NSPDE/ANA 2022 - Lecture 11

Last lecture:
- Strongly consistent FEM for advection-dominated problems
- Intro to Parabolic problems

Today:
- Energy method
- Explicit Euler method

References:
- Morton-Mayers
- Larsson-Thomee

$u_t + \mathcal{L}u = f$

$(u_t, v) + \mathcal{A}(u,v) = (f,v)$

Theorem: if $\mathcal{A}$ is cont. and coercive in $V \times V$, e.g. $H^1_0(\Omega)$,
$f \in L^2(I; L^2(\Omega))$, $u_0 \in L^2(\Omega)$. Then
$\exists!\ u \in L^2(I; V) \cap \mathcal{C}^0(\bar{I}; L^2)$ sol. of (P_W)
and

$$u_t \in L^2(I, V')$$

$$\text{(EE)} \quad \max_{t\in \bar{I}} \|u(t)\|_0^2 + \int_I \|u(t)\|_V^2 \leq \|u_0\|_{L^2(\Omega)}^2 + \frac{1}{\alpha_0}\int_I \|f(t)\|^2$$

"energy estimate"

that is, the problem is well-posed ?

Proof:

- existence is via FAEDO-GALERKIN method, i.e. considering sequence of spaces generated by an orthonormal basis and showing that the assoc. Galerkin solutions $u_h \longrightarrow u$ solution of cont. problem.
- uniqueness follows from (EE)
- proof of (EE): test problem with $v = u(t)$

$$(u_t, u) + a(u, u) = (f, u)$$

- $(u_t, u) = \int_\Omega u_t u = \frac{1}{2}\int_\Omega (u^2)_t = \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}$
- $a(u,u) \geq \alpha_0 \|u\|_1^2$
- $(f, u) \overset{\text{Schwarz}}{\leq} \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}$

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall \varepsilon > 0$$

$\varepsilon = \frac{1}{2\alpha_0}$
$$\leq \frac{1}{2\alpha_0}\|f\|^2_{L^2(\Omega)} + \frac{\alpha_0}{2}\|u\|^2_{H^1(\Omega)}$$

$$\frac{1}{\cancel{2}}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \frac{\alpha_0}{\cancel{2}}\|u\|^2_{H^1(\Omega)} \leq \frac{1}{\cancel{2}\alpha_0}\|f\|^2_{L^2(\Omega)}$$

$\forall\, t \in (0,T]$

$$\|u(t)\|^2_{L^2(\Omega)} + \alpha_0 \int_0^t \|u\|^2_{H^1(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} + \frac{1}{\alpha_0}\int_0^t \|f\|^2_{L^2(\Omega)}$$

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FD methods for parabolic pbms

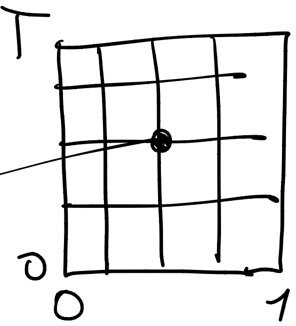
1 Explicit Euler method

model pbm: 1D heat eq.

$$\begin{cases} u_t = a\, u_{xx} & (0,1)\times(0,T] \\ u(x,0) = u_0(x) & (0,1) \\ u(0,t) = 0 = u(1,t) \end{cases}$$

space-time grid: $N_t, N_x \in \mathbb{N}$, $k = T/N_t$, $h \in 1/N_x$

$$(x_i, t_n) = (hi, kn)$$

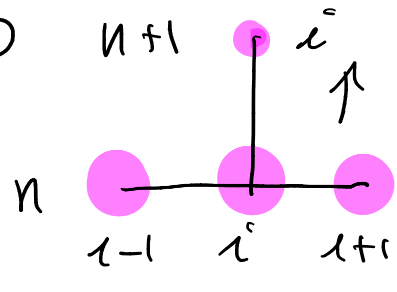


Discrete sol.

$$U_i^n \approx u(x_i, t_n)$$

Explicit (forward) Euler method

$$\delta^t_{k,+} U_i^n - a_i^n \left(\delta^x_h\right)^2 U_i^n = 0$$



that is,

$$\frac{U_i^{n+1}-U_i^n}{k} - a_i^n \, \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0$$

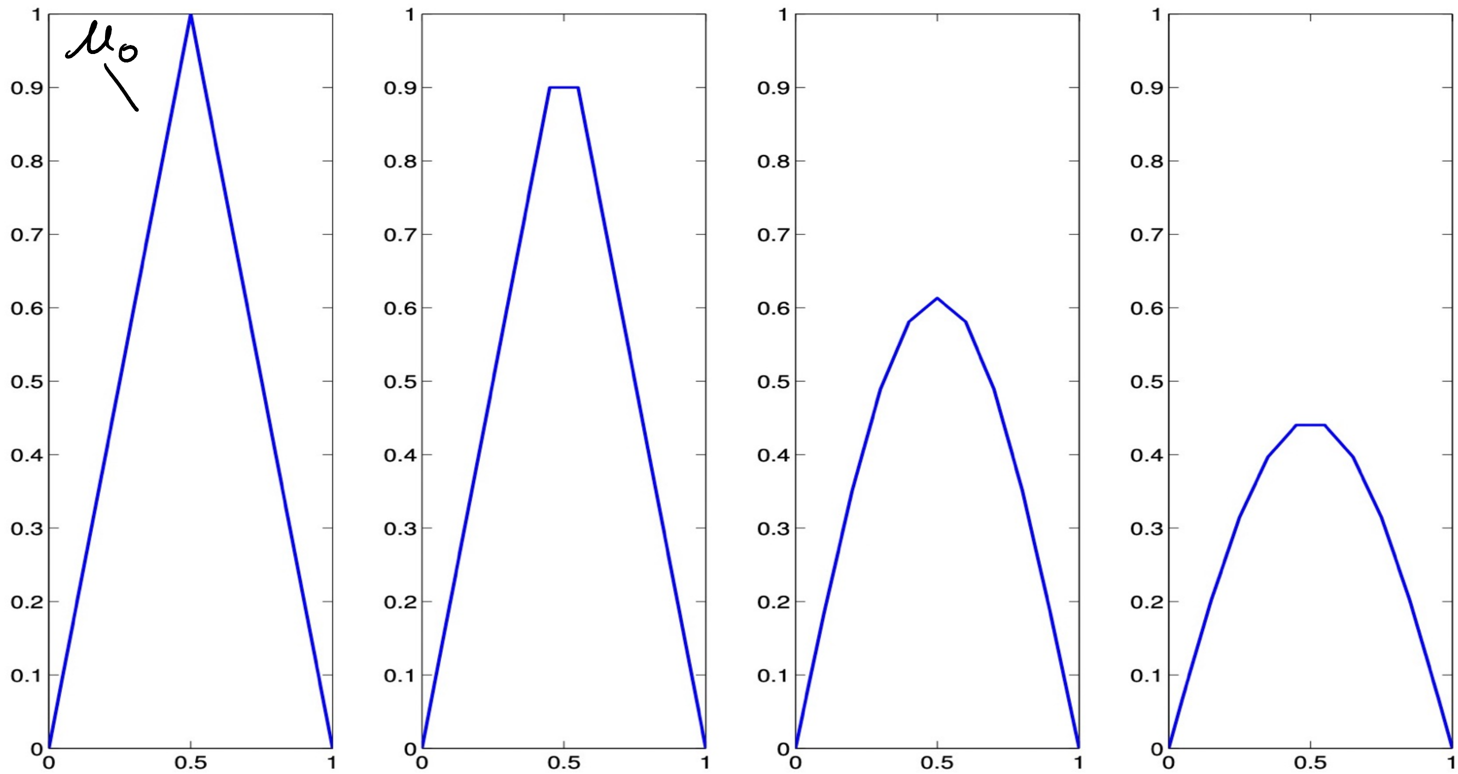
that is, $\mu = k/h^2$ courant number : $U_i^0 = u_0(x_i)$,

$$\begin{cases} U_i^{n+1} = a_i^n \mu U_{i+1}^n + a_i^n (1-2\mu) U_i^n + a_i^n \mu U_{i-1}^n \\ U_0^{n+1} = 0 = U_{M_x}^{n+1} \end{cases}$$

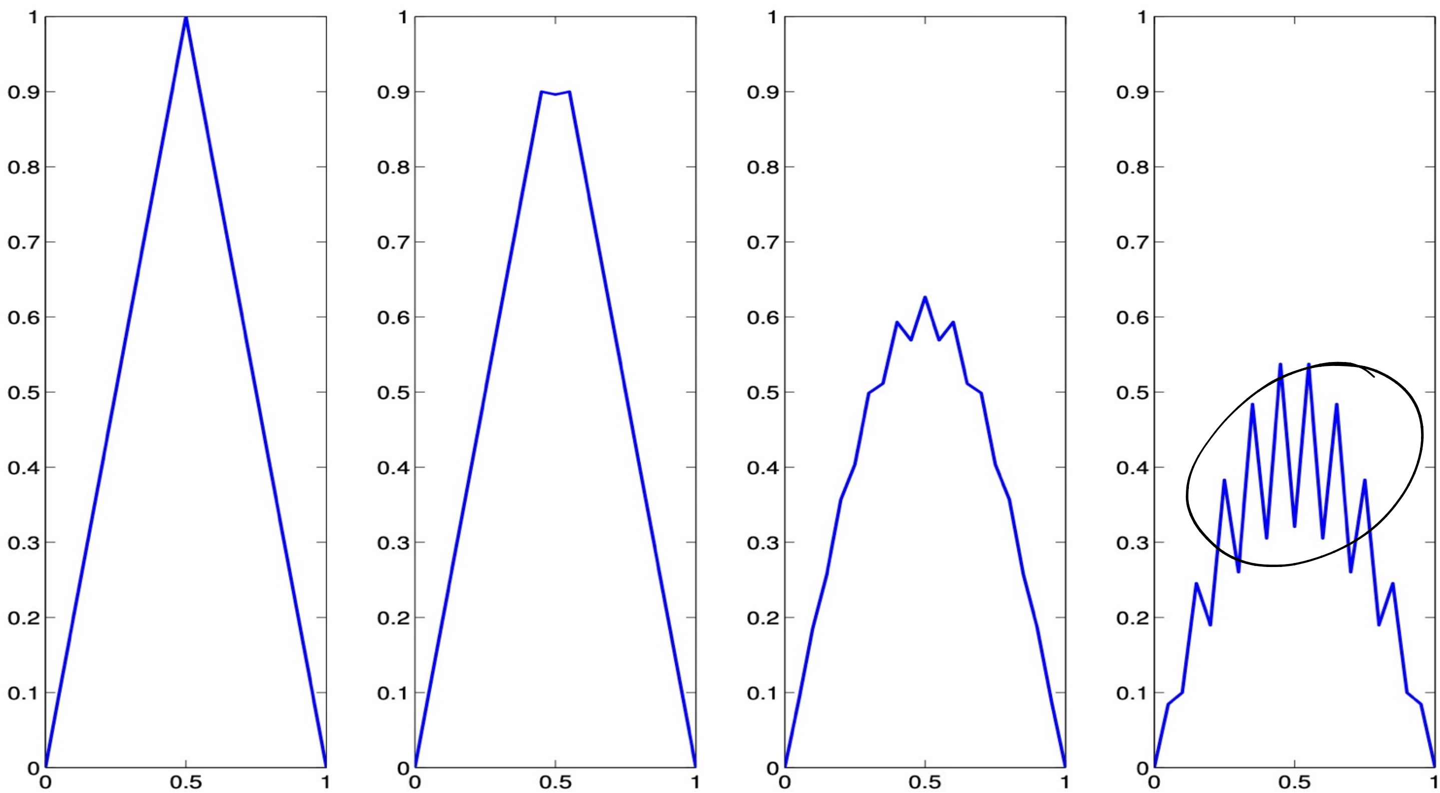
$\forall n = 0, \dots, M_t - 1$

$\forall i = 1, \dots, M_x - 1$

h=0.05, k=0.00125 => (a=1) mu=0.5



h=0.05, k=0.0013 => (a=1) mu=0.52



Analysis of Exp. Eul

Notation

- $U^n = (U_0^n, \dots, U_{N_x}^n)$
- $\|U^n\|_{\infty,h} = \max_{0 \le i \le N_x} |U_i^n|$
- solution operator $E_k U^n = U^{n+1}$
- discrete error $e_i^n = u_i^n - U_i^n$

  where $u_i^n := u(x_i, t_n)$

- Truncation error

$$T(x,t) := \delta_{k,t}^{t} u(x,t) - (\delta_h^x)^2 u(x,t)$$

$$T_i^n = T(x_i, t_n), \quad T_M = \max_{i,n} |T_i^n|$$

Lemma (consistency): For the Exp. Eul. method, assuming data smooth enough, initial and boundary conditions are compatible,

$$|T_i^n| \le \frac{k}{2} M_{tt} + \frac{h^2}{12} \bar{a} M_{xxxx} \qquad \mu = \frac{k}{h^2}$$

$$= \frac{k}{2} \left(M_{tt} + \frac{1}{6\mu} \bar{a} M_{xxxx} \right)$$

where $\bar{a} = \max |a(x,t)|$

$M_{tt} = \max |u_{tt}|$, $M_{xxxx} = \max |u_{xxxx}|$

Proof: by Taylor exp of $\delta^t_{k,+}$ and $(\delta^x_h)^2$:

$$\delta^t_{k,+} u^n_i = \frac{u^{n+1}_i - u^n_i}{k} = \frac{1}{k}\left(\cancel{u^n_i} + k(u_t)^n_i + \frac{k^2}{2} u_{tt}(x_i, \rho) - \cancel{u^n_i}\right)$$

$\exists \rho_n (t_n, t_{n+1})$

$$(\delta^x_h)^2 u^n_i = \frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{h^2}$$

$$= \frac{1}{h^2}\Big(\cancel{u^n_i} + \cancel{h(u_x)^n_i} + \frac{h^2}{2}(u_{xx})^n_i + \cancel{\frac{h^3}{6}(u_{xxx})^n_i} + \frac{h^4}{24} u_{xxxx}(\xi_1, t_n)$$

$$-2\cancel{u^n_i}$$

$$+\cancel{u^n_i} - \cancel{h(u_x)^n_i} + \frac{h^2}{2}(u_{xx})^n_i - \cancel{\frac{h^3}{6}(u_{xxx})^n_i} + \frac{h^4}{24} u_{xxxx}(\xi_2, t_n)$$

$\exists\ \xi_1 \in (x_{i-1}, x_i)$

$\xi_2 \in (x_i, x_{i+1})$

$$= (u_{xx})^n_i + \frac{h^2}{24}\Big(u_{xxxx}(\xi_1, t_n) + u_{xxxx}(\xi_2, t_n)\Big)$$

Theorem (Convergence): Under the same assumptions, if $\bar{a}\mu \le 1/2$ for all h, k then

$$\|e^n\|_{\infty,h} \le T\left(\frac{k}{2} M_{tt} + \frac{h^2}{12} M_{xxx}\right),$$

$\forall n = 1, 2, \dots, M_t$.

Proof: $\forall i \quad e_i^n = u_i^n - U_i^n$

$$e_i^{n+1} - a_i^n\left(\mu e_{i+1}^n - (1-2\mu) e_i^n + \mu e_{i-1}^n\right) = k T_i^n \quad \forall n, i$$

$$\Rightarrow |e_i^{n+1}| \leq a_i^n \mu |e_{i+1}^n| + \underbrace{(1 - a_i^n 2\mu)}_{\text{positive by assumption}} |e_i^n| + a_i^n \mu |e_{i-1}^n| + k |T_i^n|$$

sum to 1

$$\Rightarrow \|e^{n+1}\|_{\infty,h} \leq \|e^n\|_{\infty,h} + k T_M$$

$$\underbrace{\leq \cdots \leq}_{\text{apply some formula } n\text{-times}} \|e^0\|_{\infty,h} + (n+1) k T_M$$

that is $\|e^n\|_{\infty,h} \leq T \, T_M$ $\square$

comments:

- valid as $h, k \to 0$ which could happen indep. (that is, along different space-time refinement path) as long as $a\mu < 1/2$
- if μ kept fixed, say $a\mu = 1/2$ then $O(k)$
- the courant number cond is sufficient in the theorem for convergence

– it is actually necessary by

Proposition (Stability): The Exp Eul method is stable $\Longleftrightarrow$ $a\mu \le 1/2$.

a constant $\quad \left(\Leftarrow \ \|U^n\|_{\infty,h} \le \|u_0\|_{\infty,h} \right)$ $\quad \|(E_h U_0)^n\|_{\infty,h} \le \|u_0\|_{\infty,h}$

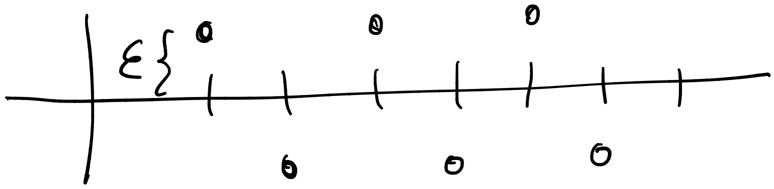
Proof: ① "$\Leftarrow$"

$$|U_i^{n+1}| \le a\mu |U_{i+1}^n| + |1-a\,2\mu|\,|U_i^n| + a\mu |U_{i-1}^n|$$

use assump $(1 - a\,2\mu)$

$$\le \|U^n\|_{\infty,h} \le \dots \le \|U^0\|_{\infty,h}$$

② "$\Rightarrow$" Let $U_i^0 = (-1)^i \varepsilon \qquad \varepsilon > 0$



$$U_i^1 = \left(a\mu(-1)^{i-1} + (1-2a\mu)(-1)^i + a(-1)^{i+1} \right)\varepsilon$$

$$= (-1)^i (1 - a\,4\mu)\,\varepsilon$$

$$\Rightarrow U_i^n = (1-4a\mu)^n (-1)^i \varepsilon$$

$$\|U_i^n\|_{\infty,h} = \underbrace{|1-4a\mu|}^n \varepsilon$$

for $a\mu > 1/2 \qquad 1-4a\mu > 1 \qquad \|U_i^n\|_{\infty,h} \xrightarrow{n\to+\infty} \infty$

for any given ε.

$\square$

- the $-1, 1, -1, 1 \ldots$ mode used in proof sawtooth function is the highest mode carried by scheme and the most likely to get amplified.
- Also in consideration of roundoff errors, instability always exp for unstable schemes

A "different" stability analysis:

Von Neumann stability

Consider again (P_0) $\mathbb{R} \times \mathbb{R}^+$: $\begin{cases} u_t - u_{xx} = 0 & \mathbb{R} \times \mathbb{R}^+ \\ u(\cdot, x) = u_0 & \text{in } \mathbb{R} \end{cases}$

we saw that

$$u(x,t) = \sum_{\gamma=1}^{\infty} \hat{u}_\gamma^0 \, e^{-\lambda_\gamma t} \, \varphi_\gamma(x)$$

similarly discrete solution has representation:

$$U_i^n = \sum_{-\infty}^{\infty} a_\gamma \, \chi_\gamma^n \, e^{i \gamma (ih)}$$

i = imaginary unit

so, to analyse stability, von Neumann tests scheme with single mode

$$U_i^0 = e^{i\gamma(ih)}$$

giving

$$U_i^1 = \mu e^{i\gamma(i+1)h} + (1-2\mu) e^{i\gamma ih} + \mu e^{i\gamma(i-1)h}$$

$$= \left(\mu e^{i\gamma h} + 1 - 2\mu + \mu e^{-i\gamma h}\right) e^{i\gamma ih}$$

$$\mu e^{i\gamma h} - 2\mu + \mu e^{-i\gamma h} = \mu\left(e^{i\gamma h/2} - e^{-i\gamma h/2}\right)^2 = -4\mu \sin^2\left(\frac{\gamma h}{2}\right)$$

$$= \underbrace{\left(1 - 4\mu \sin^2\left(\frac{\gamma h}{2}\right)\right)}_{\lambda(\gamma)} e^{i\gamma ih}$$

amplification factor associated to discrete wave number γ

so

$$U_i^n = \lambda(\gamma)^n \underbrace{e^{i\gamma ih}}_{U_i^0}$$

$\Rightarrow$ need $|\lambda(\gamma)| \leq 1$ for stability

$$\Leftrightarrow \quad -1 \leq 1 - 4\mu \sin^2\left(\frac{\gamma h}{2}\right) \leq 1$$

$$\Leftrightarrow \quad 0 \leq 2 - 4\mu \sin^2\left(\frac{\gamma h}{2}\right) \leq 2$$

$$\Leftrightarrow \quad 0 \leq 1 - 2\mu \sin^2\left(\frac{\gamma h}{2}\right) \leq 1$$

$$\Leftrightarrow \quad \boxed{0 \le \mu \le 1/2}$$

Evaluation of error norms

$$|I_h u - u_h|^2_{H^1(\Omega)} = \int_\Omega \nabla(I_h u - u_h)\cdot\nabla(I_h u - u_h)$$

$$= \int_\Omega \nabla\Big(\sum_i (u^i - u_h^i)\varphi_i\Big)\cdot\nabla\Big(\sum_j (u^j - u_h^j)\varphi_j\Big)$$

$$= \sum_i \sum_j (u^i - u_h^i)\Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\Big)(u^j - u_h^j)$$

$$= (U - U_h)^T A (U - U_h)$$

expect $|u - u_h|_{H^1}$ same order as $|u - I_h u|$

$$|u - u_h|^2_{H^1} = \int_\Omega \nabla(u - u_h)\cdot\nabla(u - u_h)$$

$$= \sum_{T \in \mathcal{T}_h} \int_T \nabla(u - \underbrace{u_h}_{\sum u_h^i \varphi^i})\cdot\nabla(u - u_h)$$