NSPDE/ANA 2022 - Lecture 5

Last lecture:

- General elliptic problems DMP
- FD for Poisson in 2D
- Analysis (stability, consistency, convergence)
- Compact FD formulas
- FD for general elliptic problems
- Intro to functional spaces

Today:
- Sobolev spaces
- Weak formulation of elliptic problems
- Regularity of the solution

References:
- Quarteroni
- Larsson & Thomee

Recap

- Dual space: Let $(V, \|\cdot\|)$ a normed space, $V' = \mathcal{L}(V; \mathbb{R})$, the dual space of V, is the space of all linear continuous functionals normed with operator norm $\|L\|_{V'} := \sup_{v \in V \setminus \{0\}} \frac{|L(v)|}{\|v\|}$

- Duality: $\langle \cdot, \cdot \rangle : V' \times V \to \mathbb{R}$ bilinear form by $\langle L, v \rangle := L(v)$

- Hilbert space: $(H, (\cdot,\cdot))$ inner product space complete w.r.t. $\|\cdot\| = (\cdot,\cdot)^{1/2}$

- L^p-spaces: for $1 \leq p \leq \infty$

$$L^p(\Omega) = \left\{ f:\Omega \to \mathbb{R} : f \text{ Lebesgue measurable}, \text{ and } \int_\Omega |f|^p \, dx < \infty \quad p<\infty \text{ or } \sup_{x\in\Omega} |f(x)| < \infty \quad p=+\infty \right\}$$

$\Omega \subset \mathbb{R}^d$

with norm

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, dx \right)^{1/p} \qquad p<+\infty$$

$$\|f\|_{L^\infty} := \sup_{x\in\Omega} |f(x)| \qquad p=+\infty$$

$\left(L^p(\Omega), \|\cdot\|_{L^p(\Omega)} \right)$ normed space

- Important examples:

$p=2$, $L^2(\Omega)$ is also Hilbert

$$(u,v) = \int_\Omega u\, v \, dx$$

($\to$ Schwarz ineq.:

$$|(u,v)| \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \text{)}$$

- . Weak derivatives

$$C_0^\infty(\Omega) \underset{\text{with right topology}}{\rightsquigarrow} \mathcal{D}(\Omega) \xrightarrow[\text{dual}]{\text{take}} \mathcal{D}'(\Omega)$$ space of distributions

$$T \in \mathcal{D}'(\Omega) : \quad T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$$
$$\varphi \longrightarrow \langle T, \varphi \rangle := T(\varphi)$$

Note: $L^p(\Omega) \subsetneq \mathcal{D}'(\Omega)$, $p \in [1, +\infty]$

indeed, $f \in L^p(\Omega)$, $\quad f(\varphi) = \int_\Omega f \varphi$

(an example of distribution not in L^p is the Dirac delta $\delta(v) = v(0)$

For $T \in \mathcal{D}'(\Omega)$, $DT \in \mathcal{D}'(\Omega)$ given by

$$\langle DT, \varphi \rangle = -\langle T, \varphi' \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

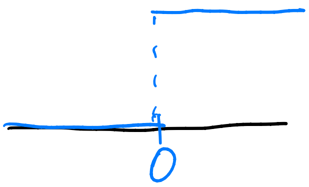
is the weak derivative of T.

And weak derivatives of any order can be defined in the same way

But, there is no guarantee that if, e.g.

$T \in L^2 \Rightarrow DT \in L^2$

example: $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ Heaviside function



for $x \in \mathbb{R}$.

$H \in L^2_{loc}$ (L^2 on every bounded domain)

$$DH = \delta \notin L^2_{loc}$$

- Sobolev spaces: $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $p \in [1, +\infty]$

$$W^{k,p}(\Omega) = \{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), \forall |\alpha| \leq k \}$$

Important example: $p = 2$, $k \in \mathbb{N}$

$$W^{k,2}(\Omega) = H^k(\Omega)$$

again this is Hilbert w.r.t.

$$(v, w)_k = \sum_{|\alpha| \leq k} (D^\alpha v, D^\alpha w) \rightsquigarrow \text{norm!}$$

In particular,

$$H^1(\Omega) = \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega) \}$$

with I.P.

$$(v, w)_1 = (v, w) + (\nabla v, \nabla w)$$

$\int v_x w_x + \int v_y w_y \cdots$

and norm

$$\|v\|_1 = \left(\|v\|^2_{L^2} + \|\nabla v\|^2_0 \right)^{1/2}$$

Finally, we introduce

$$H^1_0(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \}$$

$$H^1_\Gamma(\Omega) = \{ v \in H^1(\Omega) : v|_\Gamma = 0 \}$$

where $\Gamma \subset \partial\Omega$.

In. Product for $H^1_0(\Omega)$ / $H^1_\Gamma(\Omega)$?

Poincaré Inequality: $\Omega \in \mathbb{R}^d$ bounded and open, $\Gamma \subseteq \partial\Omega$, Lipschitz. Then $\exists\, C_\Omega > 0$:

$$\int_\Omega v^2 \, dx \leq C_\Omega \int_\Omega |\nabla v|^2 \, dx$$

$$\|v\|_{L^2}^2 \leq C_\Omega \|\nabla v\|_{L^2}^2$$

for all $v \in H^1_\Gamma(\Omega)$.

Hence $\left(H^1_\Gamma(\Omega), (\cdot,\cdot)_{H^1_\Gamma} \right)$

$$(u,v)_{H^1_\Gamma} := \int_\Omega \nabla u \cdot \nabla v$$

is IP space, normed with

$$\|v\|_{H^1_\Gamma} := \|\nabla v\|_{L^2(\Omega)} =: |v|_{H^1(\Omega)}$$

(indeed recall, $\|v\|_{H^1}^2 = \|v\|_{L^2} + |v|_{H^1(\Omega)}$)

Weak formulation for elliptic PDE boundary value problems

Model Problem: Poisson $\Omega \subset \mathbb{R}^d$ open, bounded

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Recall associated bilinear form:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

also, the linear form

$$(f, v) = \int_\Omega f \, v \, dx \qquad \text{for } f \in L^2(\Omega)$$

weak formulation: given $f \in L^2(\Omega)$, find

$$u \in H_0^1(\Omega) : \quad a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)$$

$\llcorner$:= weak solution

Comments:

(1) If u is classical sol. $\Rightarrow$ u is weak solution

proof: suppose u is classical sol. of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then, actually $f \in C(\Omega) \subset L^2(\Omega)$

By testing $\quad -\Delta u \, v = f v \quad v \in H_0^1(\Omega)$

$$\therefore \quad -\int_\Omega \Delta u \, v = \int_\Omega f v$$

$$\therefore \quad \int_\Omega \nabla u \, \nabla v - \int_{\partial\Omega} \nabla u \cdot n \, \underset{0}{\underset{\parallel}{v}} = \int_\Omega f v$$

$$\therefore \quad \int_\Omega \nabla u \cdot \nabla v = \int f v \quad \forall v \in H_0^1(\Omega)$$

$\therefore$ u is weak solution !

$\square$

(2) The viceversa is not true as $f \in L^2(\Omega) \not\Rightarrow f \in C(\Omega)$
(weak problem more general !)

(3) Suppose $\begin{cases} f \in L^2(\Omega) \\ \Omega \text{ convex} \end{cases} \Rightarrow u \in H_0^2(\Omega)$

$$\Downarrow$$

strong form $\quad -\Delta u = f$ in $L^2(\Omega)$

(so, in $\mathcal{D}'(\Omega)$)

not a classical sol. as $u \in C^2(\Omega)$

to get a classical sol., need

$$\begin{cases} f \in H^2(\Omega) \\ \partial\Omega \text{ smooth enough} \end{cases} \Rightarrow u \in H^4(\Omega) \hookrightarrow C^2(\Omega)$$

"Sobolev embedding"

Sobolev embedding: $\Omega \in \mathbb{R}^d$ bounded, smooth (or polygonal). Then

$$H^k \subset C(\bar{\Omega}) \quad \text{if } k > d/2$$

		$k=1$
$d=1$	$k > 1/2$	$H^1 \subset C$
$d=2$	$k > 1$	NO
$d=3$	$k > 3/2$	NO

(4) In general, it is much easier to prove well-posedness of weak formulations, particularly for nonlinear problems.

Theorem ($\exists!$ for weak Poisson):
Let $f \in L^2(\Omega)$ given. The $\exists!$

$$u \in H_0^1(\Omega) : \underbrace{a(u,v)}_{\int_\Omega \nabla u \cdot \nabla v} = (f,v) \quad \forall v \in H_0^1(\Omega)$$

<u>Proof</u>: application of

Theorem (Riesz representation):
Let $(H, (\cdot,\cdot))$ Hilbert. Then

$\forall \; T \in H'$, $\exists!\, u \in H$: $T(v) = (u, v)$ $\forall v \in H$

Moreover, $\|T\|_{H'} = \|u\|_H$.

uniqueness: let u_1, u_2 be weak solutions, let $w = u_1 - u_2$. Then

$$0 = a(u_1 - u_2, v) \quad \forall v \in H_0^1(\Omega)$$

$$\overset{!}{=} \int_\Omega \nabla(\underbrace{u_1 - u_2}_{w}) \cdot \nabla v \, dx$$

Take $v = w$ $\Rightarrow$ $= \int_\Omega \nabla w \cdot \nabla w = 0$ $\Rightarrow$ $\underline{\underline{w \equiv 0}}$

$$= \|w\|_{H_0^1(\Omega)}$$

Existence:

- $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = (u, v)_{H_0^1}$

- $(f, v) = \int_\Omega f v \, dx$

$$\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Poincare ineq.

$$\leq \|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} = \|v\|_{H_0^1}$$

Hence $v \longrightarrow (f,v)$ is continuous

$$\underset{\ell(v)}{\overset{(f,v)}{\|}} \qquad \ell \in (H_0^1)'$$

Riesz, $\exists!\, u \in H_0^1(\Omega)$: $\ell(v) = (u,v)_{H_0^1} = a(u,v) = (f,v)$

see book by GRISVARD on regularity of solutions to elliptic problems

Regularity of the solution:

① Let $\Omega \in \mathbb{R}^2$, $\partial\Omega$ smooth

If $f \in H^s(\Omega)$, then $u \in H^{s+2}(\Omega)$

and $\|u\|_{H^{s+2}} \le C \|f\|_{H^s}$

e.g. • $s=0 \Rightarrow \|u\|_{H^2} \le C\|f\|_{L^2}$

② Ω is polygonal and convex, then

again $u \in H^2$.