NSPDE/ANA 2022 - Lecture 15

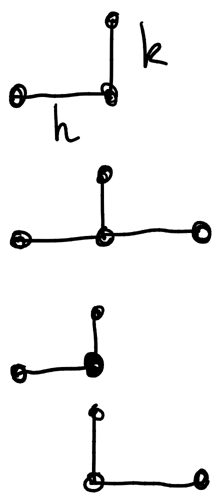
Last lecture:
- higher-order time-stepping: BDF, discontinuous Galerkin
- Intro to Hyperbolic problems
- Upwind method for linear transport
- CFL, stability

Today:
- Error analysis of Upwind schemes
- Higher-order schemes: Lax-Wendroff, Leap Frog
- Numerical dissipation, numerical dispersion
- Discretisation of the wave equation

References:
- Morton-Mayers
- Larsson-Thomee

$\nu = k/h$

$$\begin{cases} u_t + a u_x = 0 \\ u(0,x) = u_0(x) \end{cases} \qquad \mathbb{R} \times (0,T] \qquad \underline{a > 0}$$



stability $\nu \le 1$

unstable always

UPWIND METHOD

$$U_i^{n+1} = \begin{cases} (1-a_i^n \nu) U_i^n + a_i^n \nu U_{i-1}^n & a > 0 \\ (1-a_i^n \nu) U_i^n + a_i^n \nu U_{i+1}^n & a < 0 \end{cases}$$

CFL / von Neumann : $|a_i^n| \nu \le 1 \quad \forall i, n$

CONSISTENCY : truncation error

$$|T_i^n| \le \frac{1}{2} \left(k M_{tt} + h M_{xx} \right)$$

CONVERGENCE: If $0 \le |a_i^n|\nu \le 1$

- coeffs are all positive
- add up to 1

$\Rightarrow$ max princ. analysis

$$\Rightarrow \underset{\max_i |e_i^{n+1}|}{\underset{\shortparallel}{E^{n+1}}} \le E^n + \Delta t\, T^n \le \dots \le n\Delta t \max T^n$$

$$\Rightarrow E^n \le T \max_n T^n$$

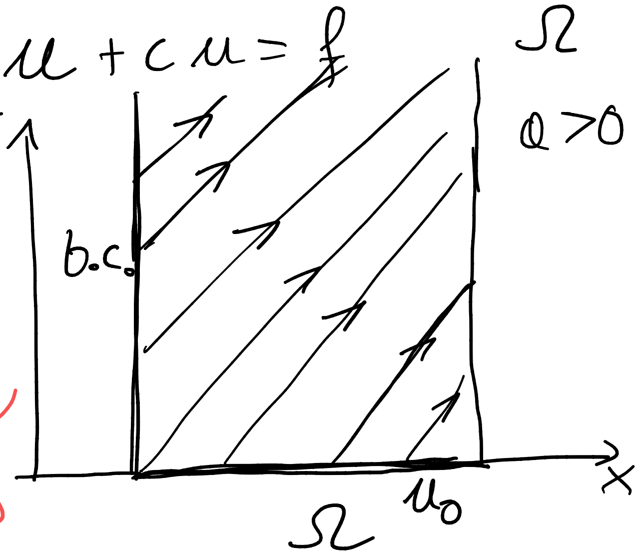
hence upwind method is conv. of $\mathcal{O}(h,k)$.

Initial/Boundary value problems

- general transport eq. $\bar{a} \cdot \bar{\nabla} u + c u = f$

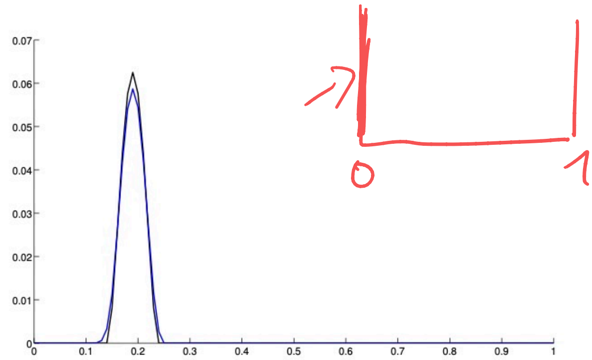
$$\bar{a} = (\underset{\underset{t}{\uparrow}}{1}, \underset{\underset{\bar{x}}{\uparrow}}{a})$$

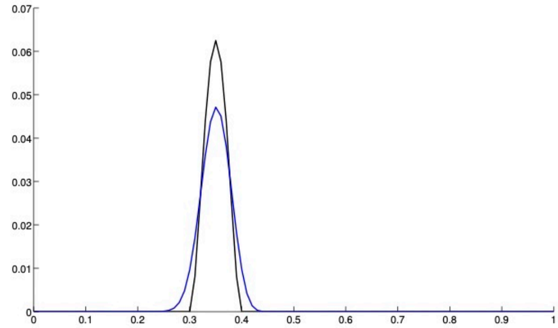
$$\Gamma_- = \{ x \in \partial\Omega : \underline{n}(x) \cdot a(x) < 0 \}$$ inflow boundary



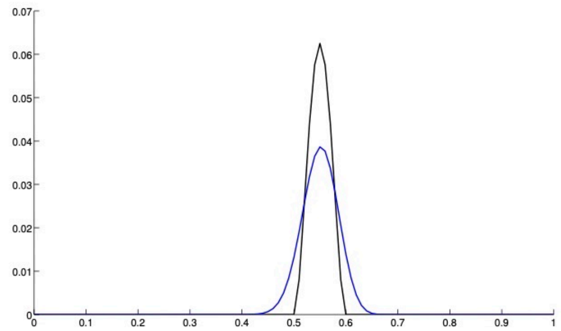
$u_t + \quad = 0$ in $(0,1] \times [0,1]$ $\rightarrow$ need $\underline{\nu \leq 0.5}$ for stability

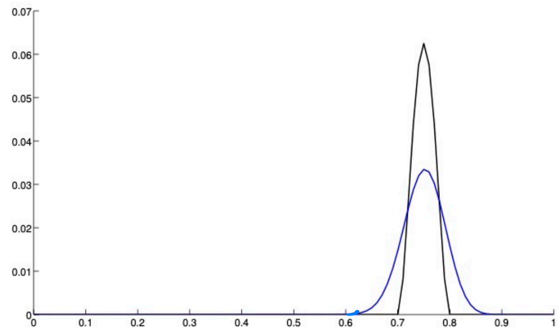
$N_t = 250$; $N_x = 100$ $\Rightarrow$ $\nu = 0.4$





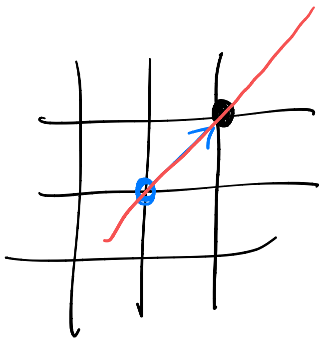
$u(t,0) = 0$

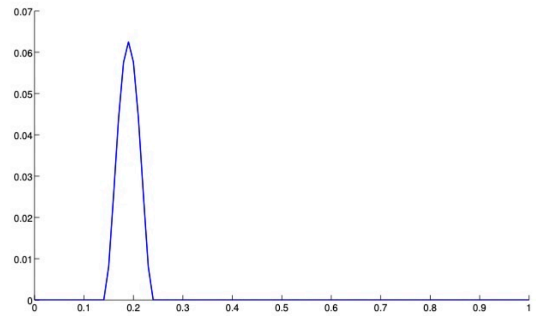


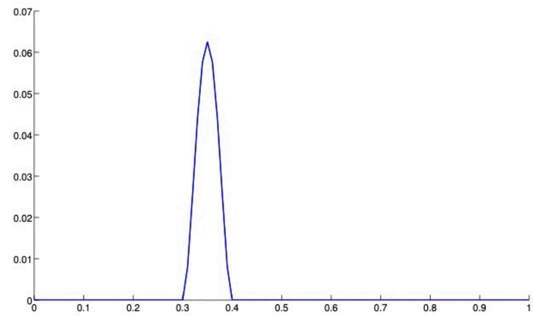


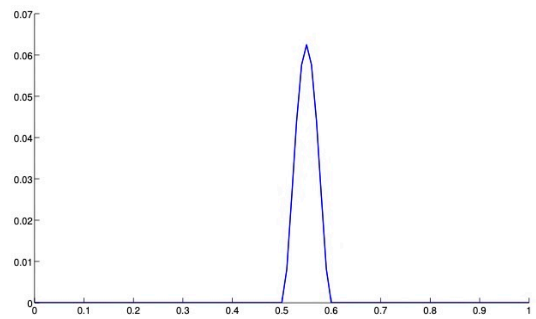
$\underline{\nu = 0.5}$ $\Rightarrow$ $(1 - \overset{2}{\cancel{0}}\nu) = 0$

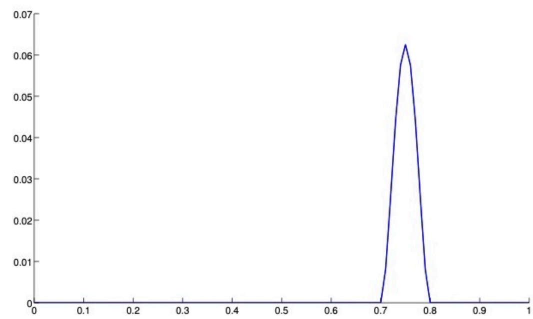
$U_i^{n+1} = U_{i-1}^{n}$











- upwind is dissipative
- upwind has no phase lag (not dispersive)

- recall : $a>0$

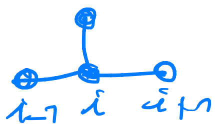
$$a_i^n \, \delta^x_{h,-} U_i^n = a_i^n \, \delta^x_{2h} U_i^n - \frac{a_i^n h}{2} \left(\delta^x_h\right)^2 U_i^n$$

$$a\, u_x \approx a\, u_x \underbrace{- \frac{a h}{2} u_{xx}}_{\text{dissipative term}}$$

instead, central diff. method uses

$$a_i^n \, \delta^x_{2h} U_i^n + O$$

Proposition: The upwind method is monotone if $|a|\nu \le 1$ $\left(\min_i U_i^0 \le U_i^n \le \max_i U_i^n\right)$ among the 3-points schemes



if is the least dissipative.

$\Rightarrow$ higher-order methods cannot be monotone

Lax-Wendroff:

$$\frac{u_i^{n+1} - u_i^n}{k} + a \frac{u^n_{i+1} - u^n_{i-1}}{2h} - \frac{k (a_i^n)^2}{2} \overbrace{\frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{h^2}}^{\text{cent. diff.}} = 0$$

upwind $\frac{|a|h}{2} = \frac{|a|k}{2\nu} = \frac{a^2 k}{2\underbrace{|a|\nu}_{\leq 1}} \geq \frac{a^2 k}{2}$ LW

$\Rightarrow$ expect LW to be less dissipative

$$u_i^{n+1} = \frac{a_i^n \nu}{2}\left(1 - a_i^n \nu\right) u_{i+1}^n + \left(1 - {a_i^n}^2 \nu^2\right) u_i^n$$

$$- \frac{a_i^n \nu}{2}\left(1 - a_i^n \nu\right) u_{i-1}^n$$

- CFL/ von Neumann : $u_i^n = \lambda^n e^{i \xi i h}$

$$\lambda = 1 - a^2\nu^2 \sin^2\left(\tfrac{1}{2}\xi h\right) - i a \nu \sin(\xi h)$$

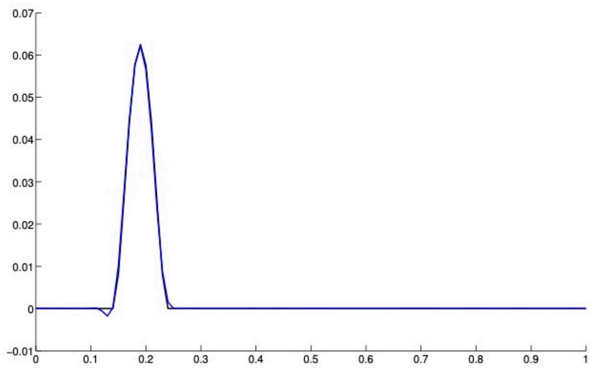
$$|\lambda| \leq 1 \Leftrightarrow |4a^2\nu^2(1 - a^2\nu^2)| \leq 1 \Leftrightarrow |a|\nu \leq 1$$

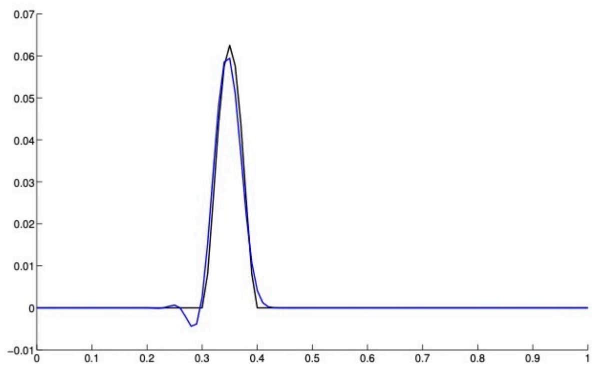
- Consistency : $|T_i^n| \leq \frac{k^2}{2} M_{ttt} + |a| \frac{h^2}{3} M_{xxx}$

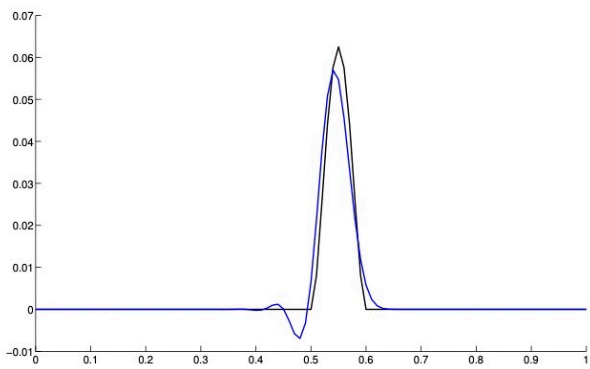
- Convergence : ... $\mathcal{O}(h^2, k^2)$ if $|a|\nu \leq 1$.

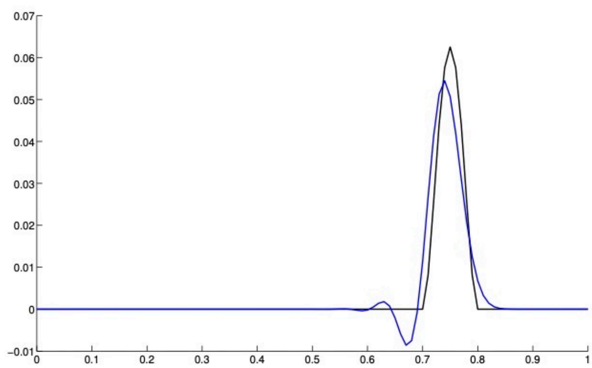
$u_t + 2u_x = 0$ in $(0,1] \times [0,1]$

$M_t = 250$, $M_x = 100$









from truncation error analysis

UPWIND

$\frac{k}{2} u_{tt} + \frac{h}{2} a\, u_{xx}$

use $u_{tt} = a^2 u_{xx}$

$= \frac{1}{2}\left[k a^2 + h a\right] u_{xx}$

LW

$c_1 k^2 u_{ttt} - c_2 h^2 u_{xxx}$

$\Rightarrow$ the two schemes approximate at higher order a PDE of the kind:

$u_t + a u_x = \underbrace{\mu u_{xx}}_{\text{dissipative term}}$

$a_t + a u_x = \underbrace{\mu u_{xxx}}_{\text{dispersive}}$

numerical dispersion/dissipation

Any time-dep. PDE with const. coeffs will admit plane-wave solutions:

$$u(t,x) = e^{i(\xi x + \omega t)} \qquad \xi \in \mathbb{R},\ \omega \in \mathbb{C}$$

ξ = wave number, ω = frequency

The PDE imposes a relationship:

$\omega = \omega(\xi)$ dispersion relation example: (1) $u_t + a u_x = 0$ $\qquad \omega = -a\xi$

or $u(0,x) = e^{-i\xi x} \Rightarrow u(t,x) = e^{-i\xi(at-x)}$

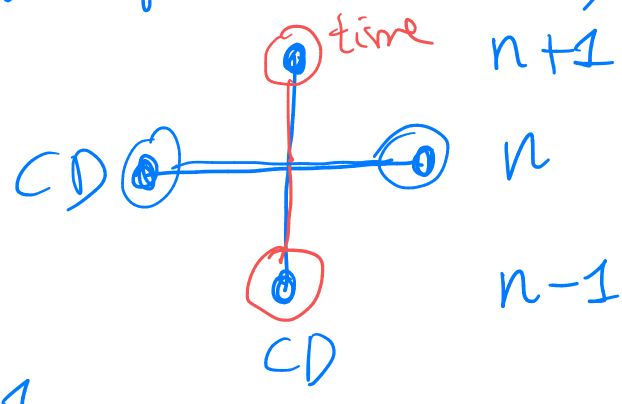
in this case, the phase velocity $\omega/\xi = -a$ is indep. of wave number

(2) $u_{tt} - u_{xx} = 0$ $\qquad \omega^2 = \xi^2 \quad \omega = \pm\xi$

(see Morton-Mayers) the dispersion relation for the numerical scheme (see von Neumann analysis) is trigonometric instead of polynomial. The analysis of the dispersion properties of the num. method is by matching the discrete and continuous dispersion relations.

Leap-Frog (non dissipative scheme)

$$\frac{U_i^{n+1} - U_i^{n-1}}{2k} + a\,\frac{U_{i+1}^n - U_{i-1}^n}{2h} = 0$$



CFL / von Neumann: $|a|\nu \le 1$

analysis of dispersion relation reveals that LP produces waves travelling in the wrong

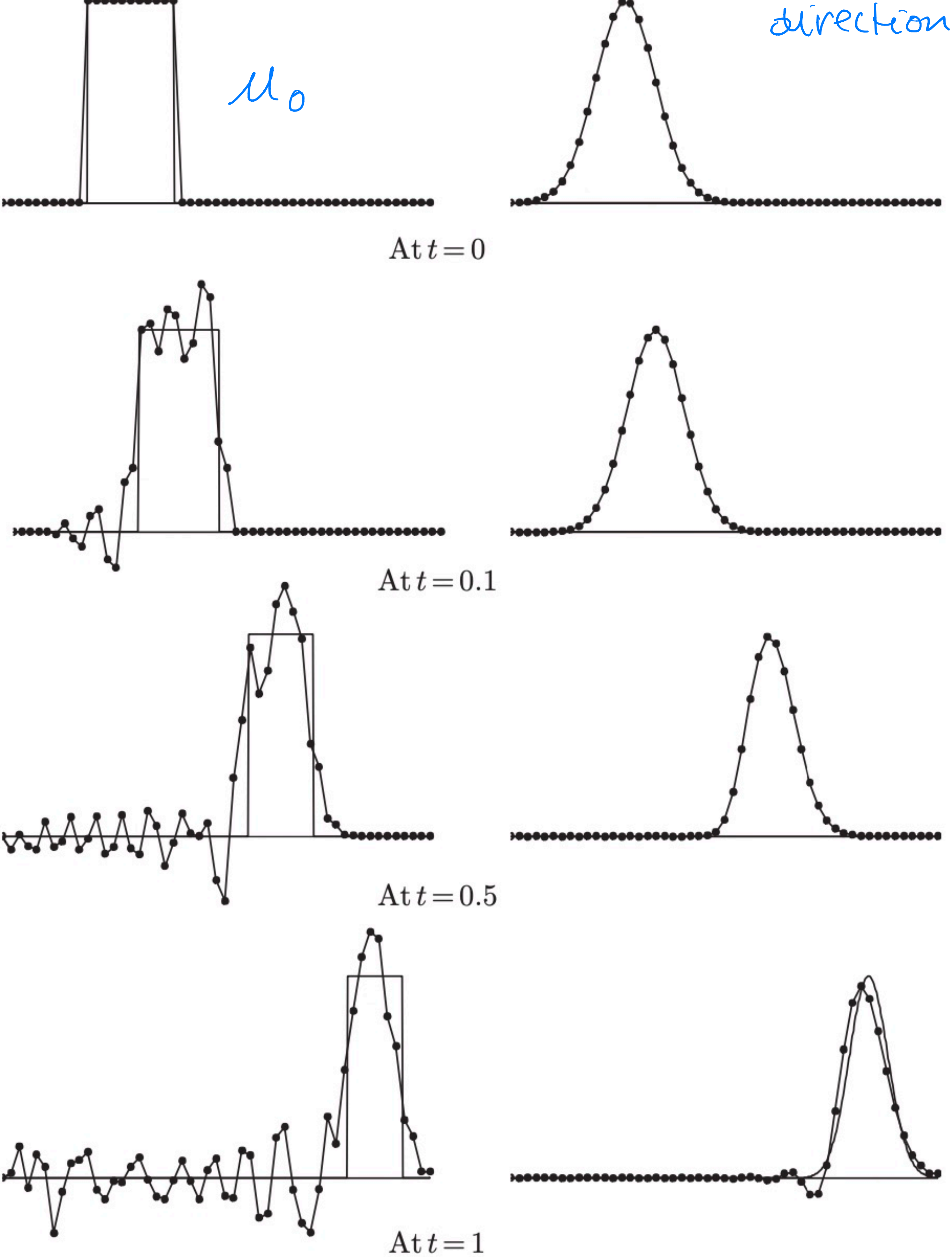
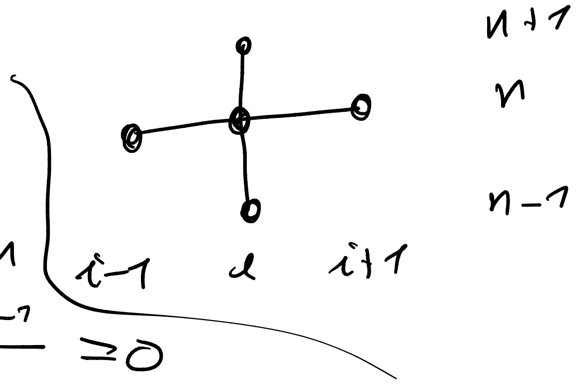


Fig. 4.15. Linear advection by the leap-frog scheme with $\Delta t = \Delta x = 0.02$ for (a) a square pulse and (b) Gaussian initial data.

LEAP FROG for wave eq.

$$\begin{cases} u_{tt} - u_{xx} = 0 & (0,T] \times [0,1] \\ u(0,x) = u_0(x) & [0,1] \\ u_t(0,x) = v_0(x) & \\ u(t,0) = u(t,1) = 0 & \end{cases}$$



$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{k^2} - \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0$$

CFL / von neumann: $\nu \le 1$

exercise:
- write wave eq. as 1st order system
- discretise the system by LF (use staggered grids)
- show that resulting scheme gives back LF applied to wave eq directly.