NSPDE/ANA 2022 - Lecture 7

Last lecture:
- Other boundary conditions
- General elliptic problems
- Simmetric case: Dirichlet principle
- General case: Lax Milgram lemma
- The method of Galerkin

Today:
- The Finite Element Method (FEM) in 1D
- Error analysis
- Conditioning of the FE matrix
- FEM in more dimensions

References:
- Quarteroni
- Larsson & Thomee

FEM

1. FEM two points bvp

$$\begin{cases} \mathcal{L}u = -(a u')' + cu = f & \Omega = (0,1) \\ u(0) = 0 = u(1). \end{cases}$$

$f \in L^2(\Omega)$, $a(x) > 0$, $c \geq 0$

weak prob: Find $u \in H_0^1(\Omega)$:

$$A(u,v) := \int_\Omega a u' v' + \int_\Omega c\, uv = \int_\Omega f v = \ell(v) \quad \forall v \in H_0^1(\Omega)$$

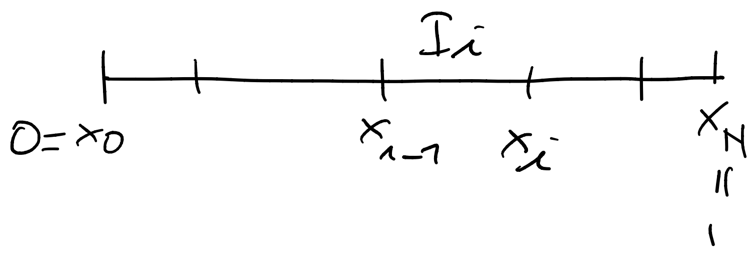
FE spaces.

- Fix partition of Ω, $I_i = (x_{i-1}, x_i)$

$x_i = x_{i-1} + h_i$

$x_0 = 0$

$h = \max_i h_i$

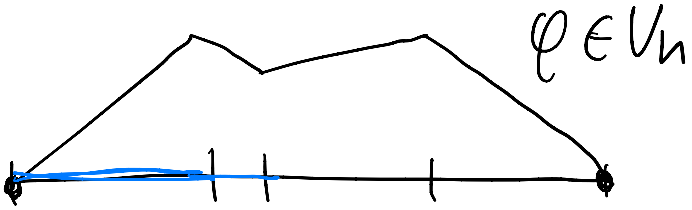


- Fix finite dim. space

$$V_h = V_h^k = \{ v \in C_0(\bar{\Omega}) : v|_{I_i} \in \mathbb{P}^k(I_i) : i = 1, \dots, H \}$$

$h \in H$

example: $k=1$



Notice $V_h \subset V$.

FE method (C^0-conforming):

Find $u_h \in V_h$: $\mathcal{A}(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h$

FE in algebraic form

fix a basis $\{\varphi_j\}_{j=1}^m$ of V_h, then

$$u_h = \sum_{j=1}^m U_j \varphi_j$$

Hence, testing discrete prb. with test fun,

Find $\{U_j\}_{j=}^m$ such that

$$\sum_{j=1}^{m} U_j \Big(\underbrace{\int_\Omega a\, \varphi_j' \varphi_i'}_{S_{ij}} + \underbrace{\int_\Omega c\, \varphi_j \varphi_i}_{M_{ij}} \Big) = \ell(\varphi_i) \qquad \forall i = 1, \dots, m$$

with:

$$A := S + M$$

$$F = \big(\ell(\varphi_i) \big)$$

$$\Leftrightarrow \quad \boxed{AU = F}$$

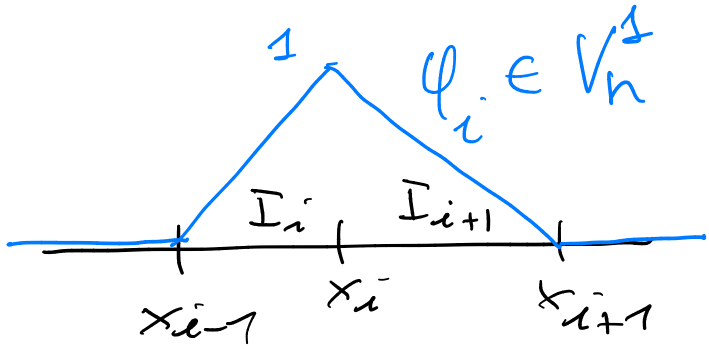
The "assembly" of A and F requires the eval. of integrals. Practical + efficient assembly is achieved by selecting "good" basis (local support):

(1) $k = 1$ (linear FEM)

φ_i select:

$i = 1, \dots, N-1$ (internal nodes)

gives basis



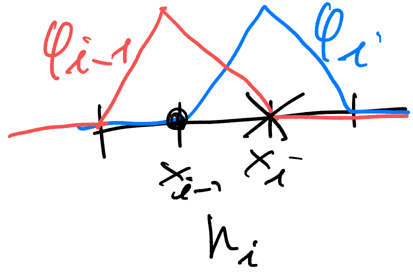
$$\varphi_i(x) = \begin{cases} 0 & I_j \quad j < i \\ \dfrac{x - x_{i-1}}{h_i} & I_i \\ \dfrac{x_{i+1} - x}{h_{i+1}} & I_{i+1} \\ 0 & I_j \quad j > i+1 \end{cases}$$

"hat function"

then

$$\mathrm{supp}(\varphi_i) \cap \mathrm{supp}(\varphi_j) = 0 \qquad |i - j| > 1$$

example: $a = 1$ $\quad \int_0^1 \varphi_i' \varphi_{i-1}'$



$$= \int_{x_{i-1}}^{x_i} \varphi_i' \varphi_{i-1}' \, dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right) \left(-\frac{1}{h_i}\right) dx = -\frac{1}{h_i^2} h_i = -\frac{1}{h_i}$$

$$= \int_0^1 \varphi_i' \varphi_{i+1}'$$

$$\int_0^1 \varphi_i' \varphi_i' = \frac{2}{h_i}$$

$$(h_i \equiv h) \qquad S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

= FD matrix apart from scaling $1/h$ instead of $1/h^2$

Right-hand side is discretised differently:

- FD $\quad F_i = f(x_i)$
- FEM $\quad F_i = \int_0^1 f(x) \varphi_i(x) \, dx$

$$= \int_{x_{i-1}}^{x_{i+1}} f(x) \varphi_i(x) \, dx \sim h$$

Question: how we compute $\int f \varphi$

(similarly $\int (a \varphi_i' \varphi_j' + b \varphi_i \varphi_j)$)

answer: use appropriate quadrature formulas

exercise: If $\int f \varphi_i$ approximated by trapezoidal rule, then also rhs of FEM $\equiv$ rhs of FD

fact: In 1D for $a \equiv 1$, ($c \equiv 0$), if $\int f \varphi_i$ computed exactly, then

$$u_h(x_i) = u(x_i) \quad !$$

(nodally exact)

Assembly: (of A, S, F)

assembly of system is by loop over the elements (intervals)

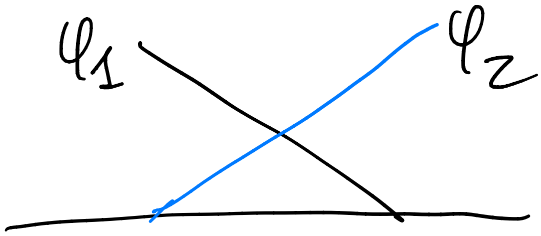
for $i = 1 : N-1$

A_{I_i} local matrix $\begin{pmatrix} \int_{I_i} (\varphi_{i-1}')^2 & \int_{I_i} \varphi_{i-1}' \varphi_i' \\ \int_{I} \varphi_i' \varphi_{i-1}' & \int_{I_i} (\varphi_i')^2 \end{pmatrix}$

distribute terms in A right places

For the computation of A_{I_i} use quadrature rules on ref. elemt

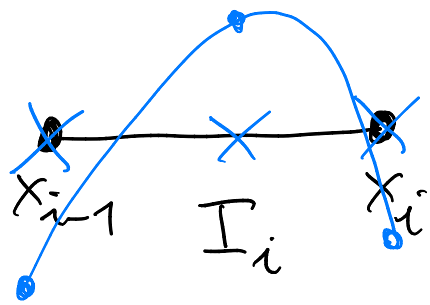
$I = (0,1)$



use affine map from $I \longrightarrow I_i$

(2) $k > 1$

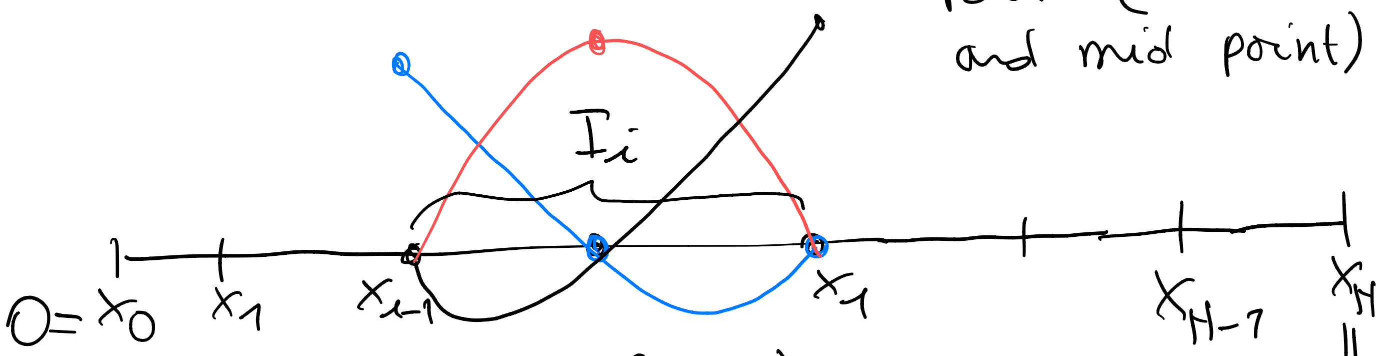
- $k = 2$



fix "local" basis of 3 quadratic functions,

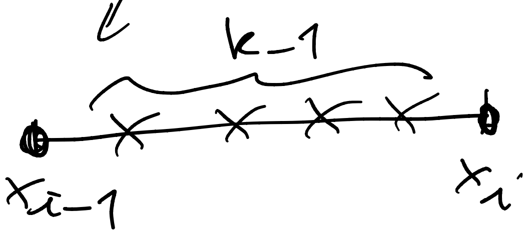
$$\varphi_\ell(x_m) = \delta_{\ell m}$$

where $\ell = 1,2,3$, x_m are the three points (end points and mid point)



$$\dim V_h^2 = (N-1) + N$$

- k general $\quad \dim V_h^k = (N-1) + N(k-1)$



Analysis of FEM:

use abstract analysis results:

cont const

① Lemma of Céa: $\|u-u_h\|_V \le \frac{\gamma}{\alpha_0} \inf_{v_h \in V_h} \|u-v_h\|_V$

coerc. const

$$V = H^1_0(\Omega)$$

$$\|v\|_V^2 = \int_0^1 (v')^2\,dx \qquad \left(\text{or } + \int_0^1 v^2\,dx\right)$$

$$|v|_{H^1}^2 \qquad\qquad + \|v\|_{L^2}^2$$

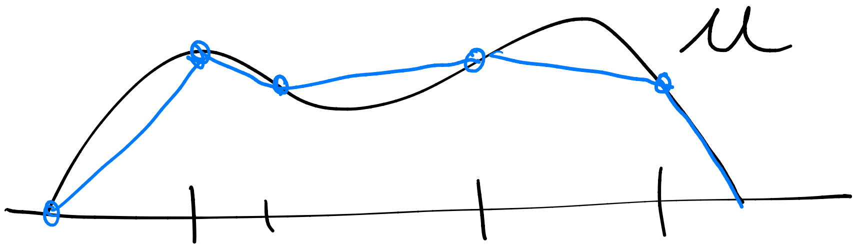
$$\|v\|_\Omega^2 = \int_0^1 a\,(v')^2\,dx + \int_0^1 c\,v^2\,dx$$

② $\inf_{v_h \in V_h} \|u-v_h\|_V^2 \le \|u - I_h u\|_V^2$

for some "special" choice $I_h u \in V_h^k$

$$= \sum_{i=1}^{M-1} \|u - I_h u\|_{I_i}^2$$

For instance, I_h can be the interpolation op.



assuming $u \in C^0(\Omega)$.

Def (FEM interpolant): $I_h : Z^0 \to V_h$

as: $\forall v \in Z^0$, $I_h v \in Z^0 \cap V_h$:

$I_h v|_{I_i} \in \mathbb{P}^k(I_i)$ Lagrange interpolant

Theorem (LT): $\left(\| v - I_h v \|_{L^2(I_i)} \leq C h_i^{k+1} |v|_{H^{k+1}(I_i)} \right)$ $\forall v \in H^{k+1}(I_i)$

(see Heltai)

$\| (v - I_h v)' \|_{L^2(I_i)} \leq C h_i^k |v|_{H^{k+1}(I_i)}$

ex: $k = 1$

L^2 $O(h_i^2)$

H^1 $O(h_i)$

from which "global" interp. error bounds also follow

Theorem (A apriori error bound for FEM):

If $u \in H^{k+1}(\Omega)$, then $\left(\sum_{i=1}^{N-1} h_i^k |u|^2_{H^{k+1}(\Omega)} \right)^{1/2}$

$$\| (u - u_h)' \|_{L^2(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

$$\| u - u_h \|_{L^2(\Omega)} \leq C h^{k+1} |u|_{H^{k+1}(\Omega)}$$

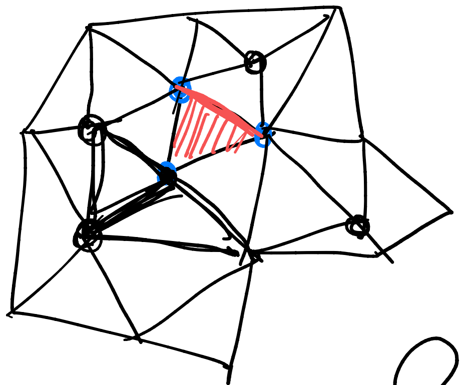
Moreover, if $u \in H^s(\Omega)$ $\quad 1 \le s \le k+1$,

$$\|(u-u_h)'\|_{L^2(\Omega)} \le C h^s |u|_{H^s(\Omega)}$$

FEM in more dimensions

FE spaces ($\mathbb{R}^2$ for example)

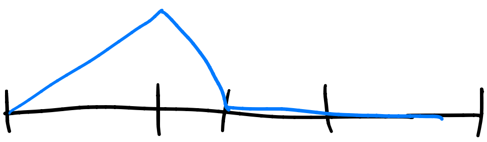
Ω polygon



Ω

linear functions defined by value at 3 points!

$d=1$



- Let $\mathcal{T}_h$ a triangulation of Ω

$$\mathcal{T}_h = \{T\}, \quad T \text{ triangle}$$

$$\bar{\Omega} = \bigcup T \quad + \quad \mathcal{T}_h \text{ is "regular"}$$

$$\bar{T}_i \cap \bar{T}_j = \begin{cases} \emptyset \\ 1 \text{ vertex} \\ 1 \text{ edge} \end{cases}$$

- $V_h^k = \{ v \in C^0(\Omega) : v|_T \in \mathbb{P}^k(T), \ \forall T \in \mathcal{T}_h \}$

Proposition : $v \in H^1(\Omega) \Longleftrightarrow \begin{cases} v_{|T} \in H^1(T) \quad \forall T \in \mathcal{T}_h \\ \text{if } F = T_1 \cap T_2 \text{ then} \\ (v_{|T_1})_{|F} = (v_{|T_2})_{|F} \end{cases}$

Given Ω and $\mathcal{T}_h$ regular triangulation