# NSPDE/ANA 2022 - Lecture 4

**Last lecture:**
- A priori analysis: stability, error estimate
- General elliptic problems

**Today:**
- General elliptic problems DMP
- FD for Poisson in 2D
- Analysis (stability, consistency, convergence)
- Compact FD formulas
- FD for general elliptic problems
- Intro to functional spaces

general elliptic problems, $\Omega \subset \mathbb{R}^d$

$$\mathcal{L}u = \underbrace{-\sum_{ij}^{d} D_i(a_{ij}(x) D_j u)}_{\text{diff.}} + \underbrace{\sum_{i=1}^{d} D_i(b_i u)}_{\text{adv.}} + \underbrace{c\,u}_{\text{reac}}$$

$\Downarrow$

$A = (a_{ij})$ positive def. $\Rightarrow$ $\mathcal{L}$ elliptic

- max princ
- uniqueness for the b. v. p.
- a priori bounds

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ c \geq 0 \\ \max_{\Omega} |u| \leq \max_{\partial\Omega} |u| + C \sup_{\Omega} |f| \end{cases}$$

Suppose $\mathcal{L}u = f$ + b.c. has been discretised on a grid $\bar{\Omega}_h$, and assume:

- $\forall P \in \Omega_h \qquad \mathcal{L}_h U_P = f_P + g_P$ discrete equations
- and $\qquad \mathcal{L}_h U_P = c_P U_P - \sum_K c_K U_K$

  where $K$ ranges through neighbours.

- and $\begin{cases} c_K & \text{non neg.} \\ c_P \geq \sum_K c_K \end{cases}$

then, if $U$ grid function, satisfies

$$\mathcal{L}_h U_P \leq 0 \qquad \forall P \in \Omega_h$$

we have

$$\max_{P \in \Omega_h} U_P \leq \max \left\{ \max_{Q \in \partial\Omega_h} U_Q , 0 \right\}$$

$\mathcal{L}_h U_P \geq 0$

min

Proof: (mimic 1D proof for $u''$ )

exercise !

see Morton and Mayers.

# FD in higher dimensions

Starting from simple Poisson, $d=2$.

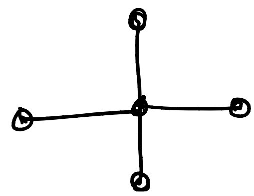
$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- Fix $N_x \times N_y$ grid $(x_i, y_j)$ : $x_i = i h_x$, $y_j = j h_y$ with $h_x = 1/N_x$, $h_y = 1/N_y$
- Define grid functions $U = \{U_{ij}\}$ $U_{ij} \approx u(x_i, y_j)$
- $\Delta u(x_i, y_j) \approx \delta^2_{h_x} U_{ij} + \delta^2_{h_y} U_{ij}$

$$= \frac{1}{h_x^2}\left(U_{i+1,j} - 2U_{ij} + U_{i-1,j}\right) + \frac{1}{h_y^2}\left(U_{i,j+1} - 2U_{ij} + U_{i,j-1}\right)$$

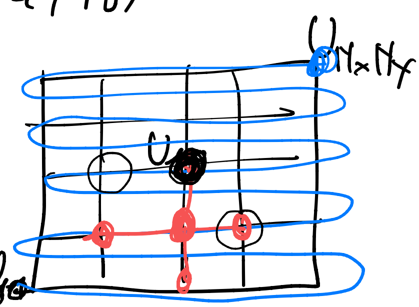
special case: $h_x = h_y = h$

$$= \frac{1}{h^2}\left(U_{i+1,j} + U_{i,j+1} - 4U_{ij} + U_{i,j-1} + U_{i-1,j}\right)$$




$$\begin{cases} U_{ij} = 0 & \text{if } (x_i, y_j) \in \partial\Omega \\ -\left(\delta^2_{h_x} U_{ij} + \delta^2_{h_y} U_{ij}\right) = f_{ij} & (x_i, y_j) \in \Omega \end{cases}$$

## Matrix form




- $\underline{U} = \{U_{11}, U_{21}, \ldots, U_{N_x-1\,1}, U_{12}, \ldots, U_{N_x-1\,N_y-2}\}$

- $F = \{f_{11}, \dots \}$
- $$A = -\begin{pmatrix} B & I & & 0 \\ I & B & \ddots & \\ & \ddots & \ddots & I \\ 0 & & I & B \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 1 & & \\ 1 & -4 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \end{pmatrix}_{N-1 \times N-1}$$

$N_x = N_y = N$

$I = (N-1) \times (N-1)$ identity

- sol. $A \underline{U} = F$

- DMP: if $\mathcal{L}_h V_{ij} \le 0 \ \forall i,j : (x_i, y_j) \in \Omega$

then $\max\limits_{(x_i,y_j) \in \bar{\Omega}} V_{ij} = \max\limits_{(x_i,y_j) \in \partial\Omega} V_{ij}$

because general theorem hypothesis are satisfied.

<u>Stability</u>: $\forall V$ discrete function,

$$\|V\|_{\ell_\infty(\Omega_h)} \le \|V\|_{\ell_\infty(\partial\Omega_h)} + \frac{1}{8} \|\mathcal{L}_h V\|_{\ell_\infty(\Omega_h)}$$

$\uparrow$ grid points

proof is like 1D case, using $w = x + y - x^2 - y^2$

as comparison function

Given FD sol. satisfies $\mathcal{L}_h U_{ij} = f_{ij} \Rightarrow$ a priori

Stability bound

**Lemma** (truncation error): If $u \in C^4(\Omega) \cap C^0(\bar{\Omega})$, then

$$|T(x)| = |\mathcal{L}u(x) - \mathcal{L}_h u(x)|$$

$(h_x = h_y = h)$

$$\le \frac{h^2}{12}\left(M_{xxxx} + M_{yyyy}\right)$$

with $M_{xxxx} = \max_{\Omega} |u_{xxxx}|$, $M_{yyyy} = \dots$ .

Proof: ex.

**Theorem** (convergence): Under some assumptions, for $h > 0$, we have

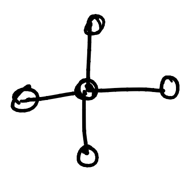
$$|u_{i,j} - U_{ij}| \le \frac{h^2}{96}\left(M_{xxxx} + M_{yyyy}\right)$$

$u_{i,j} = u(x_i, y_j)$

Proof: as 1D case (exercise).

Comments:

- 5-points scheme is $2^{nd}$ order

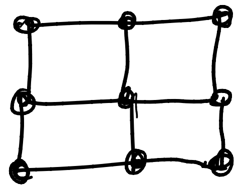


stencil

- in general, higher order schemes demand larger stencils
- however, sometime "compact" FD schemes

can be obtained with higher-order conv.

For instance, there exists a 9 points scheme, with stencil



which is $4^{th}$ order accurate!

(see Iserles)

Obtained by discretising, instead of $\Delta u = f$, the perturbed problem

$$\left(\Delta + \frac{1}{12} h^2 \Delta^2\right) u = f + \frac{1}{12} h^2 \Delta f$$

Looking at discretising a perturbed PDE is a recurrent idea of numerical PDEs.

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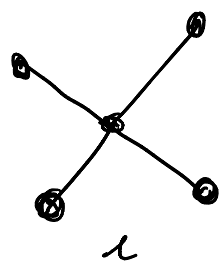
More general problems:

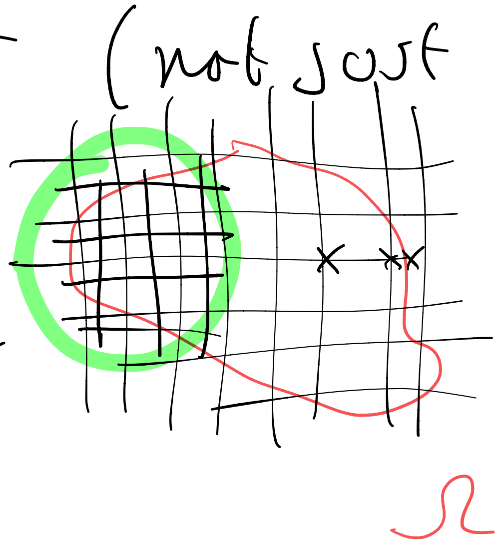
- general operator $\mathcal{L}$,
  - low order terms or in 1D
  - $u_{xy}(x_i, y_j) \approx \delta^{y}_{2h} \delta^{x}_{2h} U_{ij}$ ,



$i$

- 3D problem, just add 1 variable
- more general boundary cond (follow 1D recipe
- more general domains (not just box type domains)

we'll need the following:

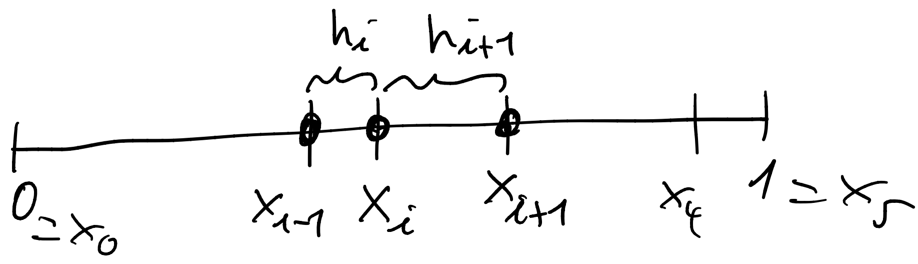



Non uniform grids

$$\begin{cases} -u'' = f & \text{in } [0,1] \\ u(0) = 0 = u(1) \end{cases}$$

Let $h_i$, $i = 1, \dots, N$ : $\sum_i h_i = 1$

$x_i = x_{i-1} + h_i$




Method of undetermined coefficients: look for $\alpha, \beta, \gamma \in \mathbb{R}$ coeffs of 3 points formula:

$$u''(x_i) \approx \alpha u_{i+1} + \beta u_i + \gamma u_{i-1}$$

expand by Taylor and try to zero as many terms as possible (ex.), gives

$$-\left(\frac{2}{h_i(h_i+h_{i+1})}U_{i+1} - \frac{2}{h_i h_{i+1}}U_i + \frac{2}{h_i(h_i+h_{i+1})}U_{i-1}\right) = f_i$$

note: $h_i = h_{i+1}$ gives back the uniform grid method!

- Truncation error bound:

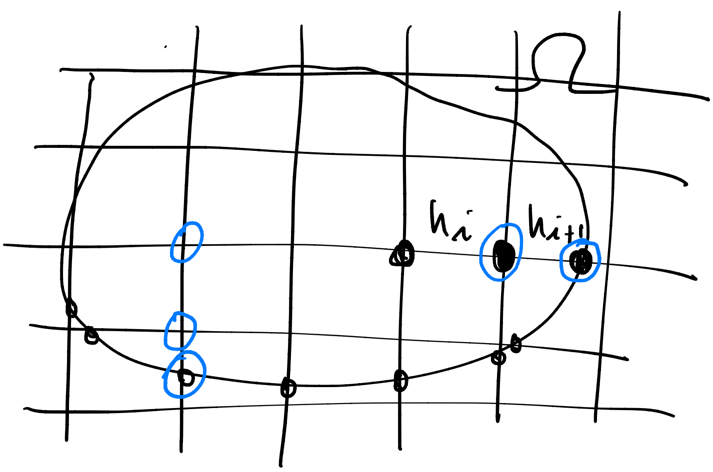
$$|T_i| \le \frac{2}{3} h_{max} M_3 + \frac{1}{12}\frac{h_{max}^3}{h_{min}} M_4$$

(if $u \in C^4$)

$\rightarrow$ $O(h)$ in general, but needs

$$\frac{h_{max}^3}{h_{min}} \longrightarrow 0$$ for convergence!

2D problems on convex domains




$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

$\Omega_h$ = set of internal nodes

$\partial\Omega_h$ = set of boundary nodes (obtained by cutting grid with $\supset \Omega$ )

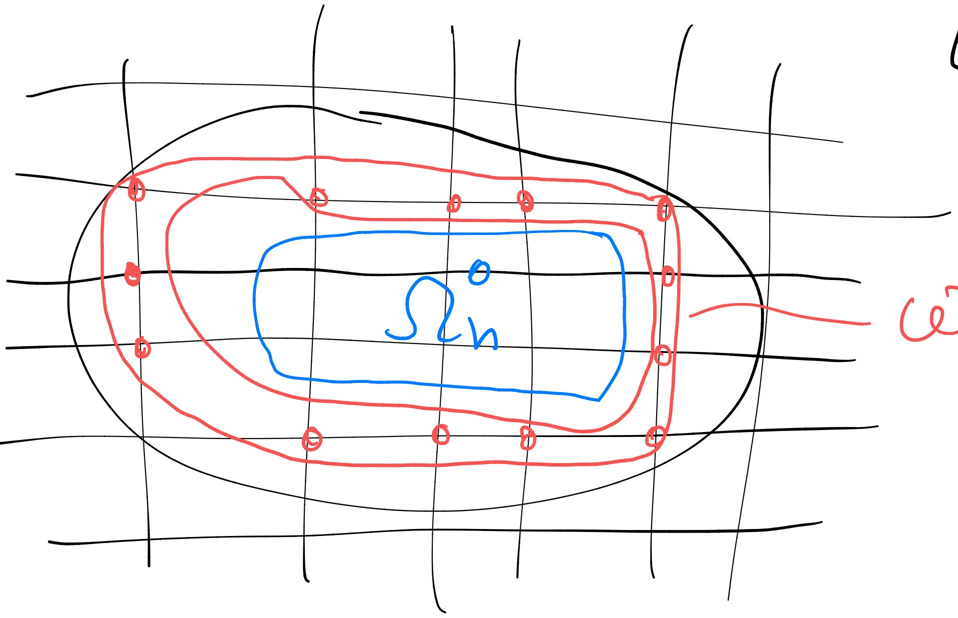
internal not near $\parallel$ boundary

FD method

$\nearrow O(h^2)$

- use 5 point formula internally in $\Omega_h^0$
- modified by nonuniform scheme near the boundary, at $\omega_h \equiv \{(x_i, y_j)$ internal but near boundary$\} \to O(h)$

Truncation error would be $\begin{cases} \text{quadratic at } \Omega_h^0 \\ \text{linear at } \omega_h \end{cases}$




It is possible to show that

$$|u_{ij} - U_{ij}| \le \frac{d^2}{96} h^2 M_4 \qquad \forall i,j$$

$= \max |u_{xxxx}|$

$d$ = radius of a sphere containing $\Omega$

$\Rightarrow$ lower-order truncation error at boundary does not necessarily reduce order of method.

See also Larsson – Thomée for curved domain scheme based on the idea of interpolation.

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## Notions of Functional spaces

motivation: take, for simplicity,

$$\mathcal{L}u := -\Delta u = f \qquad \Omega$$

classical setting: $u \in C^2$, $f \in C^0$

to $\mathcal{L}$ I can associate a bilinear form:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v$$

indeed,

- multiply by "test function" $v$ : $v|_{\partial\Omega} = 0$

$$\Delta u \, v = f v$$

- integ. over $\Omega$ :

$$-\int_\Omega \Delta u \, v = \int_\Omega f v$$

- use Gauss-Green formula

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} \underset{\substack{\| \\ 0}}{v} = \int_\Omega f v$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v$$

$\forall v$ in the "right space" and $v|_{\partial\Omega} = 0$

space ?

non-smooth $f$

$\Omega \subset \mathbb{R}^d$ open

$C^k(\bar{\Omega})$ k-times cont. diff.

"Banach" w.r.t. $\|v\|_{C^k(\bar{\Omega})} = \max_{|\alpha| \le k} \|D^\alpha v\|_{C(\bar{\Omega})}$

multi index

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} \qquad \alpha = (\alpha_1, \dots, \alpha_d)$$

$$C_0^k(\Omega) = \{ f \in C^k(\Omega) : f \text{ has compact support in } \Omega \}$$

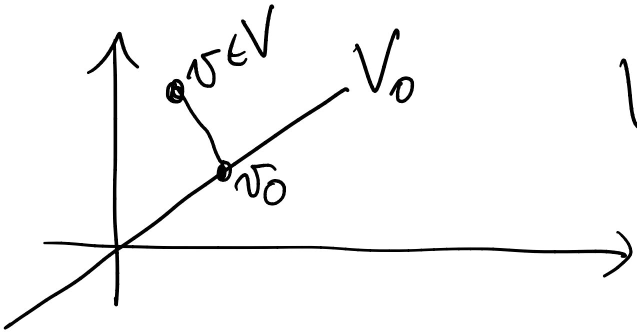
inner product

Hilbert space: $(H, (\cdot, \cdot))$

$H$ inner product space which is complete w.r.t. $\|\cdot\|_H = (\cdot, \cdot)^{1/2}$

**Properties:** 

- $d(v,w) = \|v-w\|_H$
- $|(v,w)| \le \|v\|_H \|w\|_H$ Schwarz ineq.
- $\|v+w\|_H \le \|v\|_H + \|w\|_H$ triangle ineq.
- if $V_0$ closed linear subspace, then $\forall v \in V$, $v = v_0 + w$ $\exists!\, w \in V$: $v_0 \in V_0$




- $(w, z) = 0$ $\forall z \in V_0$
- $= \|w\| = \min_{z \in V_0} \|v - z\|$
$= \|v - v_0\|$

**Dual space:** Let $(V, \|\cdot\|)$ a normed space, $V' = \mathcal{L}(V; \mathbb{R})$, the dual space of $V$, is the space of all linear continuous functionals $L : V \longrightarrow \mathbb{R}$
$\hookrightarrow$ (i.e. bounded: $|L(v)| \le C \|v\|$)

The dual space is normed w.r.t. the operator norm

$$\|L\|_{V'} := \sup_{v \in V \setminus \{0\}} \frac{|L(v)|}{\|v\|}$$

- $\langle \cdot , \cdot \rangle : V' \times V \longrightarrow \mathbb{R}$ bilinear form by

$\langle L, v \rangle :\Leftarrow L(v)$ "duality pairing"