

NSPDE/ANA 2022 - Lecture 14

Last lecture:

- Semidiscrete FEM for parabolic problems
- Error analysis
- Fully discrete FD-FEM

Today:

- higher-order time-stepping: BDF, discontinuous Galerkin
- Intro to Hyperbolic problems
- Upwind method for linear transport
- CFL, stability

References:

- Morton-Mayers
- Larsson-Thomee

Higher-order Time Stepping

1) BDF formulas :

2nd order BDF

$$\begin{aligned}\bar{D}u^{n+1} &= \sum_{\Delta t_j} u^{n+1} + \frac{1}{2} \Delta t \left(\sum_{\Delta t_j}^2 u^{n+1} \right) \\ &= \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{2} \Delta t \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} \\ &= \frac{1}{\Delta t} \left(\frac{3}{2} u^{n+1} - 2u^n + \frac{1}{2} u^{n-1} \right)\end{aligned}$$

$$(\text{Taylor}) \quad \Delta u^{n+1} = u_t(t_{n+1}) + O(\Delta t^2)$$

BDF2-FEM

- $u_h^n = \Pi_h u_0 \in V_h^k$
- $(\int_{\Delta t} u_h^n, v_h) + R(u_h^n, v_h) = (f^1, v_h) \quad \forall v_h \in V_h^k$
- $(\int_{\Delta t} u_h^{n+1}, v_h) + R(u_h^{n+1}, v_h) = (f^{n+1}, v_h) \quad \forall v_h \in V_h^k$
 $n = 1, 2, \dots$

Theorem : if u smooth enough ,

$$\|u_h^n - u(t_n)\|_{L^2} \leq C h \left(\|u_0\|_{H^{k+1}} + \int_0^{t_n} \|u_t\|_{k+1} \right) \\ + C \Delta t \underbrace{\int_0^{\Delta t} \|u_{tt}\|_{L^2} + (\Delta t)^2 \int_0^{t_n} \|u_{ttt}\|_{L^2}}_{\text{take } \Delta t \text{ small enough}}$$

Discontinuous

Ideas : use FEM in time also

- in & try to maintain the approach of time stepping

Continuous problem in weak form :

$$t \in (0, T], \quad u(t) \in V \quad (= H_0^1)$$

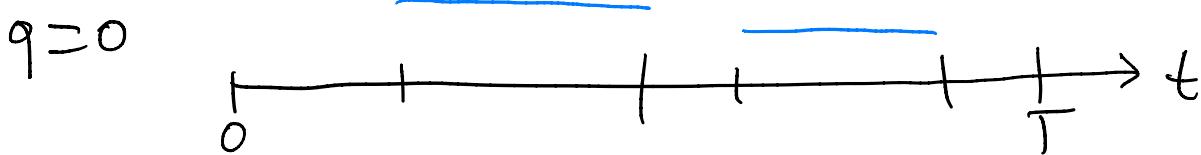
$$\begin{cases} f(u_t, v) + g(u, v) = (f, v) & \forall v \in V \\ u(0) = u_0 \end{cases}$$

Let $0 = t_0 < t_1 < \dots < t_n < \dots \bar{T} = t_{N_t}$
 $\Delta t_n = t_{n+1} - t_n$, $\Delta t := \max_n \Delta t_n$.
 $J_n := (t_n, t_{n+1}]$.

Let $q \in \mathbb{N} \cup \{\infty\}$

$$\rightarrow W_{\Delta t}^q = \left\{ v: [0, T] \rightarrow V : v(t) \in P_q(V), \forall n=0, 1, \dots, N_t \right\}$$

example



\cap

$L^2(I; V)$

$W_{\Delta t}^q = \text{DG PER SPACE}$

Derivation of DG-time stepping method

From weak formulation, test now with $v = v(t)$ such that $v(T) = 0$, and integrate by parts in time: v smooth enough

$$\begin{aligned} \int_0^T (u_t, v) dt &= - \int_0^T (e_t v_t) + (e_t, v) \Big|_{t=0}^{t=T} - (u, v) \\ &= - \int_0^T (u, v_t) - (u(0), v(0)) \end{aligned}$$

$$\int_0^T [F(u, v_t) + R(u, v)] dt = (u_0, v(0)) + \int_0^T (f, v) dt$$

Consider $u_{\Delta t} \in W_{\Delta t}^q$ and integrate by parts again :

$$\begin{aligned} -\int_0^T (u_{\Delta t}, v_t) &= -\sum_{n=0}^{N_t-1} \int_{J_n} (u_{\Delta t}, v_t) \\ &= +\sum_{n=0}^{N_t-1} \left(\int_{J_n} u_{\Delta t}, v \right) + (u_{\Delta t}, v) \Big|_{J_n} \\ &= + \int_0^T \left(\int_{J_n} u_{\Delta t}, v \right) + (u_{\Delta t}^0, v) \end{aligned}$$

$$\text{Def } \lfloor v^n \rfloor := v^n_+ - v^n_- \quad \text{JUMP}$$

$$v^n_+ = \lim_{z \rightarrow 0^+} v(t_n + z)$$

$$v^n_- = \lim_{z \rightarrow 0^-} v(t_n + z)$$

$$+ \sum_{n=1}^{N_t-1} (\lfloor u_{\Delta t}^n \rfloor, v^n)$$



D6 scheme : find $u_{\Delta t} \in W_{\Delta t}^q$:

$$\left\{ \begin{aligned}
 & \int_0^T \left[\left(\sum_{\Delta t} u_{\Delta t}, v \right) + R(u_{\Delta t}, v) \right] \\
 & + \sum_{n=1}^{N_t-1} \left(L[u_{\Delta t}^n], v^n \right) + (u_{\Delta t}^0, v^0) \\
 & = \int_0^T (f, v) + (u^0, v^0)
 \end{aligned} \right.$$

equivalently, $\forall n=0, 1, \dots, N_t-1$

$$\begin{aligned}
 & \int_{J_n} \left[\left(\sum_{\Delta t} u_{\Delta t}, v \right) + R(u_{\Delta t}, v) \right] + (u_{\Delta t, +}^n, v^n) \\
 & = \int_{J_n} (f, v) + (u_{\Delta t, -}^n, v^n)
 \end{aligned}$$

with $u_{\Delta t, -}^0 = u^0$

*local
initial
condition
weakly
imposed*

- case $q=0$ DG \equiv Implicit Euler ?

error estimate:

$$\|u(T) - u_{\frac{Nt}{\Delta t}}^{Nt}\| \leq C \left(\sum_{n=0}^{N-1} \Delta t_n^{2(q+1)} \int_{J_n} \left| \frac{d^{q+1} u}{dt^{q+1}} \right|_{H^1} dt \right)^{1/2}$$

e.g. for $q=0$ the bound only requires $\frac{d u(t)}{dt}$ in H^1 in contrast with Backward (Implicit) Euler analysis which requires 2nd derivative in time.

HYPERBOLIC PDES

model problems:

- ① linear transport: $u_t + \alpha u_x = 0$
- ② conservation laws: $u_t + (f(u))_x = 0$
- ③ wave equation: $u_{tt} - c u_{xx} = 0$ (cc0)

Recall: for

$$\begin{cases} u_t + \alpha u_x = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

We know that information travels along the one family of characteristics:

$$\downarrow \text{from } (\underline{1}, \underline{a}) \cdot \nabla_{t,x} u = 0$$

stating that the derivative of u with respect to $\underline{v} = (\underline{1}, \underline{a})$ is zero, we have that solution is constant along

$$\frac{dx}{dt} = a(x, t)$$

\Rightarrow solution is known from initial cond travelling along the characteristic

Same applies for systems of transport equations, e.g.

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{f}(\bar{u})}{\partial x} = 0$$

for example, the incompressible Euler equations

$$\begin{cases} \frac{D u_i}{D t} = \frac{\partial u_i}{\partial t} + \underline{u} \cdot \nabla u_i \\ \nabla \cdot \underline{u} = 0 \end{cases} = g + \text{source terms}$$

$$\hookrightarrow \text{in conservative form: } \frac{D u_i}{D t} = \frac{\partial u_i}{\partial t} + \sum_j \frac{\partial u_i}{\partial x_j} (u_j^2) + \sum_{j \neq i} \frac{\partial (u_i u_j)}{\partial x_j}$$

equivalently/ ↴

$$\bar{u}_t + A(\bar{u}) \bar{u}_x = 0$$

with

$$A(\bar{u}) = \frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots \\ \vdots & \ddots & \ddots \\ \frac{\partial f_n}{\partial u_1} & \dots & \dots \end{pmatrix}$$

characteristic speeds of propagation
are the eigenvalues of A , however
in general characteristic curves and
closed formulas for solutions would
not be available

Another example is wave equation:

$$u_{tt} - u_{xx} = 0 \Leftrightarrow \begin{cases} u_t + v_x = 0 & (1) \\ u_x + v_t = 0 & (2) \end{cases}$$

$$\begin{aligned} (1)_t \Rightarrow 0 &= u_{tt} + v_{xt} \\ &\stackrel{|}{=} u_{tt} + (v_t)_x \end{aligned}$$

$$\text{use (2)} \quad | \quad = u_{tt} - u_{xx}$$

$$\overbrace{U_t + A U_x = 0}^{U = \begin{pmatrix} u \\ v \end{pmatrix}}$$

$$\textcircled{2} \Rightarrow 0 = U_{ff} - V_{xx} \quad \boxed{A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

In conclusion :

- plenty of physical problems that reduce to transport systems
- often closed solutions not available

Properties of solutions

conservation of energy :

$$\textcircled{1} \quad u_t + \alpha u_x = 0 \quad x \in \mathbb{R}, \alpha \equiv \text{const.}$$

$\|u(t)\|_{L^2(\mathbb{R})}$ is conserved.

proof: test with u and integrate in x :

$$\int_{\mathbb{R}} u_t u dx + \alpha \int_{\mathbb{R}} u_x u dx = 0$$

\uparrow

$$\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} u^2 dx + \frac{\alpha}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} u_x^2 dx = 0$$

$\underbrace{u^2}_{\begin{matrix} \uparrow \\ \rightarrow \end{matrix}}$

otherwise
 $u \notin L^2(\mathbb{R})$

$$\Rightarrow \int_{\mathbb{R}} u^2 dx = 0$$

$$\|u(t)\|_{L^2(\mathbb{R})}^2 \equiv \text{const} \equiv \|u_0\|_{L^2(\mathbb{R})}^2$$

$$\textcircled{2} \quad \text{for } -u_{tt} - u_{xx} = 0$$

$$\text{energy : } \frac{1}{2} \int_{\mathbb{R}} ((u_t)^2 + (u_x)^2) dx$$

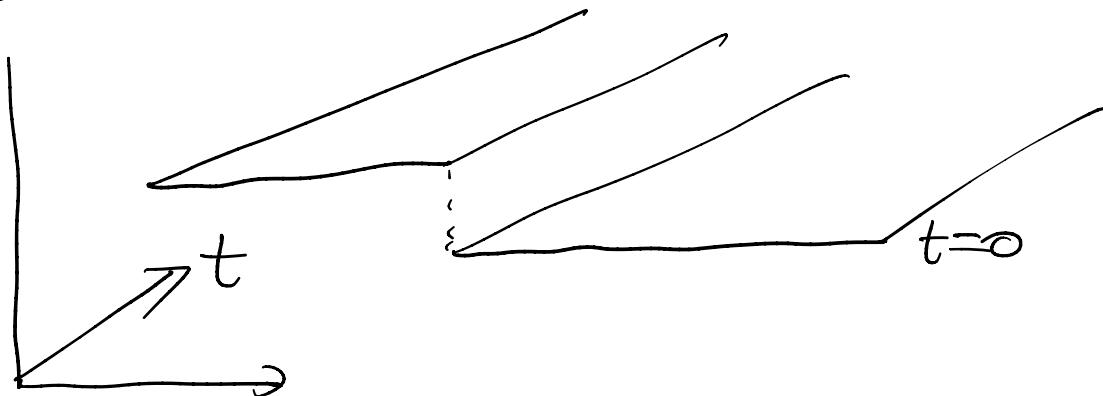
is conserved

proof : test by u_t (exercise)

Comments:

- hyperbolic \rightarrow no dispersion
- discontinuous initial data is propagated
(\Rightarrow "generalised solutions")

example: Riemann problem: piecewise constant initial cond with one step:



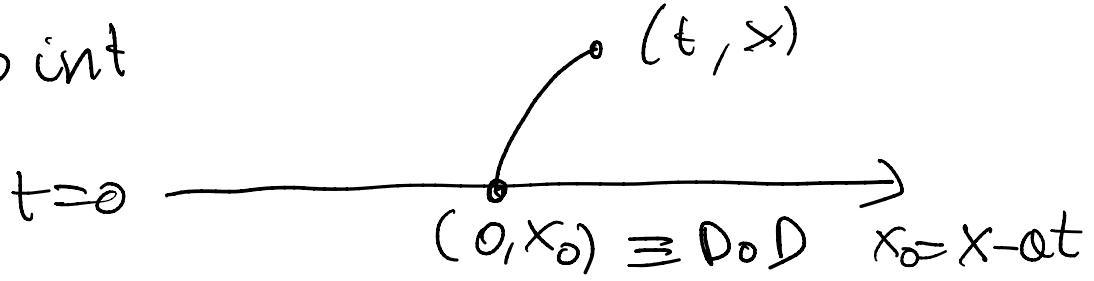
\rightarrow solution is classical only piecewise away from jumps

+ CFL condition (DoD of FD scheme must be within the DoD of the PDE)

for hyperbolic problems we have finite speed of propagation so DoD of PDE is finite.

$$u_{t+1} - u_t = c \Delta t$$

example, for the transport eq. it is just one point



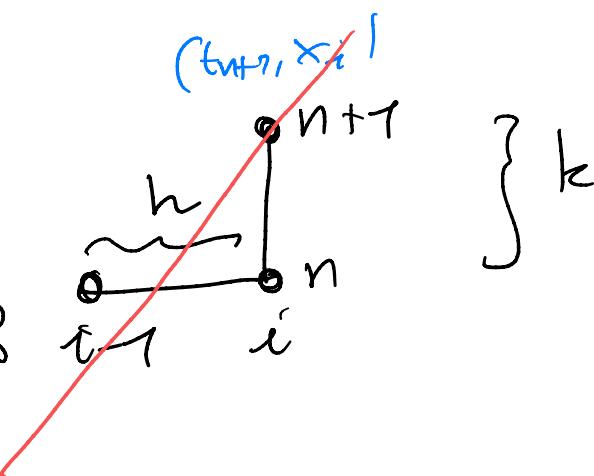
What about the FD scheme DoD:

$$\begin{cases} u_t + \alpha u_x = 0 & (0, T] \times \mathbb{R} \\ u(0, x) = u_0(x) \end{cases}$$

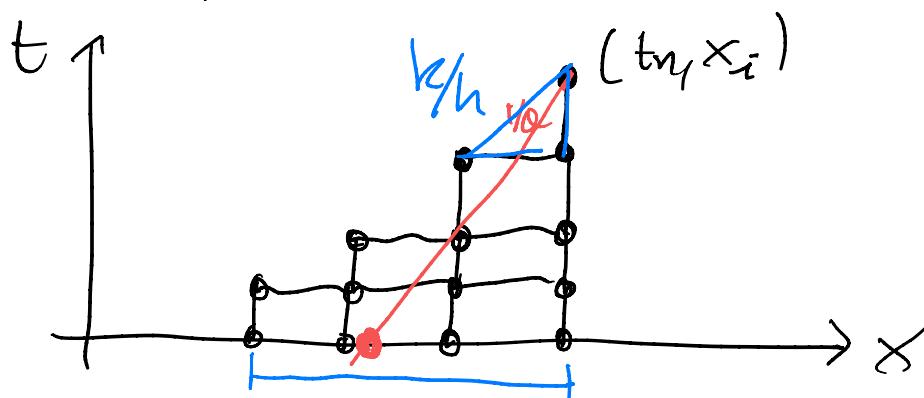
$$\alpha > 0$$

consider backward diff. in space

$$\frac{U_i^{n+1} - U_i^n}{k} + \alpha_i^n \frac{U_i^n - U_{i-1}^n}{h} = 0$$



DoD of scheme, define or base of the FD pyramid



$$\text{CFL: } \frac{k}{h} \leq 1/\alpha \quad \Leftrightarrow \quad \alpha D \leq 1$$

$$D := k/h$$

much less stringent requirement wrt.
parabolic case \Rightarrow "explicit methods
competitive"

The above is an instance of "UPWIND" method

$$\begin{cases} U_i^{n+1} = (1 - \alpha_i^n V) U_i^n + \alpha_i^n V U_{i-1}^n & \forall n=0, 1, \dots \\ U_i^0 = M_0(x_i) \end{cases}$$

STABILITY by von Neumann analysis:

plug $U_i^n = \lambda^n e^{i\gamma h}$

$$\lambda = (1 - \alpha_i^n V) + \alpha_i^n V e^{-i\gamma h}$$

$$= (1 - \alpha_i^n V) + \alpha_i^n V \cos(\gamma h) - i \alpha_i^n V \sin(\gamma h)$$

$$|\lambda|^2 = 1 - 4 \alpha_i^n V (1 - \alpha_i^n V) \sin^2(\frac{1}{2}\gamma h)$$

$$|\lambda| \leq 1 \text{ for stability} \Leftrightarrow \alpha_i^n V \leq 1$$

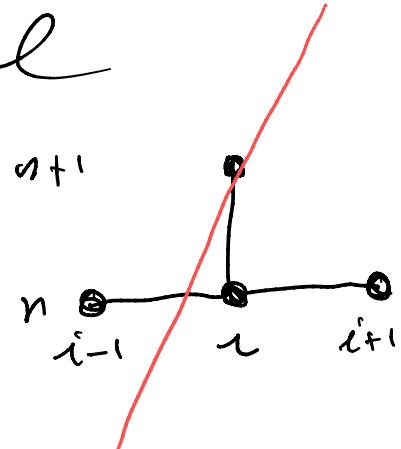
(identical to
CFL cond.)

In general, CFL is only necessary.

For example, the scheme

$$\frac{U_i^{n+1} - U_i^n}{k} + \alpha_i^n \frac{U_{i+1}^n - U_{i-1}^n}{2h} = 0$$

center diff.



$$\text{CFL} \quad \alpha V \leq 1$$

however, von Neumann gives

$$\lambda = \frac{v}{2} e^{-ikh} + 1 - \frac{v}{2} e^{ikh} \quad (\alpha = 1)$$
$$\Rightarrow 1 - iv \sin kh$$

$$|\lambda| = \sqrt{1 + v^2 \sin^2 kh} > 1 \text{ always}$$

\Rightarrow The CD scheme is always
unstable !!!