NSPDE/ANA 2022 - Lecture 12

Last lecture:
- Energy method
- Explicit Euler method

Today:
- The CFL condition
- Lax Equivalence Theorem
- The theta method
- l2 analysis

References:
- Morton-Mayers
- Larsson-Thomee

The CFL condition
(Courant, Friedrichs, Levy 1928)

CFL Key idea: necessary condition for convergence of a FD scheme is that the domain of dependence of the FD solution must lay within the domain of dependence of the PDE.

Domain of dependence (DoD) for the PDE $u_t = F(u, \underline{x}, t)$ at the point $(\underline{x}, t)$ is the set $X(\underline{x}, t)$ of all points in the initial data manifold that effect the solution $u(\underline{x}, t)$.

- hyperbolic problems: (eg $u_t = u_x$, $u_{tt} = c^2 u_{xx}$, $u_t = (\frac{1}{2} u^2)_x$, ...) information travels at finite speed along characteristics, hence the DoD of PDE is finite.

  For $u_t = u_x$ is just 1 point!

- parabolic problems: (eg $u_t = u_{xx}$) DoD is typically infinite (info travels at ∞ speed)

Domain of Dependence of FD scheme at (x,t) is the set $X_k(x,t)$ in grid at $t=0$ such that $U(x,t)$ depends on all and only the points $x_j \in X_k(x,t)$.

The LIMITING NUMERICAL DoD $X_0(x,t)$ is set of all limit points of $X_k(x,t)$ as $k \to 0$.

Theorem: A necessary cond for convergence of a FD scheme applied to a parabolic pbm is that $k = o(h)$ as $k \to 0$.

For instance Explicit Euler we had $\mu = k/h^2 \le 1/2$ can convergence: within CFL but perhaps too restrictive.

LAX EQUIVALENCE

model problem: IVP

$$\begin{cases} u_t = Au \\ u(0) = u_0 \end{cases}, \quad A \text{ some operator}$$

$A : B \longrightarrow B$ Banach space

well posed

$S_k : B \to B$ characterises FD scheme

$$v^{n+1} = S_k v^n$$

say that S_k is OF ORDER p if $\forall t$

$$\underbrace{\| u(t+k) - S_k(u(t)) \|_B}_{\text{truncation error}} = O(k^{p+1}) \quad \text{as } k \to 0$$

(scheme applied to exact sol

and CONSISTENT if $p > 0$.

Further, S_k is CONVERGENT, if $\forall t$

$$\lim_{\substack{k \to 0 \\ nk = t}} \| S_k^n u_0 - u(t) \| = 0$$

for all initial values u_0. Finally, S_k is STABLE if

$$\| S_k^n \| \leq C \quad \forall n, k.$$

note: • Stability has nothing to do with the PDE ! It is a property of the discrete operator of not amplifying errors as the computation progresses.

• Consistency: errors are small at the time they are introduced

Lax equivalence: let S_k provide consistent approximation to a well-posed linear IV problem. Then S_k is convergent $\iff$ it is stable.

It is of order p if

$$\| S_k^n(u_0) - u(t) \| = O(k^p) \quad \text{as} \quad \begin{array}{l} k \to 0 \\ nk = t \end{array}$$

uniformly $\forall t$.

Comments: - applies to linear problems only
- applies to a fixed IV problem

(differently from Dahlquist Equivalence Theorem for ODEs)

As indicated by CFL, explicit methods for parabolic problems have very restrictive condition that $k = o(h^2)$ e.g. $k/h^2 \le 1/2$ for Exp. Eul.

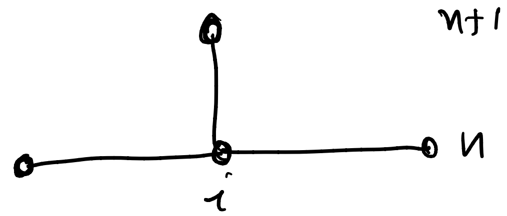
hence we look at implicit methods

Take again:

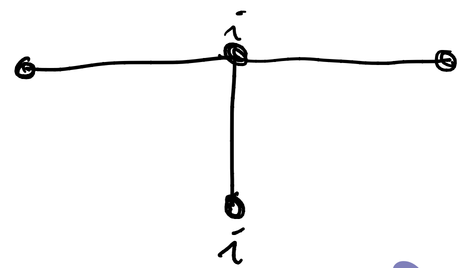
$$\begin{cases} u_t = u_{xx} & (0,1)\times(0,T] \\ u(x,0) = u_0(x) & (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

The ϑ- method:

$$\frac{U_i^{n+1}-U_i^n}{k} = \left(\vartheta\left(\delta_h^x\right)^2 U_i^{n+1} + (1-\vartheta)\left(\delta_h^x\right)^2 U_i^n\right)$$

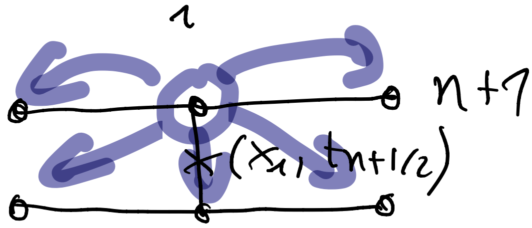


- $\vartheta = 0$ EXP EUL



- $\vartheta = 1$ IMP EUL n+1

n



- $\vartheta = 1/2$ CRANK-NICOLSON (CN)

All ϑ-methods for $\vartheta > 0$ are implicit, require solution of linear system at each step

example: can show (by Thomas algorithm) that solution of the Imp. Euler system for the 1D heat eq. takes about twice as many operations than those of Exp Euler

rewrite ϑ-method by $\mu = k/h^2$:

$$-\vartheta\mu\, U_{i+1}^{n+1} + (1+2\vartheta\mu) U_i^{n+1} - \mu\vartheta U_{i-1}^{n+1} \qquad \forall n \quad \forall i$$

$$= (1-\vartheta)\mu\, U_{i+1}^{n} + (1-2(1-\vartheta)\mu) U_i^{n} + (1-\vartheta)\mu\, U_{i-1}^{n}$$

that is, $\forall n$, solve: $B U^{n+1} = A U^n$ with

$$B = \mathrm{tridiag}\left(-\vartheta\mu,\ (1+2\vartheta\mu),\ -\vartheta\mu\right)$$

$$A = \mathrm{tridiag}\left((1-\vartheta)\mu,\ (1-2(1-\vartheta)\mu),\ (1-\vartheta)\mu\right)$$

so $U^{n+1} = E_k U^n := B^{-1} A U^n$

Lemma: Truncation error of the ϑ-method (consistency) is $O(k, h^2)$ unless $\vartheta = 1/2$ (CN) in which case it is $O(k^2, h^2)$

proof: for CN,

$$T_i^{n+1/2} := \delta_t u(x_i, t_{n+1/2}) - \frac{1}{2}\left(\delta_h^x\right)^2 \left(u(x_i, t_{n+1}) + u(x_i, t_n)\right)$$

then proof is by Taylor at $(x_i, t_{n+1/2})$

Stability

by max principle type analysis

$$(1+2\vartheta\mu)\, U_i^{n+1} = \vartheta\mu\left(U_{i+1}^{n+1} + U_{i-1}^{n+1}\right) + (1-\vartheta)\mu\left(U_{i+1}^n + U_{i-1}^n\right) + \left(1 - 2(1-\vartheta)\mu\right) U_i^n$$

if $1 - 2(1-\vartheta)\mu \geq 0 \iff \mu(1-\vartheta) \leq 1/2$

then all coeffs are nonnegative $\Rightarrow$

$$(1+2\vartheta\mu)\, |U_i^{n+1}| \leq 2\vartheta\mu \|U^{n+1}\|_{\infty,h} + \|U^n\|_{\infty,h}$$

$\forall i$, hence

$$\|U^{n+1}\|_{\infty,h} \leq \|U^n\|_{\infty,h} \quad \text{if } \mu(1-\vartheta) \leq 1/2$$

For instance

- $\vartheta = 0$ gives back $\mu \le 1/2$
- $\vartheta = 1$ uncond. stable ($\Rightarrow$ imp. euler solution is always monotone!)
- $\vartheta = 1/2$ stable in l_∞ $\Leftrightarrow$ $\mu \le 1$ (= monotone)

asking too much?

(same for $\vartheta \in [1/2, 1)$)

☆ could we have a less stringent stability condition in a different norm?

First, lets check

Von Neumann stability:

plug $U_i^n = \lambda(\xi)^n e^{i\xi i h}$ into

ϑ method:

$$-\vartheta\mu \cancel{\lambda}^{n+1}\left(e^{i\xi(\cancel{i}+1)h} + e^{i\xi(\cancel{i}-1)h}\right) + (1+2\vartheta\mu)\cancel{\lambda}^{n}\cancel{e^{i\xi i h}}$$

$$= (1-\vartheta)\mu \cancel{\lambda}^{n}\left(e^{i\xi(\cancel{i}+1)h} + e^{i\xi(\cancel{i}-1)h}\right) + (1-2(1-\vartheta)\mu)\cancel{\lambda}^{n}\cancel{e^{i\xi i h}}$$

use $-e^{i\xi h} + 2 - e^{-i\xi h} = 4\sin^2\left(\frac{1}{2}\xi h\right)$

$\vdots$

$$\left(1+4\vartheta\mu\sin^2\left(\frac{\gamma h}{2}\right)\right)\lambda(\gamma)=1-4\mu(1-\vartheta)\sin^2\left(\frac{\gamma h}{2}\right)$$

$$\Rightarrow \quad \lambda(\gamma)=\frac{1-4\mu(1-\vartheta)\sin^2(\gamma h/2)}{1+4\mu\vartheta\sin^2(\gamma h/2)}$$

stability requirement $|\lambda(\gamma)|\leq 1$

$$\Leftrightarrow \quad \mu(1-2\vartheta)\leq 1/2$$

hence, if $\vartheta<1/2$, stability iff $\mu(1-2\vartheta)\leq 1/2$

eg, for $\vartheta=0$ $\mu\leq 1/2$ as already known.

REMARK: NO stability requirement for $\vartheta\geq 1/2$!

Von Neumann is MUCH less restrictive than l_∞ analysis $\vartheta>0$!

l_2 - analysis

based Fourier Analysis (assume $h=1/N_x$)

Notation: let $V, W\in\mathbb{R}^{N_x+1}$

$$(V, W) := h \sum_{i=0}^{M_x} V_i W_i \quad ; \quad \|V\|_{2,h} := (V,V)^{1/2}$$

$\llcorner$ discrete l_2- IP and norm

$$l_2^h = \{ V \in \mathbb{R}^{M_x+1} : \|V\|_{2,h} < +\infty \}$$

$$l_2^{h,0} := \{ V \in l_2^h : V_0 = V_{M_x} = 0 \}$$

$$l_2^{h,0} = \text{span} \langle \varphi_j \rangle_{j=1}^{M_x - 1}$$

$$\varphi_{j,i} = \sqrt{2} \sin(\pi j i h) \qquad \forall i = 0, 1, .., M_x$$

Properties of the basis $\{\varphi_j\}$:

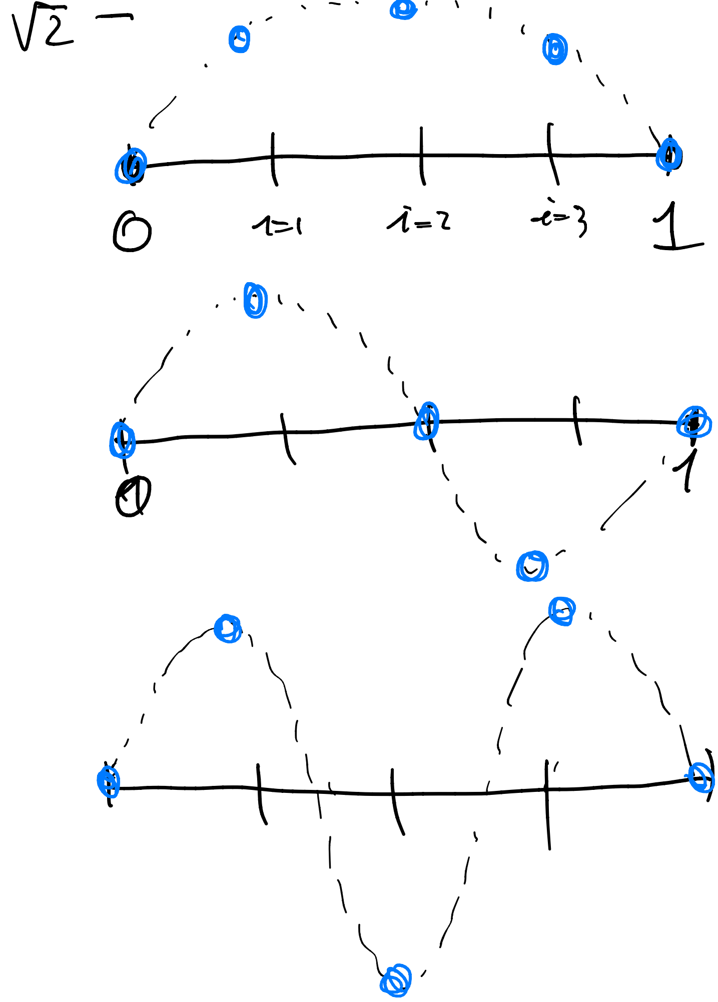
- $(\varphi_l, \varphi_m) = \delta_{lm}$

- φ_j are eigenfunctions of the discrete space operator of the 2nd centred FD:

$$-(\delta_h^x)^2 \varphi_{j,i} = \underbrace{\frac{2}{h^2} \left(1 - \cos(\pi j h) \right)}_{\text{eigenvalue}} \varphi_{j,i}$$

example: $M_x = 4$ $\qquad i = 1, 2, 3$

$h = 1/4$



$\varphi_1 = \sqrt{2}\sin(\pi i h)$

$\varphi_2 = \sqrt{2}\sin(2\pi i h)$

$\varphi_3 = \sqrt{2}\sin(3\pi i h)$

Given initial data U^0, let

$$U^0 = \sum_{j=1}^{M_x-1} \hat{U}_j^0 \, \varphi_j \qquad \hat{U}_j^0 = (U^0, \varphi_j)$$

ℓ_2-stability for Exp Eul

$$U_i^1 = U_i^0 + k(\delta_h^2) U_i^0$$

φ_j is eigenfunction

$$= \sum_{j=1}^{M_x-1} \hat{U}_j^0 \underbrace{\left(1 - 2\mu\left(1 - \cos(\pi j h)\right)\right)}_{E(\pi j h)} \varphi_{j,i}$$

$$U_i^n = \sum_{j=1}^{M_x-1} \hat{U}_j^0 \, \tilde{E}(\pi j h)^n \, \varphi_{j,i}$$

$$\|U_1^n\|_{2,h} = \left(\sum \left(\hat{U}_\gamma^0 \, E(\pi\gamma h)^n \right)^2 \right)^{1/2}$$

$$\uparrow \qquad \le \max_\gamma |E(\pi\gamma h)^n| \, \|\hat{U}^0\|_{2,h}$$

stability requires $\Rightarrow |E(\pi\gamma h)| \le 1$

$\Updownarrow$ (exercise) $\qquad = |1 - 2\mu + 2\mu\cos(\pi\gamma h)|$

$$\Longleftrightarrow \boxed{\mu \le 1/2}$$

Implicit Euler (LT): $E(\pi\gamma h) = \dfrac{1}{1 + \underbrace{2\mu(1-\cos(\pi\gamma h))}_{\ge 0}}$

unconditionally stable

ϑ-method: $E(\pi\gamma h) = \dfrac{1 - 2(1-\vartheta)\mu(1-\cos(\pi\gamma h))}{1 + 2\vartheta\mu(1-\cos(\pi\gamma h))}$

uncond. stable ?

Theorem: ($l_\infty - l_2$ error estimate) (time, space) The CN method is convergent $\forall \mu$ and

$$\max_n \|u^n - U^n\|_{2,h} = \mathcal{O}(k^2, h^2).$$

$$u^n = \{ u(x_i, t_n) \}$$

proof : stability + truncation error est : (ex)

$$\| T^{n+1/2} \|_{2,h} \le C \left(k^2 M_{tt} + h^2 M_{xxxx} \right)$$

see LT.