NSPDE/ANA 2022 - Lecture 6

Last lecture:
- Sobolev spaces
- Weak formulation of elliptic problems
- Regularity of the solution

Today: more on elliptic problems in weak form
- Other boundary conditions
- General elliptic problems
- Simmetric case: Dirichlet principle
- General case: Lax Milgram lemma
- The method of Galerkin

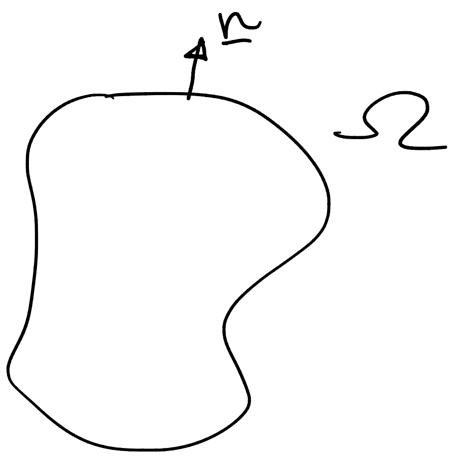
References:
- Quarteroni
- Larsson & Thomee

1. Poisson with other b.c.

Prob. 2.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \underline{n}} = g & \text{on } \partial\Omega \end{cases}$$

Neumann ("natural b.c.")



$\underline{n} = \underline{n}(\underline{x})$ outward normal at a.e. $\underline{x} \in \partial\Omega$.

(homog Dirichlet : $u \in H^1_0$: $(\nabla u, \nabla v) = (f, v)$ $\forall v \in H^1_0$)

test with v and integrate:

$$(f,v)=(-\Delta u,v) = (\nabla u,\nabla v)-\int_{\partial\Omega}\underbrace{n\cdot\nabla u}_{=g}\, v$$

weak form:

Find $u\in H^1(\Omega)$: $\int_\Omega \nabla u\cdot\nabla v = \int_\Omega f\, v + \int_{\partial\Omega} g\, v$ $\quad\forall v\in H^1(\Omega)$

Prob. 3

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \end{cases} \qquad \Gamma_D\cup\Gamma_N = \partial\Omega$$

Find $u\in H^1_{\Gamma_D}(\Omega)$: $\int_\Omega \nabla u\cdot\nabla v = \int_\Omega f\, v + \int_{\Gamma_N} g\, v$ $\quad\forall v\in H^1_{\Gamma_D}(\Omega)$

Prob. 4:

$$\begin{cases} -\Delta u = f \in L^2(\Omega) & \text{in } \Omega \\ u = g \in H^{1/2}(\partial\Omega) & \text{on } \partial\Omega \end{cases}$$

Note that the space $\{v\in H^1(\Omega): v|_{\partial\Omega}=g\}$ is affine!

Idea: use Trace Theorem: Given $\Omega\subset\mathbb{R}^d$ bounded, open, Lipschitz / polygonal boundary,

$$\exists\ \gamma_0 : H^1(\Omega)\longrightarrow H^{1/2}(\partial\Omega)$$

(the space of traces of $H^1(\Omega)$)

linear and continuous such that

$$\gamma_0 v = v_{|\partial\Omega} \quad \forall v \in H^1(\Omega) \cap C^0(\Omega)$$

well defined

so $\exists\, C^*$: $\underbrace{\|\gamma_0 v\|_{L^2(\partial\Omega)} \leq C^* \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)}_{\text{continuity}}$

Thus, given $g \in H^{1/2}(\partial\Omega)$ $\exists\, \tilde{g} \in H^1(\Omega) : \gamma_0 \tilde{g} = g$ and transform non homog Dirichlet Pbm:

1) Find $\tilde{u} \in H^1_0(\Omega)$: $\begin{cases} -\Delta \tilde{u} = \underbrace{f + \Delta \tilde{g}} & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \partial\Omega \end{cases}$

2) set $u = \tilde{u} + \tilde{g}$

2 General elliptic problem

$$\begin{cases} \mathcal{L}u := \underbrace{-\sum_{i,j=1}^{d} D_i(a_{ij}(x) D_j u) + \sum_{i=1}^{d} D_i(b_i u)}_{-\nabla\cdot(A\nabla u) + \nabla\cdot(\underline{b}u)} + cu = f \\ u = 0 \end{cases}$$

weak form: test with $v \in H^1_0(\Omega)$:

$A(x) = (a_{ij})_{ij}$

$$\int_\Omega (A(x)\nabla u)\cdot\nabla v - \int_\Omega (u\,\underline{b})\cdot\nabla v + \int_\Omega c\,u\,v =: a(u,v)$$

$$\cancel{-\int_{\partial\Omega} A\nabla u\cdot n\,\underset{0}{\underset{\|}{v}}} + \cancel{\int_{\partial\Omega} \underline{b}\cdot n\,u\,\underset{0}{\underset{\|}{v}}} = \int_\Omega f v$$

Find $u \in H^1_0(\Omega)$: $a(u,v) = (f,v)$ $\forall v \in H^1_0(\Omega)$

- Neumann b.c.: $\frac{\partial u}{\partial n_L} := A\nabla u\cdot n - \underline{b}\cdot\underline{n}\,u$ conormal derivative

$$\frac{\partial u}{\partial n_L} = g$$

$\longrightarrow$ Find $u \in H^1(\Omega)$: $a(u,v) = (f,v) + (g,v)$

- Robin: $\frac{\partial u}{\partial n_L} + \mu u = g$
- Mixed: $\begin{cases} \frac{\partial u}{\partial n_L} = g_N & \Gamma_N \\ u = g_D & \Gamma_D \end{cases}$

3 Well posedness of the weak formulations (abstract)

PBM: $\begin{cases} \mathcal{L} u = f & \text{in } \Omega \\ \mathcal{B} u = 0 & \text{on } \partial\Omega \end{cases}$

where $\mathcal{L}$ linear diff. op, f given, $\mathcal{B}$ boundary operator (affine)

Find $u \in V : a(u,v) = \ell(v) \quad \forall v \in V$

"trial"   "test"

V Hilbert

$\ell : V \to \mathbb{R}$ linear

$a : V \times V \to \mathbb{R}$ bilinear

Assumption: a is coercive:

$$\exists \alpha_0 > 0 : a(v,v) \geq \alpha_0 \|v\|_V^2$$

and a is continuous:

$$\exists \gamma > 0 : a(w,v) \leq \gamma \|w\|_V \|v\|_V$$

example: 2nd order elliptic pbms $\equiv$ coercive in 2nd order term

here we assume coercivity, instead, for the whole operator (stronger condition!)

In particular,

assumptions

ex. 1: $a(u,v) = (\nabla u, \nabla v)$ $V = H^1_0(\Omega)$

$$\|v\|_{H^1_0(\Omega)} = \|\nabla v\|_{L^2}$$

$$(= a(v,v))$$

ex. 2: $a(u,v) = (\nabla u, \nabla v) + (u,v)$ $V = H^1(\Omega)$

$$\longrightarrow \|v\|_{H^1}^2 = \|v\|_{L^2} + \|\nabla v\|_{L^2}$$

not sym !

$$(= a(v,v))$$

$$V = H_0^1(\Omega)$$

ex3 : $a(u,v) = (A \nabla u, \nabla v) + (b \cdot \nabla u, v) + (c u, v)$

assuming op. is elliptic

$$a(v,v) = (A \nabla v, \nabla v) + (b \cdot \nabla v, v) + (c v, v)$$

$$\vee\!/$$

$$\alpha \|\nabla v\|$$

$$\int_\Omega (b \cdot \nabla v) v = \int_\Omega \frac{1}{2}(b \cdot \nabla v^2) = -\int_\Omega v^2 \frac{(\nabla \cdot b)}{2}$$

$$+ \Big(\underbrace{\big(c - \tfrac{1}{2} \nabla \cdot b\big)}_{\geq 0} v, v \Big)$$

assume $c - \frac{1}{2} \nabla \cdot b \geq 0$

$$\geq \alpha \|\nabla v\|^2 \qquad \forall v \in H_0^1(\Omega)$$

Poincare

$$\geq \alpha \; \frac{1}{C_\Omega} \|v\|^2$$

$$\Rightarrow \begin{cases} \frac{1}{\alpha} a(v,v) \geq \|\nabla v\|^2 \\ \frac{C_\Omega}{\alpha} a(v,v) \geq \|v\|^2 \end{cases}$$

$$\Rightarrow \left(\frac{1 + C_\Omega}{\alpha}\right) a(v,v) \geq \left(\|\nabla v\|^2 + \|v\|^2\right.$$

$$\therefore \quad a(v,v) \geq \underbrace{\frac{\alpha}{1+C_\Omega}}_{=:\alpha_0} \|v\|^2_{H^1}$$

$\Rightarrow$ coercivity in H^1 under the extra condition $c - \frac{1}{2}\nabla\cdot b \geq 0$.

(Extra cond is sufficient but <u>not</u> necessary.)

ex: show continuity for general 2nd order elliptic pbm in H^1_0

Well posedness

CASE 1 : $Q(\cdot,\cdot)$ symmetric (e.g. Poisson)

then Q defines an inner product so as done for Poisson, can just use Riesz representation, to get

Theorem: V Hilbert, $Q(\cdot,\cdot)$ is symmetric cont. and coercive bilinear form in $V \times V$, $\ell(\cdot)$ cont. lin op. in V, the the problem : $u \in V : Q(u,v) = \ell(v) \quad \forall v \in V$

is well posed, that is $\exists!$ solution $u \in V$ and

$$\|u\|_V \leq \|\ell\|_{V'}$$

Theorem: u is the sol. of the above $\Longleftrightarrow$ u is the unique solution of the minimisation (variational) problem:

Find $u \in V$: $J(u) \leq J(v) \quad \forall v \in V$

where $J(v) := \frac{1}{2} Q(v,v) - \ell(v)$.

Prof: "$\Rightarrow$" $\forall w \in V$ $J(u+w)$ ...... $\geq J(u)$

(hint) "$\Leftarrow$" let $t \in \mathbb{R}$, use $0 = \frac{d}{dt} J(u+tw)\Big|_{t=0}$

example: $Q(u,v) = (A\nabla u, \nabla v) + (cu, v)$, $\ell(v) = (f,v)$

$\exists!\, u \in H_0^1(\Omega)$ is the unique minimizer of

$$J(v) = \underbrace{\underbrace{\frac{1}{2}\int \left(A\,|\nabla v|^2 + c\,v^2\right)}_{\text{elastic energy}} - \underbrace{\int f\,v}_{\text{load potential}}}_{\text{Dirichlet energy (potential)}}$$

Dirichlet principle

CASE 2: not necessarily sym. problems

(e.g. problems with non-zero advect.)

Lemma (Lax-Milgram): given V Hilbert,

$Q(\cdot,\cdot): V \times V \longrightarrow \mathbb{R}$ bilinear form $\forall v, w \in V$

* continuous: $\exists \gamma > 0 : Q(w,v) \leq \gamma \|w\|_V \|v\|_V$
* coercive: $\exists \alpha_0 > 0 : Q(v,v) \geq \alpha_0 \|v\|_V^2 \quad \forall v \in V$

$\ell(\cdot): V \to \mathbb{R}$ linear functional, $\|\ell\|_{V'}$

& continuous: $\exists c_0 > 0: |\ell(v)| \leq c_0 \|v\| \quad \forall v \in V$

Then $\exists! \, u \in V: \; a(u,v) = \ell(v) \quad \forall v \in V$, and

$$\|u\| \leq \frac{1}{\alpha_0} \|\ell\|_{V'}$$

Proof: (see in books, or Helton course)

Just the continuous dependence (a priori / stability):

$$\underbrace{\alpha_0 \|u\|^2 \leq}_{\text{coerc.}} a(u,u) \underset{\text{testing with } u \text{ itself}}{=} \ell(u)$$

$$\Rightarrow \quad \|u\| \leq \frac{1}{\alpha_0} \frac{\ell(u)}{\|u\|} \leq \frac{1}{\alpha_0} \|\ell\|_{V'} \qquad \square$$

comments:

- ☆ uses Riesz
- ☆ but it is proper generalisation.

4. The Galerkin method

= restrict problem into finite dimensional subspaces

Galerkin method: given $u \in V : a(u,v) = \ell(v)$ $\forall v \in V$,

Consider $V_n < V$ ($V_n \subset V$ as a subspace)

with $\dim V_n = n$,

Find $u_n \in V_n : a(u_n, v) = \ell(v)$ $\forall v \in V_n$

Ritz-Galerkin method: Galerkin method symetric case, i.e. look for the minimizer from V_n.

e.g. for V separable, $\{\varphi_i\}$ countable basis,

Ritz: minimize over $\mathrm{span}\langle \varphi_1, \dots, \varphi_n \rangle$

Well posedness of Galerkin method follows imediately by Lax Milgram given V is still a Hilbert (sub) space

$\Rightarrow \exists!\, u_n \in V_n$ sol. and $\|u_n\|_V \leq \alpha_0 \|\ell\|_{V'}$

Algebric point of view for Galerkin:

suppose $V_n = \text{span}\langle \varphi_1, \dots, \varphi_n \rangle$

$$u_n(x) = \sum_{j=1}^{n} U_j \varphi_j(x)$$

unknown coeffs

Gal method $\iff$ find $\{U_j\}_{j=1}^{n}$:

$$\sum_{j=1}^{n} U_j \underbrace{a(\varphi_j, \varphi_i)}_{A_{ji}} = \underbrace{\ell(\varphi_i)}_{F_i} \quad \forall i = 1, \dots, n$$

!! ☺

proof: by linearity

$$\Updownarrow$$

$$\boxed{AU = F}$$

algebric form of the Galerkin method for $V = \text{span}\langle \varphi_1 \dots, \varphi_n \rangle$

moreover

- a coercive $\Rightarrow A$ pos def $\rightarrow$ invert, can $\exists!$ sol.
- a sym $\Rightarrow A$ sym

- a coerc + sym $\Rightarrow$ A SPD   sym + pos def

If a sym, u is the minimizer, u_n minim. from the subspace, i.e.

$u_n \in V_n : a(u_n, v) = \ell(v) \quad \forall v \in V_n$ $AU = F$

$\Updownarrow$ $\Leftrightarrow$ $\Updownarrow$

U min.

$J(u_n) \leq J(v) \quad \forall v \in V_n$ $\phi(V) = \frac{1}{2} V^T A V - V^T F$

Questions:

Q_1 : How good is the Galerkin sol.

Q_2 : does it converge ?

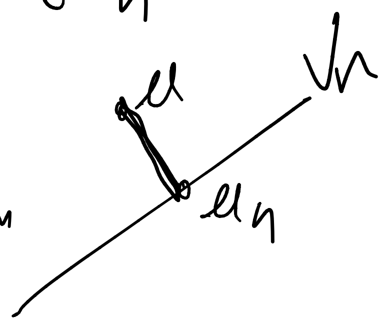
Proposition (consistency): It holds

$$a(u - u_n, v) = 0 \quad \forall v \in V_n$$

(Galerkin orthogonality)

"Full consistency"

Proof: $\begin{cases} a(u,v) = \ell(v) \\ a(u_n,v) = \ell(v) \end{cases}$ $\forall v \in V_n$



sym case

Lemma of Céa: Under the assump. of Lax–Milgram,

$$\|u-u_n\|_V \le \frac{\gamma}{\alpha_0} \inf_{v_n \in V_n} \|u-v_n\|_V$$

"quasi optimality"

Proof:

$$\alpha_0 \|u-u_n\|_V^2 \le a(u-u_n, u-u_n)$$

Gal orth: $= a(u-u_n, u)$

$\forall v_n \in V_n$ $\qquad = a(u-u_n, u-v_n)$

Cont. $\le \gamma \|u-u_n\|_V \|u-v_n\|_V$

Proposition: Assume $a(\cdot,\cdot)$ is also sym.
then:

- $\|u-u_n\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min_{v_n \in V_n} \|u-v_n\|$
- $\|u-u_n\|_a = \min_{v_n \in V_n} \|u-v_n\|$
- u_n orthogonal projection of u into V_n w.r.t. $(w,v)_a = a(w,v)$.

"Theoretical" Galerkin method:

- fix $\{\varphi_j\}_{j=1}^{n} \subset \{\varphi_j\}$ complete orthonormal basis of V
- consider sequence of Galerkin methods with $V_n = \text{span}\langle \varphi_1, \dots, \varphi_n \rangle$
- Given that $\lim_{n \to +\infty} V_n$ is dense in V

$$\forall v \in V, \ \forall \varepsilon > 0, \ \exists n > 0 : \ \exists v_n \in V_n : \ \|v - v_n\|_V \leq \varepsilon$$

Hence, by Céa such a method would be convergent!

Problem: construction of the orthonormal basis?

"Practical" Galerkin method: use some (not nec. orthonormal) but convenient finite dim. spaces and basis

finite elements (piecewise polynomials)

Spectral methods (global polynomials)

wavelets