following property: If (x_i) is an X-generic sequence of points in X such that $\hat{h}_\iota(x_i) \to \hat{h}_\iota(X)$, then:

$$\frac{1}{\#O_\nu(x_i)} \sum_{x \in O_\nu(x_i)} f(x) \overset{i\to\infty}{\longrightarrow} \int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} f\, \mu_\nu, \tag{2}$$

for every compactly supported continuous function $f\colon X^{\mathrm{an}}_{\mathbb{C}_\nu} \to \mathbb{R}$.

It may be possible that (REC) is true by vacuity if $\hat{h}_\iota(X) > 0$, as not necessarily all such X will contain an X-generic sequence of points (x_i) such that $\hat{h}_\iota(x_i) \to \hat{h}_\iota(X)$. The main theorem of the paper shows the validity of (REC) in the case in which $\hat{h}_\iota(X) = 0$ and X contains an X-generic sequence (x_i) of points with $\hat{h}_\iota(x_i) \to 0$. Actually, only the second hypothesis is needed:

Theorem 1.3. *Let $X \subseteq A$ be a non-degenerate irreducible subvariety containing an X-generic sequence (x_i) of points with $\hat{h}_\iota(x_i) \to 0$. Then, we have:*

$$\hat{h}_\iota(X) = \lim_{k\to\infty} \left(\frac{h_{\overline{\mathcal{O}}(1)}(\overline{X}_k)}{n^{2k}} \right) = 0.$$

Furthermore, for each Archimedean place $\nu \in \Sigma_\infty(K)$, there exists a measure μ_ν on $X^{\mathrm{an}}_{\mathbb{C}_\nu}$ such that (2) holds for every continuous function $f\colon X^{\mathrm{an}}_{\mathbb{C}_\nu} \to \mathbb{R}$ with compact support.

The measures constructed in this theorem satisfy a very interesting property: outside a proper subvariety, every point is either "very equidistributed" or "of big height". Formally:

Theorem 1.4. *Let $X \subseteq A$ be a non-degenerate subvariety containing an X-generic sequence (x_i) such that $\widehat{h}_\iota(x_i) \to 0$. Fix an embedding $K \hookrightarrow \mathbb{C}$. The measure μ on $X(\mathbb{C})$ given by Theorem 1.3 has the following property: For each continuous function with compact support $f\colon X(\mathbb{C}) \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a proper subvariety $Z_{f,\varepsilon} \subseteq X$ and some $\delta_\varepsilon > 0$ such that for every closed point $x \in X \setminus Z_{f,\varepsilon}$, we have either*

$$\left| \frac{1}{\#O(x)} \sum_{z \in O(x)} f(z) - \int_{X(\mathbb{C})} f\, d\mu \right| < \varepsilon$$

or $\widehat{h}_\iota(x) \geq \delta_\varepsilon$.

Proof. The proof is by contradiction: suppose that there exists some compactly supported continuous $f\colon X(\mathbb{C}) \to \mathbb{R}$ and some $\varepsilon > 0$ such that, for all $\delta > 0$, the set $D_\delta \subseteq X$ consisting of closed points $x \in X$ such that $\widehat{h}_\iota(x) < \delta$ and $\left| \frac{1}{\#O(x)} \sum_{z\in O(x)} f(z) - \int_{X(\mathbb{C})} f\, d\mu \right| \geq \varepsilon$ is Zariski-dense in X. Notice that the closed subvarieties $Z \subsetneq X$ defined over $\overline{\mathbb{Q}}$ are countable since they are defined by a finite number of polynomials with coefficients in a countable set. Thus, we can order them in a sequence (Z_k).

For any positive integer i, since $D_{1/i}$ is Zariski-dense in X we can choose a point $x_i \in D_{1/i}$ that is outside the closed subvariety $\bigcup_{k=1}^{i} Z_k$. Then (x_i) is an X-generic sequence, and $\widehat{h}_\iota(x_i) \to 0$ by the definition of the $D_{1/i}$. But this violates Theorem 1.3, since $\left|\frac{1}{\#O(x_i)} \sum_{z \in O(x_i)} f(z) - \int_{X(\mathbb{C})} f\, d\mu\right| \geq \varepsilon$ for all i. $\square$

1.2. Reductions

We will prove Theorem 1.3 through the next two sections. However, we start by remarking that we can do some reductions in the hypothesis of the theorem. As in this theorem we only care about things happening in some fixed non-degenerate irreducible subvariety $X \subseteq A$, we can suppose, reducing S if necessary, that $\pi(X) = S$.

Next, we show that we can assume that the line bundle $\iota^*\mathcal{O}(1)$ on A is fiberwise symmetric, that is: for all closed point $s \in S$, $\iota^*\mathcal{O}(1)|_s \sim [-1]^*\iota^*\mathcal{O}(1)|_s$. Consider the composition of the immersion

$$\iota \times (\iota \circ [-1]) : A \hookrightarrow \mathbb{P}^N_K \times \mathbb{P}^N_K$$

with the Segre embedding $\mathcal{S}_{N,N} \colon \mathbb{P}^N_K \times \mathbb{P}^N_K \hookrightarrow \mathbb{P}^{(N+1)^2-1}_K$. We get a new projective immersion $\iota' \colon A \hookrightarrow \mathbb{P}^{(N+1)^2-1}_K$. Notice that:

$$(\iota')^*\mathcal{O}(1) = (\iota \times (\iota \circ [-1]))^* S^*_{N,N}\mathcal{O}(1) \sim (\iota \times (\iota \circ [-1]))^*\mathcal{O}(1,1) = \iota^*\mathcal{O}(1) \otimes [-1]^*\iota^*\mathcal{O}(1),$$

which is fiberwise symmetric. (we used Proposition 0.1, item (a) in the equivalence above). Now, let $x \in X$ be a closed point. Then the heights $\hat{h}_\iota(x)$ and $\hat{h}_{\iota'}(x)$ are the ordinary projective heights $h(\iota(x))$ and $h(\iota'(x))$. By Proposition 0.1, item (b), we then have:

$$\hat{h}_{\iota'}(x) = h(\iota'(x)) = h(S_{N,N}(\iota(x), \iota(-x))) = h(x) + h(-x) = \hat{h}_\iota(x) + \hat{h}_\iota(-x) = 2\hat{h}_\iota(x).$$

In particular, given an X-generic sequence (x_i) in X with $\hat{h}_\iota(x_i) \to 0$, we have:

$$\hat{h}_{\iota'}(x_i) = 2\hat{h}_\iota(x_i) \to 0,$$

so that we can replace ι with ι', in order to assume that $\iota^*\mathcal{O}(1)$ is fiberwise symmetric.

Our last reduction is to show we can assume that $\iota^*\mathcal{O}(1)$ is fiberwise of type $(\delta_1, \delta_2, \dots, \delta_g) \in \mathbb{Z}^g$ with $\delta_1 \geq 3$ (see [16, Section 3.1] for the definition of type). In order to do this, consider the composition ι' of the immersion $\iota : A \hookrightarrow \mathbb{P}^N_K$ with the Veronese embedding of degree 3: $\nu_3 \colon \mathbb{P}^N_K \hookrightarrow \mathbb{P}^{\binom{N+3}{N}-1}_K$. Notice that $(\iota')^*\mathcal{O}(1) = \iota^*\nu_3^*\mathcal{O}(1) \sim \iota^*\mathcal{O}(3)$, which is of the desired form since this corresponds to multiplying each δ_i by 3. Given a closed point $x \in X$, we have:

$$\hat{h}_{\iota'}(x) = h(\iota'(x)) = h(\nu(\iota(x))) = 3h(\iota(x)) = 3\hat{h}_\iota(x),$$

where the third equality above follows from Proposition 0.1, item (c). In particular, given an X-generic sequence (x_i) in X with $\hat{h}_\iota(x_i) \to 0$, we have:

$$\hat{h}_{\iota'}(x_i) = 3\hat{h}_\iota(x_i) \to 0,$$

so that we can replace ι with ι'.

These three extra suppositions will always be assumed to hold in the next two sections.

1.3. Uniformity Results

In this subsection, we will present uniform generalizations of the classical conjectures in Diophantine geometry stated in the introduction. More precisely, uniformity is about finding bounds that work for several curves simultaneously. In the case of the Manin-Mumford conjecture, it depends only on the genus:

Theorem 1.5. *(Uniform Manin-Mumford Conjecture) For each $g \geq 2$, there exists an integer $c(g) \geq 1$ such that the following assertion is true: For every algebraic curve C defined over $\mathbb{C}$ and every embedding $C \hookrightarrow J$ of this curve in its Jacobian, we have:*

$$\#(C(\mathbb{C}) \cap J(\mathbb{C})_{\mathrm{tors}}) \leq c(g).$$

This is a direct consequence of the following uniform version of Mordell-Lang:

Theorem 1.6. *(Uniform Mordell-Lang Conjecture) For each $g \geq 2$, there exists a constant $c(g) > 0$ such that the following assertion is true: For every algebraic curve C defined over $\mathbb{C}$, every embedding $C \hookrightarrow J$ of this curve in its Jacobian and every subgroup $\Gamma \subseteq J(\mathbb{C})$ of finite rank ρ, we have:*

$$\#(C(\mathbb{C}) \cap \Gamma) \leq c(g)^{1+\rho}.$$

Of course, this also gives us a uniform version of the Faltings's Theorem/Mordell Conjecture:

Theorem 1.7. *(Uniform Faltings's Theorem/Mordell Conjecture) For each $g \geq 2$, there exists a constant $c(g) > 0$ such that the following assertion is true: For every algebraic curve C defined over $\mathbb{C}$, every embedding $C \hookrightarrow J$ of this curve in its Jacobian and every number field K, we have:*

$$\#C(K) \leq c(g)^{1+\mathrm{rank}(J(K))}.$$

Theorem 1.6 can be proved using a combination of results in [6] and a uniform version of Bogomolov conjecture (see [14] for more details about this). As the classical Bogomolov conjecture follows from equidistribution results, we would expect the same to hold for the uniform version. Indeed, that is the case: it follows from Theorem 1.3.

In order to state the uniform Bogomolov conjecture, we will need some facts about moduli spaces. First of all, moduli spaces are generally constructed as quotients, and automorphisms of spaces are an obstacle in the construction of these quotients. That is why we sometimes endow the spaces we are interested in with *level structures*, which make the automorphism group smaller. For a definition of level structures, see [16, Section 8.3]. Similarly, we have the notion of a Jacobi structure for smooth algebraic curves. Besides, we call an abelian variety *principally polarized* if its polarization is of type $(1, \dots, 1)$.

Let $n \geq 3$. Then there is a smooth quasi-projective variety $\mathscr{A}_{g,n}$ that is a moduli space for principally polarized abelian varieties of dimension g with symplectic level n structure. We denote by $\pi_{g,n} \colon \mathscr{B}_{g,n} \to \mathscr{A}_{g,n}$ the universal family of abelian varieties over $\mathscr{A}_{g,n}$. We can also construct a smooth quasi-projective variety $\mathscr{M}_{g,n}$ which is a moduli space for smooth algebraic curves with Jacobi structure of level n (see [1, Section XVI.2] for a definition). We denote by $\pi'_{g,n} \colon \mathscr{C}_{g,n} \to \mathscr{M}_{g,n}$ the universal family over $\mathscr{M}_{g,n}$.

Associating to a curve its Jacobian induces a functor $\tau_{g,n}\colon \mathscr{M}_{g,n} \to \mathscr{A}_{g,n}$, which we call the *Torelli map with level n structure.* For every $s \in \mathscr{M}_{g,n}(\overline{\mathbb{Q}})$ and points p, q in the fiber $\mathscr{C}_{g,n,s}(\overline{\mathbb{Q}})$, the difference $q-p$ can be considered as a $\overline{\mathbb{Q}}$-point of the Jacobian $\mathrm{Jac}(\mathscr{C}_{g,n,s})$, which is canonically a subvariety of $\mathscr{B}_{g,n}$. By fixing an immersion $\iota\colon \mathscr{B}_{g,n} \to \mathbb{P}^N_{\mathbb{Q}}$, we can assign a height $\widehat{h}_\iota(q-p)$ to this difference. We can finally state:

Theorem 1.8. *(Uniform Bogomolov conjecture) Let $g \geq 2$ and $n \geq 3$ be integers and $\iota\colon \mathscr{B}_{g,n} \to \mathbb{P}^N_{\mathbb{Q}}$ be an immersion. There exist constants $\varepsilon(g,n), c(g,n) > 0$ for which the following assertion is true: For each $s \in \mathscr{M}_{g,n}(\overline{\mathbb{Q}})$ and all $p \in \mathscr{C}_{g,n,s}(\overline{\mathbb{Q}})$, we have:*

$$\#\{q \in \mathscr{C}_{g,n,s}(\overline{\mathbb{Q}})\colon \widehat{h}_\iota(q-p) \leq \varepsilon(g,n)\} < c(g,n).$$

The proof of this theorem from Theorem 1.3 is very long and technical. We refer the reader to [14, Section 4].

2. Theorem 1.3 – Convergence

In this section, we will prove the first part of Theorem 1.3, namely:

Theorem 2.1 (Theorem 1.3 - Convergence)**.** *Let $X \subseteq A$ be a non-degenerate irreducible subvariety containing an X-generic subsequence (x_i) of points with $\hat{h}_\iota(x_i) \to 0$. Then, we have:*

$$\hat{h}_\iota(X) = \lim_{k\to\infty} \left(\frac{h_{\overline{\mathcal{O}}(1)}(\overline{X}_k)}{n^{2k}} \right) = 0.$$

2.1. Proving the Result

The key to showing this result is to estimate how far the asymptotic "height" $\hat{h}_\iota$ is from the usual projective height. More precisely, we let $h_\iota(x) := h_{\overline{\mathcal{O}}(1)}(\iota(x))$ for each closed point $x \in A$. Choose an arbitrary (not necessarily open) immersion $\kappa : S \hookrightarrow \mathbb{P}^{N'}_K$ of the base S and set $h_\kappa(s) := h_{\overline{\mathcal{O}}(1)}(\kappa(s))$. The following two inequalities will be fundamental in what follows:

Lemma 2.2. *There exists a constant $c_1 = c_1(\pi, \iota, \kappa) > 0$ such that*

$$\left|\widehat{h}_\iota(x) - h_\iota(x)\right| \leq c_1 \cdot \max\{1, h_\kappa(\pi(x))\}$$

for every closed point $x \in A$.

The proof of Lemma 2.2, adapted from [6], will be done in the next subsection.

Lemma 2.3. *There exists a constant $c_2 = c_2(X, \iota, \kappa) > 1$ such that*

$$h_\kappa(\pi(x_i)) < c_2 \cdot \max\{1, \widehat{h}_\iota(x_i)\}$$

for all but finitely many $i \in \mathbb{N}$.

This lemma is a direct consequence of the following stronger result:

Proposition 2.4. *There exist constants $c > 0$ and $c' \geq 0$ and a Zariski open dense subset U of X with:*

$$\widehat{h}_\iota(x) \geq c h_\kappa(\pi(x)) - c',$$

for every closed point $x \in U$.

Indeed, this inequality is equivalent to $h_\kappa(\pi(x)) \leq \widehat{h}_\iota(x)/c + c'/c$. Choosing $c_2 > 1$ big enough we get $h_\kappa(\pi(x)) < c_2 \cdot \max\{1, \widehat{h}_\iota(x)\}$, for every $x \in U$. Taking into account that (x_i) is X-generic, this proves Lemma 2.3.

The proof of Proposition 2.4, however, is very involved. We restrict ourselves here with a brief overview and refer the reader to [6], where the proof of this result ([6, Proposition B.1]) occupies several sections. The first step to prove it is to reduce the problem through a series of dévissages, in order to assume that $A \to S$ carries a principal polarization and has level ℓ structure for some $\ell \geq 3$. This reduced problem is ([6, Theorem 1.6]), and can be deduced from ([6, Proposition 4.1]). This proposition, in turn, follows from the bounding of some intersection numbers. In the proof of this proposition, the non-degeneracy of X is essential.

The inequalities given by Lemma 2.2 and Lemma 2.3 are combined in the important next lemma:

Lemma 2.5. *For each integer $k \geq 0$, set $x_i^{(k)} := \iota\left(\left[n^k\right](x_i)\right)$ and*

$$\ell_k := \limsup_{i \to \infty} \left(\frac{h_{\overline{\mathcal{O}}(1)}\left(x_i^{(k)}\right)}{n^{2k}} \right).$$

Then, $\ell_k \in [0, \infty)$ for each integer $k \geq 0$. Furthermore, $\lim_{k\to\infty} \ell_k = 0$.

For later reference, we set $\ell := \sup_{k \geq 0} \{\ell_k\} \in [0, \infty)$.

Proof. By Proposition 1.2, we have $\widehat{h}_\iota(x_i) = n^{-2k} \widehat{h}_\iota\left(\left[n^k\right](x_i)\right)$ for all integers $i, k \geq 0$. Thus, for i large enough:

$$\begin{aligned} \left| \widehat{h}_\iota(x_i) - \frac{h_{\overline{\mathcal{O}}(1)}(x_i^{(k)})}{n^{2k}} \right| &= \frac{\left| \widehat{h}_\iota\left(\left[n^k\right](x_i)\right) - h_\iota\left(\left[n^k\right](x_i)\right) \right|}{n^{2k}} \\ &\leq \frac{c_1 \max\{1, h_\kappa(\pi(x_i))\}}{n^{2k}} \\ &\leq \frac{c_1 c_2 \max\left\{1, \widehat{h}_\iota(x_i)\right\}}{n^{2k}}, \end{aligned}$$

where the first inequality follows from Lemma 2.2 and the second one follows from Lemma 2.3 and $c_2 > 1$. By assumption, $\lim_{i\to\infty} \widehat{h}_\iota(x_i) = 0$. Hence each $\ell_k \in [0, \infty)$, and their limit when $k \to \infty$ is 0 since $n^{-2k} \to 0$. □

From now on, we denote $y_i^{(k)} := (\iota(x_i), x_i^{(k)}) = (\iota(x_i), \iota([n^k](x_i))) \in \overline{Y}_k$. As (x_i) is X-generic, we see that each $(y_i^{(k)})$ is $\overline{Y}_k$-generic. Finally, we have the following lemma:

Lemma 2.6. *For all integers $k \geq 0$, we have:*

$$0 \leq h_{\overline{\mathcal{O}}(1)}\left(\overline{X}_k\right) \leq n^{2k}\ell_k$$

Besides, for any integers $k, k_1, k_2 \geq 0$, we have[5]*:*

$$0 \leq h_{\overline{\mathcal{O}}(k_1,k_2)}\left(\overline{Y}_k\right) \leq k_1\ell_0 + k_2 n^{2k}\ell_k$$

Proof. The inequality $h_{\overline{\mathcal{O}}(1)}(\overline{X}_k) \geq 0$ follows from Proposition 1.1, and the paragraph right after it. On the other hand, by the same proposition, we have $h_{\overline{\mathcal{O}}(1)}(\overline{X}_k) \leq e_1$, where

$$e_1 \coloneqq \sup_{\substack{Z \subseteq \overline{X}_k \text{ subvariety} \\ \operatorname{codim} Z = 1}} \inf_{x \in \overline{X}_k \setminus Z} h_{\overline{\mathcal{O}}(1)}(x).$$

Since $(x_i^{(k)})$ is $\overline{X}_k$-generic, each subvariety $Z \subseteq \overline{X}_k$ of codimension 1 contains only a finite number of the $x_i^{(k)}$. Besides, by Lemma 2.5, for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $i \geq i_0 \Rightarrow h_{\overline{\mathcal{O}}(1)}(x_i^{(k)}) \leq n^{2k}\ell_k - \varepsilon$. We conclude that $\inf_{x \in \overline{X}_k \setminus Z} h_{\overline{\mathcal{O}}(1)}(x) \leq n^{2k}\ell_k$ for any Z, thus $h_{\overline{\mathcal{O}}}(\overline{X}_k) \leq e_1 \leq n^{2k}\ell_k$, as we wanted.

Now, notice that:

$$\begin{aligned} h_{\overline{\mathcal{O}}(k_1,k_2)}(y_i^{(k)}) &= h_{\mathrm{pr}_1^*\overline{\mathcal{O}}(k_1)\otimes \mathrm{pr}_2^*\overline{\mathcal{O}}(k_2)}(\iota(x_i), x_i^{(k)}) \\ &= h_{\mathrm{pr}_1^*\overline{\mathcal{O}}(k_1)}(x_i^{(0)}, x_i^{(k)}) + h_{\mathrm{pr}_2^*\overline{\mathcal{O}}(k_2)}(x_i^{(0)}, x_i^{(k)}) \\ &= h_{\overline{\mathcal{O}}(k_1)}(x_i^{(0)}) + h_{\overline{\mathcal{O}}(k_2)}(x_i^{(k)}) \\ &= k_1 h_{\overline{\mathcal{O}}(1)}(x_i^{(0)}) + k_2 h_{\overline{\mathcal{O}}(1)}(x_i^{(k)}), \end{aligned}$$

where we used the functoriality and the additivity of the Weil height (items (b) and (c) of Proposition 0.2). We conclude that the inequalities for $h_{\overline{\mathcal{O}}(k_1,k_2)}(\overline{Y}_k)$ follow from the ones already obtained, using that $(y_i^{(k)})$ is $\overline{Y}_k$-generic and applying again Proposition 1.1. $\square$

By Lemma 2.6, we then have, for all integers $k \geq 0$:

$$0 \leq \frac{h_{\overline{\mathcal{O}}(1)}(\overline{X}_k)}{n^{2k}} \leq \ell_k.$$

By Lemma 2.5, on the other hand, we have $\lim_{k\to\infty} \ell_k = 0$. Hence, we conclude that:

$$\widehat{h}_\iota(X) = \lim_{k\to\infty}\left(\frac{h_{\overline{\mathcal{O}}(1)}\left(\overline{X}_k\right)}{n^{2k}}\right) = 0,$$

as we wanted.

[5] See Appendix A.5 for the definition of $\overline{\mathcal{O}}\,(k_1, k_2)$.

2.2. Proof of Lemma 2.2

We start by stating the following classical result about abelian varieties:

Proposition 2.7 (The Seesaw Theorem)**.** *Let $f\colon V \to S$ be a proper flat morphism with geometrically integral fibers. If $L \in \operatorname{Pic}(V)$ is a line bundle on V, then:*
(1) the set $Z \coloneqq \{s \in S\colon L_s \text{ is trivial}\}$ is closed in S;
(2) $L_{Z_{\mathrm{red}}}$ is pulled back from Z_{red}, where Z_{red} denotes the unique reduced subscheme structure on Z.

In order to prove Lemma 2.2, we will first show the following result:

Proposition 2.8. *There exists a constant $c > 0$ such that, for every closed point $x \in A$, we have the inequality:*

$$|h_{\overline{\mathcal{O}}(1)}([n](x)) - n^2 h_{\overline{\mathcal{O}}(1)}(x)| \leq c \max\{1, h_\kappa(\pi(x))\}.$$

Proof. Let $L \coloneqq \iota^*\mathcal{O}(1)$ and assume this line bundle is fiberwise symmetric (remember we can restrict ourselves to this case). Then, by Mumford's formula (Proposition 0.4), for each $s \in S$ closed we have $[n]^*L|_{\pi^{-1}(s)} \sim L|_{\pi^{-1}(s)}^{\otimes n^2}$. So $([n]^*L) \otimes L^{\otimes -n^2}$ is equivalent to the trivial line bundle when restricted to each $\pi^{-1}(s)$. Proposition 2.7 then implies that $([n]^*L) \otimes L^{\otimes -n^2}$ is the pullback of a line bundle M of S: $([n]^*L) \otimes L^{\otimes -n^2} = \pi^*M$. Let $x \in A$ be a closed point and define $s \coloneqq \pi(x)$. As the trivial line bundle $\mathcal{O}_A$ is effective and base-point free, by the positivity of Weil's Height Machine (Proposition 0.2, item (e)), we have $h_{A|_s,\mathcal{O}_A|_s}(x) \geq O(1)$. Now, $\mathcal{O}_A = ([n]^*L)^{\otimes -1} \otimes L^{\otimes n^2} \otimes \pi^*M$, and using the additivity of Weil's Height Machine (Proposition 0.2, item (c)), we obtain:

$$O(1) \leq -h_{A|_s,[n]^*L|_s}(x) + n^2 h_{A|_s,L|_s}(x) + h_{A|_s,\pi^*M}(x).$$

For some choice of the heights above, we have $h_{A|_s,[n]^*L|_s}(x) = h_{\overline{\mathcal{O}}(1)}([n](x))$ and $h_{A|_s,L|_s}(x) = h_{\overline{\mathcal{O}}(1)}(x)$. Now, using the functoriality of Weil's Height Machine (Proposition 0.2, item (b)), we have $h_{A|_s,\pi^*M}(x) = h_{S,M}(\pi(x))$. Hence we obtain:

$$h_{\overline{\mathcal{O}}(1)}([n](x)) - n^2 h_{\overline{\mathcal{O}}(1)}(x) \leq O(1) + h_{S,M}(\pi(x)).$$

Let $H \coloneqq \kappa^*\mathcal{O}(1)$ be the hyperplane line bundle on S. As H is ample, there exists some positive integer m such that $H^{\otimes m} \otimes M^{\otimes -1}$ is effective and base-point free (see for example [10, Theorem A.3.2.3]). So, again using the positivity, the additivity and the functoriality of Weil's Height Machine, we get:

$$O(1) \leq h_{S,H^{\otimes m} \otimes M^{\otimes -1}} = m \cdot h_{S,H} - h_{S,M} = h_\kappa - h_{S,M}.$$

Therefore $h_{S,M}(\pi(x)) \leq h_\kappa(\pi(x)) + O(1)$, and thus that there exists a constant $c > 0$ such that:

$$h_{\overline{\mathcal{O}}(1)}([n](x)) - n^2 h_{\overline{\mathcal{O}}(1)}(x) \leq c \cdot \max\{1, h_\kappa(\pi(x))\}.$$

As we also have $\mathcal{O}_A = ([n]^*L) \otimes L^{\otimes -n^2} \otimes \pi^*M^{\otimes -1}$, we can proceed in the same fashion to get (adjusting c if needed):

$$h_{\overline{\mathcal{O}}(1)}([n](x)) - n^2 h_{\overline{\mathcal{O}}(1)}(x) \geq -c \cdot \max\{1, h_\kappa(\pi(x))\}.$$

The two inequalities obtained above combine to give the desired bound. $\square$

With the previous proposition at hand, Lemma 2.2 follows from a classical argument:

Proof of Lemma 2.2. Fix a closed point $x \in A$, and let c be the constant of the proposition above. For every $k \geq 0$, we can apply the triangle inequality to the appropriate telescopic sum to obtain:

$$\begin{aligned}\left|\frac{h_{\overline{\mathcal{O}}(1)}([n^k](x))}{n^{2k}} - h_{\overline{\mathcal{O}}(1)}(x)\right| &\leq \sum_{j=0}^{k-1}\left|\frac{h_{\overline{\mathcal{O}}(1)}([n^{j+1}](x))}{n^{2(j+1)}} - \frac{h_{\overline{\mathcal{O}}(1)}([n^j](x))}{n^{2j}}\right| \\ &= \sum_{j=0}^{k-1} n^{-2(j+1)}\left|h_{\overline{\mathcal{O}}(1)}([n^{j+1}](x)) - n^2 h_{\overline{\mathcal{O}}(1)}([n^j](x))\right|.\end{aligned}$$

Now, applying the proposition above to each j, we get:

$$\left|h_{\overline{\mathcal{O}}(1)}([n^{j+1}](x)) - n^2 h_{\overline{\mathcal{O}}(1)}([n^j](x))\right| \leq c \cdot \max\{1, h_\kappa(\pi([n^j]x))\} = c \cdot \max\{1, h_\kappa(\pi(x))\}$$

(notice that $\pi([n^j](x)) = \pi(x)$ since $[n]$ preserves fibers). So the sum above is bounded by

$$c \cdot \max\{1, h_\kappa(\pi(x))\} \cdot \sum_{j=0}^{k-1} n^{-2(j+1)} \leq c \cdot \max\{1, h_\kappa(x)\} \cdot \sum_{j=0}^{\infty} n^{-2(j+1)} = \frac{c}{n^2-1} \cdot \max\{1, h_\kappa(\pi(x))\}.$$

Let $c_1 := c/(n^2-1)$. Then, for every $k \geq 0$, the following inequality holds:

$$\left|\frac{h_{\overline{\mathcal{O}}(1)}([n^k](x))}{n^{2k}} - h_{\overline{\mathcal{O}}(1)}(x)\right| \leq c_1 \cdot \max\{1, h_\kappa(\pi(x))\}.$$

Letting k go to infinity, we obtain $|\widehat{h}_\iota(x) - h_\iota(x)| \leq c_1 \cdot \max\{1, h_\kappa(\pi(x))\}$, proving the lemma. $\square$

3. Theorem 1.3 – Equidistribution

Now, we tackle the equidistribution part of Theorem 1.3.

3.1. The Equilibrium Measure

In this subsection, we start the preparation for the desired proof with some lemmas. The central result here is Lemma 3.3, which describes - up to a proportionality factor that will remain mysterious until the very end - the equilibrium measure μ_ν. The setup of this subsection is a bit different from the general setup defined in the first section. Here, S is a smooth complex variety S, $\pi : A \to S$ is a family of complex abelian varieties of relative dimension g, $X \subseteq A$ is an irreducible complex subvariety of dimension d and $\iota : A \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ is an immersion. The rest of the notations, as $\overline{X}_k$ and $\overline{Y}_k$ for example, are defined analogously, changing K by $\mathbb{C}$. Besides, for each complex-analytic space Z, we will denote its associated complex algebraic variety $Z^{\text{an}}_{\mathbb{C}}$ simply by $Z(\mathbb{C})$.

Let ω_{FS} be the Chern form of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^N_{\mathbb{C}}$ equipped with the Fubini-Study metric[6]. For each integer $k \geq 1$, we consider the $(1,1)$-form:

$$\alpha_k := \frac{(\iota \circ [n^k])^* \omega_{\mathrm{FS}}}{n^{2k}}.$$

Then it holds that $\alpha_{k+l} = ([n^l]^* \alpha_k)/n^{2l}$. Indeed:

$$\begin{aligned} \frac{[n^l]^* \alpha_k}{n^{2l}} = \frac{[n^l]^* \left(\frac{(\iota \circ [n^k])^* \omega_{\mathrm{FS}}}{n^{2k}} \right)}{n^{2l}} &= \frac{[n^l]^* (\iota \circ [n^k])^* \omega_{\mathrm{FS}}}{n^{2k+2l}} = \frac{(\iota \circ [n^k] \circ [n^l])^* \omega_{\mathrm{FS}}}{n^{2k+2l}} \\ &= \frac{(\iota \circ [n^{k+l}])^* \omega_{\mathrm{FS}}}{n^{2(k+l)}} = \alpha_{k+l}. \end{aligned}$$

As the α_k's are, up to a constant, pullbacks of the positive form ω_{FS}, they are semipositive. Besides, as $\alpha_0 = \iota^* \omega_{\mathrm{FS}}$ is the pullback of ω_{FS} by the immersion ι, it is positive.

Guided by our previous reductions in the hypothesis of Theorem 1.3, we assume throughout this subsection:

(1) $\iota^* \mathcal{O}(1)$ is fiberwise symmetric (i.e., $\iota^* \mathcal{O}(1)|_s$ is a symmetric line bundle on $A|_s$ for every $s \in S(\mathbb{C})$),
(2) $\iota^* \mathcal{O}(1)$ is fiberwise of type $\Delta = (\delta_1, \delta_2, \ldots, \delta_g) \in \mathbb{Z}^g$ (i.e., $\iota^* \mathcal{O}(1)|_s$ is of type Δ for some, and hence for every, point $s \in S(\mathbb{C})$).

During this section, we will need to consider moduli spaces of families of abelian varieties. We remember here the facts we will use. The **Siegel upper half-space** is define by:

$$\mathscr{H}_g := \{\tau \in \mathbb{C}^{g \times g} \colon \tau = \tau^t \text{ and } \operatorname{Im}(\tau) \in \mathbb{R}^{g \times g} \text{ is positive-definite}\}.$$

It is a moduli space for polarized abelian varieties of type Δ with a symplectic basis. We will use the notation $(z, \tau) = ((z_1, \ldots, z_g)^t, (\tau_{ij}))$ for the standard coordinates of $\mathbb{C}^g \times \mathscr{H}_g$. Fixed $\tau \in \mathscr{H}_g$, let $j_\tau \colon \mathbb{R}^{2g} \to \mathbb{C}^g$ be the map $j_\tau(x) := (\tau, I_g) \cdot x$. Then j_τ is an isomorphism of $\mathbb{R}$-vector spaces. Define a lattice Λ_Δ on $\mathbb{R}^{2g}$ by:

$$\Lambda_\Delta := \begin{pmatrix} I_g & 0_{g \times g} \\ 0_{g \times g} & \operatorname{diag}(\Delta) \end{pmatrix} \cdot \mathbb{Z}^{2g}.$$

Then $j_\tau(\Lambda_\Delta) = (\tau, \operatorname{diag}(\Delta)) \cdot \mathbb{Z}^{2g}$. The lattice Λ_Δ acts freely and properly discontinuously on $\mathbb{C}^g \times \mathscr{H}_g$ by $\lambda \cdot (z, \tau) := (z + j_\tau(\lambda), \tau)$. Therefore, the quotient $\mathscr{A}_{g,\Delta} := (\mathbb{C}^g \times \mathscr{H}_g)/\Lambda_\Delta$ with respect to this action is a complex manifold. The projection $\mathrm{pr}_2 \colon \mathbb{C}^g \times \mathscr{H}_g \to \mathscr{H}_g$ induces a projection $\pi_{g,\Delta} \colon \mathscr{A}_{g,\Delta} \to \mathscr{H}_g$. This is called the *universal family of complex abelian varieties of type* Δ, as every family of complex abelian varieties of type Δ will be locally the pullback of this family by a holomorphic function.

Following [16, Section 8.7], we can define an holomorphic line bundle $L_{g,\Delta}$ on the quotient $\mathscr{A}_{g,\Delta}$. To each $Z \in \mathscr{H}_g$, let $\mathcal{A}_Z$ and H_Z be the associated abelian variety and its polarization.

See [16, Chapter 8] (especially sections 8.1 and 8.7) for the proofs of these results.

Proposition 3.1. *Denote* $(L, \|\cdot\|) := \iota^*(\mathcal{O}(1), \|\cdot\|_{\mathrm{FS}})$. *Fix* $s \in S(\mathbb{C})$. *Then there exists a Zariski neighborhood* $V \subseteq S$ *of* s *such that there are isomorphisms*

$$\psi_k \colon L|_{\pi^{-1}(V)}^{\otimes n^{2k}} \to [n^k]^* L|_{\pi^{-1}(V)},$$

[6]See appendix A.4.

for $k \geq 1$.

Proof. By hypothesis, $L|_s$ is symmetric, that is, $[-1]^*L|_{\pi^{-1}(s)} \sim L|_{\pi^{-1}(s)}$. Hence, by Proposition 0.4 (Mumford's formula), we have $[n]^*L|_{\pi^{-1}(s)} \sim L|_{\pi^{-1}(s)}^{\otimes n^2}$. So $L^{\otimes n^2} \otimes ([n]^*L)^{\otimes -1}$ is equivalent to the trivial line bundle when restricted to each $\pi^{-1}(s)$. Proposition 2.7 then implies that $L^{\otimes n^2} \otimes [n]^*L^{\otimes -1}$ is the pullback of a line bundle M of S: $L^{\otimes n^2} \otimes [n]^*L^{\otimes -1} = \pi^*M$. There exists a neighbourhood $V \subseteq S$ of s that trivializes M: $M|_V \cong \mathcal{O}_V$. Taking the pullback by π, we get $L|_{\pi^{-1}(V)}^{\otimes n^2} \otimes [n]^*L|_{\pi^{-1}(V)}^{\otimes -1} \cong \mathcal{O}_{\pi^{-1}(V)}$, hence an isomorphism $\psi_1\colon L|_{\pi^{-1}(V)}^{\otimes n^2} \to [n]^*L|_{\pi^{-1}(V)}$. For each integer $k \geq 1$, this induces further, by recurrence, an isomorphism $\psi_k : L|_{\pi^{-1}(V)}^{\otimes n^{2k}} \to \left[n^k\right]^* L\Big|_{\pi^{-1}(V)}$, given by the composition:

$$\begin{aligned} L|_{\pi^{-1}(V)}^{\otimes n^{2k}} = (L|_{\pi^{-1}(V)}^{\otimes n^2})^{\otimes n^{2(k-1)}} \xrightarrow{\psi_1^{\otimes n^{2(k-1)}}} &\quad ([n]^*L|_{\pi^{-1}(V)})^{\otimes n^{2(k-1)}} \\ = &\quad [n]^*L|_{\pi^{-1}(V)}^{\otimes n^{2(k-1)}} \xrightarrow{[n]^*\psi_{k-1}} [n^k]^*L|_{\pi^{-1}(V)}. \end{aligned}$$

That is, for every s meromorphic section of L over some open subset of $\pi^{-1}(V)$, we have $\psi_k(s^{\otimes n^{2k}}) = [n]^*\psi_{k-1}(\psi_1(s^{\otimes n^2})^{\otimes n^{2(k-1)}})$. ☐

We will also need the following result:

Proposition 3.2. *There exists a semipositive, smooth, closed (1, 1)-form* ω^{univ} *on* $\mathscr{A}_{g,\Delta}$*, called the* ***universal Betti form****, such that, for all* $m \in \mathbb{Z}$*, we have* $[m]^*\omega = m^2 \cdot \omega$*. Besides, its construction is as follows:*

- *Define the form* $\widehat{\omega}^{\text{univ}} := dd^c(2\pi \cdot \mathrm{Im}(z)^t \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z))$ *on* $\mathbb{C}^g \times \mathscr{H}_g$*. Then this smooth* $(1,1)$*-form is closed, semipositive, satisfies* $[m]^*\widehat{\omega}^{\text{univ}} = m^2 \cdot \widehat{\omega}^{\text{univ}}$ *for all* $m \in \mathbb{Z}$*, and is given by the equation:*

$$\widehat{\omega}^{\text{univ}} = i(dz - d\tau \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z))^t \wedge \mathrm{Im}(\tau)^{-1}(d\overline{z} - d\overline{\tau} \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z)),$$

  where $\wedge$ *denotes a combination of wedge product and matrix multiplication.*
- *The form* $\widehat{\omega}^{\text{univ}}$ *is invariant by the action of* Λ_Δ*, hence descends to* ω^{univ} *on* $\mathscr{A}_{g,\Delta}$ *by the quotient map* $\mathbb{C}^g \times \mathscr{H}_g \to \mathscr{A}_{g,\Delta}$*.*

Proof. See [6, Section 2]. ☐

Lemma 3.3. *There exists a semipositive, smooth, closed* $(1,1)$*-form* β *on* $A(\mathbb{C})$ *such that, for every subvariety* $X \subseteq A$ *of dimension* d *and every compactly supported continuous function* $f : X(\mathbb{C}) \to \mathbb{R}$*, we have:*

$$\lim_{k\to\infty} \int_{X(\mathbb{C})} f\alpha_k^{\wedge d} = \int_{X(\mathbb{C})} f\beta^{\wedge d}.$$

Furthermore, we have $[m]^*\beta = m^2 \cdot \beta$ *for all integers* m*.*

As we will see after, the measure $\beta^{\wedge d}$ on $X(\mathbb{C})$ is proportional to the equilibrium measure μ_ν of Theorem 1.3 for a non-degenerate subvariety X.

Proof. Denote $(L, \|\cdot\|) := \iota^*(\mathcal{O}(1), \|\cdot\|_{\mathrm{FS}})$. Fix $x_0 \in X$. By Proposition 3.1, there is a neighborhood $V \subseteq S$ of $\pi(x_0)$ such that we have an isomorphism $\psi_k \colon L|_{\pi^{-1}(V)}^{\otimes n^{2k}} \to [n^k]^* L|_{\pi^{-1}(V)}$ for each $k \geq 1$.

The Hermitian metric $\|\cdot\| \colon L(\mathbb{C}) \to \mathbb{R}_+$ induces, by pullback, a Hermitian norm

$$[n^k]^* \|\cdot\| \colon ([n^k]^* L)(\mathbb{C}) \to \mathbb{R}_+ .$$

Using the isomorphisms ψ_k, for each k we obtain a smooth Hermitian metric $\|\cdot\|_k : L(\mathbb{C})|_{\pi^{-1}(V)} \to \mathbb{R}_+$ by demanding

$$\|s\|_k^{n^{2k}} := [n^k]^* \left\| \psi_k(s^{\otimes n^{2k}}) \right\|$$

for any meromorphic section s of $L(\mathbb{C})|_{\pi^{-1}(V)}$. This metric is indeed Hermitian since it is derived from the Fubini-Study one.

Let $K \subset V(\mathbb{C})$ be a compact subset containing $\pi(x_0)$. We will now show that the metrics $\|\cdot\|_k$ converge uniformly on $\pi^{-1}(K)$ to a Hermitian metric $\|\cdot\|_\infty$ as $k \to \infty$. For convenience, we will consider this sequence starts with $\|\cdot\|_0 := \|\cdot\|$. We start by showing that the metrics $\|\cdot\|_k$ can be defined by recurrence by $\|\cdot\|_k = ((\cdot)^{\otimes n^2})^* \psi_1^* [n]^* \|\cdot\|_{k-1}^{1/n^2}$. Indeed, let s be any meromorphic section of L on some open subset if $\pi^{-1}(V)$. Then:

$$\begin{aligned} ([n]^* \|\psi_1(s^{\otimes n^2})\|_{k-1}^{1/n^2})^{n^{2k}} &= [n]^* [n^{k-1}]^* \| [n]^* \psi_{k-1}(\psi_1(s^{\otimes n^2})^{\otimes n^{2(k-1)}}) \| \\ &= [n^k]^* \|\psi_k(s^{\otimes n^{2k}})\| = \|s\|_k^{n^{2k}}, \end{aligned}$$

as we wanted. For simplicity, we will denote $\psi := \psi_1 \circ (\cdot)^{\otimes n^2}$, so that $\|\cdot\|_k = \psi^* [n]^* \|\cdot\|_{k-1}^{1/n^2}$. Define the continuous function $h := \log \frac{\|\cdot\|_1}{\|\cdot\|}$. Now, for any $k \geq 1$:

$$\begin{aligned} \|\cdot\|_k = \psi^*[n]^* \|\cdot\|_{k-1}^{1/n^2} \Rightarrow \log \|\cdot\|_k &= \frac{1}{n^2} \psi^* [n]^* \log \|\cdot\|_{k-1} \\ &= \left(\frac{1}{n^2} \psi^* [n]^* \right)^2 \log \|\cdot\|_{k-2} \\ &\vdots \\ &= \left(\frac{1}{n^2} \psi^* [n]^* \right)^{k-1} \log \|\cdot\|_1 \\ &= \left(\frac{1}{n^2} \psi^* [n]^* \right)^{k-1} (h + \log \|\cdot\|) \\ &= \left(\frac{1}{n^2} \psi^* [n]^* \right)^{k-1} h + \left(\frac{1}{n^2} \psi^* [n]^* \right)^{k-1} \log \|\cdot\| \\ &= \left(\frac{1}{n^2} \psi^* [n]^* \right)^{k-1} h + \log \|\cdot\|_{k-1}, \end{aligned}$$

where the equality in the last line follows from the same recurrence procedure as the beginning of the calculation above. Using induction once more, we obtain:

$$\log \|\cdot\|_k = \log \|\cdot\| + \sum_{j=0}^{k-1} \left(\frac{1}{n^2}\psi^*[n]^*\right)^j h.$$

As h is continuous and we are working over a compact set K, $\|h\|_{\sup} < \infty$. Furthermore, we have $\left\|\left(\frac{1}{n^2}\psi^*[n]^*\right)^j\right\|_{\sup} \leq \frac{1}{n^{2j}}\|h\|_{\sup}$, so we conclude that $\sum_{j=0}^{k-1}\left(\frac{1}{n^2}\psi^*[n]^*\right)^j h$ converges absolutely and uniformly to a bounded continuous function h_0. Define $\|\cdot\|_\infty := e^{h_0}\|\cdot\|$. Then it is easy to see that $\|\cdot\|_\infty$ is a metric, and that $\|\cdot\|_k \to \|\cdot\|_\infty$ by the definition of convergence of metrics on line bundles. From this convergence and the fact that $\|\cdot\|_k = \psi^*[n]^*\|\cdot\|_{k-1}^{1/n^2}$, we get the relation $\|\cdot\|_\infty = \psi^*[n]^*\|\cdot\|_\infty^{1/n^2}$.

Let $U \subset X(\mathbb{C})$ be an open (euclidean) neighborhood of x satisfying $\overline{U} \subset \pi^{-1}(K)$. We choose U small enough in order to assure the existence of a local frame s of $L(\mathbb{C})$ over U. Notice that:

$$n^{2k}\alpha_k = (\iota \circ [n^k])^* c_1(||.||_{\mathrm{FS}}) = c_1([n^k]^*\iota^*\|\cdot\|_{\mathrm{FS}}) = c_1([n^k]^*\|\cdot\|).$$

As s is a local frame of L over U and ψ_k is an isomorphism, $\psi_k(s^{\otimes n^2})$ is a local frame of $[n^k]^*L$ over U. Hence we have:

$$dd^c \log \|s\|_k^{-1} = n^{-2k} dd^c \log \left(\left[n^k\right]^* \left\|\psi_k(s^{\otimes n^2})\right\|\right)^{-1} = n^{-2k} c_1([n^k]^*\|\cdot\|)|_U = \alpha_k|_U$$

(remember the local expression for the Chern form of a line bundle given in appendix A.4)). We define $\beta|_U := dd^c \log \|s\|_\infty^{-1}$. This is a $(1,1)$-current on U, and as $\|\cdot\|_k \to \|\cdot\|_\infty$, for any continuous function $f : X(\mathbb{C}) \to \mathbb{R}$ with compact support in U, we have:

$$\int_{X(\mathbb{C})} f\alpha_k^{\wedge d} \xrightarrow{k\to\infty} \int_{X(\mathbb{C})} f\left(\beta|_U\right)^{\wedge d}$$

(this follows, for example, from [5, Corollary 2.6]). Now, by construction, the currents $\beta|_U$ glue to a $(1,1)$-current β on A, and using partitions of unity is then easy to see that, for every compactly supported continuous function $f\colon X(\mathbb{C}) \to \mathbb{R}$:

$$\int_{X(\mathbb{C})} f\alpha_k^{\wedge d} \xrightarrow{k\to\infty} \int_{X(\mathbb{C})} f\beta^{\wedge d}.$$

So it suffices to show that β is actually a smooth differential form which is semipositive, closed, and such that $[m]^*\beta = m^2 \cdot \beta$ for all $m \in \mathbb{Z}$. Everything will follow from the fact that $2 \cdot \beta$ is the *Betti form* in $A(\mathbb{C})$, that is, it is locally given by the pullback of the universal Betti form from Proposition 3.2. Fix a point $s \in S(\mathbb{C})$. As L is fiberwise of type Δ, there exists a neighborhood $U \subseteq S(\mathbb{C})$ of s such that there exists a holomorphic map $c\colon U \to \mathscr{H}_g$ for which $\pi^{-1}(U)$ is the pullback of the family $\pi_{g,\Delta} : \mathscr{A}_{g,\Delta} \to \mathscr{H}_g$. Let $\varphi\colon \pi^{-1}(U) \to \mathscr{A}_{g,\Delta}$

be the corresponding map. So we have the following pullback diagram:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, two heads, "\pi"] & \mathscr{A}_{g,\Delta} \arrow[d, two heads, "\pi_{g,\Delta}"] \\
U \arrow[r, "c"] & \mathscr{H}_g
\end{tikzcd}$$

Let $Z_s \coloneqq c(s) \in \mathscr{H}_g$. Then we have an induced map $A|_s \to \mathcal{A}_{Z_s} = \mathscr{A}_{g,\Delta}|_{Z_s}$ between abelian varieties, which we also call φ.

The line bundles $\varphi^* L_{g,\Delta}^{\otimes 2}|_{Z_s}$ and $L(\mathbb{C})^{\otimes 2}|_s$ are isomorphic, since their associated Appel-Humbert data agree (see [16, Corollary 2.3.7 and Lemma 8.7.1]). Analogously as in the proof of Proposition 3.1, we can shrink U in order to assume $\varphi^* L_{g,\Delta}^{\otimes 2} \sim L(\mathbb{C})^{\otimes 2}|_{\pi^{-1}(U)}$.

We can give a smooth Hermitian metric $\|\cdot\|'$ to the trivial line bundle $\mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g}$ by assigning, for every holomorphic function f in some open set:

$$\|f\|' = e^{-\pi \cdot \mathrm{Im}(z)^t \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z)} |f|.$$

Notice that, as 1 is a global section, the Chern form of its metrized line bundle is:

$$dd^c \log \|1\|'^{-1} = dd^c(\pi \cdot \mathrm{Im}(z)^t \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z)) = \widehat{\omega}^{\mathrm{univ}}/2.$$

Consider $[n]\colon \mathbb{C}^g \times \mathscr{H}_g \to \mathbb{C}^g \times \mathscr{H}_g$ given by multiplication by n in $\mathbb{C}^g$. As pullbacks and tensor products of trivial line bundles are trivial, we have

$$\mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g}^{\otimes n^2} = [n]^* \mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g} = \mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g}$$

We will show that, in fact, we have $\left(\mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g}, \|\cdot\|'\right)^{\otimes n^2} = [n]^* \left(\mathcal{O}_{\mathbb{C}^g \times \mathscr{K}_g}, \|\cdot\|'\right)$. Indeed, let f be some section of $\mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g}$. Then f corresponds to $f \otimes 1 \otimes \cdots \otimes 1$ in the isomorphism $\mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g} \sim \mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g}^{\otimes n^2}$. So:

$$\|f\|'^{\otimes n^2} = \|f\|' \|1\|' \cdots \|1\|' = e^{-n^2 \pi \cdot \mathrm{Im}(z)^t \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z)} |f|.$$

On the other hand, f corresponds to $[n]^* f$ in $[n]^* \mathcal{O}_{\mathbb{C}^g \times \mathscr{H}_g}$. As $[n](z, \tau) = (nz, \tau)$, we obtain:

$$[n]^* \|[n]^* f\| = \|f\| \circ [n] = e^{-\pi \cdot \mathrm{Im}(nz)^t \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(nz)} |f| = e^{-n^2 \pi \cdot \mathrm{Im}(z)^t \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z)} |f|,$$

so we get the desired equality. From the explicit automorphy factors describing $L_{g,\Delta}$ (see [16, Section 8.7]), it is possible to read off that $\|\cdot\|'$ descents to a metric $\|\cdot\|_{g,\Delta}$ on $L_{g,\Delta}$. Taking the pullback by φ, we obtain then a metric $\|\cdot\|'_\infty$ in $L(\mathbb{C})|_{\pi^{-1}(U)}$ satisfying $(L(\mathbb{C})|_{\pi^{-1}(U)}, \|\cdot\|'_\infty)^{\otimes n^2} \sim [n]^*(L(\mathbb{C})|_{\pi^{-1}(U)}, \|\cdot\|'_\infty)$. Now let s be a section of $L(\mathbb{C})|_{\pi^{-1}(U)}$. We have:

$$\psi^*[n]^* \|s\|'_\infty = [n]^* \|\psi_1(s^{\otimes n^2})\|'_\infty = \|s^{\otimes n^2}\|'^{\otimes n^2}_\infty = \|s\|'^{n^2}_\infty.$$

So we get the relation $\|\cdot\|'_\infty = \psi^*[n]^* \|\cdot\|'^{1/n^2}_\infty$, the same relation satisfied by $\|\cdot\|_\infty$. Consider the function $g \coloneqq \log \frac{\|\cdot\|_\infty}{\|\cdot\|'_\infty}$. Then we have:

$$\psi^*[n]^* g = \log \frac{\psi^*[n]^* \|\cdot\|_\infty}{\psi^*[n]^* \|\cdot\|'_\infty} = \log \frac{\|\cdot\|_\infty^{n^2}}{\|\cdot\|'^{n^2}_\infty} = n^2 \cdot g.$$

But as $[n]$ is surjective, g and $\psi^*[n]^*g = n^2 \cdot g$ assume the same values. In particular, these functions have the same supremum. As g is continuous and bounded, we conclude that $g = 0$, that is, $\|\cdot\|_\infty = \|\cdot\|'_\infty$.

This shows that $\|\cdot\|_{\infty,\pi^{-1}(U)}$ is the pullback by φ of the metric over $L_{g,\Delta}$ that we constructed, so it is also the pullback of the metric $\|\cdot\|'$ over $\mathbb{C}^g \times \mathscr{H}_g$. In terms of its Chern forms, we conclude that $\beta|_{\pi^{-1}(U)}$ is the pullback of $\widehat{\omega}^{\mathrm{univ}}/2$. In particular, it is a smooth differential form which is semipositive, closed and such that $[m]^*\beta = m^2 \cdot \beta$, as we wanted. $\square$

The next lemma gives an alternative characterization of the degeneracy locus of X, in terms of β:

Lemma 3.4. *For each $x \in X^{\mathrm{sm}}(\mathbb{C})$, we have $\mathrm{rank}_{\mathrm{Betti}}(X, x) = 2d$ if and only if $(\beta|_X^{\wedge d})_x \neq 0$.*

Proof. Let $U \subseteq S(\mathbb{C})$ be a connected open neighborhood of $\pi(x)$ such that we have a Betti map $b\colon \pi^{-1}(U) \to (\mathbb{R}/\mathbb{Z})^{2g}$. By the proof of Lemma 3.3, the form $2\cdot\beta$ is locally the pullback of the form

$$\widehat{\omega}^{\mathrm{univ}} = i(dz - d\tau \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z))^t \wedge \mathrm{Im}(\tau)^{-1}(d\overline{z} - d\overline{\tau} \cdot \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z))$$

that appears in Proposition 3.2. So there exists an open subset $V \subseteq A(\mathbb{C})$ containing x and a holomorphic map $c\colon V \to \mathbb{C}^g \times \mathscr{H}_g$ such that $2 \cdot \beta|_V = c^*\widehat{\omega}^{\mathrm{univ}}$. We can suppose $V \subseteq \pi^{-1}(U)$ and V simply connected, so we get a restriction map $b|_V\colon V \to (\mathbb{R}/\mathbb{Z})^{2g}$ that lifts to a map $\tilde{b}\colon V \to \mathbb{R}^{2g}$. Denote $\tilde{b} = (b_1, \ldots, b_{2g})^t$, and let $\mathrm{diag}(\Delta) \coloneqq \mathrm{diag}(\delta_1, \ldots, \delta_g)$.

It is possible to arrange c so that we have the equality $z \circ c = (\tau \circ c, \mathrm{diag}(\Delta)) \cdot \tilde{b}$ of functions $V \to \mathbb{C}^g$ (compare [16, Section 8.1]). For simplicity, we will denote $z = z \circ c$ and $\tau = \tau \circ c$ from now on. Notice in particular that with this notation the formula for $2 \cdot \beta|_V$ is the same as the one for $\widehat{\omega}^{\mathrm{univ}}$. So we have the equality of vectors:

$$\begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} = \begin{pmatrix} \tau_{11}b_1 + \cdots + \tau_{1g}b_g + \delta_1 b_{g+1} \\ \vdots \\ \tau_{g1}b_1 + \cdots + \tau_{gg}b_g + \delta_g b_{2g} \end{pmatrix}. \tag{3}$$

Taking the imaginary part:

$$\mathrm{Im}(z) = \begin{pmatrix} \mathrm{Im}(\tau_{11})b_1 + \cdots + \mathrm{Im}(\tau_{1g})b_g \\ \vdots \\ \mathrm{Im}(\tau_{g1})b_1 + \cdots + \mathrm{Im}(\tau_{gg})b_g \end{pmatrix} = \mathrm{Im}(\tau) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix}.$$

So $\mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(z) = (b_1, \ldots, b_g)^t$, and we obtain a simplified expression for $2 \cdot \beta|_V$:

$$2 \cdot \beta|_V = i \left(dz - d\tau \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} \right)^t \wedge \left(d\overline{z} - d\overline{\tau} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} \right).$$

Let $B_1(0)^d \coloneqq \{(w_1, \ldots, w_d) \in \mathbb{C}^d\colon \max\{|w_1|, |w_2|, \ldots, |w_d|\} < 1\}$ be the open ball with center 0 and radius 1 for the supremum norm. As X is a complex variety of dimension d, we can choose a local chart $\chi\colon B_1(0)^d \to X(\mathbb{C})$ such that $\chi(0) = x$. We want to calculate the pullback $2 \cdot \chi^*\beta|_V$. We denote by $(w_1, \ldots, w_d)$ the standard coordinates on $\mathbb{C}^d$. Besides, for simplicity, for each function f on $X(\mathbb{C})$ we will write f (resp. $\partial f/\partial w_l, \partial f/\partial \overline{w}_l$) instead of $f \circ \chi$ (resp. $\partial(f \circ \chi)/\partial w_l, \partial(f \circ \chi)/\partial \overline{w_l}$). Then on $B_1(0)^d$ we have $dz_j = \sum_{l=1}^g \frac{\partial z_j}{\partial w_l} dw_l$ and $d\tau_{jk} = \sum_{l=1}^g \frac{\partial \tau_{jk}}{\partial w_l} dw_l$. Thus:

$$\begin{aligned}
dz - d\tau \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} &= \begin{pmatrix} dz_1 \\ \vdots \\ dz_g \end{pmatrix} - \begin{pmatrix} d\tau_{11} & \cdots & d\tau_{1g} \\ \vdots & \ddots & \vdots \\ d\tau_{g1} & \cdots & d\tau_{gg} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} \\
&= \begin{pmatrix} dz_1 - b_1 d\tau_{11} - \cdots - b_g d\tau_{1g} \\ \vdots \\ dz_g - b_1 d\tau_{g1} - \cdots - b_g d\tau_{gg} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{l=1}^g \left(\frac{\partial z_1}{\partial w_l} - \frac{\partial \tau_{11}}{\partial w_l} \cdot b_1 - \cdots - \frac{\partial \tau_{1g}}{\partial w_l} \cdot b_g \right) dw_l \\ \vdots \\ \sum_{l=1}^g \left(\frac{\partial z_g}{\partial w_l} - \frac{\partial \tau_{g1}}{\partial w_l} \cdot b_1 - \cdots - \frac{\partial \tau_{gg}}{\partial w_l} \cdot b_g \right) dw_l \end{pmatrix} \\
&= J \cdot dw,
\end{aligned}$$

where J is the $(g \times d)$-matrix

$$J = \left(\frac{\partial z_j}{\partial w_l} - \sum_{k=1}^g \frac{\partial \tau_{jk}}{\partial w_l} \cdot b_k \right)_{\substack{1 \le j \le g \\ 1 \le l \le d}}.$$

of complex-valued real-analytic functions on $B_1(0)^d$. Taking the complex conjugate, we obtain:

$$d\overline{z} - d\overline{\tau} \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} = \overline{J} \cdot d\overline{w}.$$

So we get:

$$2 \cdot \chi^* \beta|_V = i(J \cdot dw)^t \wedge \operatorname{Im}(\tau)^{-1}(\overline{J} \cdot d\overline{w}).$$

By the definition of $\mathscr{H}_g$, $\operatorname{Im}(\tau)(0)^{-1} \in \mathbb{R}^{g \times g}$ is symmetric and positive definite. So by the spectral theorem there exists an orthogonal matrix $O \in \mathbb{R}^{g \times g}$ such that $\operatorname{Im}(\tau)(0)^{-1} = O^t \cdot D \cdot O$ with $D \in \mathbb{R}^{g \times g}$ a diagonal matrix with diagonal entries $a_1, \ldots, a_g \in \mathbb{R}_+^*$. Therefore:

$$\begin{aligned}
2 \cdot (\chi^* \beta)_0 &= i(J(0) \cdot dw)^t \wedge O^t \cdot D \cdot O(\overline{J(0)} \cdot d\overline{w}) \\
&= i(O \cdot J(0) \cdot dw)^t \wedge D \cdot (O \cdot \overline{J(0)} \cdot d\overline{w}) \\
&= i \sum_{j=1}^g a_j (O \cdot J(0) \cdot dw)_j \wedge (O \cdot \overline{J(0)} \cdot d\overline{w})_j.
\end{aligned}$$

Let r be the rank of $J(0)$. So it is also the rank of the matrix $O \cdot J(0)$, since O is invertible. Hence, by the rank decomposition of a matrix, we can find an invertible matrix $H \in \mathbb{R}^{d \times d}$ such that:

$$O \cdot J(0) \cdot H = \begin{pmatrix} I_r & 0_{r \times (d-r)} \\ 0_{(g-r) \times r} & 0_{(g-r) \times (d-r)} \end{pmatrix}.$$

Consider another system of coordinates $(v_1, \ldots, v_d)$ such that $w = H \cdot v$. So $dw = H \cdot dv$, thus writing $2 \cdot (\chi^* \beta)_0$ in these coordinates we get:

$$\begin{aligned} 2 \cdot (\chi^* \beta)_0 &= i \sum_{j=1}^{g} a_j (O \cdot J(0) \cdot dw)_j \wedge (O \cdot \overline{J(0)} \cdot d\overline{w})_j \\ &= i \sum_{j=1}^{g} a_j (O \cdot J(0) \cdot H \cdot dv)_j \wedge (O \cdot \overline{J(0)} \cdot H \cdot d\overline{v})_j \\ &= i \sum_{j=1}^{r} a_j \, dv_j \wedge d\overline{v}_j. \end{aligned}$$

Since the matrix $\operatorname{diag}(a_1, \ldots, a_d) \in \mathbb{R}^{d \times d}$ is positive-definite, this proves that $(\beta|_X^{\wedge d})_x \neq 0$ if and only if $r = d$, that is, if the matrix $J(0)$ has maximal rank d (see Proposition A.1, item (a)).

On the other hand, by definition $\operatorname{rank}_{\mathrm{Betti}}(X, x)$ is the rank of the $2g \times 2g$ complex matrix:

$$B := \begin{pmatrix} \frac{\partial b_1}{\partial w_1} & \frac{\partial b_1}{\partial w_2} & \cdots & \frac{\partial b_1}{\partial w_d} & \frac{\partial b_1}{\partial \overline{w}_1} & \frac{\partial b_1}{\partial \overline{w}_2} & \cdots & \frac{\partial b_1}{\partial \overline{w}_d} \\ \frac{\partial b_2}{\partial w_1} & \frac{\partial b_2}{\partial w_2} & \cdots & \frac{\partial b_2}{\partial w_d} & \frac{\partial b_2}{\partial \overline{w}_1} & \frac{\partial b_2}{\partial \overline{w}_2} & \cdots & \frac{\partial b_2}{\partial \overline{w}_d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial b_{2g}}{\partial w_1} & \frac{\partial b_{2g}}{\partial w_2} & \cdots & \frac{\partial b_{2g}}{\partial w_d} & \frac{\partial b_{2g}}{\partial \overline{w}_1} & \frac{\partial b_{2g}}{\partial \overline{w}_2} & \cdots & \frac{\partial b_{2g}}{\partial \overline{w}_d} \end{pmatrix}$$

when evaluated at 0. Deriving (3) with respect to each w_l, we get for each $1 \leq j \leq g$:

$$\frac{\partial z_j}{\partial w_l} = \sum_{k=1}^{g} \frac{\partial \tau_{jk}}{\partial w_l} \cdot b_k + \sum_{k=1}^{g} \tau_{jk} \cdot \frac{\partial b_k}{\partial w_l} + \delta_j \cdot \frac{\partial b_{g+j}}{\partial w_l}.$$

Doing the same for the $\overline{w}_l$, we get:

$$0 = \frac{\partial z_j}{\partial \overline{w}_l} = \sum_{k=1}^{g} \tau_{jk} \cdot \frac{\partial b_k}{\partial \overline{w}_l} + \delta_j \cdot \frac{\partial b_{g+j}}{\partial \overline{w}_l}.$$

These two equations and their conjugates will be very useful in what follows. Consider the $2g \times 2g$ complex matrix:

$$M := \begin{pmatrix} \tau & \operatorname{diag}(\Delta) \\ \overline{\tau} & \operatorname{diag}(\Delta) \end{pmatrix}.$$

Notice that:

$$\begin{pmatrix} \tau & \operatorname{diag}(\Delta) \\ \overline{\tau} & \operatorname{diag}(\Delta) \end{pmatrix} \cdot \begin{pmatrix} I_g & 0_{g\times g} \\ -\operatorname{diag}(\Delta)^{-1} \cdot \overline{\tau} & I_g \end{pmatrix} = \begin{pmatrix} \tau - \overline{\tau} & \operatorname{diag}(\Delta) \\ 0_{g \times g} & \operatorname{diag}(\Delta) \end{pmatrix},$$

hence $\det M = \det(\tau - \overline{\tau}) \cdot \det(\operatorname{diag}(\Delta)) = \det(2i \operatorname{Im}(\tau)) \cdot \delta_1 \cdots \delta_g \neq 0$, since each $\delta_j > 0$ and as $\tau \in \mathscr{H}_g$ its imaginary part is positive-definite. So M is invertible.

We will compute the product $M \cdot B$. There are four cases. For $1 \leq j, l \leq g$:

$$\begin{aligned}
(MB)_{jl} &= \sum_{k=1}^{g} \tau_{jk} \frac{\partial b_k}{\partial w_l} + \delta_j \cdot \frac{\partial b_{g+j}}{\partial w_l} = \frac{\partial z_j}{\partial w_l} - \sum_{k=1}^{g} \frac{\partial \tau_{jk}}{\partial w_l} \cdot b_k = J_{jl};\\
(MB)_{j(g+l)} &= \sum_{k=1}^{g} \tau_{jk} \frac{\partial b_k}{\partial \overline{w}_l} + \delta_j \cdot \frac{\partial b_{g+j}}{\partial \overline{w}_l} = 0;\\
(MB)_{(g+j)l} &= \sum_{k=1}^{g} \overline{\tau}_{jk} \frac{\partial b_k}{\partial w_l} + \delta_j \cdot \frac{\partial b_{g+j}}{\partial w_l} = 0;\\
(MB)_{(g+j)(g+l)} &= \sum_{k=1}^{g} \overline{\tau}_{jk} \frac{\partial b_k}{\partial \overline{w}_l} + \delta_j \cdot \frac{\partial b_{g+j}}{\partial w_l} = \frac{\partial \overline{z}_j}{\partial \overline{w}_l} - \sum_{k=1}^{g} \frac{\partial \overline{\tau}_{jk}}{\partial \overline{w}_l} \cdot b_k = \overline{J_{jl}}.
\end{aligned}$$

So we have the equality:

$$M \cdot B = \begin{pmatrix} J & 0_{g\times d} \\ 0_{g\times d} & J \end{pmatrix} \Rightarrow M(0) \cdot B(0) = \begin{pmatrix} J(0) & 0_{g\times d} \\ 0_{g\times d} & J(0) \end{pmatrix}.$$

Since $M(0)$ is invertible, we conclude that the rank of $B(0)$ equals $2 \cdot \operatorname{rank} J(0)$. So $\operatorname{rank}_{\mathrm{Betti}} = 2d$ if and only if the rank of $J(0)$ is d, finishing the proof. □

Lemma 3.3 has a useful consequence:

Lemma 3.5. *If X is non-degenerate, then $\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right) \gg_{X,\iota} n^{2kd}$ for all integers $k \geq 0$.*

Proof. Let $U \subseteq S(\mathbb{C})$ be a relatively compact open. As X is non-degenerate, there exists a point $x_0 \in \pi^{-1}(U) \cap X(\mathbb{C})^{\mathrm{sm}}$ such that $\operatorname{rank}_{\mathrm{Betti}}(X, x_0) = 2d$. By Lemma 3.4, this is equivalent to $(\beta^{\wedge d})_x \neq 0$. Since β is semipositive, so is $\beta^{\wedge d}$, by Proposition A.1, item (a). We then have $\int_{\pi^{-1}(U)\cap X(\mathbb{C})} \beta^{\wedge d} > 0$.

By the integral convergence given by Lemma 3.3 above, we obtain

$$\int_{\pi^{-1}(U)\cap X(\mathbb{C})} \alpha_k^{\wedge d} \geq C_{X,\iota}$$

for $k \gg_{X,\iota} 1$ and some constant $C_{X,\iota} > 0$. Consider the maps

$$X \xhookrightarrow{\mathrm{id}\times[n^k]} X \times [n^k](X) \xhookrightarrow{\iota\times\iota} \mathbb{P}^N_{\mathbb{C}} \times \mathbb{P}^N_{\mathbb{C}}.$$

The composition is given by $\iota \times (\iota \circ [n^k])\colon X \to \mathbb{P}^N_{\mathbb{C}} \times \mathbb{P}^N_{\mathbb{C}}$. Let M be the line bundle on $X(\mathbb{C})$ given by $(\iota \times (\iota \circ [n^k]))^* \mathcal{O}(1,1)$. Notice that

$$c_1(\mathcal{O}(1,1)) = c_1(\mathrm{pr}_1^* \mathcal{O}(1)) + c_1(\mathrm{pr}_2^* \mathcal{O}(1)) = \mathrm{pr}_1^* \omega_{\mathrm{FS}} + \mathrm{pr}_2^* \omega_{\mathrm{FS}},$$

hence:

$$c_1(M) = (\iota \times (\iota \circ [n^k]))^* (\mathrm{pr}_1^* \omega_{\mathrm{FS}} + \mathrm{pr}_2^* \omega_{\mathrm{FS}}) = \iota^* \omega_{\mathrm{FS}} + (\iota \circ [n^k])^* \omega_{\mathrm{FS}} = \alpha_0 + n^{2k} \alpha_k.$$

So, by Wirtinger formula (equation (9)), we obtain $\deg_M X = \int_{X(\mathbb{C})} (\alpha_0 + n^{2kd} \cdot \alpha_k)^{\wedge d}$. Furthermore, notice that M is the pullback of the hyperplane line bundle of $\overline{Y}_k(\mathbb{C})$ by a function of degree 1, hence[7] $\deg_M X = \deg(\overline{Y}_k(\mathbb{C}))$. Since α_0 is positive and α_k is semipositive, Proposition A.1, item (b), guarantees that the form $\alpha_0^{\wedge r} \wedge \alpha_k^{\wedge(d-r)}$ is semipositive, for each $0 \leq r \leq d$. As the integral of a semipositive form is non-negative, we have:

$$\begin{aligned}
\deg\left(\overline{Y}_k(\mathbb{C})\right) = \deg_M X &= \int_{X(\mathbb{C})} \left(\alpha_0 + n^{2kd} \cdot \alpha_k\right)^{\wedge d} \\
&\geq n^{2kd} \int_{X(\mathbb{C})} \alpha_k^{\wedge d} \\
&\geq n^{2kd} \int_{\pi^{-1}(U) \cap X(\mathbb{C})} \alpha_k^{\wedge d} \\
&\gg_{X,\iota} n^{2kd},
\end{aligned}$$

□

We will need a converse bound of the one given by the previous lemma.

Lemma 3.6. *For each integer $k \geq 0$, we have*

$$\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right) \leq \deg_{\mathcal{O}\left(n^{2k},1\right)}\left(\overline{Y}_k\right) \ll_{X,\iota} n^{2kd}$$

.

Proof. Let H be the hyperplane line bundle on $\overline{Y}_k$. Then, by definition, $\deg_{\mathcal{O}(1,1)}(\overline{Y}_k) = (H^d \cdot \mathcal{O}(1,1))_{\overline{Y}_k}$ and $\deg_{\mathcal{O}(n^{2k},1)}(\overline{Y}_k) = (H^d \cdot \mathcal{O}(n^{2k},1))_{\overline{Y}_k}$. So the first inequality is equivalent to

$$(H^d \cdot \mathcal{O}(1,1))_{\overline{Y}_k} \leq (H^d \cdot \mathcal{O}(n^{2k},1))_{\overline{Y}_k} \iff (H^d \cdot \mathcal{O}(n^{2k}-1,0))_{\overline{Y}_k} \geq 0,$$

by linearity. Now, this last inequality follows from Proposition B.2, since $\mathcal{O}(n^{2k}-1,0)$ is nef. This is because $\mathcal{O}(n^{2k}-1)$ and $\mathcal{O}$ are nef, the pullback of a nef line bundle by a proper map is nef and the tensor product of nef line bundles is nef. Hence it remains to prove the second inequality.

Let $s \in S(\mathbb{C})$ be an arbitrary point and write again $(L, \|\cdot\|) := \iota^*(\mathcal{O}(1), \|\cdot\|_{\mathrm{FS}})$. By Proposition 3.1, there is an open neighborhood $V \subseteq S$ of s and an isomorphism

$$\psi_k \colon L|_{\pi^{-1}(V)}^{\otimes n^{2k}} \to [n^k]^* L|_{\pi^{-1}(V)}.$$

Let $Z_0, Z_1, \ldots, Z_N$ be the standard coordinates on $\mathbb{P}^N_{\mathbb{C}}$. Then the isomorphism ψ_k^{-1} shows the existence of homogeneous polynomials Q_i of degree n^{2k} such that

$$\left[n^k\right]^* \iota^* Z_i = Q_i\left(\iota^* Z_0, \iota^* Z_1, \ldots, \iota^* Z_N\right)$$

For each $i \in \{0, 1, \ldots, N\}$, let $H_i \subseteq \mathbb{P}^N_{\mathbb{C}} \times \mathbb{P}^N_{\mathbb{C}}$ be the hypersurface defined by

$$\mathrm{pr}_2^* Z_i = Q_i\left(\mathrm{pr}_1^* Z_0, \mathrm{pr}_1^* Z_1, \ldots, \mathrm{pr}_1^* Z_N\right)$$

[7]see appendix B.1.

It is not so difficult to see that $\deg_{\mathcal{O}\left(n^{2k},1\right)}(H_i) \ll_\iota n^{2k}$. By the definition of $\overline{Y}_k$, we have:

$$\overline{Y}_k \cap \left(\iota\left(\pi^{-1}(V)\right) \times \mathbb{P}^N_{\mathbb{C}}\right) = \left(\left(\overline{X}_0 \cap \iota\left(\pi^{-1}(V)\right)\right) \times \mathbb{P}^N_{\mathbb{C}}\right) \cap H_0 \cap H_1 \cap \cdots \cap H_N.$$

Making $s \in S(\mathbb{C})$ (and hence V) vary, we obtain the equality:

$$\overline{Y}_k = (\overline{X}_0 \times \mathbb{P}^N_{\mathbb{C}}) \cap H_0 \cap H_1 \cap \cdots \cap H_N.$$

As $\overline{Y}_k$ has dimension d, there exist integers $0 \leq i_1 < \cdots < i_d \leq N$ such that $\overline{Y}_k$ is an irreducible component of

$$\left(\overline{X}_0 \times \mathbb{P}^N_{\mathbb{C}}\right) \cap H_{i_1} \cap \cdots \cap H_{i_d}.$$

Using a multiprojective version of Bézout's theorem (see for example [21, Section 3]), we deduce that

$$\deg_{\mathcal{O}\left(n^{2k},1\right)}\left(\overline{Y}_k\right) \ll_d \prod_{j=1}^{d} \deg_{\mathcal{O}\left(n^{2k},1\right)}\left(H_{i_j}\right) \ll_{X,\iota} n^{2kd}$$

□

The last two lemmas together give us immediately:

Lemma 3.7. *For each integer $k \geq 0$,*

$$n^{2kd} \ll_{X,\iota} \deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right) \leq \deg_{\mathcal{O}\left(n^{2k},1\right)}\left(\overline{Y}_k\right) \ll_{X,\iota} n^{2kd}.$$

We will also need the following estimate, that complements Lemma 3.3.

Lemma 3.8. *Let $0 \leq l \leq d$. For every smooth, compactly supported $2l$-form γ on $X(\mathbb{C})$, we have:*

$$\int_{X(\mathbb{C})} \alpha_k^{\wedge(d-l)} \wedge \gamma \ll_{X,\iota,\gamma} 1$$

Proof. Let $\emptyset \neq U \subseteq S(\mathbb{C})$ be a simply connected Euclidean open subset. Then we have a real-analytic isomorphism $a : A(\mathbb{C})|_U \to (\mathbb{R}/\mathbb{Z})^{2g} \times U$ that is a group homomorphism on each fiber over U. Let $u_1, \ldots, u_{2g}$ be the pullbacks by the projection $(\mathbb{R}\,/\,\mathbb{Z})^{2g} \times U \to (\mathbb{R}\,/\,\mathbb{Z})^{2g}$ of the standard real-analytic coordinates on $(\mathbb{R}/\mathbb{Z})^{2g}$. Take U small enough so that there exists a system of local real analytic coordinates on U as well and we write $u'_1, \ldots, u'_{2s}$ for its pullback by the projection $(\mathbb{R}/\mathbb{Z})^{2g} \times U \to U$. So $u_1, \ldots, u_{2g}, u'_1, \ldots, u'_{2s}$ form a system of real-analytic coordinates on $A(\mathbb{C}))|_U$, and we can write:

$$\alpha_0|_U = \sum_{1 \leq i < j \leq 2g} g_{i,j} du_i \wedge du_j + \sum_{1 \leq i \leq 2s} h_i du'_i \wedge \gamma_i$$

for 1-forms γ_i and smooth complex-valued functions $g_{i,j}, h_i$ on $A(\mathbb{C})|_U$.

Suppose first that the support of γ is a compact subset K contained in $U \subset S(\mathbb{C})$. By compactness:

$$\sup_{x \in K \cup [n^k](K)} \left(\max\left\{\left|g_{i,j}(x)\right|, \left|h_i(x)\right|\right\}\right) \ll_{X,\iota,K} 1$$

As $\alpha_k = n^{-2k}[n^k]^*\alpha_0$:

$$\alpha_k^{\wedge(d-l)} \wedge \gamma = n^{-2k(d-l)} \left(\left[n^k\right]^* \alpha_0^{\wedge(d-l)} \wedge \gamma \right)$$

Since each $[n]^*du_i = n \cdot du_i$ and $[n]^*du_i' = du_i'$ (because the morphism $[n]$ corresponds to multiplication by n also in the tangent spaces), we see using the equality above that the coefficients of $\alpha_k^{\wedge(d-l)} \wedge \gamma$ are all $\ll_{X,\iota,K} 1$ on K, and using the hypothesis $\operatorname{Supp}\gamma = K \subseteq U$ we get the desired inequality.

In the general case, notice that for each $s \in S(\mathbb{C})$ we can choose an open $U \ni s$ as above, and using a partition of unity we reduce the problem to the case done above. □

3.2. Equidistribution

With the help of the lemmas of the previous subsection, we can finally prove the equidistribution part of Theorem 1.3. Fix a place $\nu \in \Sigma_\infty(K)$, and let $f \in \mathscr{C}_c^0(X_{\mathbb{C}_\nu}^{\mathrm{an}})$. We start by noticing that every continuous function with compact support can be uniformly approximated by smooth functions with compact support (this is a generalization of Stone-Weierstrass for analytic spaces, which can be proved from the classical statement using a partition of unity argument). So we can assume, without loss of generality, $f \in \mathscr{C}_c^\infty(X_{\mathbb{C}_\nu}^{\mathrm{an}})$. Let $f_!$ be the extension by 0 of f on $\overline{X}_{0,\mathbb{C}_\nu}^{\mathrm{an}}$. For each integer $k \geq 1$, we define $f_k := f_! \circ \phi_{1,\mathbb{C}_\nu}^{\mathrm{an}} \in \mathscr{C}_c^\infty(\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}})$. Given $\lambda \in \mathbb{R}$, we denote by $\overline{\mathcal{O}}(1,1;\lambda)$ the $\mathscr{C}^\infty$-Hermitian line bundle:

$$\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_K}(\lambda n^{2k} f_k) = \overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k} f_k\right) \in \widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathscr{Y}_k).$$

Notice that, ignoring metrics, this line bundle is just the restriction of $\mathcal{O}(1,1)$ to $\mathscr{Y}_K$. In particular, the corresponding line bundle on $\overline{Y}_k$ is just $\mathcal{O}(1,1)|_{\overline{Y}_k}$.

Let μ_k be the measure on $\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}$ given by the restriction of the (d,d)-form

$$\frac{(\mathrm{pr}_1^*\,\omega_{\mathrm{FS}} + \mathrm{pr}_2^*\,\omega_{\mathrm{FS}})^{\wedge d}}{n^{2kd}}.$$

with $\mathrm{pr}_i : \mathbb{P}_{\mathbb{C}_\nu}^N \times \mathbb{P}_{\mathbb{C}_\nu}^N \to \mathbb{P}_{\mathbb{C}_\nu}^N$.

Lemma 3.9. *Assume that $|\lambda| < 1$. Then, there exists a constant $C_1 = C_1(X,\iota,f) > 0$ such that:*

$$\left| h_{\overline{\mathcal{O}}(1,1;\lambda)}\left(\overline{Y}_k\right) - h_{\overline{\mathcal{O}}(1,1)}\left(\overline{Y}_k\right) - \frac{\delta_\nu \lambda n^{2k(d+1)}}{[K:\mathbb{Q}] \deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k \mu_k \right| \leq C_1 |\lambda|^2 n^{2k}$$

for every integer $k \geq 0$.

Proof. By the definition of the height, we get the equalities:

$$h_{\overline{\mathcal{O}}(1,1;\lambda)}(\overline{Y}_k) = \frac{(\overline{\mathcal{O}}(1,1;\lambda)|_{\mathscr{Y}_k})^{d+1}}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}$$

and

$$h_{\overline{\mathcal{O}}(1,1)}(\overline{Y}_k) = \frac{(\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k})^{d+1}}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}.$$

By the multilinearity and the commutativity of the arithmetic intersection numbers (Proposition B.3), and from $\overline{\mathcal{O}}(1,1;\lambda) = \overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k} f_k\right)$, we get:

$$(\overline{\mathcal{O}}(1,1;\lambda)|_{\mathscr{Y}_k})^{d+1} = \sum_{j=0}^{d+1} \binom{d+1}{j} (\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k})^{d+1-j} \cdot (\overline{\mathcal{O}}(\lambda n^{2k} f_k))^j.$$

Hence:

$$h_{\overline{\mathcal{O}}(1,1;\lambda)}(\overline{Y}_k) - h_{\overline{\mathcal{O}}(1,1)}(\overline{Y}_k) = \sum_{j=1}^{d+1} \binom{d+1}{j} \frac{(\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k})^{d+1-j} \cdot (\overline{\mathcal{O}}(\lambda n^{2k} f_k))^j}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}.$$

Now, by Proposition B.3, item (c), we have:

$$\begin{aligned}
(\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k})^d \overline{\mathcal{O}}(\lambda n^{2k} f_k) &= \delta_\nu \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} \lambda n^{2k} f_k c_1(\overline{\mathcal{O}}(1,1))^{\wedge d} \\
&= \delta_\nu \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} \lambda n^{2k} f_k (\mathrm{pr}_1^* \omega_{\mathrm{FS}} + \mathrm{pr}_2^* \omega_{\mathrm{FS}})^{\wedge d} \\
&= \delta_\nu \lambda n^{2k(d+1)} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k \, \mu_k
\end{aligned}$$

Thus:

$$\begin{aligned}
&\left| h_{\overline{\mathcal{O}}(1,1;\lambda)}\left(\overline{Y}_k\right) - h_{\overline{\mathcal{O}}(1,1)}\left(\overline{Y}_k\right) - \frac{\delta_\nu \lambda n^{2k(d+1)}}{[K:\mathbb{Q}]\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k \mu_k \right| \\
= &\left| \sum_{j=2}^{d+1} \binom{d+1}{j} \frac{(\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k})^{d+1-j} \cdot (\overline{\mathcal{O}}(\lambda n^{2k} f_k))^j}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} \right|
\end{aligned}$$

By Lemma 3.7, $\deg_{\mathcal{O}(1,1)}(\overline{Y}_k) \gg_{X,\iota} n^{2kd}$, so it suffices to prove that:

$$\left| \sum_{j=2}^{d+1} \binom{d+1}{j} (\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k})^{d+1-j} \cdot (\overline{\mathcal{O}}(\lambda n^{2k} f_k))^j \right| \ll_{X,\iota,f} |\lambda|^2 n^{2k(d+1)}.$$

Again by Proposition B.3, item (c), for each $2 \leq j \leq d+1$ we have:

$$\begin{aligned}
&(\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k})^{d+1-j} \overline{\mathcal{O}}(\lambda n^{2k} f_k)^j \\
= &\ \delta_\nu \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} \lambda n^{2k} f_k (dd^c \lambda n^{2k} f_k)^{\wedge(j-1)} \wedge c_1(\overline{\mathcal{O}}(1,1))^{\wedge(d+1-j)} \\
= &\ \delta_\nu \lambda^j n^{2kj} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k (dd^c f_k)^{\wedge(j-1)} \wedge (\mathrm{pr}_1^* \omega_{\mathrm{FS}} + \mathrm{pr}_2^* \omega_{\mathrm{FS}})^{\wedge(d+1-j)}.
\end{aligned}$$

Since $\delta_\nu \in \{1,2\}$ and $|\lambda| < 1$ by assumption, it is enough to show that:

$$\left| \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k (dd^c f_k)^{\wedge(j-1)} \wedge (\mathrm{pr}_1^* \omega_{\mathrm{FS}} + \mathrm{pr}_2^* \omega_{\mathrm{FS}})^{\wedge(d+1-j)} \right| \ll_{X,\iota,f} n^{2k(d+1-j)}.$$

Now, notice that the clearly commutative diagrams:

$$\begin{tikzcd} X \times [n^k](X) \arrow[r, hook, "\iota\times[n^k]\circ\iota"] \arrow[d, "\mathrm{pr}_1"] & \mathbb{P}^N_{\mathbb{C}_\nu} \times \mathbb{P}^N_{\mathbb{C}_\nu} \arrow[r, two heads, "\mathrm{pr}_1"] & \mathbb{P}^N_{\mathbb{C}_\nu} \\ X \arrow[urr, hook, "\iota"'] & & \end{tikzcd}$$

and

$$\begin{tikzcd} X \times [n^k](X) \arrow[r, hook, "\iota\times[n^k]\circ\iota"] \arrow[d, "\mathrm{pr}_1"] & \mathbb{P}^N_{\mathbb{C}_\nu} \times \mathbb{P}^N_{\mathbb{C}_\nu} \arrow[r, two heads, "\mathrm{pr}_2"] & \mathbb{P}^N_{\mathbb{C}_\nu} \\ X \arrow[urr, hook, "\iota\circ[n^k]"'] & & \end{tikzcd}$$

imply that:

$$\begin{aligned} & \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k(dd^c f_k)^{\wedge(j-1)} \wedge (\mathrm{pr}_1^*\omega_{\mathrm{FS}} + \mathrm{pr}_2^*\omega_{\mathrm{FS}})^{\wedge(d+1-j)} \\ = & \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} (f_! \circ \phi^{\mathrm{an}}_{1,\mathbb{C}_\nu})(dd^c(f_! \circ \phi^{\mathrm{an}}_{1,\mathbb{C}_\nu}))^{\wedge(j-1)} \wedge ((\phi^{\mathrm{an}}_{1,\mathbb{C}_\nu})^*\iota^*\omega_{\mathrm{FS}} + (\phi^{\mathrm{an}}_{1,\mathbb{C}_\nu})^*(\iota\circ[n^k])^*\omega_{\mathrm{FS}})^{\wedge(d+1-j)} \\ = & \int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} f(dd^c f)^{\wedge(j-1)} \wedge (\iota^*\omega_{\mathrm{FS}} + (\iota\circ[n^k])^*\omega_{\mathrm{FS}})^{\wedge(d+1-j)} \\ = & \int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} f(dd^c f)^{\wedge(j-1)} \wedge (\alpha_0 + n^{2k}\alpha_k)^{\wedge(d+1-j)}, \end{aligned}$$

where in the second equation we used the change of variables formula with respect to $\phi^{\mathrm{an}}_{1,\mathbb{C}_\nu} : \overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu} \to \overline{X}^{\mathrm{an}}_{0,\mathbb{C}_\nu}$ and the fact that $f_! = f$ inside $X^{\mathrm{an}}_{\mathbb{C}_\nu}$ and 0 outside. Now, since f is a smooth function with compact support, we can write this integral as a sum:

$$\sum_{l=j}^{d} n^{2k(d-l)} \int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} \alpha_k^{\wedge(d-l)} \wedge \gamma_l,$$

where each γ_l is a smooth, compactly supported $2l$-form on $X^{\mathrm{an}}_{\mathbb{C}_\nu}$ depending on f. Finally, by Lemma 3.8 we conclude that this sum is $\ll_{X,\iota,f} n^{2k(d+1-j)}$, as desired. $\square$

Lemma 3.10. *For every integer $k \geq 0$, we have*

$$\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda)) - \overline{\mathcal{O}}(1,1;\lambda)^{d+1} \gg_{X,f,\iota,\ell} -|\lambda|^2 n^{2(d+1)k}.$$

Proof. Since f is continuous and $X^{\mathrm{an}}_{\mathbb{C}_\nu}$ is compact (as X is proper), there exists a constant $\sigma = \sigma(X,f)$ be a constant such that $f(x)+\sigma \geq 0$ for all $x \in X^{\mathrm{an}}_{\mathbb{C}_\nu}$. We define:

$$\overline{\mathcal{O}}(1,1;\lambda;\sigma) \coloneqq \overline{\mathcal{O}}(1,1;\lambda) \otimes \overline{\mathcal{O}}\left(\lambda n^{2k}\sigma\right) = \overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k}(f_k+\sigma)\right)$$

By equation (11):

$$\begin{aligned} \widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda;\sigma)) - \widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda)) &= \overline{\mathcal{O}}(1,1;\lambda;\sigma)^{d+1} - \overline{\mathcal{O}}(1,1;\lambda)^{d+1} \\ \Rightarrow \widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda)) - \overline{\mathcal{O}}(1,1;\lambda)^{d+1} &= \widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda;\sigma)) - \overline{\mathcal{O}}(1,1;\lambda;\sigma)^{d+1}, \end{aligned}$$

so it suffices to bound the difference on the right-hand side from below. Since ω_{FS} is a strictly positive $(1,1)$-form and $f_!$ has compact support, a compacity argument shows that there exists some integer number $q = q(X, f) > 0$ such that the $(1,1)$-form $q \cdot \omega_{\mathrm{FS}} + dd^c f_!$ on $\overline{X}^{\mathrm{an}}_{0,\mathbb{C}_\nu}$ is strictly positive. Consequently, the pullback form $q \cdot \mathrm{pr}_1^* \omega_{\mathrm{FS}} + dd^c f_k$ on $\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}$ is semipositive. We can write:

$$\overline{\mathcal{O}}(1,1;\lambda;\sigma) = \left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k}\left(f_k + \sigma\right)\right)\right) \otimes \left(\overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}}\right)^{\otimes -1}.$$

Therefore, by Lemma B.4, item (c), $\mathrm{vol}_\chi(\overline{\mathcal{O}}(1,1;\lambda;\sigma))$ is bounded from below by:

$$\begin{aligned}&\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k}\left(f_k + \sigma\right)\right)\right)^{d+1} \\ &-(d+1)\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k}\left(f_k + \sigma\right)\right)\right)^{d} \cdot \left(\overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}}\right).\end{aligned}$$

By the multilinearity and the commutativity of the arithmetic intersection numbers (Proposition B.3), subtracting $\overline{\mathcal{O}}(1,1;\lambda;\sigma)^{d+1}$ from the above expression results in

$$-\sum_{i=2}^{d+1}\binom{d+1}{i}\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k}\left(f_k + \sigma\right)\right)\right)^{d+1-i} \cdot \left(\left(-q\lambda n^{2k}\right) \cdot \overline{\mathcal{O}}(1,0)\Big|_{\mathscr{Y}_k}\right)^{i}. \quad (4)$$

We claim that the absolute value of the expression above is $\ll_{X,f} |\lambda|^2 n^{2(d+1)k}$. For this purpose, we expand the intersection number

$$\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k} \otimes \overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}} \otimes \overline{\mathcal{O}}\left(\lambda n^{2k}\left(f_k + \sigma\right)\right)\right)^{d+1-i} \cdot \left(\overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}}\right)^{i} \qquad (5)$$

using again Proposition B.3, and estimate the intersection number

$$\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k}\right)^{j_1} \cdot \left(\overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}}\right)^{j_2} \cdot \left(\overline{\mathcal{O}}\left(\lambda n^{2k}\left(f_k + \sigma\right)\right)\right)^{j_3},\ j_1 + j_2 + j_3 = d+1, \qquad (6)$$

in the general term of this expansion. Notice that only intersection numbers with $j_2 \geq 2$ appear in the expansion of (5), so we may assume this freely in the following. If $j_3 \geq 1$, then, by Proposition B.3, item (c), and by the linearity of Chern classes, (6) equals:

$$q^{j_2}\left(\lambda n^{2k}\right)^{j_2+j_3}\int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} (f_k + \sigma)\left(\mathrm{pr}_1^*\omega_{\mathrm{FS}} + \mathrm{pr}_2^*\omega_{\mathrm{FS}}\right)^{\wedge j_1} \wedge \left(\mathrm{pr}_1^*\omega_{\mathrm{FS}}\right)^{\wedge j_2} \wedge \left(dd^c f_k\right)^{\wedge(j_3-1)}$$

Using the change of variables formula with respect to $\phi^{\mathrm{an}}_{1,\mathbb{C}_\nu} : \overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu} \to \overline{X}^{\mathrm{an}}_{0,\mathbb{C}_\nu}$ as in the proof of Lemma 3.9, we can rewrite this as:

$$q^{j_2}\left(\lambda n^{2k}\right)^{j_2+j_3}\int_{\overline{X}^{\mathrm{an}}_{0,\mathbb{C}_\nu}} (f_! + \sigma)\left(\alpha_0 + n^{2k}\alpha_k\right)^{\wedge j_1} \wedge \alpha_0^{\wedge j_2} \wedge \left(dd^c f_!\right)^{\wedge(j_3-1)}.$$

Notice that this integral is non-negative thanks to the hypothesis on σ. Since $f_!$ is a smooth function with compact support, we can proceed similarly to the proof of Lemma 3.9 to conclude, using Lemma 3.8 with respect to α_0, that:

$$q^{j_2}(\lambda n^{2k})^{j_2+j_3}\int_{\overline{X}^{\mathrm{an}}_{0,\mathbb{C}_\nu}} (f_! + \sigma)(\alpha_0 + n^{2k}\alpha_k)^{\wedge j_1} \wedge \alpha_0^{\wedge j_2} \wedge (dd^c f_!)^{\wedge(j_3-1)} \ll_{X,f} |\lambda|^2 n^{2(d+1)k},$$

since $|\lambda| < 1$ and $j_2 \geq 2$. So we conclude that the absolute value of (6) is $\ll_{X,f} |\lambda|^2 n^{2(d+1)k}$ for $j_3 \geq 1$. The absolute value of the corresponding terms in (4) is likewise $\ll_{X,f} |\lambda|^2 n^{2(d+1)k}$.

It remains to consider the case $j_3 = 0$. In this case, (6) becomes:

$$\left(\overline{\mathcal{O}}(1,1)|_{\mathscr{Y}_k}\right)^j \cdot \left(\overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}}\right)^{(d+1)-j}.$$

We will need to use the fact that the intersection numbers:

$$\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k}\right)^j \cdot \left(\overline{\mathcal{O}}\left(n^{2k},0\right)\Big|_{\mathscr{Y}_k}\right)^{(d+1)-j}, \; 0 \leq j \leq d-1 \tag{7}$$

are non-negative. It is possible to show this by means of the equation (10), using the fact that both $\overline{\mathcal{O}}(1,0)$ and $\overline{\mathcal{O}}(1,1)$ are globally generated by sections having norm ≤ 1 everywhere and using a recursion formula for the arithmetic intersection numbers (see [12]). Compare the proof of [28, Lemma 5.3(i)]. It is hence sufficient to find an upper bound. With a similar argument and using Proposition B.3, each of the intersection numbers in (7) can be bounded from above by $\left(\overline{\mathcal{O}}\left(n^{2k},1\right)|_{\overline{\mathscr{Y}}_k}\right)^{d+1}$. Notice that, by Lemma 2.6:

$$\frac{\left(\overline{\mathcal{O}}\left(n^{2k},1\right)|_{\mathscr{Y}_k}\right)^{d+1}}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}\left(n^{2k},1\right)}\left(\overline{Y}_k\right)} = h_{\overline{\mathcal{O}}\left(n^{2k},1\right)}\left(\overline{Y}_k\right) \leq n^{2k}\left(\ell_0+\ell_k\right)$$
$$\Rightarrow \quad (\overline{\mathcal{O}}(n^{2k},1)|_{\mathscr{Y}_k})^{d+1} \leq [K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(n^{2k},1)}(\overline{Y}_k) n^{2k}(\ell_0+\ell_k).$$

Since $\deg_{\mathcal{O}(n^{2k},1)}(\overline{Y}_k) \ll_{X,\iota} n^{2kd}$ by Lemma 3.7, we obtain:

$$\left(\overline{\mathcal{O}}\left(n^{2k},1\right)\Big|_{\overline{y}_k}\right)^{(d+1)} \ll_{X,\iota} n^{2kd} n^{2k}(\ell_0+\ell_k) = n^{2(d+1)k}(\ell_0+\ell_k) \leq 2n^{2(d+1)k}\ell.$$

So we obtain that for $0 \leq j \leq d-1$, $\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k}\right)^j \cdot \left(\overline{\mathcal{O}}\left(n^{2k},0\right)\big|_{\mathscr{Y}_k}\right)^{(d+1)-j} \ll_{X,\iota,\ell} n^{2(d+1)k}$.
Consequently:

$$\begin{aligned}
& \left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k}\right)^j \cdot \left(\overline{\mathcal{O}}(1,0)|_{\mathscr{Y}_k}^{\otimes q\lambda n^{2k}}\right)^{(d+1)-j} \\
= \quad & \left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k}\right)^j \cdot \left(\overline{\mathcal{O}}(n^{2k},0)|_{\mathscr{Y}_k}^{\otimes q\lambda}\right)^{(d+1)-j} \\
= \quad & q^{(d+1)-j}\lambda^{(d+1)-j}\left(\overline{\mathcal{O}}(1,1)\big|_{\mathscr{Y}_k}\right)^j \cdot \left(\overline{\mathcal{O}}(n^{2k},0)|_{\mathscr{Y}_k}^{\otimes q\lambda}\right)^{(d+1)-j} \\
\ll_{X,\iota,\ell} \quad & |\lambda|^2 n^{2(d+1)k},
\end{aligned}$$

where we used Proposition B.3 and the fact that $j \leq d-1$. This concludes the proof. ☐

The following lemma shows we are almost finished:

Lemma 3.11. *We have:*

$$\limsup_{i\to\infty}\left|\frac{1}{\#O_\nu\left(x_i\right)}\sum_{y\in O_\nu(x_i)} f(y) - \frac{\delta_\nu n^{2kd}}{\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)}\int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k\mu_k\right| \xrightarrow{k\to\infty} 0.$$

Proof. Fix $\lambda \in (0,1)$ and $\varepsilon > 0$. Applying Lemma B.4 item (a) to $\mathcal{X} = \mathscr{Y}_k$ and $\overline{\mathscr{L}} = \overline{\mathcal{O}}(1,1;\lambda)$, we find a positive integer N_0 and a nonzero section $s \in \Gamma(\mathscr{Y}_k, \overline{\mathcal{O}}(1,1;\lambda)^{\otimes N_0})$ such that:

$$\delta_\nu \log \|s_\nu\|_\nu^{(\infty)} \leq \left(-\frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda))}{(d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} + \varepsilon \right) N_0$$

and $\log \|s_\nu\|_\mu^{(\infty)} \leq 0$ for all other $\mu \in \Sigma_\infty(K)\backslash\{\nu\}$. Notice that this can be stated as:

$$\frac{\log \|s(x)\|_\nu^{1/N_0}}{[K:\mathbb{Q}]} \leq -\frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda))}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} + \varepsilon$$

for every point $x \in \overline{Y}_{\mathbb{C}_\nu}^{\mathrm{an}}$ and $\log \|s(x)\|_\mu \leq 0 \, (\mu \in \Sigma_\infty(K)\backslash\{\nu\})$ for every point $x \in \overline{Y}_{\mathbb{C}_\mu}^{\mathrm{an}}$. So we get:

$$\begin{aligned}
& -\frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda))}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} \\
= \;& -\frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda)) - \overline{\mathcal{O}}(1,1;\lambda)^{d+1}}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} - \frac{\overline{\mathcal{O}}(1,1;\lambda)^{d+1}}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} \\
= \;& -\frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda)) - \overline{\mathcal{O}}(1,1;\lambda)^{d+1}}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} - h_{\overline{\mathcal{O}}(1,1;\lambda)}(\overline{Y}_k).
\end{aligned}$$

Now, by Lemma 3.7 and Lemma 3.10, we get:

$$-\frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{O}}(1,1;\lambda)) - \overline{\mathcal{O}}(1,1;\lambda)^{d+1}}{[K:\mathbb{Q}](d+1)\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} \ll_{X,f,\iota,[K:\mathbb{Q}],\ell} \frac{\lambda^2 n^{2(d+1)k}}{n^{2kd}} = \lambda^2 n^{2k}.$$

So there exists some constant $C_2 = C_2(X, f, \iota, [K:\mathbb{Q}], \ell) > 0$ such that

$$\frac{\log \|s(x)\|_\nu^{1/N_0}}{[K:\mathbb{Q}]} \leq -h_{\overline{\mathcal{O}}(1,1;\lambda)}\left(\overline{Y}_k\right) + C_2 \lambda^2 n^{2k} + \varepsilon$$

for every point $x \in \overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}$. Now, by Lemma 3.9:

$$\begin{aligned}
& h_{\overline{\mathcal{O}}(1,1;\lambda)}\left(\overline{Y}_k\right) - h_{\overline{\mathcal{O}}(1,1)}\left(\overline{Y}_k\right) - \frac{\delta_\nu \lambda n^{2k(d+1)}}{[K:\mathbb{Q}]\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k \mu_k \geq -C_1 \lambda^2 n^{2k} \\
\Rightarrow \;& -h_{\overline{\mathcal{O}}(1,1;\lambda)}\left(\overline{Y}_k\right) \leq C_1 \lambda^2 n^{2k} - h_{\overline{\mathcal{O}}(1,1)}\left(\overline{Y}_k\right) - \frac{\delta_\nu \lambda n^{2k(d+1)}}{[K:\mathbb{Q}]\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k \mu_k.
\end{aligned}$$

By Lemma 2.6, $h_{\overline{\mathcal{O}}(1,1)}(\overline{Y}_k) \geq 0$, hence we obtain the inequality:

$$\frac{\log \|s(x)\|_\nu^{1/N_0}}{[K:\mathbb{Q}]} \leq -\frac{\delta_\nu \lambda n^{2k(d+1)}}{[K:\mathbb{Q}]\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k \mu_k + C_3 \lambda^2 n^{2k} + \varepsilon, \tag{8}$$

for every $x \in \overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}$, where we denote $C_3 := C_1 + C_2$. Let $x \in \overline{Y}_k \setminus |\mathrm{div}(s)|$ be a closed point. Then by (1) we have:

$$\begin{aligned} h_{\overline{\mathcal{O}}(1,1;\lambda)^{\otimes N_0}}(x) &= \frac{1}{[K(x):\mathbb{Q}]}\left(\log \#\left(\mathcal{O}(1,1)|_x \,/\, (s|_x)\right) - \sum_{\mu\in\Sigma_\infty(K)} \sum_{y\in O_\mu(x)} \delta_\mu \log\|s(y)\|_\mu\right) \\ &\geq -\frac{1}{[K(x):\mathbb{Q}]} \sum_{\mu\in\Sigma_\infty(K)} \sum_{y\in O_\mu(x)} \delta_\mu \log\|s(y)\|_\mu \\ &\geq -\frac{1}{[K(x):\mathbb{Q}]} \sum_{y\in O_\nu(x)} \delta_\nu \log\|s(y)\|_\nu, \end{aligned}$$

since by hypothesis $\log\|s(y)\|_\mu \leq 0$ for $\mu \in \Sigma_\infty(K)\setminus\{\nu\}$ and $y \in \overline{Y}_{\mathbb{C}_\mu}^{\mathrm{an}}$. Remember that we have $\#O_\nu(x) = [K(x):K]$. Therefore, if we let $y_0 \in O_\nu(x)$ be the element of $O_\nu(x)$ with the biggest norm, we obtain:

$$\begin{aligned} h_{\overline{\mathcal{O}}(1,1;\lambda)^{\otimes N_0}}(x) &\geq -\frac{1}{[K(x):\mathbb{Q}]} \sum_{y\in O_\nu(x)} \delta_\nu \log\|s(y)\|_\nu \\ &\geq -\frac{[K(x):K]\delta_\nu \log\|s(y_0)\|_\nu}{[K(x):\mathbb{Q}]} \\ &= -\frac{\delta_\nu \log\|s(y_0)\|_\nu}{[K:\mathbb{Q}]}. \end{aligned}$$

Using (8):

$$\begin{aligned} h_{\overline{\mathcal{O}}(1,1;\lambda)}(x) &= \frac{h_{\overline{\mathcal{O}}(1,1;\lambda)^{\otimes N_0}}(x)}{N_0} \geq -\frac{\delta_\nu \log\|s(y_0)\|_\nu^{1/N_0}}{[K:\mathbb{Q}]} \\ &\geq \frac{\delta_\nu^2 \lambda n^{2k(d+1)}}{[K:\mathbb{Q}]\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k \mu_k - \delta_\nu C_3 \lambda^2 n^{2k} - \delta_\nu \varepsilon. \end{aligned}$$

Now, $h_{\overline{\mathcal{O}}(1,1;\lambda)}(x) = h_{\overline{\mathcal{O}}(1,1)}(x) + h_{\overline{\mathcal{O}}(\lambda n^{2k} f_k)}(x)$. Expanding $h_{\overline{\mathcal{O}}(\lambda n^{2k} f_k)}(x)$ using again (1) with the section 1, we get:

$$h_{\overline{\mathcal{O}}(\lambda n^{2k} f_k)}(x) = -\frac{1}{[K(x):\mathbb{Q}]} \sum_{\mu\in\Sigma_\infty(K)} \sum_{y\in O_\mu(x)} \delta_\mu \log\|1\|_\mu(y).$$

By the definition of $\overline{\mathcal{O}}(\lambda n^{2k} f_k)$, for $\mu \neq \nu$ we have $\log\|1\|_\mu(y) = \log 1 = 0$. Besides, $\log\|1\|_\nu(y) = \log(e^{-\lambda n^{2k} f_k(y)}) = -\lambda n^{2k} f_k(y)$, so:

$$h_{\overline{\mathcal{O}}(\lambda n^{2k} f_k)}(x) = \frac{\delta_\nu \lambda n^{2k}}{[K(x):\mathbb{Q}]} \sum_{y\in O_\nu(x)} f_k(y) = \frac{\delta_\nu \lambda n^{2k}}{[K:\mathbb{Q}]\cdot\#O_\nu(x)} \sum_{y\in O_\nu(x)} f_k(y).$$

Hence:

$$\begin{aligned}
& h_{\overline{\mathcal{O}}(1,1)}(x) + \frac{\delta_\nu \lambda n^{2k}}{[K:\mathbb{Q}]\cdot \# O_\nu(x)} \sum_{y\in O_\nu(x)} f_k(y) \\
\geq\ & \frac{\delta_\nu^2 \lambda n^{2k(d+1)}}{[K:\mathbb{Q}] \deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k \mu_k - \delta_\nu C_3 \lambda^2 n^{2k} - \delta_\nu \varepsilon \\
\Rightarrow\ & h_{\overline{\mathcal{O}}(1,1)}(x) + \frac{\delta_\nu}{[K:\mathbb{Q}]} \left(\frac{\lambda n^{2k}}{\# O_\nu(x)} \sum_{y\in O_\nu(x)} f_k(y) - \frac{\delta_\nu \lambda n^{2k(d+1)}}{\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k \mu_k \right) \\
\geq\ & -\delta_\nu C_3 \lambda^2 n^{2k} - \delta_\nu \varepsilon \\
\Rightarrow\ & \frac{1}{\# O_\nu(x)} \sum_{y\in O_\nu(x)} f_k(y) - \frac{\delta_\nu n^{2kd}}{\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k \mu_k \\
\geq\ & -\frac{[K:\mathbb{Q}] h_{\overline{\mathcal{O}}(1,1)}(x)}{\delta_\nu \lambda n^{2k}} - [K:\mathbb{Q}] C_3 \lambda - \frac{[K:\mathbb{Q}]\varepsilon}{\lambda n^{2k}}.
\end{aligned}$$

Since the sequence $(y_i^{(k)}) = ((\iota(x_i), x_i^{(k)}))$ is $\overline{Y}_k$-generic, there exists some integer i_0 such that $y_i^{(k)} \notin |\operatorname{div}(s)|$ for all $i \geq i_0$. Let $\tilde{y} \in O_\nu(y_i^{(k)})$. Then $y := \phi^{\mathrm{an}}_{1,\mathbb{C}_\nu}(\tilde{y}) \in O_\nu(x_i^{(k)})$ is such that $f_k(\tilde{y}) = f(y)$. On the other hand, given $y \in O_\nu(x_i^{(k)})$, the surjectivity of $\phi_{1,\mathbb{C}_\nu}$ implies the existence of some $\tilde{y} \in O_\nu(y_i^{(k)})$ such that $y = \phi^{\mathrm{an}}_{1,\mathbb{C}_\nu}(\tilde{y})$. Furthermore, since $\phi_{1,\mathbb{C}_\nu}$ is birational and $(y_i^{(k)})$ is generic, such a $\tilde{y}$ is unique for i big enough. As we showed in the proof of Lemma 2.6, $h_{\overline{\mathcal{O}}(1,1)}(y_i^{(k)}) = h_{\overline{\mathcal{O}}(1)}(x_i^{(0)}) + h_{\overline{\mathcal{O}}(1)}(x_i^{(k)})$. So substituting $x = y_i^{(k)}$ in the inequality above for i big enough, we obtain:

$$\begin{aligned}
& \frac{1}{\# O_\nu(x_i)} \sum_{y\in O_\nu(x_i)} f(y) - \frac{\delta_\nu n^{2kd}}{\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k \mu_k \\
\geq\ & -\frac{[K:\mathbb{Q}](h_{\overline{\mathcal{O}}(1)}(x_i^{(0)}) + h_{\overline{\mathcal{O}}(1)}(x_i^{(k)}))}{\delta_\nu \lambda n^{2k}} - [K:\mathbb{Q}] C_3 \lambda - \frac{[K:\mathbb{Q}]\varepsilon}{\lambda n^{2k}}.
\end{aligned}$$

Now, by Lemma 2.5, we have

$$\limsup_{i\to\infty} h_{\overline{\mathcal{O}}(1)}(x_i^{(0)}) = \ell_0 \in [0,\infty)$$

and

$$\limsup_{i\to\infty} \frac{h_{\overline{\mathcal{O}}(1)}(x_i^{(k)})}{n^{2k}} = \ell_k \in [0,\infty),$$

hence choosing an appropriate constant $C_4 = C_4(X, \iota, f, [K:\mathbb{Q}], \ell) > 0$, we get an inequality:

$$\begin{aligned} &\liminf_{i\to\infty}\left(\frac{1}{\#O_\nu(x_i)}\sum_{y\in O_\nu(x_i)} f(y) - \frac{\delta_\nu n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}\int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k\mu_k\right) \\ \geq\ & -C_4\lambda - \frac{[K:\mathbb{Q}]\varepsilon}{\lambda n^{2k}}. \end{aligned}$$

Working with $-\lambda$ instead of λ and adjusting C_4, we obtain a similar inequality:

$$\begin{aligned} &\limsup_{i\to\infty}\left(\frac{1}{\#O_\nu(x_i)}\sum_{y\in O_\nu(x_i)} f(y) - \frac{\delta_\nu n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}\int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k\mu_k\right) \\ \leq\ & C_4\left(\lambda + \lambda^{-1}\ell_k\right) + \frac{[K:\mathbb{Q}]\varepsilon}{\lambda n^{2k}} \end{aligned}$$

Notice the appearance of an extra term $\lambda^{-1}\ell_k$ multiplying the constant C_4. It appears because, when working with $-\lambda$, we have to use the opposite inequality of Lemma 2.6, namely $h_{\overline{\mathcal{O}}(1,1)}(\overline{Y}_k) \leq \ell_0 + n^{2k}\ell_k$, so when choosing the corresponding C_3 we will obtain a term $C_3(\lambda^2 n^{2k} + n^{2k}\ell_k)$ instead of only $C_3\lambda^2 n^{2k}$. Combining the two inequalities, we get:

$$\begin{aligned} &\limsup_{i\to\infty}\left|\frac{1}{\#O_\nu(x_i)}\sum_{y\in O_\nu(x_i)} f(y) - \frac{\delta_\nu n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}\int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k\mu_k\right| \\ \leq\ & C_4\left(\lambda + \lambda^{-1}\ell_k\right) + \frac{[K:\mathbb{Q}]\varepsilon}{\lambda n^{2k}} \end{aligned}$$

In order to conclude the proof, we only need to show that, for every $\varepsilon_0 > 0$, there exists $k_0 > 0$ such that the right side above is less than ε_0 for every $k \geq k_0$. Choose $\lambda > 0$ such that $\lambda < \min\{\varepsilon_0/3C_4, 1\}$. Let $\varepsilon := \lambda\varepsilon_0/(3[K:\mathbb{Q}])$, By Lemma 2.5, $\lim_{k\to\infty}\ell_k = 0$, so there exists an integer $k_0 = k_0(\varepsilon_0)$ such that

$$\ell_k < \varepsilon_0\lambda/3C_4 < \min\left\{\varepsilon_0^2/9C_4^2, \varepsilon_0/3C_4\right\}$$

for all $k \geq k_0$. Therefore, for every $k \geq k_0$, the right side of the inequality above is:

$$C_4(\lambda + \lambda^{-1}\ell_k) + [K:\mathbb{Q}]\varepsilon < C_4\left(\frac{\varepsilon_0}{3C_4} + \lambda^{-1}\frac{\varepsilon_0\lambda}{3C_4}\right) + \frac{\varepsilon_0}{3n^{2k}} \leq \varepsilon_0,$$

as we wanted. □

With Lemma 3.11 on hand, it is clear our interest in the asymptotic behavior of the expression:

$$\frac{\delta_\nu n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}\int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k\mu_k.$$

This is the content of the following two lemmas:

Lemma 3.12. *We have:*

$$\int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k\mu_k \overset{k\to\infty}{\longrightarrow} \int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} f\beta^{\wedge d}$$

where β *is the smooth* $(1,1)$*-form on* $X_{\mathbb{C}_\nu}^{\mathrm{an}}$ *introduced in Lemma 3.3.*

Proof. Notice that:

$$\int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k\mu_k = \int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k\frac{(\mathrm{pr}_1^*\,\omega_{\mathrm{FS}} + \mathrm{pr}_2^*\,\omega_{\mathrm{FS}})^{\wedge d}}{n^{2kd}} = n^{-2kd}\int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} f(\alpha_0 + n^{2k}\alpha_k)^{\wedge d},$$

where we proceeded by changing variables in the same manner as in the proof of Lemma 3.9. Again by the same reasoning as in the proof of Lemma 3.9, we write:

$$\int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} f(\alpha_0 + n^{2k}\alpha_k)^{\wedge d} = \sum_{l=0}^{d} n^{2k(d-l)}\int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} \alpha_k^{\wedge(d-l)}\wedge\gamma_l,$$

where each γ_l is a smooth, compactly supported $2l$-form on $X_{\mathbb{C}_\nu}^{\mathrm{an}}$ depending on f. Notice further that $\gamma_0 = f$. By Lemma 3.8, $\int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} \alpha_k^{\wedge(d-l)}\wedge\gamma_l \ll_{X,\iota,f} 1$, so that in the limit only the term with $l=0$ matters, and we get:

$$\lim_{k\to\infty}\int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k\mu_k = \lim_{k\to\infty}\int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} f\alpha_k = \int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} f\beta^{\wedge d},$$

where we used Lemma 3.3 in the last equality. □

The last step missing before concluding Theorem 1.3 is studying the asymptotic behavior of $\frac{n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}$. As the lemma below shows, this expression converges to a positive number. Curiously, the proof of this lemma also uses the distribution result given by Lemma 3.11.

Lemma 3.13. *The limit:*

$$\kappa := \lim_{k\to\infty}\left(\frac{n^{2kd}}{\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)}\right)$$

exists in $(0,\infty)$.

Proof. Choose some test function $f\in\mathscr{C}_c^\infty(X_{\mathbb{C}_\nu})$ so that $I := \int_{X_{\mathbb{C}_\nu}^{\mathrm{an}}} f\beta^{\wedge d} > 0$ (this is possible since $\beta^{\wedge d}$ is semipositive). For any $k\geq 0$, denote

$$\begin{aligned}
\kappa_k &:= \frac{n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)},\\
I_k &:= \int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k\mu_k,\\
J_k &:= \kappa_k I_k = \frac{n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)}\int_{\overline{Y}_{k,\mathbb{C}_\nu}^{\mathrm{an}}} f_k\mu_k.
\end{aligned}$$

Also, for any $i \geq 0$ denote $S_i := \frac{1}{\#O_\nu(x_i)} \sum_{y \in O_\nu(x_i)} f(y)$. Then, by Lemma 3.11, we have:

$$\limsup_{i \to \infty} |S_i - J_k| \overset{k\to\infty}{\longrightarrow} 0.$$

Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, we have $\limsup_{i\to\infty}|S_i - J_k| < \varepsilon/3$. Let $k, k' \geq k_0$. Then there exists $i \in \mathbb{N}$ such that $|S_i - J_k|, |S_i - J_{k'}| < \varepsilon/2$, and we get $|J_{k'} - J_k| < \varepsilon$. So (J_k) is a Cauchy sequence and converges to some $J \in \mathbb{R}$. By Lemma 3.12, we know that $I_k \to I > 0$. As each $\kappa_k > 0$, we obtain $J \geq 0$. Besides, from $\kappa_k I_k \to J \geq 0$ and $I_k \to I > 0$ we get that $\kappa_k \to \kappa := J/I \geq 0$.

Now we only have to prove that κ cannot be 0. By Lemma 3.7, there is a constant $C > 0$ such that $\deg_{\mathcal{O}(1,1)}(\overline{Y}_k) \leq Cn^{2kd}$ for all k big enough. Then $\kappa_k = \frac{n^{2kd}}{\deg_{\mathcal{O}(1,1)}(\overline{Y}_k)} \geq 1/C > 0$ for all k big enough, showing that $\kappa \geq 1/C > 0$, as we wanted. $\square$

We can finally conclude the proof of Theorem 1.3. Define $\mu_\nu := \delta_\nu \kappa \cdot \beta^{\wedge d}$. Then, by Lemma 3.12 and Lemma 3.13:

$$\frac{n^{2kd}}{\deg_{\mathcal{O}(1,1)}\left(\overline{Y}_k\right)} \int_{\overline{Y}^{\mathrm{an}}_{k,\mathbb{C}_\nu}} f_k \mu_k \overset{k\to\infty}{\longrightarrow} \int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} f \mu_\nu.$$

Let us use the notations of the proof of Lemma 3.13. Then we found $J = \int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} f \mu_\nu$. Let $\varepsilon > 0$. By Lemma 3.11, there exists a $k_0 \in \mathbb{N}$ such that we have $|J_{k_0} - J| < \varepsilon/2$ and $\limsup_{i\to\infty} |S_i - J_{k_0}| < \varepsilon/3$. So there exists some $i_0 \in \mathbb{N}$ such that, for all $i \geq i_0$, we have $|S_i - J_{k_0}| < \varepsilon/2$. Thus, for all $i \geq i_0$:

$$|S_i - J| \leq |S_i - J_{k_0}| + |J_{k_0} - J| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that $S_i \to J$, that is:

$$\frac{1}{\#O_\nu(x_i)} \sum_{x \in O_\nu(x_i)} f(x) \overset{i\to\infty}{\longrightarrow} \int_{X^{\mathrm{an}}_{\mathbb{C}_\nu}} f\, \mu_\nu,$$

proving Theorem 1.3.

A. Hermitian Line Bundles

Complex line bundles can be thought of as families of complex vector spaces. Hence, it makes sense to define a metric over a line bundle as being a continuous choice of metrics for each fiber. The same can be done in order to define a Hermitian product over a line bundle.

As we will see, these constructions are way useful, as they allow us to define important geometric concepts such as Chern forms and (semi)positivity.

A.1. Metrics on Line Bundles

Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $\mathscr{C}_X$ be the sheaf of (real) continuous functions on X. In order to have the notion of an absolute value of functions of X, we assume that there is a morphism of sheaves $\mathcal{O}_X \to \mathscr{C}_X$, written $f \mapsto |f|$, satisfying $|fg| = |f||g|$,

$|1| = 1$ and $|f + g| \leq |f| + |g|$ (notice that for X an analytic space or a real manifold, there exists an obvious such morphism).

Definition. Given a line bundle L over X, we define a (continuous) **metric** $\|\cdot\|$ on L as being the datum, for any open set $U \subseteq X$ and any section $s \in \Gamma(U, L)$, of a continuous function $\|s\|_U : U \xrightarrow{x \mapsto \|s(x)\|_U} \mathbb{R}_+$, satisfying the following properties:

(1) For any open set $V \subseteq U$, $\|s\|_V$ is the restriction to V of the function $\|s\|_U$;
(2) For any function $f \in \mathcal{O}_X(U)$, $\|fs\| = |f| \cdot \|s\|$;
(3) If s is a local frame on U, then $\|s\|$ doesn't vanish at any point of U.

We denote the pair $(L, \|\cdot\|)$ by $\overline{L}$, and call it a **metrized line bundle**.

We define an **isometry** of metrized line bundles as being an isomorphism of line bundles that respect the metrics.

Finally, if each function $\|s\|_U$ is actually of class $\mathscr{C}^\infty$, we say that $\|\cdot\|$ is a **smooth metric**, or a **$\mathscr{C}^\infty$-metric**.

Notice that defining a metric $\|\cdot\|$ on $\mathcal{O}_X$ is the same as choosing a continuous function $\|\cdot\|_X : X \to \mathbb{R}_+^*$, what is in turn equivalent to choosing a continuous function $h : X \to \mathbb{R}$ and setting $\|1\|_X = e^{-h}$. In particular, $\mathcal{O}_X$ can be given the *trivial metric*, for which $\|1\| = 1$.

Another thing that we would expect is the triangular inequality: if $s, t \in \Gamma(U, L)$ and $x \in U$, we would like to get

$$\|(s + t)(x)\|_U \leq \|s(x)\|_U + \|t(x)\|_U.$$

This is easy to do once we choose U small enough so that it trivializes L. Then, identifying s and t with functions on $\Gamma(U, \mathcal{O}_U)$, we obtain by (2):

$$\begin{aligned} \|(s + t)(x)\|_U = |(s + t)(x)| \cdot \|1\|_U(x) &\leq |s(x)| \cdot \|1\|_U(x) + |t(x)| \cdot \|1\|_U(x) \\ &= \|s(x)\|_U + \|t(x)\|_U. \end{aligned}$$

The intuition behind the definition of a metric is that of a continuous choice of metrics for each fiber. In order to formalize this, let $(L, \|\cdot\|)$ be a metrized line bundle over X. Fix $x \in X$. Then we can define $\|\cdot\|_x : L_x \to \mathbb{R}_+$ such that, for every $s = [(s, U)] \in L_x$, we have $\|s\|_x \coloneqq \|s(x)\|_U$. By (1), this does not depend on the choice of the representative for s. Besides, by (2) it is easy to see that for any $f \in \mathcal{O}_{X,x}$ we have $\|fs\|_x = |f| \|s\|_x$. Also, notice that, for $s, t \in L_x$, by the triangle inequality proven above we have:

$$\|s + t\|_x \leq \|s\|_x + \|t\|_x.$$

Let us now study when $\|\cdot\|_x$ vanishes. Let $s \in L_x$. and denote by the same letter some representative $s \in \Gamma(U, L)$. We can suppose that U trivializes L, and identify s as being a function $\in \Gamma(U, \mathcal{O}_U)$. We then get:

$$\|s\|_x = |s(x)| \cdot \|1\|_U(x).$$

Since by property (3) $\|1\|_U(x) \neq 0$, we conclude that $\|s\|_x = 0$ if and only if $s(x) = 0$, or equivalently if $s = 0$ in the fiber $L|_x \coloneqq L_x \otimes_{\mathcal{O}_{X,x}} \mathrm{k}(x) = L_x/(\mathfrak{m}_{X,x} \cdot L_x)$.

We will now show that $\|\cdot\|_x$ descends to a function on $L|_x$. Let $s, t \in L_x$ corresponding to the same element in $L|_x$. Then $t = s + u$ for some $u \in \mathfrak{m}_{X,x} \cdot L_x$, and:

$$\|t\|_x = \|s + u\|_x \leq \|s\|_x + \|u\|_x = \|s\|_x,$$

since $\|u\|_x = 0$ from what we have just seen. Using $s = t - u$, we get the inverse inequality, and thus $\|s\|_x = \|t\|_x$, proving that $\|\cdot\|_x$ descends to a function on the fiber $L|_x$, that we will also denote $\|\cdot\|_x \colon L|_x \to \mathbb{R}_+$. This function satisfies the following properties:

(1) If $s, t \in L|_x$, then $\|s + t\|_x \leq \|s\|_x + \|t\|_x$;
(2) If $s \in L|_x$ and $f \in$ k(x), then $\|fs\|_x = |f| \cdot \|s\|$
(3) If $s \in L|_x$, then $\|s\|_x = 0 \iff s = 0$.

So $\|\cdot\|_x$ is really a k(x)-metric on $L|_x$, as we wanted. When X is an analytic space, each k$(x) = \mathbb{C}$, so we have really a $\mathbb{C}$-metric for each fiber. Similarly, for real manifolds we have a $\mathbb{R}$-metric for each fiber.

Suppose X is an analytic space, and let L be a line bundle on X, equipped with two norms $\|\cdot\|_1$, $\|\cdot\|_2$. It is clear that any two metrics in a complex vector space of dimension 1 differ by a multiplicative constant. As each fiber of L is a vector space as such, we see that for any open set $U \subseteq X$, the continuous function $\|s\|_1/\|s\|_2 : U \to \mathbb{R}_+$ is independent of the choice of a non-vanishing meromorphic section s of L in U. This gives us a continuous function $\|\cdot\|_1/\|\cdot\|_2 \colon X \to \mathbb{R}_+$. This allows us to define the notion of convergence of metrics:

Definition. Let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of metrics on L. We say that $\|\cdot\|_n$ **converges** to a metric $\|\cdot\|_\infty$ on L if $\log \frac{\|\cdot\|_n}{\|\cdot\|_0} \to \log \frac{\|\cdot\|_\infty}{\|\cdot\|_0}$.

The tensor product of two metrized line bundles L and M can be given fiberwise a natural metric such that $\|s \otimes t\| = \|s\|\|t\|$ if s and t are local sections of L and M respectively. Similarly, the dual $L^\vee$ of a metrized line bundle $\overline{L}$ given in each fiber $L^\vee|_x$ by:

$$\|f\| = \sup_{0 \neq s \in L|_x} \frac{|f(s)|}{\|s\|}.$$

This choice of metric makes the isomorphism $L \otimes L^\vee \cong \mathcal{O}_X$ an isometry. We can also consider pull-backs of metrized line bundles. Let $\varphi \colon Y \to X$ be a morphism of locally ringed spaces such that $|\varphi^* f| = |f| \circ \varphi$ for any $f \in \mathcal{O}_X$. Let L be a metrized line bundle on X. Then, there is a canonical metric $\varphi^*\|\cdot\|$ on $\varphi^* L$ such that $\varphi^*\|\varphi^* s\| = \|s\| \circ \varphi$ for any section $s \in \Gamma(U, L)$ (as always, this can be defined fiberwise).

A.2. Hermitian Products on Line Bundles

As one would expect, a *Hermitian metric* $\|\cdot\|$ on L is a metric coming from a *Hermitian product* $\langle \cdot, \cdot \rangle$ on L, that is, $\|s\|^2 = \langle s, s \rangle$.

Definition. Given a line bundle L over X, we define a **Hermitian product** on L as being the datum, for any open set $U \subseteq X$ and any sections $s, t \in \Gamma(U, L)$, of a continuous function $\langle s, t \rangle_U \colon U \to \mathbb{C}$, satisfying the following properties:

(1) For any open set $V \subseteq U$, $\langle s, t \rangle_V$ is the restriction to V of the function $\langle s, t \rangle_U$;
(2) For any function $f \in \mathcal{O}_X(U)$ and sections $s_1, s_2, t \in \Gamma(U, L)$,

$$\langle fs_1 + s_2, t \rangle_U = f\langle s_1, t \rangle_U + \langle s_2, t \rangle_U;$$

(3) $\langle t, s \rangle_U = \overline{\langle s, t \rangle_U}$.

We denote the pair $(L, \langle\cdot,\cdot\rangle)$ by $\overline{L}$ and call it a **Hermitian line bundle**.

In order for the Hermitian product to induce a metric, we have to suppose further that it is **positive definite**, that is:

(4) The function $\langle s, s\rangle_U$ has values in $\mathbb{R}_+$. Furthermore, if s is a local frame on U, then $\langle s, s\rangle_U$ does not vanish at any point of U.

The tensor product of two Hermitian line bundles L and M has a natural Hermitian product such that $\langle s_1 \otimes t_1, s_2 \otimes t_2\rangle = \langle s_1, s_2\rangle\langle t_1, t_2\rangle$ if s_1, s_2 and t_1, t_2 are local sections of L and M respectively. Similarly, the dual of a Hermitian line bundle is naturally Hermitian. We can also consider pull-backs of Hermitian line bundles. Let $\varphi\colon Y \to X$ be a morphism of locally ringed spaces. Let L be a metrized line bundle on X. Then, there is a canonical Hermitian product on φ^*L such that

$$\langle \varphi^* s, \varphi^* t\rangle_{\varphi^{-1}(U)} = \langle s, t\rangle_U \circ \varphi|_{\varphi^{-1}(U)}$$

for any section $s \in \Gamma(U, L)$. As in the case of metrics, all of these constructions can be made fiberwise in a natural way.

Notice that defining a Hermitian product $\langle\cdot,\cdot\rangle$ on $\mathcal{O}_X$ is the same as choosing a continuous function $\langle 1, 1\rangle\colon X \to \mathbb{R}$. In particular, $\mathcal{O}_X$ can be given the *trivial Hermitian product*, for which $\langle 1, 1\rangle = 1$. Notice that the trivial Hermitian product induces the trivial metric.

Now we define what will be the most important Hermitian line bundle for us. Let $X = \mathbb{P}^n_{\mathbb{C}}$ for some $n \geq 1$, and let $\pi\colon\ \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$ be the canonical projection. Consider the line bundle $\mathcal{O}(1)$ on X. For each open set $U \subseteq \mathbb{P}^n_{\mathbb{C}}$, $\Gamma(U, \mathcal{O}(1))$ can be identified with the set of analytic functions on $\pi^{-1}(U)$ which are homogeneous of degree 1.

Definition. Given $U \subseteq \mathbb{P}^n_{\mathbb{C}}$ open and $s \in \Gamma(U, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1))$, we define

$$\|s\|_{\mathrm{FS}}([x_0\colon\ \cdots : x_n]) = \frac{|s(x_0, \ldots, x_n)|}{\sqrt{|x_0|^2 + \cdots + |x_n|^2}},$$

where we identify s with the analytic function homogeneous of degree 1 in U it corresponds to. We call this datum the **Fubini-Study metric** of $\mathcal{O}(1)$.

It is easy to see that this is indeed a well-defined smooth metric on $\mathcal{O}(1)$. It is in fact a Hermitian metric, as it comes from the Hermitian product given by

$$\langle s, t\rangle_{\mathrm{FS}}([x_0\colon\ \cdots : x_n]) = \frac{\langle s(x_0, \ldots, x_n), t(x_0, \ldots, x_n)\rangle}{\sqrt{|x_0|^2 + \cdots + |x_n|^2}}$$

To give an example, consider the global section $x + 2y$ in $\mathbb{P}^1_{\mathbb{C}}$. We have $\|x + 2y\|_{\mathrm{FS}} = |x + 2y|/\sqrt{|x|^2 + |y|^2}$. Notice that this is a well-defined function on $\mathbb{P}^1_{\mathbb{C}}$, as $x + 2y$ is homogeneous of degree 1.

A.3. (Semi)positive differential forms

Let X be a complex manifold of dimension n. We remember the following definitions concerning differential forms:

Definition. A differential (p, p)-form ω in X is said to be **real** if it satisfies $\overline{\omega} = \omega$.

For example, the $(1,1)$-form $i\,dz_1 \wedge d\overline{z}_1 + (5-3i)i\,dz_1 \wedge d\overline{z}_2 + (5+3i)i\,dz_2 \wedge d\overline{z}_1 - 2i\,dz_2 \wedge d\overline{z}_2$ is real. More generally, it is easy to see that a $(1,1)$-form $\omega = \sum \omega_{jk} i\,dz_j \wedge d\overline{z}_k$ is real if and only if the coefficient matrix (ω_{jk}) is pointwise Hermitian, and that a (n,n)-form $\omega = f dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ is real if and only if:

$$\begin{cases} f \text{ is a real-valued function, if } n \text{ is even;} \\ (-i)f \text{ is a real-valued function, if } n \text{ is odd.} \end{cases}$$

Notice that, if we write $\omega = f dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ in real coordinates, we get:

$$\begin{aligned} \omega &= f dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \\ &= (-1)^{\frac{n(n-1)}{2}} f (dz_1 \wedge d\overline{z}_1) \wedge \cdots \wedge (dz_n \wedge d\overline{z}_n). \end{aligned}$$

Now, for each $1 \leq j \leq n$, we have:

$$dz_j \wedge d\overline{z}_j = (dx_j + i\,dy_j) \wedge (dx_j - i\,dy_j) = -2i\,dx_j \wedge dy_j,$$

hence:

$$\begin{aligned} \omega &= (-1)^{\frac{n(n-1)}{2}} (-2i)^n f dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= (-i)^n (-1)^{\frac{n(n-1)}{2}} 2^n f dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \end{aligned}$$

Notice that:

$$(-i)^n (-1)^{\frac{n(n-1)}{2}} = \begin{cases} 1, \text{ if } n \text{ is even;} \\ -i, \text{ if } n \text{ is odd.} \end{cases}$$

Hence, by the definition of the integral of a form:

$$\int \omega = \begin{cases} 2^n \int f, \text{ if } n \text{ is even;} \\ 2^n \int (-i) f, \text{ if } n \text{ is odd.} \end{cases}$$

We also need the notion of (semi)positive forms:

Definition. A real differential $(1,1)$-form $\omega = \sum \omega_{jk} i\,dz_j \wedge d\overline{z}_k$ is said to be **positive** (resp. **semipositive**) if the coefficient matrix (ω_{jk}) is pointwise a positive definite (resp. positive semidefinite) Hermitian form. A real differential (n,n)-form $\omega = f dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ is said to be **positive** (resp. **semipositive**) if:

$$\begin{cases} \text{n is even and } f \text{ assumes only positive (resp. non-negative) values.} \\ \text{n is odd and } (-i)f \text{ assumes only positive (resp. non-negative) values.} \end{cases}$$

It is possible to show that this definition does not depend on the choice of coordinates. Also, it is possible to prove that the pullback of a semipositive form by a morphism is always semipositive, and the pullback of a positive form by an immersion is always positive.

The following result, which relates (semi)positive $(1,1)$-forms with (semi)positive (n,n)-forms, is very important:

Proposition A.1. *(a) Let ω be (semi)positive real differential $(1,1)$-form. Then $\omega^{\wedge n}$ is a (semi)positive real differential (n,n)-form.*

(b) *Let ω, η be two (semi)positive real differential $(1,1)$-forms, at least one of them positive. Then, for every $0 \leq r \leq n$, the form $\omega^{\wedge r} \wedge \eta^{\wedge(n-r)}$ is a (semi)positive real differential (n,n)-form.*

Proof. (a) For each point $x \in X$ we can find a local system of coordinates $z_1, \dots, z_n$ around x and write $\omega = \sum i\,\omega_{jk} dz_j \wedge d\overline{z}_k$, with (ω_{jk}) being a Hermitian matrix on each point. As Hermitian matrices are always diagonalizable by the spectral theorem, we can choose this system of coordinates so that $(\omega_{jk}(x))$ is a diagonal matrix, let us say $\operatorname{diag}(a_1, \dots, a_n)$. So $\omega = \sum i\, a_j \cdot dz_j \wedge d\overline{z}_j$. Then:

$$\begin{aligned}
\omega^{\wedge n} &= \left(\sum i\, a_{j_1} \cdot dz_{j_1} \wedge d\overline{z}_{j_1}\right) \wedge \dots \wedge \left(\sum i\, a_{j_n} \cdot dz_{j_n} \wedge d\overline{z}_{j_n}\right) \\
&= \sum i^n\, a_{j_1} \cdots a_{j_n} \cdot dz_{j_1} \wedge d\overline{z}_{j_1} \wedge \dots \wedge dz_{j_n} \wedge d\overline{z}_{j_n} \\
&= \sum i^n (-1)^{\frac{n(n-1)}{2}} a_{j_1} \cdots a_{j_n} \cdot dz_{j_1} \wedge \dots \wedge dz_{j_n} \wedge d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}_{j_n} \\
&= i^n (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)^2 a_{\sigma(1)} \cdots a_{\sigma(n)} \cdot dz_1 \wedge \dots \wedge dz_n \wedge d\overline{z}_1 \wedge \dots \wedge d\overline{z}_n \\
&= i^n (-1)^{\frac{n(n-1)}{2}} n! \cdot a_1 \cdots a_n \cdot dz_1 \wedge \dots \wedge dz_n \wedge d\overline{z}_1 \wedge \dots \wedge d\overline{z}_n.
\end{aligned}$$

Notice that:

$$i^n (-1)^{\frac{n(n-1)}{2}} = \begin{cases} 1, & \text{if } n \text{ is even;} \\ i, & \text{if } n \text{ is odd.} \end{cases}$$

As $1 \cdot 1 = (-i) \cdot i = 1$, the positivity (resp. semipositivity) of $\omega^{\wedge n}$ at x is by definition equivalent to the fact that $a_1 \cdots a_n > 0$ (resp. $a_n \geq 0$). But $(\omega_{jk}(x))$ is positive (semi)definite by hypothesis, hence each $a_1, \dots, a_n$ is positive (resp. non-negative), proving the assertion (notice furthermore that $a_1 \cdots a_n = \det(\omega_{jk}(x))$).

(b) For each point $x \in X$ we can find a local system of coordinates $z_1, \dots, z_n$ around x and write $\omega = \sum i\,\omega_{jk} dz_j \wedge d\overline{z}_k$ and $\eta = \sum i\,\eta_{jk} dz_j \wedge d\overline{z}_k$, with (ω_{jk}) and (η_{jk}) being Hermitian matrices on each point. The idea is to reproduce the proof of (a), but in order to do that we would like to find a local system of coordinates that diagonalize both forms $(\omega_{jk}(x))$ and $(\eta_{jk}(x))$. Fortunately, it is always possible to simultaneously diagonalize two Hermitian forms if at least one of them is positive definite, as we prove in Lemma A.2 below[8]. So we can choose our local system so that $(\omega_{jk}(x)) = \operatorname{diag}(a_1, \dots, a_n)$ and $(\eta_{jk}(x)) = \operatorname{diag}(b_1, \dots, b_n)$, for some $a_1, \dots, a_n, b_1, \dots, b_n$ positive (resp. non-negative) real numbers. From now on, the

[8]**Careful**: given a matrix A, to diagonalize A as a form is to find some invertible matrix S such that S^*AS is diagonal, whereas to diagonalize A as an operator is to find some invertible matrix S such that $S^{-1}AS$ is diagonal. Two operators can be simultaneously diagonalized if and only if they commute, while this is not true for forms. Lemma A.2 refers to simultaneous diagonalization of forms only.

rest of the proof is essentially the same calculation as in (a):

$$\begin{aligned}
& \omega^{\wedge r} \wedge \eta^{\wedge(n-r)} \\
= \; & \Big(\sum i\, a_{j_1} \cdot dz_{j_1} \wedge d\overline{z}_{j_1}\Big) \wedge \cdots \wedge \Big(\sum i\, a_{j_r} \cdot dz_{j_r} \wedge d\overline{z}_{j_r}\Big) \\
& \wedge \Big(\sum i\, b_{j_{r+1}} \cdot dz_{j_{r+1}} \wedge d\overline{z}_{j_{r+1}}\Big) \wedge \cdots \wedge \Big(\sum i\, b_{j_n} \cdot dz_{j_n} \wedge d\overline{z}_{j_n}\Big) \\
= \; & \sum i^n\, a_{j_1} \cdots a_{j_r} b_{j_{r+1}} \cdots b_{j_n} \cdot dz_{j_1} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge dz_{j_n} \wedge d\overline{z}_{j_n} \\
= \; & \sum i^n (-1)^{\frac{n(n-1)}{2}} a_{j_1} \cdots a_{j_r} b_{j_{r+1}} \cdots b_{j_n} \cdot dz_{j_1} \wedge \cdots \wedge dz_{j_n} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_n} \\
= \; & i^n (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)^2 a_{\sigma(1)} \cdots a_{\sigma(r)} b_{\sigma(r+1)} \cdots b_{\sigma(n)} \cdot dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \\
= \; & i^n (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)} \cdots a_{\sigma(r)} b_{\sigma(r+1)} \cdots b_{\sigma(n)} \cdot dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n.
\end{aligned}$$

From the analysis we did about $i^n(-1)^{\frac{n(n-1)}{2}}$ on the previous item, we conclude that the (semi)positivity of the form above is equivalent to the positivity (resp. non-negativity) of the sum:

$$\sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)} \cdots a_{\sigma(r)} b_{\sigma(r+1)} \cdots b_{\sigma(n)}.$$

But this follows from the fact that the a_i, b_i are all positive (resp. non-negative). □

Lemma A.2. *Let $A, B \in \mathbb{C}^{n\times n}$ be two Hermitian matrices and suppose B is positive definite. Then there exists an invertible matrix $S \in \mathbb{C}^{n\times n}$ such that S^*AS is diagonal and $S^*BS = I$.*

Proof. As B is Hermitian, the spectral theorem guarantees the existence of a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $B = U^{-1}\operatorname{diag}(a_1, \ldots, a_n)U$ for some real numbers $a_1, \ldots, a_n$. As B is also positive definite, we have $a_1, \ldots, a_n > 0$. Define

$$B^{1/2} := U^{-1}\operatorname{diag}(\sqrt{a_1}, \ldots, \sqrt{a_n})U.$$

Taking conjugate transposes and using that U is unitary, we see easily that $B^{1/2}$ is Hermitian, and in the same manner we conclude that $B^{-1/2}AB^{-1/2}$ is Hermitian. Using again the spectral theorem, we find a unitary matrix $V \in \mathbb{C}^{n\times n}$ such that

$$V^{-1}(B^{-1/2}AB^{-1/2})V$$

is diagonal. We claim that $S := B^{-1/2}V$ satisfies the desired conditions. Indeed:

$$S^*BS = V^*(B^{-1/2})^*BB^{-1/2}V = V^{-1}B^{-1/2}BB^{-1/2}V = V^{-1}V = I,$$

and

$$S^*AS = V^*(B^{-1/2})^*AB^{-1/2}V = V^{-1}(B^{-1/2}AB^{-1/2})V$$

is diagonal by the choice of V. □

A.4. Chern forms

To each smooth metrized line bundle over a smooth analytic space X of dimension n, we can associate its *Chern form*. In order to do that, we use the notations $d := \partial + \overline{\partial}$ and $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$. Notice that we have $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

Let $\overline{L}$ be a metrized line bundle over X, and let $(U_i)_{i\in I}$ be a trivializing open cover for L such that U_i is homeomorphic to an open set of $\mathbb{C}^n$. Fix a local frame $s_i \in U_i$ and a system of local coordinates $(z_1^i, \dots, z_n^i)$ for each $i \in I$. Then we can define, for each $i \in I$, the real $(1,1)$-form $\omega_i := (dd^c)_i \log \|s_i\|^{-1}$ on U_i, where $(dd^c)_i$ indicates that the differential operators ∂ and $\overline{\partial}$ are taken with respect to the chosen local system.

For each, $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, let g_{ij} be the *transition function* of i, j, that is, the holomorphic function on $U_i \cap U_j$ such that $s_i = g_{ij}s_j$. Then on $U_i \cap U_j$ we have $\|s_i\| = \|g_{ij}s_j\| = |g_{ij}|\|s_j\|$, by property (2) of a metric. Hence, on $U_i \cap U_j$:

$$\omega_i = (dd^c)_i \log \|s_i\|^{-1} = (dd^c)_i \log (|g_{ij}|\|s_j\|)^{-1} = (dd^c)_i \log \|s_j\|^{-1} - (dd^c)_i \log |g_{ij}|.$$

We need the following lemma:

Lemma A.3. *Let $g\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Then $dd^c \log |g| = 0$.*

Proof. Let $z_1, \dots, z_n$ be the usual complex coordinates. Expanding $dd^c = \frac{i}{\pi}\partial\overline{\partial}$, we obtain:

$$dd^c \log |g|^{-1} = \frac{i}{\pi}\partial\overline{\partial}(-\log|g|) = -\frac{i}{\pi}\partial_i\overline{\partial}_i \log |g| = -\frac{i}{\pi}\sum_{1\leq k,l\leq n} \frac{\partial^2(\log |g|)}{\partial z_k \partial \overline{z_l}} dz_k \wedge d\overline{z_l}.$$

Fixed $1 \leq k, l \leq n$, we compute:

$$\frac{\partial^2(\log |g|)}{\partial z_k \partial \overline{z_l}} = \frac{\partial}{\partial z_k}\left(\frac{1}{|g|}\cdot\frac{\partial(|g|)}{\partial \overline{z_l}}\right) = \frac{1}{|g|}\cdot\frac{\partial^2(|g|)}{\partial z_k \partial \overline{z_l}} - \frac{1}{|g|^2}\cdot\frac{\partial(|g|)}{\partial z_k}\cdot\frac{\partial(|g|)}{\partial \overline{z_l}}.$$

Notice that $|g| = g\overline{g}$. As g is holomorphic, we have $\frac{\partial g}{\partial \overline{z_l}} = \frac{\partial \overline{g}}{\partial z_j} = 0$, so:

$$\frac{\partial(|g|)}{\partial z_k} = \frac{\partial(g\overline{g})}{\partial z_k} = \overline{g}\frac{\partial g}{\partial z_k}$$

Similarly, $\frac{\partial(|g|)}{\partial \overline{z_l}} = g\frac{\partial \overline{g}}{\partial \overline{z_l}}$. So we have:

$$\begin{aligned}
\frac{\partial^2(\log |g|)}{\partial z_k \partial \overline{z_l}} &= \frac{1}{|g|}\cdot\frac{\partial}{\partial z_k}\left(g\frac{\partial \overline{g}}{\partial \overline{z_l}}\right) - \frac{1}{|g|^2}\cdot\overline{g}\frac{\partial g}{\partial z_k}\cdot g\frac{\partial \overline{g}}{\partial \overline{z_l}} \\
&= \frac{1}{|g|}\cdot\frac{\partial g}{\partial z_k}\cdot\frac{\partial \overline{g}}{\partial \overline{z_l}} + \overline{g}\frac{\partial^2 \overline{g}}{\partial z_k \partial \overline{z_l}} - \frac{1}{|g|}\cdot\frac{\partial g}{\partial z_k}\cdot\frac{\partial \overline{g}}{\partial \overline{z_l}} \\
&= \overline{g}\frac{\partial^2 \overline{g}}{\partial z_k \partial \overline{z_l}} = 0,
\end{aligned}$$

since the function $\frac{\partial \overline{g}}{\partial \overline{z_l}}$ is antiholomorphic. □

Using the lemma above, we conclude that $\omega_i = (dd^c)_i \log \|s_j\|^{-1} = (dd^c)_j \log\|s_j\|^{-1} = \omega_j$, since dd^c does not depend on the choice of coordinates. So $\omega_i = \omega_j$ on $U_i \cap U_j$, and this gives us a well-defined real $(1,1)$-form over X.

Definition. Let $\overline{L}$ be a smooth metrized line bundle over X. The real $(1,1)$-form constructed above is denoted by $c_1(\overline{L})$, and is called the **Chern form** or **curvature** of $\overline{L}$.

The Chern form of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{C}}$ equipped with the Fubini-Study metric is denoted by ω_{FS}.

For $0 \leq i \leq n$, let $U_i \subseteq \mathbb{P}^n_{\mathbb{C}}$ be the open set given by $x_i \neq 0$. Then the U_i cover $\mathbb{P}^n_{\mathbb{C}}$. Over U_0, we can take $z_1 = x_1/x_0, \dots z_n = x_n/x_0$ as a system of local coordinates. Then a direct calculation (see [4] for example) shows that:

$$\omega_{\mathrm{FS}}|_U = \frac{i}{2\pi}\sum_{j=1}^{n}\frac{1}{1+\|z\|^2}dz_j \wedge dz_k - \frac{i}{2\pi}\sum_{j,k=1}^{n}\frac{z_k\overline{z_j}}{(1+\|z\|^2)^2}dz_j \wedge dz_k,$$

where we denote $z = (z_1, \dots, z_n)$ and $\|z\|^2 \coloneqq \sum_{j=1}^{n}|z_j|^2$. By this expression, it is clear that ω_{FS} is positive at $z = 0$. Using the invariance of ω_{FS} with respect to some transitive action, we can prove that ω_{FS} is a positive form (see [27, Lemma 3.16]).

Using the Chern form, we can define the concept of positivity for metrized line bundles:

Definition. Let $\overline{L}$ be a metrized line bundle over X. We say $\overline{L}$ is **positive** (resp. **semipositive**) if $c_1(\overline{L})$ is a positive $(1,1)$-form.

For example, $(\mathcal{O}(1), \|\cdot\|_{\mathrm{FS}})$ is a positive metrized line bundle.

A.5. Hermitian line bundles on arithmetic varieties

Definition. Let $\mathcal{X}$ be a flat, integral, and projective $\mathcal{O}_K$-scheme. A **($\mathscr{C}^\infty$-)Hermitian line bundle** $\overline{\mathscr{L}}$ on $\mathcal{X}$ is the datum $(\mathscr{L}, \{\|\cdot\|_\nu\}_{\nu\in\Sigma_\infty(K)})$ of a line bundle $\mathscr{L}$ on $\mathcal{X}$ and, for each $\nu \in \Sigma_\infty(K)$, of a $\mathscr{C}^\infty$-Hermitian metric $\|\cdot\|_\nu \colon \mathscr{L}^{\mathrm{an}}_{\mathbb{C}_\nu} \to \mathbb{R}_+$ that is invariant under $\mathrm{Gal}(\mathbb{C}_\nu/K_\nu)$.

We say that $\overline{\mathscr{L}}$ is **vertically semipositive** if L is relatively nef with respect to $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ and for each $\nu \in \Sigma_\infty(K)$, the $\mathscr{C}^\infty$-Hermitian metric $\|\cdot\|_\nu$ is semipositive.

Two Hermitian line bundles $\overline{\mathscr{L}}$ and $\overline{\mathscr{M}}$ are called **isometric** if there is an isomorphism $\mathscr{L} \cong \mathscr{M}$ that is an isometry in all Archimedean places. The **arithmetic Picard group** $\widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$ is the set of isometry classes of $\mathscr{C}^\infty$ Hermitian line bundles on $\mathcal{X}$.

The trivial line bundle $\mathcal{O}_{\mathcal{X}}$ can be given the trivial metric $|\cdot|_\nu$ in each Archimedean place, turning it into a Hermitian line bundle $\overline{\mathcal{O}}_{\mathcal{X}}$. Moreover, fixed a place $\nu_0 \in \Sigma_\infty(K)$ and a function $f \in \mathscr{C}^\infty(\mathcal{X}^{\mathrm{an}}_{\mathbb{C}_{\nu_0}})$, we define $\overline{\mathcal{O}}_{\mathcal{X}}(f) \coloneqq (\mathcal{O}_{\mathcal{X}}, \|\cdot\|_\nu)$, where $\|\cdot\|_{\nu_0} = e^{-f}|\cdot|_{\nu_0}$ and $\|\cdot\|_\nu = |\cdot|_\nu$ for $\nu \neq \nu_0$.

Given the line bundle $\mathcal{O}(1)$ in $\mathbb{P}^N_{\mathcal{O}_K}$, we can equip it with the Fubini-Study metric in each Archimedean place, giving rise to the $\mathscr{C}^\infty$-Hermitian line bundle $\overline{\mathcal{O}}(1) = (\mathcal{O}(1), \|\cdot\|_{\mathrm{FS},\nu})$. Since $\mathcal{O}(1)$ is very ample and the Chern form of a Fubini-Study metric is a strictly positive Hermitian form, we conclude that $\overline{\mathcal{O}}(1)$ is vertically semipositive.

Given two Hermitian line bundles $\overline{\mathscr{L}}$ and $\overline{\mathscr{M}}$, we can define a Hermitian structure on the line bundles $\mathscr{L} \otimes \mathscr{M}$ and $\mathscr{L}^{\otimes -1}$ by doing so for each Archimedean place. This turns $\widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$ into a group, as we would expect. In particular, given $\overline{\mathscr{L}} \in \widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$ and

$f \in \mathscr{C}^\infty(\mathcal{X}^{\mathrm{an}}_{\mathbb{C}_{\nu_0}})$, we define $\overline{\mathscr{L}}(f) \coloneqq \overline{\mathscr{L}} \otimes \overline{\mathcal{O}}_{\mathcal{X}}(f)$. We can also define a structure of Hermitian line bundle on $\mathcal{O}(k)$, for every integer k, by $\overline{\mathcal{O}}(k) \coloneqq \overline{\mathcal{O}}(1)^{\otimes k}$.

Similarly, a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of flat, integral, projective $\mathcal{O}_K$-schemes induces a group homomorphism $f^*\colon \widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{X}) \to \widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{Y})$. Let N_1, N_2 be positive integers and consider $\mathrm{pr}_i\colon \mathbb{P}^{N_1}_{\mathcal{O}_K} \times \mathbb{P}^{N_2}_{\mathcal{O}_K} \to \mathbb{P}^{N_i}_{\mathcal{O}_K}$. We can then define, for any integers k_1, k_2, the Hermitian line bundle $\overline{\mathcal{O}}(k_1, k_2) \coloneqq \mathrm{pr}_1^*\,\overline{\mathcal{O}}(k_1) \otimes \mathrm{pr}_2^*\,\overline{\mathcal{O}}(k_2) \in \widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathbb{P}^{N_1}_{\mathcal{O}_K} \otimes \mathbb{P}^{N_2}_{\mathcal{O}_K})$.

As both the notions of relative nef and semipositivity descend to the isomorphism classes of $\widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$, so does the notion of vertical semipositivity. This allows us to define:

Definition. $L \in \widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$ is said to be **vertically integrable** if there exist vertically semipositive $\overline{\mathscr{L}}_1, \overline{\mathscr{L}}_2 \in \widehat{\mathrm{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$ such that $\overline{\mathscr{L}} = \overline{\mathscr{L}}_1 - \overline{\mathscr{L}}_2$.

Clearly, $\overline{\mathcal{O}}(1)$ is vertically integrable.

Given a $\mathscr{C}^\infty$-Hermitian line bundle $\overline{\mathscr{L}} = (\mathscr{L}, \|\cdot\|_\nu)$ on $\mathcal{X}$, we can define the line bundle $\mathscr{L}_K \coloneqq \mathscr{L} \times_S K$ on $\mathcal{X}_K \coloneqq \mathcal{X} \times_K S$. We denote by $\overline{\mathscr{L}}_K$ the data $(\mathscr{L}_K, (\|\cdot\|_\nu)_{\nu \in \Sigma_\infty(K)})$.

B. Intersection Numbers

In this appendix, we will deal with the important topic of intersection theory. Let X be a variety and $Y, Z \subseteq X$ be two closed subvarieties with codimensions m and n, respectively. Then $Y \cap Z$ is a subvariety of X, and we would expect it to have codimension $m+n$. However, this is not necessarily true for particular choices of Y and Z. The idea is that by "moving a little bit" Y and Z, we will get the "true" intersection of the two subvarieties. This is done in the classical book [8]. Here we will generalize this idea to consider intersections of line bundles, and even of line bundles over number fields.

B.1. Classical Intersection Numbers

Let X be a variety of dimension d, and let $D_1, \dots, D_d$ be irreducible divisors which *intersect properly*, that is, $\bigcap_{j=1}^d D_j$ is a finite set of points. We can then define the **intersection number** $(D_1, \dots, D_d)_X$ to be the number of points in this intersection, with some (non-negative) multiplicities that correspond to the dimension of local vector spaces. Varying each divisor by linear equivalence, it is always possible to get proper intersections that do not depend on the divisors chosen. This allows us to define intersection numbers for elements of the Picard Group. Hence, given line bundles $L_1, \dots, L_d \in \mathrm{Pic}(X)$, we can define $(L_1, \dots, L_d)_X \in \mathbb{N}$.

The intersection numbers have nice properties. For example, if $f\colon Y \to X$ is a finite morphism, we have the equality:

$$(f^*L_1, \dots, f^*L_d)_Y = \deg(f) \cdot (L_1, \dots, L_d)_X.$$

We can also define the degree of a subvariety with respect to a line bundle:

Definition. Let X be a projective variety, let $\iota\colon Z \to X$ be a subvariety of dimension r, and let $L \in \mathrm{Pic}(X)$. The **degree** of Z with respect to L is defined to be:

$$\deg_L(Z) \coloneqq (\iota^*L, \dots \iota^*L)_Z.$$

For more details, see [10, Section A.2.3].

Suppose now that X is a complex analytic space and that $\overline{L}$ is a Hermitian line bundle on X. By the Wirtinger formula (see [9, Chapter 3, section "Degree of a Variety"]), we have:

$$\int_X c_1(\overline{L})^{\wedge n} = \deg_L(X). \tag{9}$$

B.2. Snapper Intersection Numbers

Let X be a proper scheme of dimension d. Let $\mathscr{F}$ be a coherent sheaf on X and $L_1, \ldots, L_t$ be line bundles on X. Let $s \coloneqq \dim \operatorname{Supp} \mathscr{F}$. Snapper proved that the Euler characteristic $\chi(\mathscr{F} \otimes L_1^{\otimes n_1} \otimes \cdots \otimes L_t^{\otimes n_t})$ is a polynomial in $n_1, \ldots, n_t$ of total degree s, with rational coefficients, which assumes integer values whenever $n_1, \ldots, n_t$ are integers.

Definition. Let $L_1, \ldots, L_t$ be line bundles on X, and let $\mathscr{F}$ be a coherent sheaf on X such that $\dim \operatorname{Supp} \mathscr{F} \leq t$. The **Snapper intersection number** $(L_1 \cdots L_t \cdot \mathscr{F})_X$ of $L_1, \ldots, L_t$ with $\mathscr{F}$ is the coefficient of the monomial $n_1 \cdots n_t$ in the polynomial $\chi(\mathscr{F} \otimes L_1^{\otimes n_1} \otimes \cdots \otimes L_t^{\otimes n_t})$.

The Snapper intersection numbers have the following properties:

Proposition B.1. *The Snapper intersection number $(L_1 \cdots L_t \cdot \mathscr{F})_X$ is an integer, and this is a symmetric t-linear form in $L_1, \ldots, L_t$. Besides:*

(a) $(L_1 \cdots L_t \cdot \mathscr{F})_X = 0$ *if* $\dim \operatorname{Supp} \mathscr{F} < t$, *and* $(\mathscr{F})_X = h^0(X, \mathscr{F})$ *if* $\dim \operatorname{Supp} \mathscr{F} = t = 0$.

(b) If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ *is an exact sequence, then:*

$$(L_1 \cdots L_t \cdot \mathscr{G})_X = (L_1 \cdots L_t \cdot \mathscr{F})_X + (L_1 \cdots L_t \cdot \mathscr{H})_X.$$

(c) Let Y be a subvariety of X containing $\operatorname{supp} \mathscr{F}$. *Denote $L_{iY} \coloneqq L_i \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Then we have:*

$$(L_1 \cdots L_t \cdot \mathscr{F})_X = (L_{1Y} \cdots L_{tY} \cdot \mathscr{F})_Y$$

If $Y \subseteq X$ is a closed subscheme of dimension less or equal to t, we denote:

$$(L_1 \cdots L_t \cdot Y)_X \coloneqq (L_1 \cdots L_t \cdot \mathcal{O}_Y).$$

If $Y = X$, we denote this number simply by $(L_1 \cdots L_t)_X$. By the proposition above, we have:

$$(L_1 \cdots L_t \cdot Y)_X = (L_{1Y} \cdots L_{tY})_X$$

When X is a variety, the Snapper intersection numbers coincide with the classical ones, that is, $(L_1, \ldots, L_d)_X = (L_1 \cdots L_d)_X$. For more details about the Snapper intersection numbers, see [13, Chapter I]. We also need the following result:

Proposition B.2. *Let X be a proper variety. If $L_1, \ldots, L_t$ are nef line bundles on X and if $\mathcal{F}$ is a coherent sheaf on X such that* $\dim \operatorname{Supp} \mathcal{F} = t$, *then* $(L_1 \cdots L_t \cdot \mathcal{F})_X \geq 0$.

(see [13, Theorem III.2.1]).

B.3. Arithmetic Intersection Numbers

Let K be a number field. Let $\mathcal{X}$ be an *arithmetic variety* of relative dimension d over $\mathcal{O}_K$, that is, a flat and quasi-projective scheme over $\operatorname{Spec}\mathcal{O}_K$ whose generic fiber $\mathcal{X}_K = \mathcal{X} \times_{\mathcal{O}_K} K$ is regular. We also suppose that $\mathcal{X}$ is regular and smooth over K. Let $\overline{\mathscr{L}}_1, \dots, \overline{\mathscr{L}}_p \in \widehat{\operatorname{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$ be vertically integrable arithmetic Hermitian line bundles, and let Z be a d-cycle on $\mathcal{X}$. We want to define an arithmetic intersection number

$$\mathscr{L}_1 \cdot \mathscr{L}_2 \cdots \mathscr{L}_{d'+1} \cdot Z \in \mathbb{R}$$

which satisfies good properties. The construction of such intersection numbers is very technical, and can be done in more than one way. Here we will follow the approach of [2]. For the proofs and more details about this construction see this reference.

The set $\mathcal{X}(\mathbb{C})$ may be identified with the disjoint union $\bigsqcup_{\nu\in\Sigma_\infty(K)} \mathcal{X}(\mathbb{C}_\nu)$. We denote by $F_\infty \colon \mathcal{X}(\mathbb{C}) \to X(\mathbb{C})$ the antiholomorphic involution given by complex conjugation in each $\mathcal{X}(\mathbb{C}_\nu)$.

For any $p \in \mathbb{N}$, we let $Z_p(\mathcal{X})$ (resp. $Z^p(\mathcal{X})$) be the group of cycles of dimension p (resp. codimension p). We denote by $A^{pp}(\mathcal{X}_\mathbb{R})$ (resp. $\mathscr{D}^{pp}(\mathcal{X}_\mathbb{R})$) the set of real forms $\alpha \in A^{pp}(\mathcal{X}(\mathbb{C}))$ (resp. real currents $\alpha \in \mathscr{D}^{pp}(\mathcal{X}(\mathbb{C}))$) such that $F_\infty^*(\alpha) = (-1)^p\alpha$.

Any cycle Z in $Z^p(\mathcal{X})$ defines a current $\delta_Z \in \mathscr{D}^{pp}(\mathcal{X}_\mathbb{R})$ by integration on its set of complex points: if $Z = \sum_\alpha n_\alpha Z_\alpha$, $\delta_Z = \sum_\alpha n_\alpha \delta_{Z_\alpha(\mathbb{C})}$. A **Green current** for Z is any current $g \in \mathscr{D}^{p-1,p-1}(\mathcal{X}_\mathbb{R})$ such that $dd^c g + \delta_Z$ is a smooth form.

We let $\widehat{Z}^p(\mathcal{X})$ be the group of pairs (Z, g) where $Z \in Z^p(X)$ and g is a Green current for Z, with addition defined componentwise. Let $\widehat{R}^p(\mathcal{X}) \subseteq \widehat{Z}^p(\mathcal{X})$ be the subgroup generated by pairs of the form $(0, \partial u + \overline{\partial} v)$ and $(\operatorname{div}(f), -\log|f|^2)$, where $f \in k(Y)^*$ is a nontrivial rational function on an integral subscheme $Y \subseteq \mathcal{X}$ of codimension $p-1$, and $-\log|f|^2$ is the current on $\mathcal{X}(\mathbb{C})$ obtained by restricting forms to the smooth part of $Y(\mathbb{C})$ and integrating against the L^1 function $-\log|f|^2$.

Definition. The **arithmetic Chow group of codimension** p **of** X is $\widehat{CH}^p(\mathcal{X}) := \widehat{Z}^p(\mathcal{X})/\widehat{R}^p(\mathcal{X})$.

If $\overline{\mathscr{L}}$ is a Hermitian line bundle on $\mathcal{X}$, we can associate to this line bundle an element $\widehat{c}_1(\overline{\mathscr{L}})$ defined by $\widehat{c}_1(\overline{\mathscr{L}}) = [(\operatorname{div} s, -\log\|s\|^2)]$ for any rational section s of L over $\mathcal{X}$ of norm $\|s\|$ on $\mathcal{X}(\mathbb{C})$. We call this element the *(arithmetic) Chern form* of $\overline{\mathscr{L}}$.

The morphism $\operatorname{Spec}\mathcal{O}_K \to \operatorname{Spec}\mathbb{Z}$ induces a morphism $\widehat{CH}^1(\operatorname{Spec}\mathcal{O}_K) \to \widehat{CH}^1(\operatorname{Spec}\mathbb{Z})$. Now, it is possible to show that $\widehat{CH}^1(\operatorname{Spec}\mathbb{Z}) \cong \mathbb{R}$, so by composition we obtain a map $\widehat{\deg} \colon \widehat{CH}^1(\mathcal{X}) \to \mathbb{R}$, which is called the *arithmetic degree*. For a Hermitian line bundle $\overline{\mathscr{L}} = (\mathscr{L}, \{\|\cdot\|_\nu\}_{\nu\in\Sigma_\infty(K)})$, it is possible to show that we have:

$$\widehat{\deg}(\widehat{c}_1(\overline{L})) = \log\#(\mathscr{L}/(s)) - \sum_{\nu\in\Sigma_\infty(K)} \log\|s\|_\nu, \tag{10}$$

for every nonzero section s of $\mathscr{L}$.

In good conditions, given Green currents g_1 and g_2 for two cycles Z_1 and Z_2 respectively, it is possible to define a Green current $g_1 * g_2$ in $Z_1 \cdot Z_2$, which we call the *star product* of

g_1 and g_2. This induces a cup product:

$$\widehat{CH}^p(\mathcal{X}) \otimes \widehat{CH}^q(\mathcal{X}) \to \widehat{CH}^{p+q}(\mathcal{X}).$$

Taking several cup products is a commutative and associative operation because the same holds for star products. We can similarly define *Chow homology groups of dimension* p consisting of classes of pairs (Z, g) where $Z \in Z_p(X)$ and g is a Green current for Z. We can then define a cap product:

$$\widehat{CH}^q(\mathcal{X}) \otimes \widehat{CH}_p(\mathcal{X}) \to \widehat{CH}^{q-p+1}(\mathcal{X}).$$

There is a way of choosing Green forms for elements of $Z_p(\mathcal{X})$ so that the map above induces a biadditive pairing:

$$\widehat{CH}^q(\mathcal{X}) \times Z_p(\mathcal{X}) \xrightarrow{(x,Z)\mapsto(x|Z)} \widehat{CH}^{q-p+1}(\operatorname{Spec}\mathcal{O}_K).$$

Suppose now that $q = p$. Then each $(x|Z)$ is an element of $\widehat{CH}^1(\operatorname{Spec}\mathcal{O}_K)$, and taking its arithmetic degree we obtain a real number. Therefore, we get a biadditive pairing $\widehat{CH}^p(\mathcal{X}) \times Z_p(\mathcal{X}) \to \mathbb{R}$. Given vertically integrable Hermitian line bundles $\overline{\mathscr{L}}_1, \dots, \overline{\mathscr{L}}_p \in \widehat{\operatorname{Pic}}_{\mathscr{C}^\infty}(\mathcal{X})$ on $\mathcal{X}$, we can consider the element of $\widehat{CH}^p(\mathcal{X})$ given by $\widehat{c}_1(\overline{\mathscr{L}}_1) \otimes \cdots \otimes \widehat{c}_1(\overline{\mathscr{L}}_p)$. The biadditive pairing above then induces a linear map:

$$\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_p \cdot (-) \colon Z_p(\mathcal{X}) \to \mathbb{R}\,.$$

For any $Z \in Z_p(\mathcal{X})$, we call $\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_p \cdot Z$ the **arithmetic intersection number** of $\overline{\mathscr{L}}_1, \dots, \overline{\mathscr{L}}_p$ with Z. If $p = d+1$ and Z is the cycle $\mathcal{X}$, we denote the intersection number $\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_{d+1} \cdot \mathcal{X}$ simply by $\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_{d+1}$.

The arithmetic intersection numbers have the following property:

Proposition B.3. *Let $\overline{\mathscr{L}}_1, \dots, \overline{\mathscr{L}}_{d+1}, \overline{\mathscr{L}}'_1$ be vertically integrable Hermitian line bundles on $\mathcal{X}$. Let $0 \le p \le d+1$, and let $Z \in Z_p(\mathcal{X})$.*

(a) (Multilinearity) We have:

$$\left(\overline{\mathscr{L}}_1 \otimes \overline{\mathscr{L}}'_1\right) \cdot \overline{\mathscr{L}}_2 \dots \overline{\mathscr{L}}_p \cdot Z = \overline{\mathscr{L}}_1 \cdot \overline{\mathscr{L}}_2 \dots \overline{\mathscr{L}}_p \cdot Z + \overline{\mathscr{L}}'_1 \cdot \overline{\mathscr{L}}_2 \dots \overline{\mathscr{L}}_d \cdot Z.$$

(b) (Commutativity) For any permutation $\sigma \colon \{1, \dots, p\} \to \{1, \dots, p\}$, we have:

$$\overline{\mathscr{L}}_{\sigma(1)} \cdot \overline{\mathscr{L}}_{\sigma(2)} \cdots \overline{\mathscr{L}}_{\sigma(p)} \cdot Z = \overline{\mathscr{L}}_1 \cdot \overline{\mathscr{L}}_2 \cdots \overline{\mathscr{L}}_p \cdot Z.$$

(c) Let $\nu \in \Sigma_\infty(K)$ be an Archimedean place and let $f \in \mathscr{C}^\infty(\mathcal{X}^{an}_{\mathbb{C}_\nu})$. Then:

$$\begin{aligned}
&\overline{\mathscr{L}}_1 \cdot \overline{\mathscr{L}}_2 \cdots \overline{\mathscr{L}}_{d+1-p} \cdot \overline{\mathcal{O}}_{\mathcal{X}}(f)^p \\
= \quad & \delta_\nu \int_{\mathcal{X}^{\mathrm{an}}_{\mathbb{C}_\nu}} f\,(dd^c f)^{p-1} \wedge c_1\left(\|\cdot\|_{1,\nu}\right) \wedge c_1\left(\|\cdot\|_{2,\nu}\right) \wedge \cdots \wedge c_1\left(\|\cdot\|_{d+1-p,\nu}\right)
\end{aligned}$$

B.4. Arithmetic volumes and Minkowski's Theorem

Let $\mathcal{X}$ be a flat, integral and projective $\mathcal{O}_K$-scheme of relative dimension d over $\mathcal{O}_K$, and let $X \coloneqq \mathcal{X}_K$. Given a line bundle L on X, we can define its *volume* as being:

$$\mathrm{vol}(L) \coloneqq \limsup_{N\to\infty} \frac{h^0(X, L^{\otimes N})}{N^{d+1}/(d+1)!}.$$

Similarly, to each $\mathscr{C}^\infty$-Hermitian line bundle $\overline{\mathscr{L}} = (\mathscr{L}, \{\|\cdot\|_\nu\}_{\nu\in\Sigma_\infty(K)})$, we can associate an *arithmetic volume* $\widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}})$. In order to construct such a volume, we start by defining, for each integer $N \geq 0$, $V_{N,\mathbb{Z}} \coloneqq \Gamma(\mathcal{X}, \mathscr{L}^{\otimes N})$ and $V_{N,\mathbb{R}} \coloneqq \Gamma(\mathcal{X}, \mathscr{L}^{\otimes N})_\mathbb{R} = V_{N,\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{R}$. Then we can identify naturally $V_{N,\mathbb{Z}}$ as a subset of $V_{N,\mathbb{R}}$, and we have a injection:

$$V_{N,\mathbb{R}} \xhookrightarrow{s\mapsto(s_\nu)} \prod_{\nu\in\Sigma_\infty(K)} \Gamma(\mathcal{X}_{\mathbb{C}_\nu}, \mathscr{L}_{\mathbb{C}_\nu}^{\otimes N}),$$

The image of $V_{N,\mathbb{Z}}$ under this injection is a $\mathbb{Z}$-lattice. For each $\nu \in \Sigma_\infty(K)$, the norm $\|\cdot\|_\nu$ in $\mathscr{L}_{\mathbb{C}_\nu}$ induces a norm $\|\cdot\|_\nu^{\otimes N}$ in $\mathscr{L}_{\mathbb{C}_\nu}^{\otimes N}$. For each section $s \in V_{N,\mathbb{Z}}$, we can then define:

$$\|s_\nu\|_\nu^{(\infty)} = \max_{x\in X_{\mathbb{C}_\nu}^{\mathrm{an}}} \left\{\|s_\nu(x)\|_\nu^{\otimes N}\right\}$$

Observe that this maximum exists, since X is projective, and therefore $X_{\mathbb{C}_\nu}^{\mathrm{an}}$ is compact. This naturally induces a supremum norm in the finite product $\prod_{\nu\in\Sigma_\infty(K)} \Gamma(\mathcal{X}_{\mathbb{C}_\nu}, \mathscr{L}_{\mathbb{C}_\nu}^{\otimes N})$. By restriction, we obtain an $\mathbb{R}$-norm on $H^0\left(\mathcal{X}, \mathscr{L}^{\otimes N}\right)_\mathbb{R}$ and a seminorm on $H^0\left(\mathcal{X}, \mathscr{L}^{\otimes N}\right)$. We consider the unit ball:

$$B_N \coloneqq \left\{s \in V_{N,\mathbb{R}} \colon \forall\, \nu \in \Sigma_\infty(K),\ \|s_\nu\|_\nu^{(\infty)} \leq 1\right\}.$$

Notice that $V_{N,\mathbb{R}}$ is a locally compact group and $V_{N,\mathbb{R}}/V_{N,\mathbb{Z}}$ is compact. Hence there exists a unique Haar measure on $V_{N,\mathbb{R}}$ such that the induced quotient measure in $V_{N,\mathbb{R}}/V_{N,\mathbb{Z}}$ has total mass 1. We denote such Haar measure by $\mathrm{vol}_N(\cdot)$. If $V_{N,\mathbb{Z}} \neq 0$, then $\mathrm{vol}_N(B_N) > 0$ (see for example [3], Theorem 7.22 and Lemma 7.23(iii)). So we can define:

$$\chi_{\sup}(\overline{\mathscr{L}}^{\otimes N}) \coloneqq \begin{cases} \log \mathrm{vol}_N(B_N), & \text{if } V_{N,\mathbb{Z}} \neq 0; \\ 0, & \text{if } V_{N,\mathbb{Z}} = 0. \end{cases}$$

Finally, we define the **arithmetic volume** of $\overline{\mathscr{L}}$ as:

$$\widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}}) = \limsup_{N\to\infty} \frac{\chi_{\sup}(\overline{\mathscr{L}}^{\otimes N})}{N^{d+1}/(d+1)!}.$$

We will need the following properties of arithmetic volumes:

Lemma B.4. *(a) Let $\nu \in \Sigma_\infty(K)$ and a real $\varepsilon > 0$ be given. Assume that $\mathscr{L}_K$ is nef and big. For a sufficiently large positive integer N_0, there exists a nonzero section $s \in \Gamma(\mathcal{X}, \mathscr{L}^{\otimes N_0})$ such that*

$$\delta_\nu \log \|s_\nu\|_\nu^{(\infty)} \leq \left(-\frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}})}{(d+1)\deg_{\mathscr{L}_K}(X)} + \varepsilon\right) N_0$$

and $\log \|s_\nu\|_\mu^{(\infty)} \leq 0$ *for all other* $\mu \in \Sigma_\infty(K)\backslash\{\nu\}$.

(b) *Let* $\overline{\mathscr{L}} = (\mathscr{L}, \{\|\cdot\|_\nu\}_{\nu\in\Sigma_\infty(K)})$ *be a* $\mathscr{C}^\infty$*-Hermitian line bundle. For any* $\nu \in \Sigma_\infty(K)$ *and any real* $\lambda \in \mathbb{R}$*, which is considered as a real-valued constant function on* $\mathcal{X}^{\mathrm{an}}_{\mathbb{C}_\nu}$*, we have:*
$$\widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}}(\lambda)) = \widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}}) + \delta_\nu\lambda(d+1)\,\mathrm{vol}(\mathscr{L}_K).$$

(c) *Let* $\overline{\mathscr{L}}$ *and* $\overline{\mathscr{M}}$ *be two* $\mathscr{C}^\infty$*-Hermitian nef line bundles on* $\mathcal{X}$*. Then:*
$$\widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}} \otimes \overline{\mathscr{M}}^{\otimes -1}) \geq \overline{\mathscr{L}}^{d+1} - (d+1)\cdot(\overline{\mathscr{L}}^d \cdot \overline{\mathscr{M}}).$$

(For (a) see [15, Lemma 2.7], for (b) see [15, Lemma 2.5 (a)] and for (c) see [11, Theorem 3.5.3 and Remark 3.5.4]) Let $\overline{\mathscr{L}} = (\mathscr{L}, \{\|\cdot\|_\mu\}_{\mu\in\Sigma_\infty(K)})$ be a vertically integrable Hermitian line bundle on X, and let $\lambda \in \mathbb{R}$, considered as a real-valued constant function on $\mathcal{X}^{\mathrm{an}}_{\mathbb{C}_\nu}$. Then we have:
$$\overline{\mathscr{L}}(\lambda)^{d+1} = (\overline{\mathscr{L}} \otimes \overline{\mathcal{O}}_{\mathcal{X}}(\lambda))^{d+1} = \sum_{j=0}^{d+1} \binom{d+1}{j} \cdot \overline{\mathscr{L}}^{d+1-j} \cdot \overline{\mathcal{O}}_{\mathcal{X}}(\lambda)^j,$$
by Proposition B.3, items (a) and (b). Since $dd^c(\lambda) = 0$, by the item (c) of the same proposition we have $\overline{\mathscr{L}}^{d+1-j} \cdot \overline{\mathcal{O}}_{\mathcal{X}}(\lambda)^j = 0$ for $j \geq 2$, and
$$\overline{\mathscr{L}}^d \cdot \overline{\mathcal{O}}_{\mathcal{X}}(\lambda) = \delta_\nu \int_{X_{\mathbb{C}_\nu^{\mathrm{an}}}} \lambda \cdot c_1(\|\cdot\|_\nu)^{\wedge d} = \delta_\nu\lambda \int_{X_{\mathbb{C}_\nu^{\mathrm{an}}}} c_1(\|\cdot\|_\nu)^{\wedge d}.$$
If $\mathscr{L}_K$ is nef, this last integral is equal to $\mathrm{vol}(\mathscr{L}_K)$ (see [17, Section 2.2.C]). We conclude that, in this case:
$$\overline{\mathscr{L}}(\lambda)^{d+1} = \overline{\mathscr{L}}^{d+1} + \delta_\nu\lambda(d+1)\,\mathrm{vol}(\mathscr{L}_K).$$
Comparing this with Lemma B.4, item (b) we get:
$$\overline{\mathscr{L}}(\lambda)^{d+1} - \overline{\mathscr{L}}^{d+1} = \widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}}(\lambda)) - \widehat{\mathrm{vol}}_\chi(\overline{\mathscr{L}}) \tag{11}$$

References

[1] Arbarello, Enrico, Cornalba, Maurizio, and Griffiths, Phillip A. *Geometry of algebraic curves: volume II with a contribution by Joseph Daniel Harris*. Springer, 2011.

[2] Bost, J.-B., Gillet, Henri, and Soulé, Christophe. “Heights of projective varieties and positive Green forms”. In: *Journal of the American Mathematical Society* 7.4 (1994), pp. 903–1027.

[3] Boucksom, Sébastien and Eriksson, Dennis. “Spaces of norms, determinant of cohomology and Fekete points in non-Archimedean geometry”. In: *Advances in Mathematics* 378 (2021). Publisher: Elsevier, p. 107501.

[4] Chambert-Loir, Antoine. “Heights and measures on analytic spaces”. In: *A survey of recent results, and some remarks* (2010).

[5] Demailly, Jean-Pierre. “Monge-Ampere operators, Lelong numbers and intersection theory”. In: *Complex analysis and geometry.* Springer, 1993, pp. 115–193.

[6] Dimitrov, Vesselin, Gao, Ziyang, and Habegger, Philipp. "Uniformity in Mordell–Lang for curves". In: *Annals of Mathematics* 194.1 (2021). Publisher: Department of Mathematics, Princeton University Princeton, New Jersey, USA, pp. 237–298.

[7] Edixhoven, Bas, Van der Geer, Gerard, and Moonen, Ben. *Abelian Varieties (Preliminary version of the first chapters)*. URL: http://van-der-geer.nl/~gerard/AV.pdf.

[8] Fulton, William. *Intersection theory*. Vol. 2. Springer Science & Business Media, 2013.

[9] Griffiths, Phillip and Harris, Joseph. *Principles of algebraic geometry*. John Wiley & Sons, 2014.

[10] Hindry, Marc and Silverman, Joseph H. *Diophantine geometry: an introduction*. Vol. 201. Springer Science & Business Media, 2013.

[11] Ikoma, Hideaki. "Boundedness of the successive minima on arithmetic varieties". In: *Journal of Algebraic Geometry* 22.2 (2013). Publisher: University Press, pp. 249–302.

[12] Institut Fourier. *A. Chambert-Loir - Equidistribution theorems in Arakelov geometry and Bogomolov conjecture (part1)*. Aug. 2017. URL: https://www.youtube.com/watch?v=_bmso6Ege98 (visited on 07/14/2023).

[13] Kleiman, Steven L. "Toward a numerical theory of ampleness". In: *Annals of Mathematics* (1966). Publisher: JSTOR, pp. 293–344.

[14] Kühne, Lars. "Equidistribution in families of abelian varieties and uniformity". In: *arXiv preprint arXiv:2101.10272* (2021).

[15] Kühne, Lars. "Points of small height on semiabelian varieties". In: *Journal of the European Mathematical Society* 24.6 (2021), pp. 2077–2131.

[16] Lange, Herbert and Birkenhake, Christina. *Complex abelian varieties*. Vol. 302. Springer Science & Business Media, 2013.

[17] Lazarsfeld, Robert K. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*. Vol. 48. Springer, 2017.

[18] Liu, Qing. *Algebraic geometry and arithmetic curves*. Vol. 6. Oxford Graduate Texts in Mathe, 2002.

[19] Mumford, David, Fogarty, John, and Kirwan, Frances. *Geometric invariant theory*. Vol. 34. Springer Science & Business Media, 1994.

[20] Mumford, David, Ramanujam, Chidambaran Padmanabhan, and Manin, Jurij Ivanovič. *Abelian varieties*. Vol. 5. Oxford university press Oxford, 1974.

[21] Philippon, Patrice. "Lemmes de zéros dans les groupes algébriques commutatifs". In: *Bulletin de la Société Mathématique de France* 114 (1986), pp. 355–383.

[22] Raynaud, Michel. "Courbes sur une variété abélienne et points de torsion". In: *Inventiones mathematicae* 71.1 (1983), pp. 207–233.

[23] Raynaud, Michel. "Sous-variétés d'une variété abélienne et points de torsion". In: *Arithmetic and Geometry: Papers Dedicated to IR Shafarevich on the Occasion of His Sixtieth Birthday Volume I Arithmetic* (1983), pp. 327–352.

[24] Raynaud, Michel. *Faisceaux amples sur les schémas en groupes et les espaces homogenes*. Vol. 119. Springer, 2006.

[25] Szpiro, Lucien, Ullmo, Emmanuel, and Zhang, Shou-Wu. "Équirépartition des petits points". In: *Inventiones mathematicae* 127 (1997). Publisher: Citeseer, pp. 337–347.

[26] Ullmo, Emmanuel. "Positivité et discrétion des points algébriques des courbes". In: *Annals of mathematics* (1998). Publisher: JSTOR, pp. 167–179.

[27] Voisin, Claire. “Hodge theory and complex algebraic geometry. I, Translated from the French original by Leila Schneps”. In: *Cambridge Studies in Advanced Mathematics* 76.11 (2002). Publisher: Cambridge University Press Cambridge, p. 3.

[28] Zhang, Shouwu. “Positive line bundles on arithmetic varieties”. In: *Journal of the American Mathematical Society* 8.1 (1995), pp. 187–221.

[29] Zhang, Shouwu. “Small points and adelic metrics”. In: *Journal of Algebraic Geometry* 4.2 (1995). Publisher: Providence, RI: University Press, c1992-, pp. 281–300.

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