NSPDE/ANA 2022 - Lecture 8

Last lecture:
- The Finite Element Method (FEM) in 1D
- Error analysis

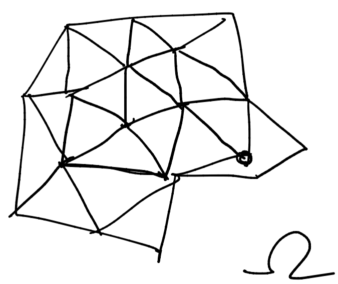
Today:
- Error analysis of FEM in more dimensions
- An a posteriori error bound
- Algebric form of FEM
- Conditioning of the FE matrix
- Generalised Galerkin method

References:
- Quarteroni
- Larsson & Thomee

others are available!

FEM in more dimensions ($\mathcal{C}^0$-conforming triangular elements)

- based on "triangulations" / general "meshes" of the problem domain Ω.



$$\mathcal{T}_h = \{ T \}, \quad T \text{ triangle (tetrahedrum)}$$

$$\bar{\Omega} = \bigcup T$$

mesh parameter: $h = \max_T \operatorname{diam}(T)$

- Galerkin method with discrete space:

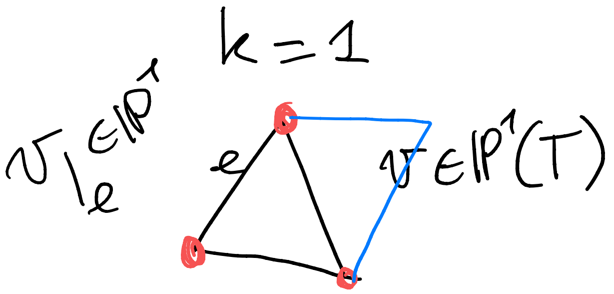
Given $\mathcal{T}_h$ and $k \in \mathbb{N}$

space of polynomials of degree $\le k$

$$V_h^k = \{ v \in \mathcal{C}^0(\Omega) : v|_T \in \mathbb{P}^k(T), \forall T \in \mathcal{T}_h \}$$

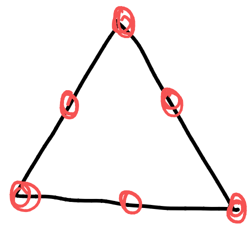
- easy to define on each T through a set of degrees of freedom (dof), in this case values at given nodes ("nodal values")

$k=1$

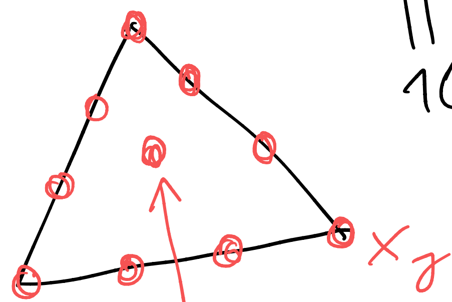


nodal values $\longrightarrow$ edge values $\longrightarrow$ elemental function $\xrightarrow{C^0}$ mesh function

$k=2$



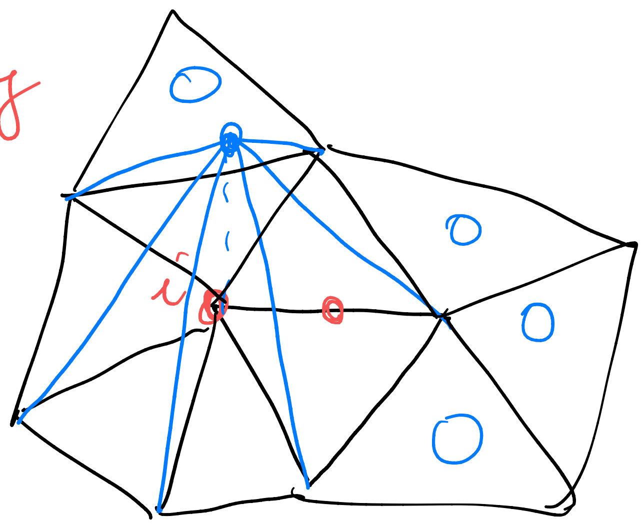
$k=3$ $\quad \dim(\mathbb{P}^3(T)) = 10$



- Lagrange basis functions $\varphi_i|_T = \varphi_T^i$

$$\varphi_T^i(x_j) := \delta_{ij}$$

example: $k=1$



FEM: Find $u_h \in V_h$: $a(u_h, v_h) = \ell(v_h)$ $\forall v_h \in V_h$

$\rightarrow$ test with basis functions:

Find $\{U_i\}$ $\left(u_h = \sum U_j \varphi_j \right)$:

Poisson

$$a(u_h, \varphi_i) = \underbrace{\ell(\varphi_i)}_{= F_i} \quad \forall \varphi_i$$

$$= \sum_j U_j \underbrace{\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i}_{S_{ij}}$$

$$\longrightarrow \quad SU = F$$

algebric form of FEM

On conditioning of FE matrix

Def: Spectral condition number

$$\chi_{sp}(A) := \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$$

max/min eigenvalues of A

note: if A symmetric $\chi_{sp}(A) \equiv \chi_2(A) = \|A\|_2 \, \|A^{-1}\|_2$

We say that A is well-conditioned if χ_{sp} is small

Prop: If $\mathcal{T}_h$ is quasi-uniform ($\exists \rho > 0$: $\min_{T \in \mathcal{T}_h} h_T \geq \rho h := \max_{T \in \mathcal{T}_h} h_T$, and A SPD ($\Leftarrow \mathcal{A}$ coercive + sym), then $\chi_{sp}(A) = \mathcal{O}(h^{-2})$

Lemma: Let $\{\mathcal{T}_h\}_h$ family of quasi uniform. Then $\exists C_1, C_2 > 0$ independent of h & $C_1 < C_2$ such that $\forall \underline{v} = \{v_i\}_{i=1}^n$: $v_h \in \sum v_i \varphi_i$, then

$$C_1 h^d |\underline{v}|^2 \leq \|v_h\|_0^2 \leq C_2 h^d |\underline{v}|^2$$

Rayleigh quotient : $R(\underline{v}) := \dfrac{A\underline{v} \cdot \underline{v}}{|\underline{v}|^2} = \dfrac{\mathcal{A}(\underline{v}, \underline{v})}{|\underline{v}|}$ $\forall \underline{v} \in \mathbb{R}^n$

$$\begin{cases} \max_{\underline{v} \neq 0} R(\underline{v}) = \lambda_{max}(A) \\ \min_{\underline{v} \neq 0} R(\underline{v}) = \lambda_{min}(A) \end{cases}$$

Proof of the proposition : $\underline{v} \in \mathbb{R}^n$

cont. of $\mathcal{A}$

$$R(\underline{v}) = \frac{A\underline{v}\cdot\underline{v}}{|\underline{v}|^2} = \frac{a(v_h,v_h)}{|\underline{v}|^2} \overset{|}{\leq} \gamma\,\frac{\|v_h\|^2_{H^1}}{|\underline{v}|^2}$$

coercivity

$$\geq \alpha_0 \frac{\|v_h\|^2_{H^1}}{|\underline{v}|^2} \geq \alpha_0 \frac{\|v_h\|^2_{L^2}}{|\underline{v}|^2}$$

by lemma

$$\geq \frac{\alpha_0 C_1 h^d \cancel{|\underline{v}|^2}}{\cancel{|\underline{v}|^2}}$$

It remains to bound $\|v_h\|^2_{H^1}$

Theorem (Inverse inequality)$_1$: Let $\{\mathcal{T}_h\}$ quasi-unif family of triang, $V_h^k \subset H_0$ corresp. FE space.

Then $\exists\, C_I > 0 : \forall v_h \in V_h^k$

$$\|\nabla v_h\|^2_{L^2} \leq C_I\, h^{-2}\, \|v_h\|^2_{L^2}$$

$$\frac{\gamma\,\|v\|^2_{H^1}}{|\underline{v}|^2} \underset{\text{inverse ineq.}}{\leq} \gamma\,\frac{(1+C_I h^{-2})\,\|v_h\|^2_{L^2}}{|\underline{v}|^2}$$

lemma

$$\leq \gamma\,(1+C_I h^{2})\cdot C_2\, h^d$$

So:

$$\alpha_0 C_1 h^d \leq R(v) \leq \gamma(1+C_I h^{-2})\, C_2\, h^d$$

$\leq \lambda_{min}, \lambda_{max} \leq$

$$\Rightarrow \chi_{sp}(A) = \frac{\lambda_{max}}{\lambda_{min}} \leq \frac{\gamma}{\alpha_0} \frac{c_2}{c_1} (1 + c_I h^{-2}) = O(h^{-2})$$

these constants are indep. of h but depend on k !

- need good solvers (direct, iterative, ..., geometric multigrid, ...)

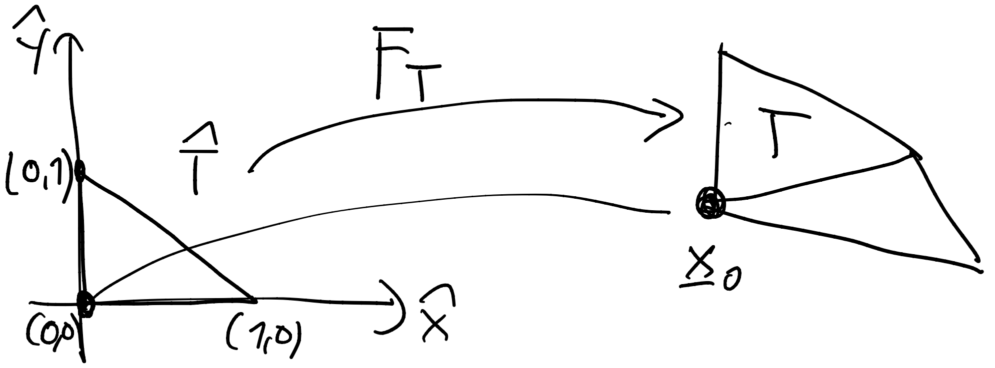
2D → direct; 3D → iterative

FEM assembly

For example,
$$\int_\Omega a \, \nabla \varphi_j \cdot \nabla \varphi_i = S_{ij}$$
$$= \sum_{T : \, T \subset \operatorname{supp}(\varphi_i) \cap \operatorname{supp}(\varphi_j)} \int_T a \, \nabla \varphi_j \cdot \nabla \varphi_i \qquad \text{local !}$$

- In general, each $\int_T$ must be computed via quadrature rule

- via mapping to a reference element $\hat{T}$



F_T affine $= J_{F_T}\,\hat{\underline{x}} + \underline{x}_0$

by fixing quadrature rule on $\hat{T}$

Hence,

$$S_{ij}^T = \int_T a(\underline{x})\nabla\varphi_i\cdot\nabla\varphi_j = \int_{\hat{T}} \hat{a}(\hat{\underline{x}})\left(J_{F_T}^{-T}\hat{\nabla}\hat{\varphi}_j\right)\cdot\left(J_{F_T}^{-T}\hat{\nabla}\hat{\varphi}_i\right)|J_{F_T}|\,d\hat{x}$$

$$\hat{a}(\hat{x}) := a(F_T\hat{x})\Big|_T$$

J_{F_T} = Jacobian matrix of F_T

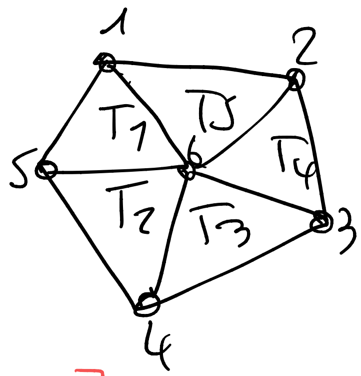
and use quadrature on reference triangle. (In particular, values of basis functions can be evaluated on the reference element once and for all!

Then $S_{ij} = \sum S_{ij}^T$

For example, $k = 1$ $\quad S_{ij}^T$ is 3×3 matrix

Given S_{ij}^{T} we need to **assemble**:

Say, mesh is



$\boxed{\text{mesh}}$: $T = \begin{bmatrix} 1 & 5 & 6 \\ 4 & 6 & 5 \\ 4 & 3 & 6 \\ & \vdots & \end{bmatrix}$ "connectivity"

$N = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_6 & y_6 \end{bmatrix}$ nodal coordinates

$S_{ij}^{T_1}$ 3×3 $\xrightarrow{\text{assemble}}$ S 6×6

6

$$S = \begin{array}{c} \begin{bmatrix} * & & & & * & * \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & * & * \\ * & & & & * & * \\ * & & & & & \end{bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \ \end{array}$$

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FEM analysis for $\Omega \subseteq \mathbb{R}^d$

- given weak form problem V Hilbert under assumptions Lax – Milgram Lemma
- FEM or Galerkin method

1. $\exists! \, u_h \in V_h^k$: $\|u_h\|_V \le \frac{1}{\alpha_0} \|\ell\|_{V'}$ linear r.h.s. coercivity

2. $a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h^k$
galerkin orthogonality (full consistency)

3. $\|u-u_h\|_V \le \gamma/\alpha_0 \inf_{v_h \in V_h} \|u-v_h\|_V$

4. $\le \gamma/\alpha_0 \, \|u - I_h u\|_V$

Pick $I_h u$ as the Lagrange interpolant

(proof: Heltai)

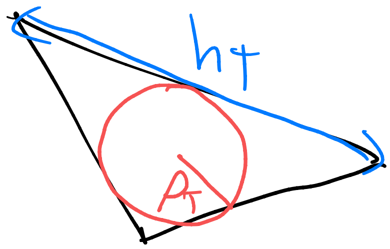
Theorem (Interpolation in $\mathbb{R}^d$). $k \in \mathbb{N}$

$\forall v \in H^{k+1}(\Omega)$, $\quad \forall \; 0 \le m \le k+1$

$$|v - I_h v|_{m,T} \le C(k) \left(\frac{h_T}{\rho_T}\right)^m h_T^{k+1-m} |v|_{k+1,T}$$

$\forall T \in \mathcal{T}_h$ mesh

with ρ_T = radius of largest circle inscribed in T



Def (shape-regularity): The triangulations $\{\mathcal{T}_h\}_h$ are shape-regular if $\exists\, \sigma \ge 1$:

$$\frac{h_T}{\rho_T} \le \sigma \qquad \forall T \in \mathcal{T}_h \quad \forall h > 0$$

($\equiv$ the minimal angle must be bounded below)

Corollary: If $\{\mathcal{T}_h\}$ is shape-regular

$$|v - I_h^T v|_{m,T} \le C(k,\sigma)\, h_T^{k+1-m} |v|_{k+1,T}$$

$\longrightarrow$ global interpolant defined:

- fix $m = 0, 1$ $\quad$ (or $v_h \notin H^m(\Omega)$ $m > 1$ in genere)

- fix $v \in H^s(\Omega)$ $\quad s \geq 2$, $d = 2, 3$.

Then by Sobolev embedding,

$$H^s(\Omega) \hookrightarrow C^0(\bar{\Omega})$$

hence interpolant is well defined:

$$I_h : H^s \longrightarrow V_h^k \qquad I_h v|_T = I_h^T v$$

local polynomial interpolant

Then

$$|v - I_h v|_{H^m(\Omega)} \leq C(k)\, h^{\ell - m}\, |v|_{H^\ell(\Omega)}$$

$(m = 0, 1)$

$$\ell = \min(k+1, s)$$

example $s = 2$, $k = 1$ $\Rightarrow$ $\begin{cases} \|v - I_h v\|_{L^2} \leq C(k)\, h^2\, |v|_{H^2} \\ |v - I_h v|_{H^1} \leq C(k)\, h\, |v|_{H^2} \end{cases}$

$\Longrightarrow$ FEM order now established:

$$|u - u_h|_{H^1(\Omega)} \leq \frac{\gamma}{\alpha_0}\, C(k)\, h^{\ell - 1}\, |u|_{H^\ell(\Omega)}$$

$$\ell = \min(k+1, s)$$

- L^2-error see Heltai
- L^∞-error bounds (LT)

A posteriori type error bounds

- (a priori) bound above is theoretical, purpose is to predict order of convergence

theoretical $\equiv$ depends on exact solution

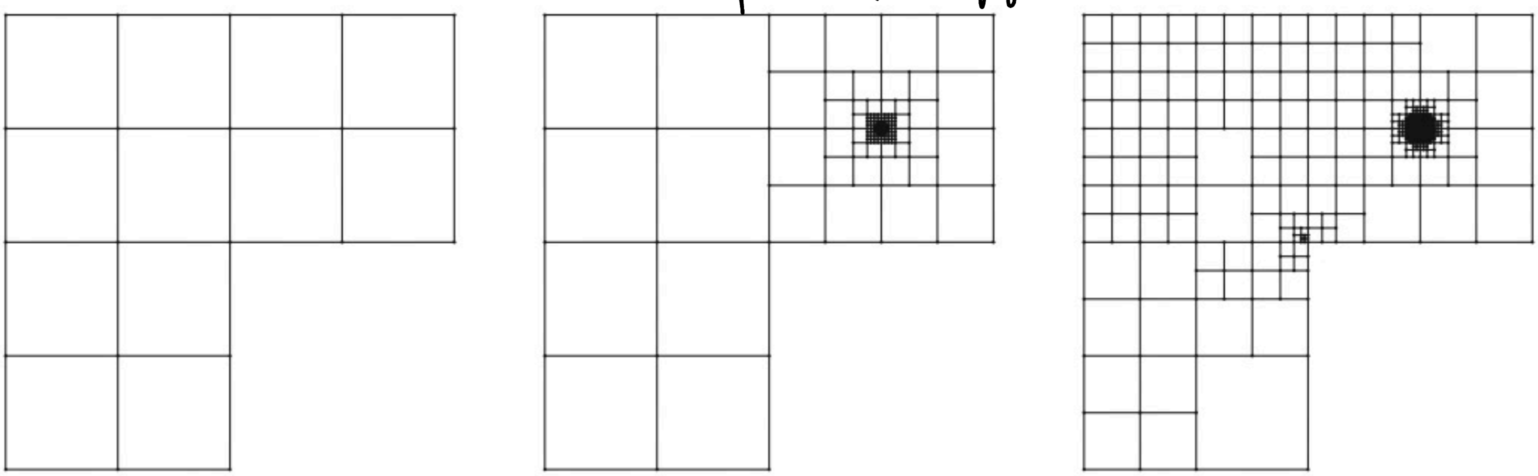
Alternatively, may try bound

$$\underbrace{|u-u_h|_{H^1} \le C\,\eta(u_h)}$$

a posteriori error bound (depends only on computable quantities)

- If $\eta^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2$ $\quad$ η_T = local error est

$\rightarrow$ can be used to improve approx where needed



motivating example:

consider linear system $AU=F$

- U_0 approx to U
- can I estimate $U-U_0$?

$$A(U-U_0)=F-AU_0$$

$$\Downarrow$$

$$U-U_0 = A^{-1}\underbrace{\underbrace{(F-AU_0)}_{\text{residual !}}}_{\text{only depend on } U_0}$$

Typical a posteriori bound for FEM:

$$|u-u_h|_{H^1} \le C\left(\sum_T h_T^2 \|f-\underbrace{\mathcal{L}u_h}_{\text{PDE}}\|_{L^2(T)}^2 + \sum_e h_e \|a[\![\nabla u_h\cdot n]\!]\|_{L^2(e)}^2\right)^{1/2}$$

$\mathcal{L}u = -\nabla\cdot(a\nabla u)$

edges of $\mathcal{T}_h$

error estimator (only dep. on u_h)

where $[\![\,\cdot\,]\!]$ = edge jump operator

(to be continued in Heltai)