Last lecture:
- Error analysis of Upwind schemes
- Higher-order schemes: Lax-Wendroff, Leap Frog
- Numerical dissipation, numerical dispersion
- Discretisation of the wave equation

Today:
- Conservation Laws
- Upwind and Lax-Wendroff
- Finite Volume method
- Implicit-Explicit (IMEX) schemes

References:
- Morton-Mayers
- Leveque *Finite Volume Methods for Hyperbolic Problems,* Cambridge, 2002.

Conservation laws

$u \equiv$ density

(1D case) $\int_{x_1}^{x_2} u(t,x)\,dx$ mass in (x_1, x_2) at time t

$$\frac{d}{dt}\int_{x_1}^{x_2} u\,dx = \underbrace{F_1(t) - F_2(t)}_{\text{fluxes into}}$$

$f = f(u,x,t)$ flux function

In particular, if $f = f(u)$

$$= - f(u)\Big|_{x_1}^{x_2} = -\int_{x_1}^{x_2} \frac{\partial f(u)}{\partial x}\,dx$$

$$\int_{x_1}^{x_2} \left[u_t + (f(u))_x \right] dx = 0 \qquad \forall t, \forall x_1, x_2$$

$$\Rightarrow \quad u_t + (f(u))_x = 0$$

examples:

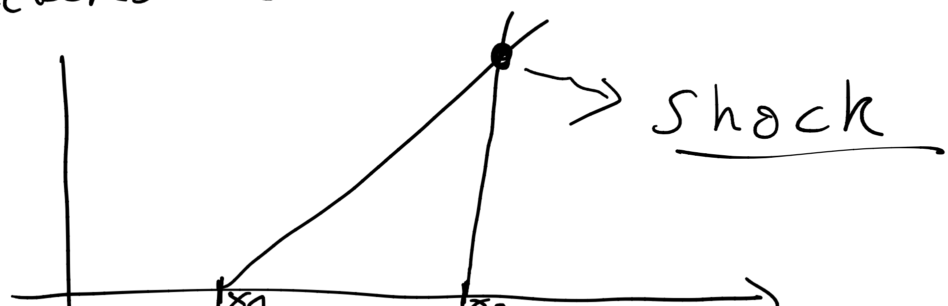
- transport eq. $f(u) = u$
- $f = -b\, u_x \quad \Rightarrow \quad u_t = b\, u_{xx}$
- $f = -b(x)\, u_x \quad \Rightarrow \quad u_t = \underbrace{b\, u_{xx}}_{\text{diff}} + \underbrace{b_x u_x}_{\text{adv}}$
- $f = \frac{1}{2} u^2 \quad \Rightarrow \quad \underline{u_t + \overset{\text{"}a(u)\text{"}}{u}\, u_x = 0}$

characteristics are still straight lines

$$\frac{dx}{dt} = a(u) = u \quad \text{constant slope}$$

$$\left(\underbrace{\frac{d}{dt} u(t, x(t))}_{\text{material deriv}} = u_t + u_x \underbrace{\frac{dx}{dt}}_{= u} = 0 \right)$$

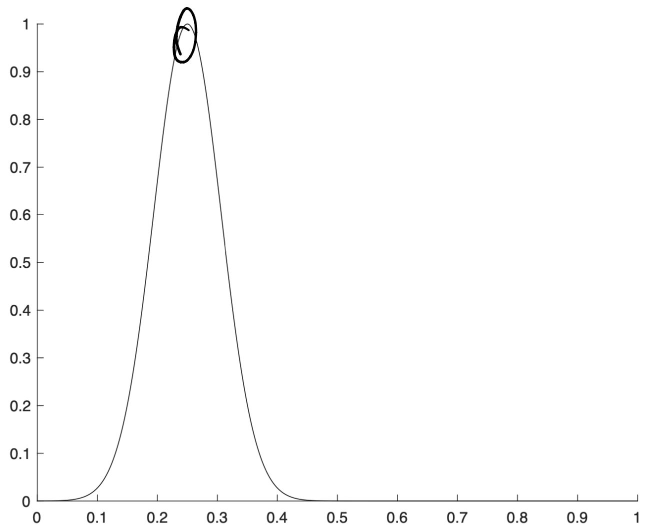
However, even if u_0 smooth, characteristics can intersect!



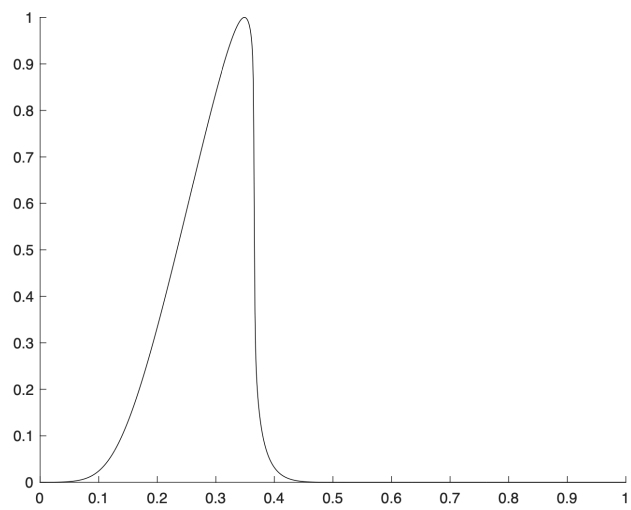
$u_0(x_1) > u_0(x_2)$

Burger's equation

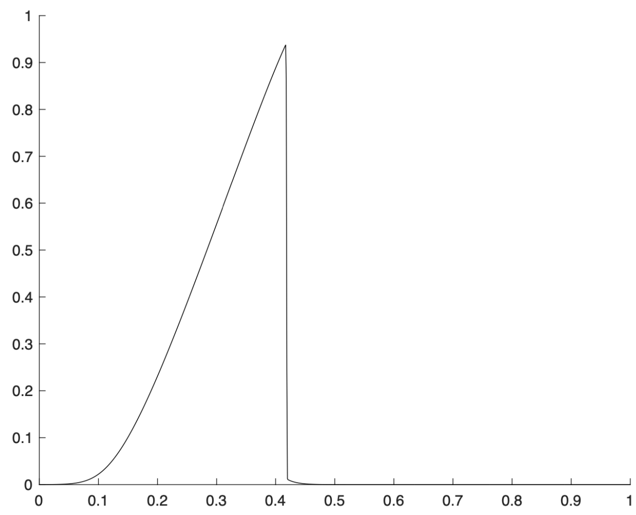
$u_t + u u_x = 0$



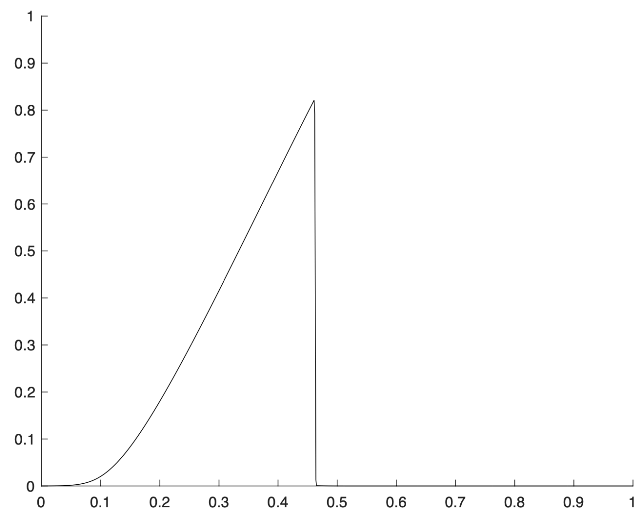
$t = 0$



$t = 0.1$



$t = 0.2$



$t = 0.3$ shock developing!

Discretisation

$$(CL)\begin{cases} u_t + (f(u))_x = 0 & t\in(0,T],\ x\in\mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$$

$$(f(u))_x = f'(u)\,u_x = a(u)\,u_x$$

$$a(u) := f'(u) \quad \text{flux}$$

Lax-Wendroff

recall: for $u_t + a u_x = 0$

$$LW: \ \delta^t_{h,t} U^n_i = -a\,\delta^x_{2h} U^n_i + \frac{k a^2}{2}(\delta^x_2)^2 U^n_i$$

$$u(t+k,x) = u(t,x) + k\,u_t(t,x) + \frac{k^2}{2} u_{tt}(t,x) + \mathcal{O}(k^3)$$

$$u_t = -a u_x, \qquad u_{tt} = a^2 u_{xx}$$

$$u_{tt} = (u_t)_t = -(a u_x)_t = -a(u_t)_x = a^2 u_{xx}$$

Do same for CL (also a nonconstant)

- $u_t = -a(u)\, u_x = -(f(u))_x$
- $u_{tt} = -(f(u))_{xt} = -(f(u))_{tx}$

$$= -\left(f'(u)\, u_t\right)_x = \left(f'(u)\,(f(u))_x\right)_x$$

$$= \left(a(u)\,(f(u))_x\right)_x$$

Hence,

$$u(t+k,x) = u(t,x) + k\, u_t(t,x) + \frac{k^2}{2} u_{tt}(t,x) + O(k^3)$$

$$= u(t,x) - k\,(f(u))_x + \frac{k^2}{2}\left(a(u)\,(f(u))_x\right)_x + O(k^3)$$

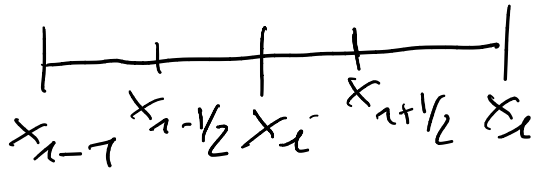
$$U_i^{n+1} = U_i^n - k\,\delta_{2h}^x f(U_i^n) + \frac{k^2}{2}\,\delta_h^x\left(a(U_i^n)\,\delta_h^x\left(f(U_i^n)\right)\right)$$

- $-k\,\delta_{2h}^x f(U_i^n) = -k\,\dfrac{f(U_{i+1}^n) - f(U_{i-1}^n)}{2h} = -\dfrac{\nu}{2}\left(f(U_{i+1}^n) - f(U_{i-1}^n) \boxed{-f(U_i^n) + f(U_i^n)}\right)$

$\nu = k/h$

$$= -\frac{\nu}{2}\left(\Delta_+^x f(U_i^n) + \Delta_-^x f(U_i^n)\right)$$

$$\begin{cases} \Delta_+^x v_i = v_{i+1} - v_i \\ \Delta_-^x v_i = v_i - v_{i-1} \end{cases}$$



- $\dfrac{k^2}{2}\,\delta_h^x\left(a(U_i^n)\,\delta_h^x\left(f(U_i^n)\right)\right)$

$$= \frac{k^2}{2}\,\delta_h^x\left(a(u_i^n)\left(\frac{f(U_{i+1/2}^n) - f(U_{i-1/2}^n)}{h}\right)\right)$$

$$= \frac{\nu^2}{2}\Big(a(u^n_{i+1/2})\big(f(U^n_{i+1})-f(U^n_i)\big)$$
$$- a(u^n_{i-1/2})\big(f(U^n_i)-f(U^n_{i-1})\big)\Big)$$

$$= \frac{\nu^2}{2}\Big(a(u^n_{i+1/2})\,\Delta^x_+ f(U^n_i) - a(u^n_{i-1/2})\,\Delta^x_- f(U^n_i)\Big)$$

evaluation of $a(u^n_{i\pm 1/2})$

suppose that $a = f'$ is continuous, then can evaluate $a(u^n_{i\pm 1/2})$ for $u^n_{i\pm 1/2} = \frac{1}{2}(u^n_{i\pm 1} + u^n_i)$

otherwise as $a = f' \Rightarrow a(u^n_{i+1/2}) \approx \underbrace{\frac{f(u^n_{i+1}) - f(U^n_i)}{U^n_{i+1} - U^n_i}}_{\text{speed of shock}}$

substitute back:

$$U^{n+1}_i = U^n_i - \frac{\nu}{2}\Big[\big(1-\nu a(u^n_{i+1/2})\big)\Delta^x_+ f(U^n_i)$$
$$+ \big(1-\nu a(u^n_{i-1/2})\big)\Delta^x_- f(U^n_i)\Big]$$

LW

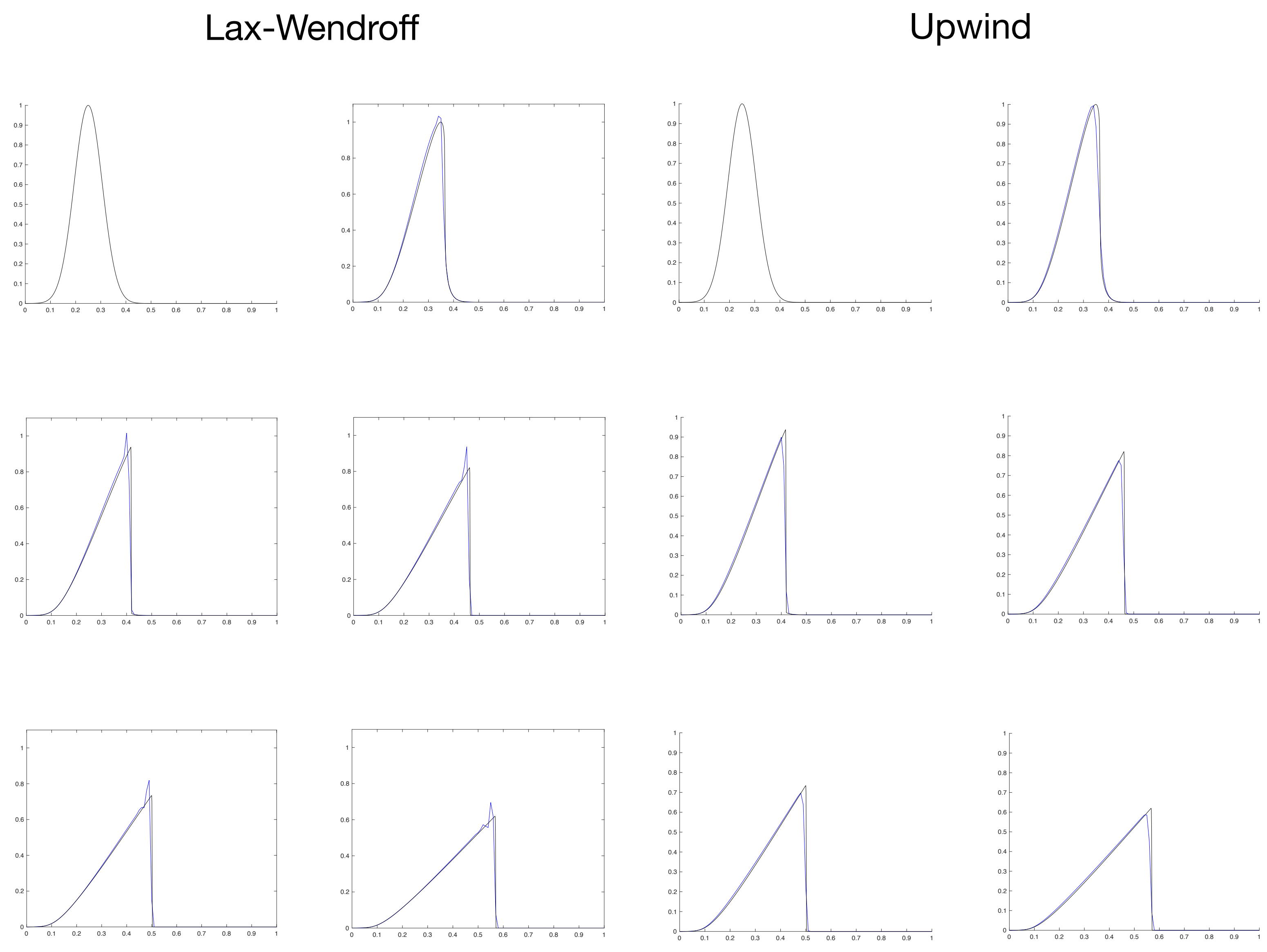
Upwind method

for $u_t + a u_x = 0$

$$U_i^{n+1} = U_i^n - k a \begin{cases} \delta_{h,-}^x U_i^n & a>0 \\ \delta_{h,+}^x U_i^n & a<0 \end{cases} = U_i^n - \frac{\nu}{2}\Big[(1-\mathrm{sign}(a))\, a\, \Delta_+^x U_i^n + (1+\mathrm{sign}(a))\, a\, \Delta_-^x U_i^n\Big]$$

for CL

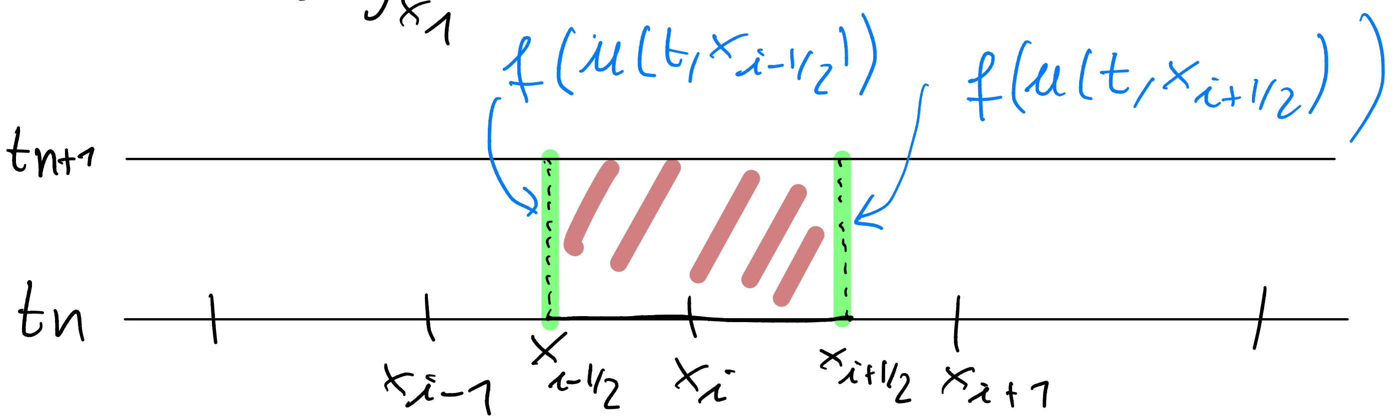
$$U_i^{n+1} = U_i^n - \frac{\nu}{2}\Big[(1-\mathrm{sign}\, a)^n_{i+1/2}\, \Delta_+^x f(U_i^n) + (1+\mathrm{sign}\, a)^n_{i-1/2}\, \Delta_-^x f(U_i^n)\Big]$$



- non smooth solutions high order does not necessarily perform better
- possible strategy: switch (adapt) method ?

Finite Volume methods

recall $\frac{d}{dt}\int_{x_1}^{x_2} u(t,x)\,dx = f(u(t,x_1)) - f(u(t,x_2))$



$C_i = (x_{i-1/2}, x_{i+1/2})$ cell/volume

integrate in (t_n, t_{n+1})

$$\underbrace{\frac{1}{h}\int_{C_1} u(t_{n+1},x)\,dx}_{\text{average at } t=t_{n+1}} - \frac{1}{h}\int_{C_1} u(t_n,x)\,dx = \frac{1}{h}\int_{t_n}^{t_{n+1}} f(u(t,x_{i-1/2}))\,dt - \frac{1}{h}\int_{t_n}^{t_{n+1}} f(u(t,x_{i+1/2}))\,dt$$

at $t=t_n$ flux over (t_n, t_{n+1})

Introduce now discrete solution (1 value per cell):

$$U_i^n \simeq \frac{1}{h}\int_{C_i} u(t_n,x)\,dx$$

this cannot be computed

also let

$$F^n_{i\pm 1/2} \simeq \frac{1}{k}\int_{t_n}^{t_{n+1}} f(t, x_{i\pm 1/2})\,dt$$

some approx of the average flux

this suggests: need to fix numerical fluxes

$$U_i^{n+1} = U_i^n - \nu \left(F^n_{i+1/2} - F^n_{i-1/2} \right)$$

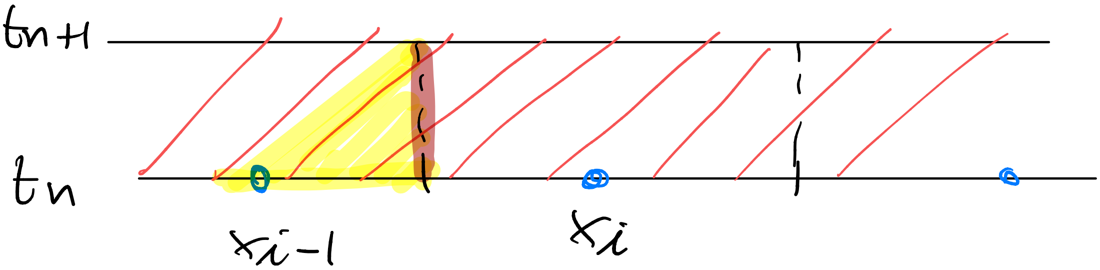
Property: discrete conservation law:

$\forall\ l < m$

$$h \sum_{i=l}^{m} U_i^{n+1} = h \sum_{i=l}^{m} U_i^n - \nu \underbrace{\left(F^n_{m+1/2} - F^n_{l-1/2} \right)}_{\text{if } = 0 \text{ no change in mass}}$$

Fixing the numerical fluxes

- given speed of propagation is finite, flux should depend only on local values



$\Rightarrow$ Def: $\mathcal{F}$ = numerical flux function, so that

$$F^n_{i+1/2} = \mathcal{F}(U_i^n, U_{i+1}^n)$$

for instance:

- for $u_t + a u_x = 0$

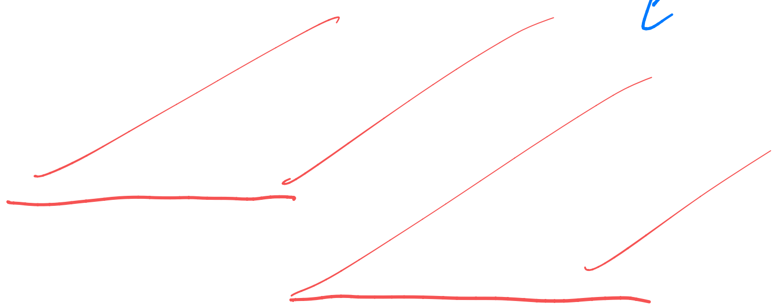
  * fix $\mathcal{F}(U^n_{i-1}, U^n_i) = a U^n_{i-1}$

  $\Rightarrow U_i^{n+1} = U_i^n - \nu a \left(U_i^n - U_{i-1}^n \right)$ upwind method!

* $F(U^n_{i-1}, U^n_i) = \frac{1}{2} a (U^n_{i-1} + U^n_i) - \frac{1}{2} \nu a^2 (U^n_i - U^n_{i-1})$ LW

- for CL, classical example is **GODUNOV'S method** which fixes the numerical fluxes as follows:

1. reconstruct piecewise polynomial solution $\bar{U}^n$ from given cell averages
2. evolve $\bar{U}^n$ exactly (by solving the Riemann problem)
3. take the average of the obtained solution as new cell averages



Advection-diffusion problems

$$\begin{cases} u_t - \varepsilon u_{xx} + a u_x = f & (0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) & \forall x \end{cases}$$

Possible methods combined all those we saw:

TIME + SPACE

- Explicit Euler + Centre Differences

$$\delta^t_{k,+} U^n_i - \varepsilon \left(\delta^x_h\right)^2 U^n_i + a^n_i \, \delta^x_{2h} U^n_i = f^n_i$$

$$\mu = \frac{k}{h^2} \; ; \; \nu = \frac{k}{h}$$

diff only — EE if $a=0$

adv only — CD if $\varepsilon=0$ always unstable!

von Neumann: stab. cond becomes $2\mu\varepsilon^2 + \frac{1}{2}a^2 k \leq \varepsilon$

- $\varepsilon \gg |a|$ $\rightsquigarrow$ $\mu\varepsilon \leq 1/2$ ($\equiv$ cond of parabolic prob)
- $\varepsilon \ll |a|$ $\rightarrow$ $k \leq \frac{2\varepsilon}{a^2}$

as $\varepsilon \to 0$ see that no k is acceptable ($\equiv$ hyperbolic cond)

- EE + upwind $a \geq 0$

$$\delta^t_{k,+} U^n_i - \varepsilon \left(\delta^x_h\right)^2 U^n_i + a^n_i \, \delta^x_{h,-} U^n_i = f^n_i$$

von Neumann: stab. cond: $\mu\varepsilon + \frac{|a|\nu}{2} \leq 1/2$

- $\varepsilon \gg |a|$ $\rightsquigarrow$ $\mu\varepsilon \leq 1/2$ (CD for parabolic
- $\varepsilon \ll |a|$ $\rightsquigarrow$ $|a|\nu \leq 1$ (upwind)

- IE + upwind (exp) $\Rightarrow$ nonsymmetric system

$$\delta^{t}_{k,-} U_i^{n+1} + \underbrace{\varepsilon (\delta_h^{x})^2 U_i^{n+1}}_{\text{implicit}} + \underbrace{a_i^n \delta^{x}_{h,-} U_i^n}_{\text{explicit}} = f_i^{n+1}$$

IMEX method (Implicit/Explicit)

gives back
- Implicit Euler for $a=0$
- upwind method for $\varepsilon=0$

von Neumann: $a_i^n \nu \leq \frac{1}{2}\left(1 + \sqrt{1+4\mu\varepsilon}\right)$

which is the hyperbolic type stability condition ($\varepsilon=0 \Rightarrow a_i^n \nu < 1/2$)

and note that diffusion helps