NSPDE/ANA 2022 - Lecture 2

Last lecture:
- Basic facts on PDEs: types of PDEs, characteristics, Cauchy problem, well posedness
- Lax-Richtmyer equivalence: stability+consistency => convergence

Today:
- The Poisson BVP
- Maximum principle
- Divided differences
- Finite Difference for two-points BVP
- A priori analysis (partially)

- Notes online
- Larsson Thomée
- Evans

Gauss-Green Theorem

$\omega \subset \mathbb{R}^d$ open and bounded, $\partial\omega \in C^1$. (boundary)

$\forall u \in C^1(\bar{\omega})$

$$\int_\omega u_{x_i}\, dx = \int_{\partial\omega} u\, \underline{n}_i \, dS \qquad i = 1, \dots, d$$

Corollaries:

- $$\int_\omega \nabla u \, dx = \int_{\partial\omega} u\, \underline{n}\, dS$$

- $\int_\omega \Delta u \, dx = \int_{\partial\omega} \frac{\partial u}{\partial \underline{n}} \, dS \qquad u \in \mathcal{C}^2(\omega)$

- $\int_\omega (\nabla u)\cdot(\nabla v)\,dx = -\int_\omega (\Delta u)\, v \, dx + \int_{\partial\omega} \frac{\partial u}{\partial \underline{n}} v \, dS$

$u, v \in \mathcal{C}^2(\omega)$

$\frac{\partial u}{\partial \underline{n}} = \underline{n} \cdot \nabla u$

Def: The function $u \in \mathcal{C}^2(\omega)$ is harmonic if

$$\Delta u = 0 \quad \text{in } \omega.$$

Theorem (mean-value formula): [Evans]

$u \in \mathcal{C}^2(\Omega)$ is harmonic if and only if

$$u(x) = ⨍_{\partial B_r(x)} u \, dS \qquad \forall x \in \Omega$$

$\left(⨍_\omega u := \frac{1}{|\omega|} \int_\omega u \right)$

$\forall \omega \in \mathbb{R}^d, \forall d$

we also have

$$u(x) = ⨍_{B_r(x)} u \, dx$$

for each ball $B_r(x) \subset \Omega$.

Theorem (Strong Maximum Principle):

Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ harmonic in Ω

Then,

(i) $\max_{\bar\Omega} u = \max_{\partial\Omega} u$ (max. princ)

(ii) If Ω connected and $\exists x_0 \in \Omega$ s.t.

$$u(x_0) = \max_{\bar\Omega} u$$

then $u \equiv \text{const}$ in Ω

(by replacing $u \to -u \Rightarrow \max \to \min$)

Proof: (ii) suppose $x_0 \in \Omega$: $u(x_0) = \max_{\bar\Omega} u =: M$.

Take $B_{x_0}(r) \subset \Omega$, then

$$M = u(x_0) = ⨍_{B_{x_0}(r)} u \, dx \le M$$

$$\Rightarrow u \equiv M \quad \text{in } B_{x_0}(r)$$

Consider the set $\{x \in \Omega : u(x) = M\}$. This set is, at the same time, open and relatively closed in Ω, so it must be equal to Ω or Ω is connected.

(ii) follows from (i).

We now consider

$$\text{BVP} \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

↑ Dirichlet boundary condition

Corollary (positiveness): Let Ω connected, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solution of BVP.

- If $g \geq 0$, then $u \geq 0$ in Ω
- If $\begin{cases} g \geq 0 \\ g(x) > 0 \ \exists x \in \partial\Omega \end{cases}$, then $u > 0$ in Ω

Corollary (uniqueness): Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$.

Then, there exists at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Proof: let $u, \tilde{u}$ solutions, consider $w_{\pm} = \pm(u - \tilde{u})$

$$\Rightarrow \quad w \equiv 0$$

Theorem [Larsson - Thomeé]: Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solve (P). Then

$$\|u\|_{C(\bar{\Omega})} \leq \|g\|_{C(\partial\Omega)} + C \|\underset{\substack{\| \\ f}}{\Delta u}\|_{C(\Omega)}$$

(continuous dependence on data)

Corollary: If $u, \tilde{u}$ solve (P) with data f, g and $\tilde{f}, \tilde{g}$, respectively, then

$$\|u - \tilde{u}\|_{C(\bar{\Omega})} \leq \|g - \tilde{g}\|_{C(\partial\Omega)} + C \|f - \tilde{f}\|_{C(\Omega)}$$

<u>Comment</u>:

Let P a problem, with sol. u

Let P_H be the discrete problem, u_H its solution

example P_H problem where data is truncated

Let $\tilde{u}$ be the exact sol. to P_H

Then

$$\|u - u_n\| \le \underbrace{\|u - \tilde{u}\|}_{\text{" problem rep. error}} + \underbrace{\|\tilde{u} - u_n\|}_{\text{numerical approx. error}}$$

Finite Difference for the two points BVP

$$(TP) \begin{cases} -u'' = f & \text{in } (a,b) \\ u(a) = 0 = u(b) & \end{cases}$$

(homogeneous D.)

Define, 2^{nd} divided difference op.

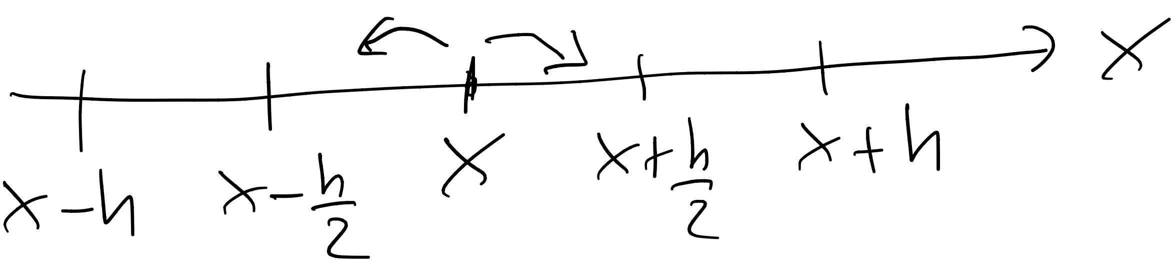
$$\delta_h^2 u(x) = \delta_h(\delta_h u(x))$$

$h \in \mathbb{R}$, $h > 0$

$$= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \approx u''(x)$$

where

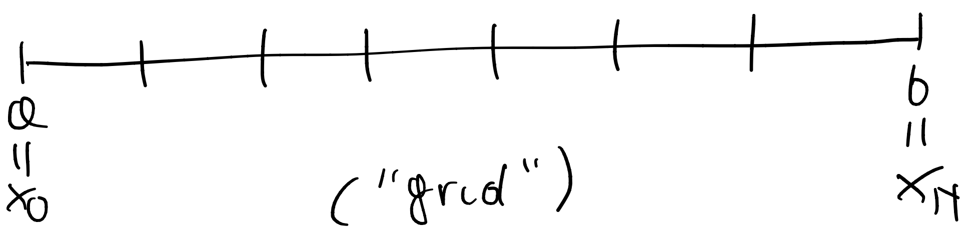
$$\delta_h(x) = \frac{u(x+1/2) - u(x-1/2)}{h} \approx u'(x)$$



Finite Difference Method:

- fix $N \in \mathbb{N}$, set $h = 1/N$

  $x_i = x_0 + h i \quad i = 0, 1, \dots, N, \quad x_0 = a$



- Define discrete function:

$$\{U_i\}_{i=0}^{N}$$

• FD method :

$$u''(x_i) \quad \begin{cases} U_0 = 0 \\ -\dfrac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = f_i := f(x_i) \\ U_N = 0 \end{cases} \quad \forall i = 1, .., N-1.$$

<u>Theorem</u>: (Discrete Maximum Principle - DMP):

Let $V = \{V_i\}_{i=0}^{N}$ grid function, and

$$\mathcal{L}_h V_i := -\frac{V_{i+1} - 2V_i + V_{i-1}}{h^2}, \quad i = 1, .., N-1$$

If $\mathcal{L}_h V_i \leq 0 \quad \forall i$, then

$$\max_{0 \leq i \leq N} V_i = \max\{V_0, V_N\}$$

Proof: By contradiction, let $V_n = \max_{1<n<N} V_i$, and suppose $V_n > V_0$ and/or $V_n > V_N$

$$0 \leq -h^2 \mathcal{L}_h V_n = V_{n+1} - 2V_n + V_{n-1}$$

$$\Rightarrow 2V_n \leq V_{n+1} + V_{n-1}$$

$$\therefore \quad V_n \leq \frac{V_{n+1} + V_{n-1}}{2} \leq V_n$$

$$\Rightarrow \quad \boxed{V_n = \frac{V_{n+1} + V_{n-1}}{2}}$$

$$\Rightarrow \quad V_n = V_{n+1} = V_{n-1}$$

by repeating the argument until reach boundary we conclude $V \equiv \text{const}$!

FINITE DIFFERENCES

Def: (Divided Difference operators)

Let $f: \mathbb{R} \to \mathbb{R}$ bounded, $h > 0$. $\forall x \in \mathbb{R}$

$$\delta_{h,+} f(x) := \frac{f(x+h) - f(x)}{h} \quad \text{FD (forward difference)}$$

$$\delta_{h,-} f(x) := \frac{f(x) - f(x-h)}{h} \quad \text{BD (backward difference)}$$

$$\delta_h f(x) := \frac{f(x+h/2) - f(x-h/2)}{h} \quad \text{CD (central difference)}$$

note: the approximation theory point of view

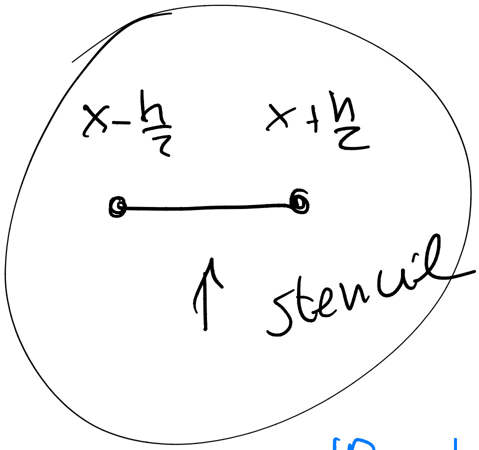
1) construct a linear interpolant

2) differentiate the interpolant

example: CD is obtained $\{x+h/2,\ x-h/2\}$

1) $P_1(y) = f(x+\frac{h}{2})\,\frac{y-(x-\frac{h}{2})}{h} + f(x-\frac{h}{2})\,\frac{(x+\frac{h}{2})-y}{h}$

2) $P_1'(x) = \frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h}$



Lemma: Let $f \in C^2([a,b])$, $a<b \in \mathbb{R}$, $h>0$

- $\left|f'(x)-\delta_{h,+}f(x)\right| \overset{\text{error identity}}{=} \left|\frac{h}{2}f''(\xi)\right| \overset{\text{error bound}}{\le} \frac{h}{2}\max_{y\in[a,b]}|f''(y)|$

$\exists\, \xi \in (x, x+h)$

- $\left|f'(x)-\delta_{h,-}f(x)\right| = \left|\frac{h}{2}f''(\xi)\right| \le \frac{h}{2}$ ''

$\exists\, \xi \in (x-h, x)$

if $f \in C^3([a,b])$, then

- $\left|f'(x)-\delta_h f(x)\right| = \left|\frac{h^2}{48}\left(f'''(\xi_1)+f'''(\xi_2)\right)\right| \le \frac{h^2}{24}\max_y |f'''(y)|$

$\exists\, \xi_1, \xi_2 \in (x-\frac{h}{2}, x+\frac{h}{2})$

Def: Given g, g_h we say g_h converges to g if $\lim_{h \to 0^+} g(x) - g_h(x) = 0$

and convergence is of order p if $\lim_{h\to 0^+} \frac{|g(x)-g_h(x)|}{h^p} \in \mathbb{R}$

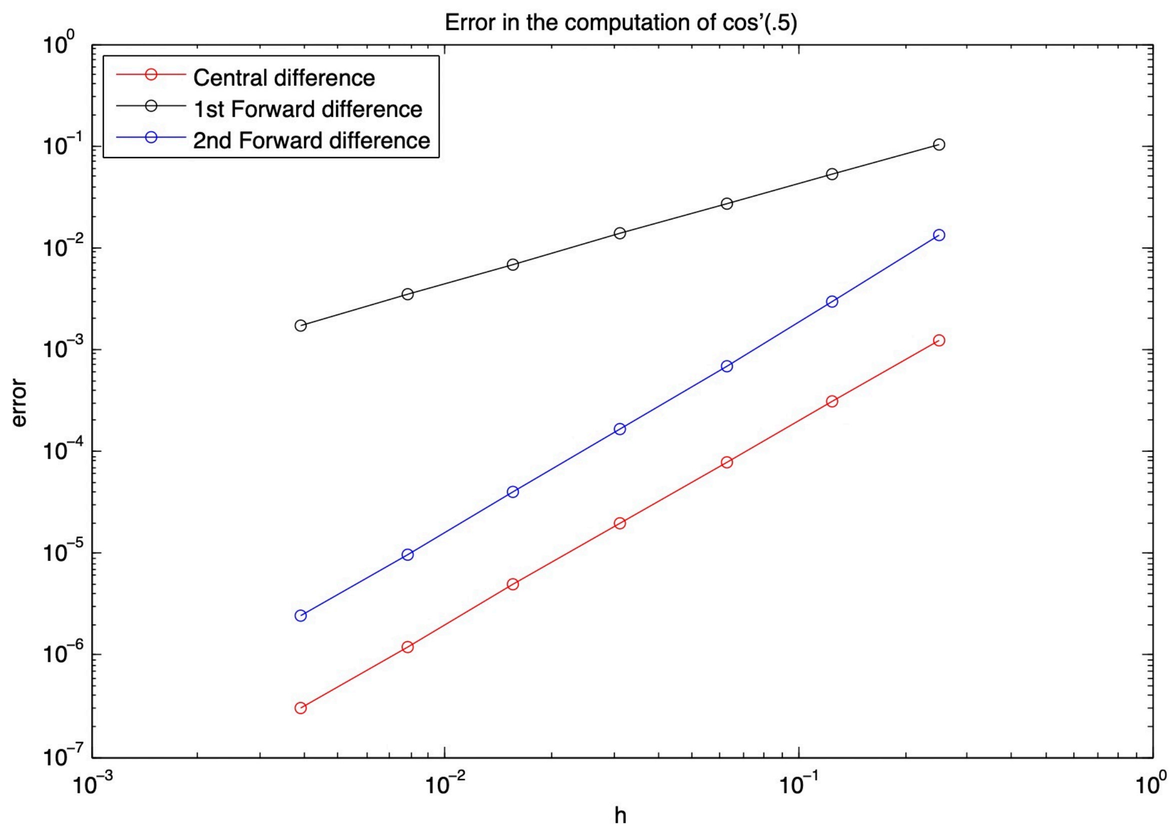
and write $|g(x) - g_h(x)| = O(h^p)$

higher order one-sided approx. of 1st derivative:

$$\delta_{h,+,2} f(x) = \frac{1}{2h}\left(-f(x+2h) + 4f(x+h) - 3f(x)\right) = \left(\delta_{h,+} - \frac{h}{2}\delta_{h,+}^2\right) f(x)$$

2nd order

$$\delta_{h,-,2} f(x) = \ldots$$



Central difference for 2^{nd} derivative:

$$\delta_h^2 f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Lemma: $f \in C^4([a,b])$, $h>0$, $x \in [a+h, b-h]$

$$\left| f''(x) - \delta_h^2 f(x) \right| = \left| \frac{h^2}{24} \left(f^{(iv)}(\xi_1) + f^{(iv)}(\xi_2) \right) \right| \le \frac{h^2}{12} \max_y \left| f^{(4)}(y) \right| = \mathcal{O}(h^2)$$

FD on discrete functions $U = \{U_i\}_{i=0}^N$ on a grid $\{x_i\}_{i=0}^N$, we can compute the FD formulas in matrix form:

FD
$$\frac{1}{h} \begin{pmatrix} -1 & 1 & & 0 \\ & -1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & -1 \end{pmatrix} U = V$$

$V[1:N-1]$ give the FD formula at the internal grid points

CD (with step $2h$)
$$\frac{1}{2h} \begin{pmatrix} 0 & 1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & 0 \end{pmatrix} U = V$$

2^{nd} Central
$$\frac{1}{h^2} \begin{pmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix} U = V$$

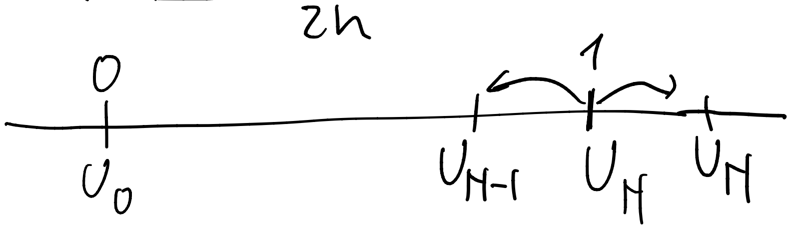
Different boundary conditions:

Consider

$$\begin{cases} -u'' = f & \text{in } \Omega = (0,1) \\ u(0) = u_0 \,, & u'(1) = 0 \end{cases}$$

$\uparrow$ $\hookrightarrow$ Neumann b.c.

$$\begin{cases} U_0 = u_0 \\ -\dfrac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = f_i \quad (f_i = f(x_i)) & i = 1, \dots, N-1 \\ \begin{cases} -\dfrac{U_{N+1} - 2U_N + U_{N-1}}{h^2} = f_N \\ \dfrac{U_{N+1} - U_{N-1}}{2h} = 0 \quad \left(u'(x_N) = 0\right) \end{cases} \end{cases} \Rightarrow 2\,\frac{U_N - U_{N-1}}{h^2} = f_N$$



or $A\tilde{U} = F$

$U = U[1:N]$

$$A = \begin{bmatrix} -2 & 1 & & & \\ \ddots & \ddots & \ddots & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 2 & -2 \end{bmatrix}, \quad F = \begin{bmatrix} -h^2 f_1 - U_0 \\ -h^2 f_2 \\ \vdots \\ -h^2 f_N \end{bmatrix}$$

More general two-points BVPs

Let $a(x) > 0$, $c(x) \geq 0$ on $\Omega = (0,1)$

$$(P)\begin{cases} \mathcal{L}u := -au'' + bu' + cu = f & \text{in } \Omega \\ u(0) = u_0 \; ; \; u(1) = u_1 \end{cases}$$

FD method. Define $a_i = a(x_i)$, $b_i = b(x_i)$, $c_i = c(x_i)$ with $\{x_i\}_{i=1}^N$ the grid points,

$$-a_i \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + b_i \frac{U_{i+1} - U_{i-1}}{2h} + c_i U_i = f_i$$

so, $\underbrace{\mathcal{L}_h U_i}_{=:} = \dfrac{-(a_i - \frac{1}{2}hb_i)U_{i+1} + (2a_i + h^2 c_i)U_i - (a_i + \frac{1}{2}hb_i)U_{i-1}}{h^2}$

A priori analysis

Def (Truncation error) Define trunc. error as

$$T(x) = \mathcal{L}u(x) - \mathcal{L}_h u(x)$$

$$T_i := T(x_i) \qquad \forall i$$

In our case,

$$T(x) = \mathcal{L}u(x) - \mathcal{L}_h u(x)$$

$$= f(x) - \left(-a(x)\,\delta_h^2 u(x) + b(x)\,\delta_{2h} u(x) + c(x)\,u(x)\right)$$

insert error eq. and use the PDE

$$= a(x)\,\frac{h^2}{24}\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right) + b(x)\frac{h^2}{12}\left(u^{(3)}(\eta_1) + u^{(3)}(\eta_2)\right)$$

$$|T(x)| \leq \left(\frac{h^2}{12}\|a\|_\infty + \frac{h^2}{6}\|b\|_\infty\right)\|u\|_{C^4}$$

$$\Rightarrow \quad T(x) = \mathcal{O}(h^2)$$

the method is consistent!