NSPDE/ANA 2022 - Lecture 14

Last lecture:
- Semidiscrete FEM for parabolic problems
- Error analysis
- Fully discrete FD-FEM

Today:
- higher-order time-stepping: BDF, discontinuous Galerkin
- Intro to Hyperbolic problems
- Upwind method for linear transport
- CFL, stability

References:
- Morton-Mayers
- Larsson-Thomee

Higher-order Time Stepping

1) BDF formulas:

2nd order BDF

$$\bar{D}u^{n+1} = \delta_{\Delta t,-} u^{n+1} + \frac{1}{2}\Delta t \left(\delta_{\Delta t,-}\right)^2 u^{n+1}$$

$$= \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{2}\Delta t \, \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$$

$$= \frac{1}{\Delta t}\left(\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}\right)$$

(Taylor) $\bar{D} u^{n+1} = u_t(t_{n+1}) + O(\Delta t^2)$

BDF2 - FEM

$f(\Delta t, x)$

- $u_h^0 = \Pi_h u_0 \in V_h^k$
- $\left(\partial_{\Delta t,-} u_h^1, v_h \right) + a(u_h^1, v_h) = (f^1, v_h) \quad \forall v_h \in V_h^k$
- $\left(\bar{D} u_h^{n+1}, v_h \right) + a(u_h^{n+1}, v_h) = (f^{n+1}, v_h) \quad \forall v_h \in V_h^k$

$n = 1, 2, \ldots$

Theorem: if u smooth enough,

$$\|u_h^n - u(t_n)\|_{L^2} \leq C h^{k+1} \left(\|u_0\|_{H^{k+1}} + \int_0^{t_n} |u_t|_{k+1} \right)$$

$$+ \underbrace{C \Delta t \int_0^{\Delta t} \|u_{tt}\|_{L^2} + (\Delta t)^2 \int_0^{t_n} \|u_{ttt}\|_{L^2}}_{\text{take } \Delta t \text{ small enough}}$$

Discontinuous

Idea: • use FEM in time also
• in a way to maintain the approach of time stepping

Continuous problem in weak form:

$\forall t \in (0, T], \quad u(t) \in V \quad (= H_0^1)$

$$\begin{cases} (u_t, v) + a(u,v) = (f,v) & \forall v \in V \\ u(0) = u_0 \end{cases}$$

Let $0 = t_0 < t_1 < \dots < t_n < \dots T = t_{N_t}$

$\Delta t_n = t_{n+1} - t_n$, $\quad \Delta t := \max_n \Delta t_n$.

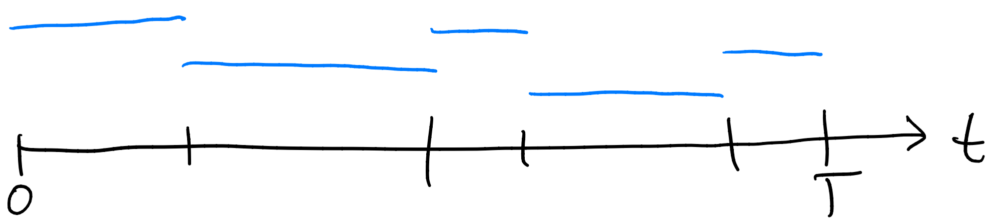
$J_n := (t_n, t_{n+1}]$.

Let $q \in \mathbb{N} \cup \{0\}$

$$W^q_{\Delta t} = \{ v : [0,T] \to V : v(t)\big|_{J_n} \in \mathbb{P}_q(V), \forall n = 0,1,\dots,N_t \}$$

$\cap L^2(I;V)$

example $q = 0$



$W^q_{\Delta t}$ = DG FEM space.

Derivation of DG-time stepping method

From weak formulation, test now with $v = v(t)$ such that $v(T) = 0$, and integrate by parts in time: (v smooth enough)

$$\int_0^T (u_t, v)\,dt = -\int_0^T (u, v_t) + (u,v)\Big|_{t=T} - (u,v)\Big|_{t=0}$$

$$= -\int_0^T (u, v_t) - (u(0), v(0)).$$

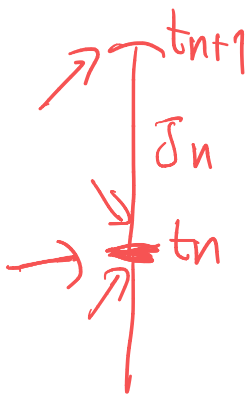
$$\int_0^T \left[(u, v_t) + \Omega(u, v)\right] dt = (u_0, v(0)) + \int_0^T (f, v)\, dt$$

Consider $u_{\Delta t} \in W^q_{\Delta t}$ and integrate by parts again:

$$-\int_0^T (u_{\Delta t}, v_t) = -\sum_{n=0}^{M_t-1} \int_{J_n} (u_{\Delta t}, v_t)$$

$= (u_{\Delta t}(t_{n+1}), v(t_{n+1})) - (u_{\Delta t}(t_n), v(t_n))$

$$= + \sum_{n=0}^{M_t-1} \int_{J_n} \left(\frac{\partial}{\partial t} u_{\Delta t}, v\right) + (u_{\Delta t}, v)_{\partial J_n}$$



$$= + \int_0^T \left(\frac{\partial}{\partial t} u_{\Delta t}, v\right) + (u^0_n, v)$$

Call $\lfloor v^n \rfloor := v^n_+ - v^n_-$ Jump

$$\begin{cases} v^n_+ = \lim_{z \to 0^+} v(t_n + z) \\ v^n_- = \lim_{z \to 0^-} v(t_t + z) \end{cases}$$

$$+ \sum_{n=1}^{M_t-1} \left(\lfloor u^n_{\Delta t} \rfloor, v^n\right)$$

$\Downarrow$

DG scheme: find $u_{\Delta t} \in W^q_{\Delta t}$:

$$\begin{cases} \int_0^T \left[\left(\frac{\partial}{\partial t} u_{\Delta t}, v \right) + a(u_{\Delta t}, v) \right] \\ + \sum_{n=1}^{N_t - 1} \left([u_{\Delta t}^n], v^n \right) + \left(u_{\Delta t}^0, v^0 \right) \\ = \int_0^T (f, v) + \left(u_h^0, v^0 \right) \end{cases}$$

equivalently, $\forall\, n = 0, 1, \dots, N_t - 1$

$$\int_{J_n} \left[\left(\frac{\partial}{\partial t} u_{\Delta t}, v \right) + a(u_{\Delta t}, v) \right] + \left(u_{\Delta t,+}^n, v^n \right)$$

PDE

$$= \int_{J_n} (f, v) + \left(u_{\Delta t,-}^n, v^n \right)$$

with $u_{\Delta t,-}^0 = u_h^0$

local initial condition weakly imposed

• case $q=0$ DG $\equiv$ Implicit Euler !

error estimate:

$$\|u(T)-u_{\Delta t}^{N_t}\| \le C\left(\sum_{n=0}^{N_t-1} \Delta t_n^{2(q+1)} \int_{J_n} \left|\frac{d^{q+1}u}{dt^{q+1}}\right|_{H^1} dt\right)^{1/2}$$

e.g. for $q=0$ the bound only requires $\frac{d\,u(t)}{dt}$ in H^1 in contrast with Backward (Implicit) Euler analysis which requires 2nd derivative in time.

HYPERBOLIC PDES

model problems:

① linear transport: $u_t + a u_x = 0$

② conservation laws: $u_t + (f(u))_x = 0$

③ wave equation: $u_{tt} - c u_{xx} = 0$ $(c<0)$

Recall: for

$$\begin{cases} u_t + a u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

we know that information travels along the one family of characteristics:

from

$$(1, a) \cdot \nabla_{t,x} u = 0$$

stating that the directional derivative of u with respect to $v = (1, a)$ is zero, we have that solution is constant along

$$\frac{dx}{dt} = a(x, t)$$

$\Rightarrow$ solution is known from initial cond travelling along the characteristics

Same applies for systems of transport equations, eg.

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{f}(\bar{u})}{\partial x} = 0$$

for example, the incompressible Euler equations

$$\begin{cases} \dfrac{D u_i}{D t} = \dfrac{\partial u_i}{\partial t} + \underline{u} \circ \nabla u_i & = g + \text{source terms} \\ \nabla \circ \bar{u} = 0 \end{cases}$$

in conservative form: $\dfrac{D u_i}{D t} = \dfrac{\partial u_i}{\partial t} + \dfrac{\partial}{\partial x_1}\left(u_1^2\right) + \sum_{j \neq i} \dfrac{\partial (u_1 u_j)}{\partial x_j}$

equivalently

$$\bar{u}_t + A(\bar{u})\,\bar{u}_x = 0$$

with

$$A(\bar{u}) = \frac{\partial \bar{f}}{\partial \bar{u}} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots \\ \frac{\partial f_2}{\partial u_1} & \cdots & \\ \vdots & & \end{pmatrix}$$

characteristic speeds of propagation are the eigenvalues of A, however in general characteristics curved and closed formulas for solutions would not be available

Another example is wave equation:

$$u_{tt} - u_{xx} = 0 \iff \begin{cases} u_t + v_x = 0 & (1) \\ u_x + v_t = 0 & (2) \end{cases}$$

$$\Updownarrow$$

$$\boxed{U_t + A\,U_x = 0 \\ U = \begin{pmatrix} u \\ v \end{pmatrix}}$$

$(1)_t \Rightarrow 0 = u_{tt} + v_{xt}$

$= u_{tt} + (v_t)_x$

use (2)

$= u_{tt} - u_{xx}$

$(2)_t \Rightarrow 0 = v_{tt} - v_{xx}$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In conclusion:

- plenty of physical problems that reduce to transport systems
- often closed solutions not available

Properties of solutions

conservation of energy:

(1) $u_t + a\, u_x = 0 \qquad x \in \mathbb{R}, \quad a \equiv \text{const.}$

$\|u(t)\|_{L^2(\mathbb{R})}$ is conserved.

proof: test with u and integrate in x:

$$\int_{\mathbb{R}} u_t u \, dx + a \int_{\mathbb{R}} u_x u \, dx = 0$$

$$\Updownarrow$$

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}} u^2 dx + \frac{a}{2} \underbrace{\int_{\mathbb{R}} \frac{\partial}{\partial x} u^2 dx}_{u^2 \Big|_{-\infty}^{+\infty} = 0} = 0$$

otherwise $u \notin L^2(\mathbb{R})$

$$\Rightarrow \frac{\partial}{\partial t} \int_{\mathbb{R}} u^2 dx = 0$$

$$\Updownarrow$$

$$\|u(t)\|^2_{L^2(\mathbb{R})} \equiv \text{const} \equiv \|u_0\|^2_{L^2(\mathbb{R})}$$

(2) for $u_{tt} - u_{xx} = 0$

energy: $\frac{1}{2} \int_{\mathbb{R}} \left((u_t)^2 + (u_x)^2 \right) dx$

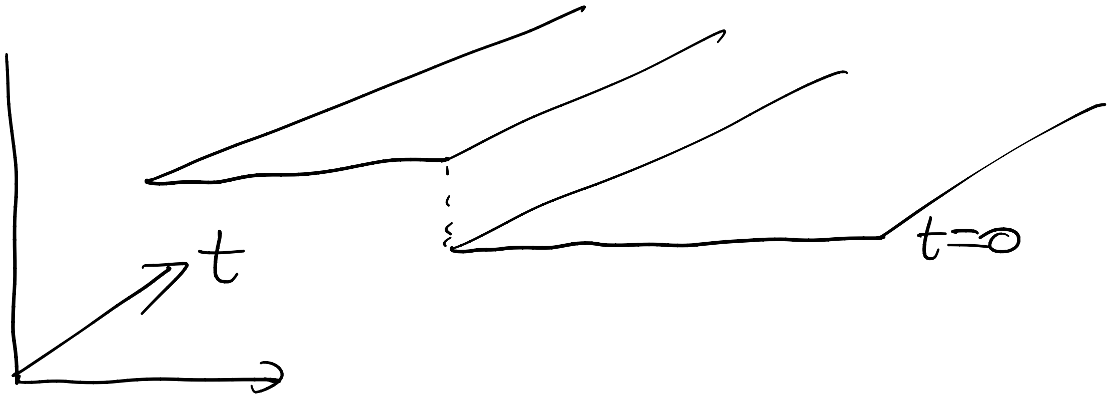
is conserved

proof: test by u_t (exercise)

Comments:

- hyperbolic → no dispersion
- discontinuous initial data is propagated ($\Rightarrow$ "generalised solutions")

example: Riemann problem: piecewise constant initial cond with one step:

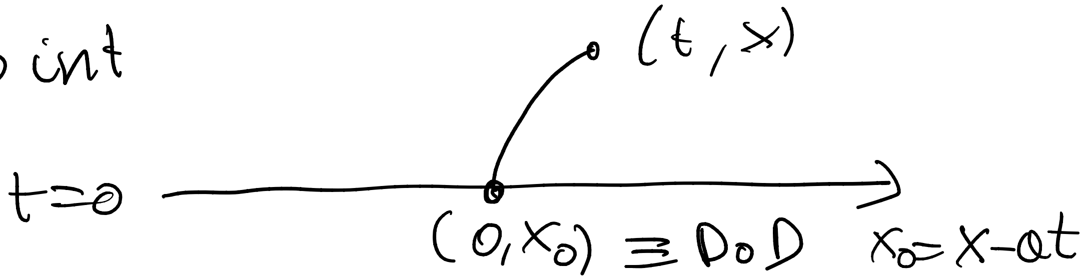


→ solution is classical only piecewise away from jumps

+ CFL condition (DoD of FD scheme must lay within the DoD of the PDE)

for hyperbolic problems we have finite speed of propagation so D.o.D of PDE is finite.

example, for the transport eq. ($u_t + a u_x = 0$) it is just one point

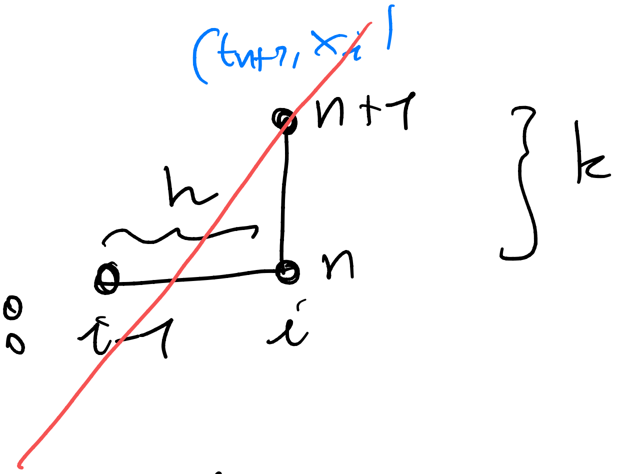


what about the FD scheme DoD:

$$\begin{cases} u_t + a u_x = 0 & (0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$$

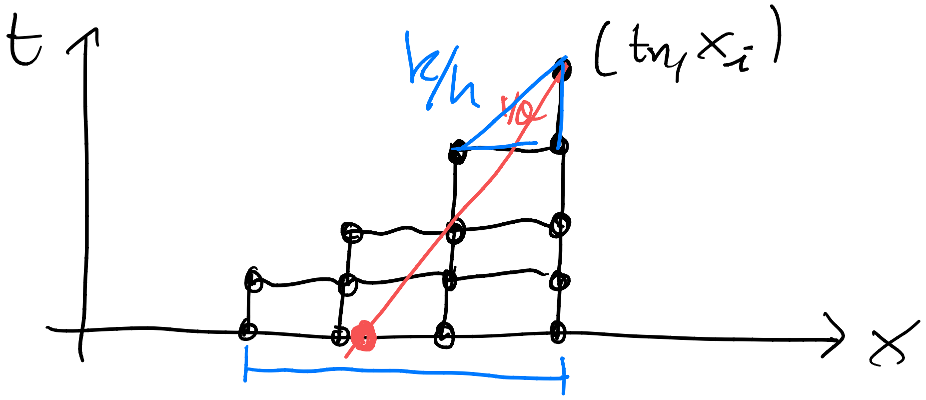
$a > 0$

consider backward diff. in space:



$$\frac{U_i^{n+1} - U_i^n}{k} + a_i^n \frac{U_i^n - U_{i-1}^n}{h} = 0$$

DoD of scheme, define as base of the FD pyramid



CFL: $k/h \le 1/a \iff a\nu \le 1$

$\nu := k/h$

much less stringent requirement w.r.t. parabolic case $\Rightarrow$ "explicit methods competitive"

The above is an instance of "UPWIND" method

$$\begin{cases} U_i^{n+1} = (1-a_i^n \nu) U_i^n + a_i^n \nu U_{i-1}^n & \forall n = 0, 1, \dots \\ U_i^0 = u_0(x_i) \end{cases}$$

STABILITY: by von Neumann analysis:

plug $U_i^n = \lambda^n e^{i \xi i h}$

$$\lambda = (1-a_i^n \nu) + a_i^n \nu e^{-i\xi h}$$

$$\overset{!}{=} (1-a_i^n \nu) + a_i^n \nu \cos(\xi h) - i\, a_i^n \nu \sin(\xi h)$$

$$|\lambda|^2 = 1 - 4 a_i^n \nu (1 - a_i^n \nu) \sin^2(\tfrac{1}{2}\xi h)$$

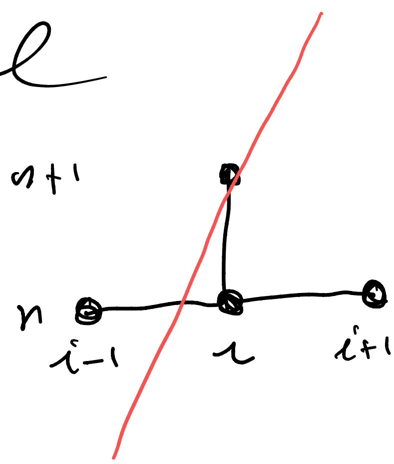
$|\lambda| \le 1$ for stability $\Longleftrightarrow$ $a_i^n \nu \le 1$

(identical to CFL cond!)

In general, CFL is only necessary.

For example, the scheme

$$\frac{U_i^{n+1} - U_i^n}{k} + a_i^n \underbrace{\frac{U_{i+1}^n - U_{i-1}^n}{2h}}_{\text{centre diff.}} = 0$$



CFL $\quad a\nu \le 1$

however, von Neumann gives

$$\lambda = \frac{\nu}{2} e^{-ikh} + 1 - \frac{\nu}{2} e^{ikh} \qquad (a = 1)$$

$$\Rightarrow 1 - i\nu \sin kh$$

$$|\lambda| = \sqrt{1 + \underbrace{\nu^2 \sin^2 kh}_{\geq 0}} > 1 \text{ always}$$

$\Rightarrow$ The CD scheme is always unstable !!!