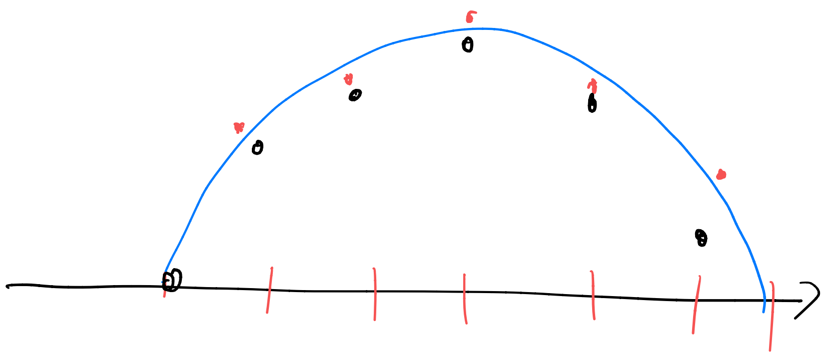
NSPDE/ANA 2022 - Lecture 13

Last lecture:
- The CFL condition
- Lax Equivalence Theorem
- The theta method
- l2 analysis

order k polynomials

Today:
- Semidiscrete FEM for parabolic problems
- Error analysis
- Fully discrete FD-FEM

References:
- Quarteroni
- Larsson-Thomee



- ℓ_∞ linked to max. princ → monotone solutions
- ℓ_2 " to von Neumann stability (equivalent)

FEM for Parabolic problems

$$(u,v) = \int_\Omega u v$$

Recall weak pbm:

find $u \in L^2(I;V) \cap C^0(\bar{I};L^2(\Omega))$

$$\begin{cases} \frac{d}{dt}(u(t),v) + a(u(t),v) = \ell(t,v) & \forall v \in V \\ u(0) = u_0 & I = (0,T) \end{cases}$$

$V = H_0^1(\Omega)$ ($\equiv$ homog Dirichlet problem)

$$a(w,v) = (a\nabla w, \nabla v) - (\underline{b} w, \nabla v) + (c w, v)$$

- $\begin{cases} \text{coercive in } V \times V \\ \text{continuous} \end{cases}$
- $\ell(t,v) = (f(t), v)$

  $f \in L^2(I; L^2(\Omega))$
- $u_0 \in L^2(\Omega)$

seen: WP above is well-posed satisfying an energy estimate

semidiscrete in space FEM

Given sequence of meshes $\mathcal{T}_h$ and $V_h^k \subset V$ FE spaces over $\mathcal{T}_h$ of order k,

$\forall t \in (0,T]$, find $u_h(t) \in V_h^k$:

$$\begin{cases} \frac{d}{dt}(u_h(0),v) + a(u_h(t),v) = \ell(t,v) & \forall v \in V_h^k \\ u_h(0) = u_{0,h} \in V_h^k & \text{approximating } u_0. \end{cases}$$

(eg the interpolant, projection)

by restriction of continuous argument in Galerkin setting,

$\exists!\, u_h$ and energy estimate for all t

$$\|u_h(t)\|_{L^2}^2 + \alpha_0 \int_0^t \|u(t)\|_{H^1}^2 \le \|u_{0,h}\|_{L^2}^2 + \frac{1}{\alpha_0}\int_0^t \|f(t)\|_0^2 \, d\tau$$

hence, we have stability if as, $h \to 0$ $\|u_{0,h}\|_{L^2} \to \|u_0\|_{L^2}$

note: by taking max over t, gives

$\underline{L^\infty - L^2}$ $\underline{L^2 - H^1}$ stability

Convergence

Def (elliptic/Ritz projection) by S. Brenner 1973.

Given $v \in V$, the ELLIPTIC PROJECTION, is the unique $R_h v \in V_h^k$:

$$\mathcal{A}(R_h v, w) = \mathcal{A}(v, w) \quad \forall w \in V_h$$

Theorem: If Ω convex, (could be more general) then $v \in \underbrace{H^{k+1} \cap H_0^1}_{V}$

$m=0,1$ $$|v - R_h v|_m \le c h^{(k+1)-m} \|v\|_{k+1}$$

see also LT Th. 5.5 proof for $k=1$

Theorem: Let $\mathcal{T}_h$ shape regular triang. (mesh) of Ω

$\mathcal{A}$ cont. and coerc

$u_0 \in H^{k+1}(\Omega)$,

u : $\frac{\partial u}{\partial t} \in L^1(I; H^{k+1}(\Omega))$

$V_h^k \subset V$ standard FE of order k

Then $\forall t \in [0,T]$, u_n (semidiscrete FEM sol):

$$\|u(t)-u_h(t)\|_{L^2} \le \|u_0-u_{0,h}\|_{L^2} + c h^{k+1}\left(\|u_0\|_{H^{k+1}} + \int_0^t \left\|\frac{\partial u}{\partial t}(\tau)\right\|_{H^{k+1}} d\tau\right)$$

$\to L^\infty L^2$

Proof:

key idea: split error in 2 components:

$$u_h(t)-u(t) = \underbrace{\overbrace{u_h(t)-R_h u(t)}^{\in V_h^k}}_{\vartheta=\vartheta(t)} + \underbrace{R_h u(t)-u(t)}_{\rho=\rho(t)}$$

$= \vartheta = \vartheta(t) + \rho = \rho(t)$

parabolic   elliptic

- bound of ρ : by elliptic proj. est:

$$\|\rho\|_{L^2} \le c h^{k+1}\|u(t)\|_{H^{k+1}} \le c h^{k+1}\left(\|u_0\|_{H^{k+1}} + \int_0^t \left\|\frac{\partial u}{\partial t}(\tau)\right\|_{H^{k+1}} d\tau\right)$$

$$u(t) = u_0 + \int_0^t \frac{du}{dt}(\tau)\, d\tau$$

- bound of ϑ : $\forall v \in V_h^k$, $\vartheta = u_h - R_h u$

$$(\vartheta_t, v) + a(\vartheta, v) = \ell(v) - \left(\frac{\partial R_h(u)}{\partial t}, v\right) - a(R_h(u), v)$$

by def of ell. proj.

$$= \ell(v) - \left(\frac{\partial R_h(u)}{\partial t}, v\right) - a(u, v)$$

$$= (u_t - R_h u_t, v)$$

$$= -(\rho_t, v)$$

Test error eq. above with $v = \vartheta$

$$\frac{1}{2}\frac{d}{dt}\|\vartheta\|^2_{L^2} + \alpha_0\|\vartheta\|^2_{H^1} \leq \|\rho_t\|_{L^2}\|\vartheta\|_{L^2}$$

In particular, or $\frac{1}{2}\frac{d}{dt}\|\vartheta\|^2_{L^2} = \|\vartheta\|_{L^2}\frac{d}{dt}\|\vartheta\|_{L^2}$,

$$\frac{d}{dt}\|\vartheta\|_{L^2} \leq \|\rho_t\|_{L^2}$$

Finally, integrate in time $(0, t)$:

$$\|\vartheta(t)\|_{L^2} \leq \|\vartheta(0)\|_{L^2} + \int_0^t \|\rho_t\|_{L^2}$$

$\square$

Remark: • L^∞-L^2 control follows by taking sup

• L^2-H^1 bound by using the $\|\vartheta\|_{H^1}$ term (or otherwise)

FULLY DISCRETE METHODS:

- FEM in space, FD in time

FEM ϑ-method

$\rightarrow V_h^k \subset V$ FEM space $\rightarrow$ time step $\Delta t = T/N_t$

$u_h^0 = u_{0,h} \in V_h^k$ $\qquad t_n = n\Delta t.$

for $n = 0, \dots, N_t - 1,$

$$\frac{1}{\Delta t}\left(u_h^{n+1} - u_h^n, v_h\right) + a\Big(\overbrace{\vartheta u_h^{n+1} + (1-\vartheta)u_h^n}^{u_h^\vartheta}, v\Big)$$

$$= \Big(\underbrace{\vartheta f(t_{n+1}) + (1-\vartheta) f(t_n)}_{f_\vartheta^{n+1}}, v\Big) \qquad \forall v \in V_h^k$$

FEM + $\begin{cases} \text{exp Eul} & (\vartheta = 0) \\ \text{imp Eul} & (\vartheta = 1) \\ \text{CN} & (\vartheta = 1/2) \end{cases}$

Stability

- Imp. Eul $(\vartheta = 1)$

testing with $v = u_h^{n+1}$

$$\frac{1}{\Delta t}\left(u_h^{n+1} - u_h^n, u_h^{n+1}\right) + \underset{\text{something positive}}{\uparrow} \leq \|f^{n+1}\|_{L^2} \|u_h^{n+1}\|_{L^2}$$

$$\|u_h^{n+1}\|_{L^2}^{\cancel{2}} - \underbrace{(u_h^n, u_h^{n+1})}_{\substack{\wedge\!| \\ \|u_h^n\|_{L^2} \|u_h^{n+1}\|_{L^2}}} \leq \Delta t \|f^{n+1}\|_{L^2} \cancel{\|u_h^{n+1}\|_{L^2}}$$

$$\Rightarrow \|u_h^{n+1}\|_{L^2} \leq \|u_h^n\|_{L^2} + \Delta t \|f^{n+1}\|_{L^2}$$

$$\|u_h^n\|_{L^2} \leq \|u_{0,h}\|_{L^2} + \Delta t \sum_{j=1}^{n} \|f^j\|_{L^2}$$

exercise: show that for CN, testing with $v = u_h^{n+1} - u_h^n$, gives

$$\|u_h^n\| \leq \|u_{0,h}\|_{L^2} + \Delta t \sum_{j=0}^{n} \|f^j\|_{L^2}$$

- both cases: unconditional stability!

In general, we have:

Theorem: Let $\mathcal{A}$ cont, coerc, $t \to \|f(t)\|_0$ bounded, and, if $0 \leq \vartheta < 1/2$ assume

coerc

$$\Delta t \left(1 + \underset{\substack{\uparrow \\ \text{constant of the} \\ \text{inverse inequality}}}{C_I^2}\, h^{-2}\right) \leq \frac{2\alpha_0}{(1-2\vartheta)\underset{\substack{\uparrow \\ \text{cont.}}}{\gamma^2}}$$

then ϑ-method in time satisfies

$$\|u_h^n\|_{L^2} \leq C_\vartheta \left(\|u_{0,h}\|_{L^2} + \sup_t \|f(t)\|_{L^2} \right)$$

Finally, one convergence result (see Quarteroni p. 135)

Theorem: Assume u_0, f, u smooth enough, $\forall n$

$$\underbrace{\|u(t_n) - u_h^n\|^2_{L^2(\Omega)}}_{\ell^\infty - L^2} + \underbrace{2\alpha_0 \Delta t \sum_{m=1}^{n} \|u(t^m) - u_h^m\|^2_{H^1}}_{\ell^2 - H^1}$$

$$\leq C(u_0, f, u) \left(h^{2k} + \begin{cases} \Delta t^2 & \vartheta \neq 1/2 \\ \Delta t^4 & \vartheta = 1/2 \end{cases} \right)$$

$\square$