NSPDE/ANA 2022 - Lecture 9

Last lecture:
- Algebric form of FEM, implementation
- Conditioning of the FE matrix
- Error analysis of FEM in more dimensions
- An a posteriori error bound

Today:
- The generalised Galerkin method
- Advection-dominated Advection-diffusion problems

References:
- Quarteroni
- Ciarlet *The Finite Element Method for Elliptic Problems*, SIAM, 2002

(One of the two 'bibles' of the FEM together with the book
Brenner & Scott The Mathematical Theory of Finite Element Methods, TAM, 2008

The Generalised Galerkin Method

These arise in presence of

Variational Crimes: modifications of the variational / weak formulation:

1 - $V_h \not\subset V$ "nonconforming" spaces (eg. non C^0 - FEM)

2 - $\Omega_h = \bigcup_{T \in \mathcal{G}_h} T \neq \Omega$

3 - $\begin{cases} a_h(u_h, v_h) \approx a(u_h, v_h) \\ \ell_h(v_h) \approx \ell(v_h) \end{cases}$

example: FEM implementation requires quadrature yielding approximations a_h, l_h of the true forms in general.

Analysis of:

find $u_h \in V_h \subset V$: $a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h$ (*)

- Galerkin orthogonality does not hold, instead (assuming a_h can be evaluated with u)

$$T_h(v_h) = a_h(u - u_h, v_h) = a_h(u, v_h) - l_h(v_h)$$

Def: Truncation error

$$T_h := \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{|T_h(v_h)|}{\|v_h\|_V}$$

- $T_h \equiv 0$ (galerkin orthog) Full consistency $\Rightarrow$ Céa lemma
- $T_h \xrightarrow{h \to 0} 0$ consistency
- T_h = same order as corresp. Gal. method strongly consistent

Strang Lemma (accounting for "inconsistency" errors)

cannot directly rely on coerc. of a. +

Def: $\mathcal{A}_h : V_h \times V_h \to \mathbb{R}$ is UNIFORMLY V_h-elliptic if $\exists\, \tilde{\alpha}_0$:

$$a_h(v_h, v_h) \geq \tilde{\alpha}_0 \|v_h\|_V^2$$

Theorem (1st Strong Lemma):

If $\ell_h \in V'$, and $\mathcal{A}_h$ unif V_h-elliptic, then

- $\exists!\; u_h$ solution to $(*)$
- $\|u_h\| \leq \frac{1}{\tilde{\alpha}_0} \|\ell_h\|_{V'}$

} by Lax-Milgram

- $$\|u-u_h\|_V \leq \inf_{v_h \in V_h} \Big[\underbrace{\left(1+\delta/\tilde{\alpha}_0\right) \|u-v_h\|_V}_{\text{Céa Term}}$$
$$\underbrace{+ \frac{1}{\tilde{\alpha}_0} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|\mathcal{A}(v_h, w_h) - \mathcal{A}_h(v_h, w_h)|}{\|w_h\|_V} \Big] + \frac{1}{\tilde{\alpha}_0} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_V}}_{\text{inconsistency terms}}.$$

Proof: $\|u-u_h\|_V \leq \|u-v_h\|_V + \|u_h - v_h\|_V \quad \forall v_h \in V_h$

$\tilde{e}_h : u_h - v_h$

$$\tilde{\alpha}_0 \|\tilde{e}_h\|_V^2 \le a_h(u_h - v_h, \underbrace{\tilde{e}_h}_{\in V_h})$$ V_h elliptic

$$= l_h(\tilde{e}_h) - a_h(v_h, \tilde{e}_h)$$ FEM

$$\underbrace{+\, a(u - v_h, \tilde{e}_h) - l(\tilde{e}_h) + a(v_h, \tilde{e}_h)}_{=0}$$ PDE

$$= a(u - v_h, \tilde{e}_h) + (a - a_h)(v_h, \tilde{e}_h) + (l_h - l)(\tilde{e}_h)$$

$$\le \gamma \|u - v_h\|_V \|\tilde{e}_h\|_V + |(a - a_h)(v_h, \tilde{e}_h)| + |(l - l_h)(\tilde{e}_h)|$$

assume $\tilde{e}_h \neq 0$

$$\|\tilde{e}_h\|_V \le \frac{\gamma}{\tilde{\alpha}_0} \|u - v_h\|_V + \frac{1}{\tilde{\alpha}_0} \left(\frac{|(a - a_h)(v_h, \tilde{e}_h)|}{\|\tilde{e}_h\|} + \frac{|(l - l_h)(\tilde{e}_h)|}{\|\tilde{e}_h\|} \right)$$

replace with sup over all $w_h \in V_h$.

$\square$

to retain optimality need inconsistency terms to converge at same speed as Céa terms

For instance, for $V_h = V_h^k$ C^0-conforming FEM

then want inconsistency $\mathcal{O}(h^k)$

for strong consistency.

example: case of quadrature, need

quadrature rule exact $\mathbb{P}_{2k-2}$

(Ciarlet)

quarteroni

convection-diffusion (-reaction)

Want to solve:

$$\begin{cases} \mathcal{L}u = -\nabla\cdot(a\nabla u) + b\cdot\nabla u + cu = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$a \geq \alpha > 0$, $c \geq 0$, $c - \frac{1}{2}\operatorname{div} b \geq 0$

then corresponding bilinear form is coercive,

the Galerkin FEM, satisfies

$$\|u - u_h\|_V \leq \gamma/\alpha_0 \inf_{v_h \in V_h} \|u - v_h\|_V$$

(Céa quasi-optimality)

Q: What if $\gamma \gg \alpha_0$, and hence $\gamma/\alpha_0 \gg 0$?

1D case convection - diff. with $a, b > 0$ constant

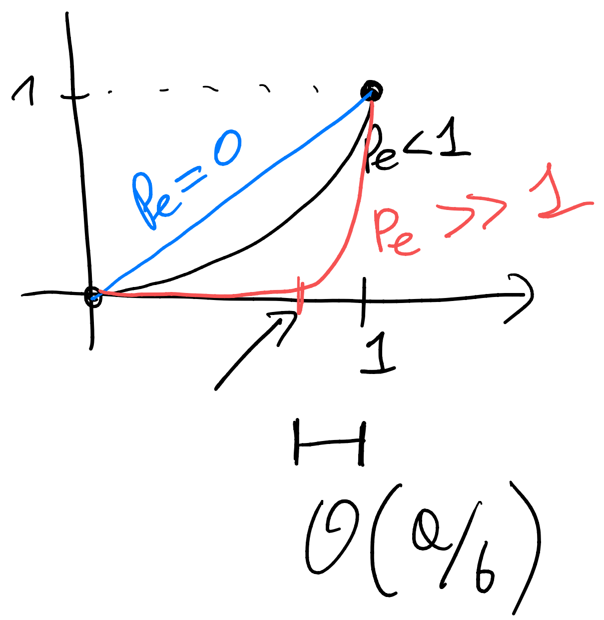
$$\begin{cases} -a u'' + b u' = 0 & \text{in } \Omega = (0,1) \\ u(0) = 0 ; u(1) = 1 \end{cases}$$

Péclet number: $Pe = \frac{bL}{2a}$ $L \doteq |\Omega| = 1$

exact solution:

$$u(x) = \frac{e^{b/a\, x} - 1}{e^{b/a} - 1}$$

monotonically increasing!



boundary layer for large Pe

This is an instance of a singularly perturbed problem:

perturbation of

- $a = 0 \longrightarrow$ hyperbolic problem $\begin{cases} bu' = 0 \\ u(0) = 0 \end{cases}$

with $a \ll 1$

$\rightarrow$ sol $u = 0$

- singular because at the limit as $a \to 0$ the gradient of sol becomes singular

Apply linear FEM: fix N, $h = 1/N$

$$AU = F \quad N \times N \text{ system:}$$

$$A = \begin{pmatrix} \ddots & \ddots & & \\ -\frac{a}{h} - \frac{b}{2} & \frac{2a}{h} & -\frac{a}{h} + \frac{b}{2} & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{a}{h} - \frac{b}{2} \end{pmatrix}$$

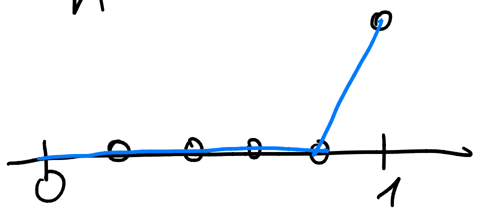
FEM / FD solution:

$Pe_h = \frac{bh}{2a}$ mesh Péclet number

- If $Pe_h = 1 \Rightarrow -\frac{a}{h} + \frac{b}{2} = 0 \Rightarrow F_j = 0 \Rightarrow U_j = 0$

$j = 0, 1, \dots, M-1$

$(U_M = 1)$



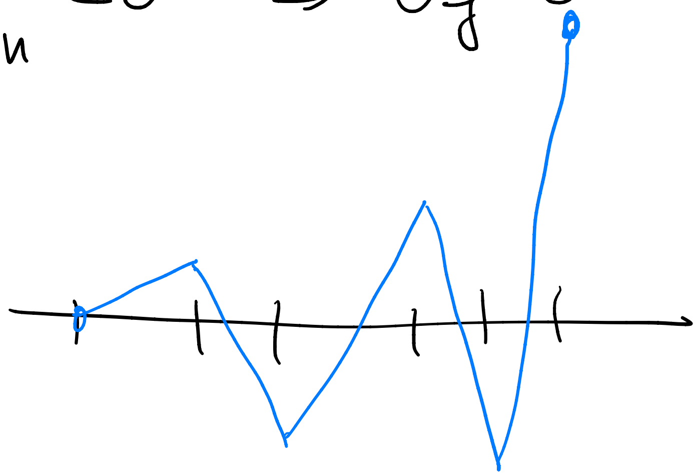
- otherwise

$$U_j = \frac{1 - \left(\frac{1+Pe_h}{1-Pe_h}\right)^j}{1 - \left(\frac{1+Pe_h}{1-Pe_h}\right)^M} \qquad \forall j$$

$Pe_h < 1$: $\frac{1+Pe_h}{1-Pe_h} > 0 \Rightarrow U_j$ monotone

$Pe_h > 1$: $\frac{1+Pe_h}{1-Pe_h} < 0 \Rightarrow U_j$ oscillates

$\longrightarrow$



"instability"

(eventually $h \to 0$ $Pe_h < 1$

$\to$ solution becomes monotone

$\longrightarrow$ method converges

In practice, problems often have

$$\begin{cases} a \approx 10^{-4} - 10^{-6} \\ b \approx 1 \end{cases} \xrightarrow{Pe_h < 1}$$

	#DOF
1D	10^4-10^6
2D	10^{12}
3D	10^{18}

behond computational capabilities in realistic applications

solutions:
- adapt the mesh (fine mesh at boundary layer)
- modify scheme