NSPDE/ANA 2022 - Lecture 3

Last lecture:
- The Poisson BVP
- Maximum principle
- Divided differences
- Finite Difference for two-points BVP
- A priori analysis: truncation error

Today:
- A priori analysis: stability, error estimate
- General elliptic probl

- Recall:

general two-points BVP:

$$\begin{cases} \mathcal{L}u := -a u'' + b u' + c u = 0 & \text{in } \Omega = (0,1), \\ u(0) = u_0 \,; \quad u(1) = u_1 \end{cases}$$

with $a > 0$, b, $c \geq 0$ smooth functions.

FD discretisation with centred formulas:

$$\underbrace{\frac{-(a_i - \frac{1}{2} h b_i) U_{i+1} + (2a_i + h^2 c_i) U_i - (a_i + \frac{1}{2} h b_i) U_{i-1}}{h^2}}_{=: \mathcal{L}_h U_i} = f_i$$

- $T(x) := \mathcal{L}u(x) - \mathcal{L}_h u(x)$ "truncation error"

- $|T(x)| \leq \left(\frac{h^2}{12} \|a\|_{\infty} + \frac{h^2}{6} \|b\|_{\infty} \right) \|u\|_{C^4}$

where $\|f\|_\tau = \sup\{|f(x)|\}$ $\overset{\shortparallel}{O(h^2)}$

$\equiv$ consistency (exact sol. satisfies the discrete scheme as $h \to 0$)

$\lightning$ does not imply convergence of the discrete solution to the exact solution.

(see Larsson_Thomee)

Theorem (DMP). Let $V = \{V_i\}$. If $\mathcal{L}_h V_i \leq 0$ $\forall i = 1, \dots, M-1$

Then,

- If $\begin{cases} c = 0 \\ b \neq 0 \end{cases}$ and $a_i \pm \frac{1}{2} h b_i \geq 0$ $\Rightarrow$ $\max_i V_i = \max\{V_0, V_M\}$
- If also $c \geq 0$ and $\shortparallel$ $\Rightarrow$ $\max_i V_i = \max\{V_0, V_M, 0\}$

Comment: DMP conditions to off-diag coeffs to be non-positive. For instance, if a is small, need h small enough!

Lemma (stability). Assume $b \equiv 0$.

Then, for any $V = \{V_i\}_{i=0}^M$ we have,

$$|V|_\infty \leq \max\{|V_0|, |V_M|\} + C\,|\mathcal{L}_h V|_\infty$$

with $C = 1/8\underline{a}$, $\underline{a} := \min_{x \in [0,1]} a(x)$.

(where $|V|_\infty := \max_i |V_i|$)

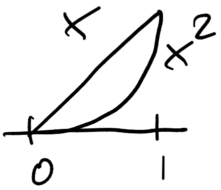
Proof (LT) using "comparison function":

$$w(x) := x - x^2$$

Let also $W = \{W_i\}$, with $W_i = w(x_i)$

1) $$\mathcal{L}_h W_i = 2a_i + c_i\,(\underbrace{\overset{x_i}{ih} - i^2h^2}_{W_i}) \qquad (-au''+cu)$$

because | w is quadratic!



$$\geq 2a_i \geq 2\underline{a}$$

$x - x^2 \geq 0$

In conclusion $\mathcal{L}_h W_i \geq 2\underline{a}$

2) Set $\tilde{V}_i^{\pm} := \pm V_i - \frac{1}{2\underline{a}} |\mathcal{L}_h V|_\infty W_i$

$$\mathcal{L}_h \tilde{V}_i^{\pm} = \pm \mathcal{L}_h V_i - \underbrace{\frac{1}{2\underline{a}} |\mathcal{L}_h V|_\infty \underset{\substack{\geq \\ 2\underline{a}}}{\mathcal{L}_h W_i}}_{\geq |\mathcal{L}_h V|_\infty} \leq 0$$

3) By DMP using $W_0 = W_M = 0$

$$\tilde{V}_i^{\pm} \leq \max\{\pm V_0, \pm V_M, 0\} \leq \max\{|V_0|, |V_M|\}$$

$$= \pm V_i - \frac{1}{2\underline{a}} |\mathcal{L}_h V|_\infty W_i$$

$$\Rightarrow \pm V_i \leq \max\{|V_0|, |V_M|\} + \frac{1}{2a} |L_h V|_\infty W_i$$

note $\quad W_i \leq 1/4$

$$\Rightarrow \boxed{|V|_\infty \leq \max\{|V_0|, |V_M|\} + \frac{1}{8a} |L_h V|_\infty}$$

Lemma (existence and uniqueness).
Let $b=0$, $a>0$, $c \geq 0$ smooth. A solution U of
$$\begin{cases} L_h U_i = f_i \\ U_0 = u_0 \,;\, U_M = u_M \end{cases}$$
exists an is unique.

Proof: - uniqueness by considering homog. Pb m:
$$\begin{cases} L_h W_i = 0 \\ W_0 = 0 = W_M \end{cases}$$
This problem, by stability est has only sol. $W = 0$

- existence follows by finite dimensionality

Theorem (error estimate): Let $b=0$, $a>0$, $c \geq 0$.
$\forall h$, the unique discrete sol. U satisfies:

$$\|e\|_\infty = \max_i |u(x_i) - U_i| \leq C h^2 |u|_{C^4(0,1)} = \mathcal{O}(h^2)$$

$\underbrace{u(x_i) - U_i}_{\text{discrete error}}$

$\left(|u|_{C^4} := \sup_x |u^{(4)}(x)| \right)$

$e_i = u(x_i) - U_i$

① use FD scheme

Proof: $\mathcal{L}_h e_i = \mathcal{L}_h (u(x_i) - U_i) = \mathcal{L}_h u(x_i) - f_i$

② use PDE

$$= \mathcal{L}_h u(x_i) - \mathcal{L} u(x_i)$$

$$= -T u(x_i) = -T_i$$

③ Truncation error est.

$$\Rightarrow |\mathcal{L}_h e_i| \overset{!}{\leq} C h^2 |u|_{C^4(0,1)}$$

④ by stability of $\mathcal{L}_h$

$$|e_i|_\infty \leq \max\{\underset{=0}{|e_0|}, \underset{=0}{|e_M|}\} + \frac{1}{8} |\mathcal{L}_h e_i| =$$

consistency + stability $\Rightarrow$ convergence

General Elliptic Problems $(d \geq 1)$

Recall: 2D case $\underbrace{a u_{xx} + 2b u_{xy} + c u_{yy}}+ \cdots = 0$

elliptic $\quad ac - b^2 > 0$
$$\overset{\parallel}{-D}$$

Look at associated quadratic form:

$$Q(x,y) = (x \ y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This has a sign,

- if $a > 0$, $ac - b^2 > 0 \Rightarrow Q$ pos. def
- if $a < 0$, $ac - b^2 < 0 \Rightarrow$ "

Def: In $\mathbb{R}^d$, define general 2^{nd} order diff. operator in divergence form:

$$\mathcal{L}u = \underbrace{-\sum_{i,j=1}^{d} D_i \left(a_{ij}(x) D_j u \right)}_{2^{nd} \text{ order terms}} + \underbrace{\sum_{i=1}^{d} D_i (b_i u)}_{1^{st} \text{ order}} + \underbrace{cu}_{0^{th} \text{ order}}$$

$$\sum_i \left(D_i \left(\sum_j a_{ij} D_j u + b_i u \right) \right)$$

could take instead

$$\sum_{i=1}^{d} \left[D_i (b_i u) + c_i D_i u \right] \quad (D_i b_i) u + b_i D_i u$$

Def: (elliptic operators): The op. $\mathcal{L}$ above is ELLIPTIC if the matrix

$$A(x) = \{a_{ij}(x)\}_{ij}$$

is positive definite a.e. in Ω, that is

$$(A(x)\bar{v}, \bar{v}) = \sum_{ij} a_{ij}(x)\, v_i v_j > 0$$

To $\mathcal{L}$, we can associate the bilinear form

coercive $\Rightarrow \mathcal{L}$ elliptic

$$a(u,v) := \sum_{ij} \int_\Omega a_{ij} D_i u\, D_j v - \sum_i \int_\Omega b_i u\, D_i v + \int_\Omega c\, u v$$

$$\int_\Omega A \nabla u \cdot \nabla v - \int_\Omega (\underline{b} \cdot \nabla v) u$$

Def: $a(\cdot,\cdot)$ is coercive if

$$\boxed{a(v,v) \geq \alpha_0 \|v\|^2} \quad \exists \alpha_0 > 0$$

$\mathcal{L}$ elliptic if its 2^{nd} order term is coercive.

Theorem: assume $\underline{b}=0$ (otherwise, see GT)

Let $\mathcal{L}$ be elliptic operator in Ω bounded.

If $\mathcal{L}u \leq 0$ in Ω, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then

- if $c \equiv 0$ then $\max\limits_{\bar{\Omega}} u = \max\limits_{\partial\Omega} u$
- if $c \geq 0$ then $\max\limits_{\bar{\Omega}} u \leq \max\{\max\limits_{\partial\Omega} u, 0\}$

and if $\mathcal{L}u = 0$ then $\max\limits_{\bar{\Omega}} |u| = \max\limits_{\partial\Omega} |u|$.

(result applies also for $\mathcal{L}u \geq 0$ with min in place of max)

Corollary: Under some assumptions, with $c \geq 0$

If $\begin{cases} \mathcal{L}u = \mathcal{L}v & \text{in } \Omega \\ u = v & \text{on } \partial\Omega \end{cases}$ then $u = v$ in Ω

($\Rightarrow$ uniqueness)

Theorem: Let $\mathcal{L}u = f$ on Ω bounded,

then $\max |u| \leq \max\limits_{\partial\Omega} |u| + C_d \sup\limits_{\Omega} |f|$.

(a priori estimate)