NSPDE/ANA 2022 - Lecture 1

LECTURES: 40 lectures + 8 computer sessions

2-5PM Tuesdays+Wednesday NO LECTURES TOMORROW!

- in person: room A-133. Wear FFP2 masks; always sanitise desk+chair
- Online: week video on; mute audio unless for asking a question

COURSE MATERIAL: see GitHub site:

https://github.com/andreacangiani/NSPDE-ANA22

ASSESSMENT: oral exam on course material + short presentation of coursework project.

BIBLIOGRAPHY:

- Morton & Mayers *Numerical Solution of Partial Differential Equations*. Cambridge, 1994.
- Larsson & Thomee *Partial Differential Equations with Numerical Methods.* Springer, 2009.
- Quarteroni *Numerical Models for Differential Equations.* Springer, 1994.
- Evans *Partial Differential Equations.* AMS 1997.
- Gilbert & Trudinger *Elliptic Partial Differential Equations of Second Order.* Springer, 1998.
- Own notes

(Core of course along the lines of Larsson Thomee book.)

IDEA OF THE COURSE:

present the fundamental ideas/methods and the interlink between different approaches.

- explore different problems to see that fundamental ideas are ubiquitous
- understand that appropriate numerical methods depend on the problem

Follows the Numerical Analysis course (Heltai+Rozza) and is complemented with Theory and Practice of FEM (Heltai) and Advanced FEM Techniques (Cangiani)

Practical sessions in Python and FEniCSx using
https://colab.research.google.com/notebooks/

PDEs' Facts

Def (PDE):
- $\Omega \subset \mathbb{R}^d$ open "Domain of Definition"
- PDE on Ω of order k:

$$F\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \dots, \frac{\partial^k u}{\partial x_d^k}\right) = 0 \qquad (*)$$

where,

$\underline{x} = (x_1, \dots, x_d)$ ("indep. variable")

$u : \Omega \to \mathbb{R}$ unknown k-times differentiable function

- F given scalar function

(systems of PDEs can also be defined, taking F, u as vector functions)

Def (Solution): Classical solution of $(*)$ is the family of all functions $u : \Omega \to \mathbb{R}$: that are k-times diff. and satisfy $(*)$

Def (types of PDEs):
- A PDE is LINEAR : coefficients in F only depend on $\underline{x}$
- A PDE is NONLINEAR if not linear
  - SEMILINEAR if coeffs of k-th order derivatives only depend on $\underline{x}$
  - QUASILINEAR if coeffs of k-th order derivatives are functions of up to the k-1-th derivatives
  - FULLY NONLINEAR if not quasilinear

examples:
- Laplace (2D) $u_{xx} + u_{yy} = 0$ linear
- inviscid Burgers : $u_t + u\, u_x = 0$ quasilinear
- KdV : $u_t + u u_x + u_{xxx} = 0$ semilinear

- Monge-Ampère : $u_{xx} u_{yy} - (u_{xy})^2 = 0$ fully nonlinear

This course: mainly linear (but not only)

motivation: - important ubiquitous ideas
- process of linearisation

nonlinear

example: (LT) $F(u) = \frac{\partial u}{\partial t} - \nabla \cdot (a(u) \nabla u) - f(u) = 0$

in 1D $\quad u_t - (a(u) u')' - f(u) = 0$

- linearisation Picard iteration :
  - Fix u^0
  - $\forall\, k \geq 0$, solve $\frac{\partial u^{k+1}}{\partial t} - \nabla \cdot (a(u^k) u^{k+1}) - f(u^k) = 0$

linear !

- Newton interation :
  - Fix u^0
  - define $u^{k+1} = u^k + \delta u^k$ where u^k solves linear PDE :

$$F'(u^k)\, \delta u^k = - F(u^k)$$

$\left(u^{k+1} = u^k - \frac{f(u^k)}{f'(u^k)}\right)$

- $\frac{\partial u^{k+1}}{\partial t} = \frac{\partial u^k}{\partial t} + \frac{\partial\, \delta u^k}{\partial t}$
- $-f(\underbrace{u^{k+1}}_{= u^k + \delta u^k}) \simeq -f(u^k) - f'(u^k)\, \delta u^k$

- $a(u^{k+1}) \simeq a(u^k) + a'(u^k)\,\delta u^k$

$$\nabla\cdot\Big(a(u^{k+1})\,\nabla u^{k+1} \simeq \big(a(u^k)+a'(u^k)\,\delta u^k\big)\big(\nabla u^k + \nabla\delta u^k\big)$$

$$= -a(u^k)\,\nabla u^k - a(u^k)\,\nabla\delta u^k - a'(u^k)\,\delta u^k\big(\nabla u^k + \cancel{\nabla\delta u^k}\big)\Big)$$

higher-order

$$\| \\ F(u^k)$$

$$\frac{\partial \delta u^k}{\partial t} - \nabla\cdot\big(a(u^k)\,\nabla\delta u^k\big) - \nabla\cdot\big(a'(u^k)(\delta u^k)\,\nabla u^k\big)$$

$$- f'(u^k)\,\delta u^k = -F(u^k)$$

$$= F'(u^k)\,\delta u^k$$

LINEAR PDEs OF 1^{st} AND 2^{nd} ORDER

① transport equation

ex: $u_x + u_y = 0$

can be solved by "method of characteristics"

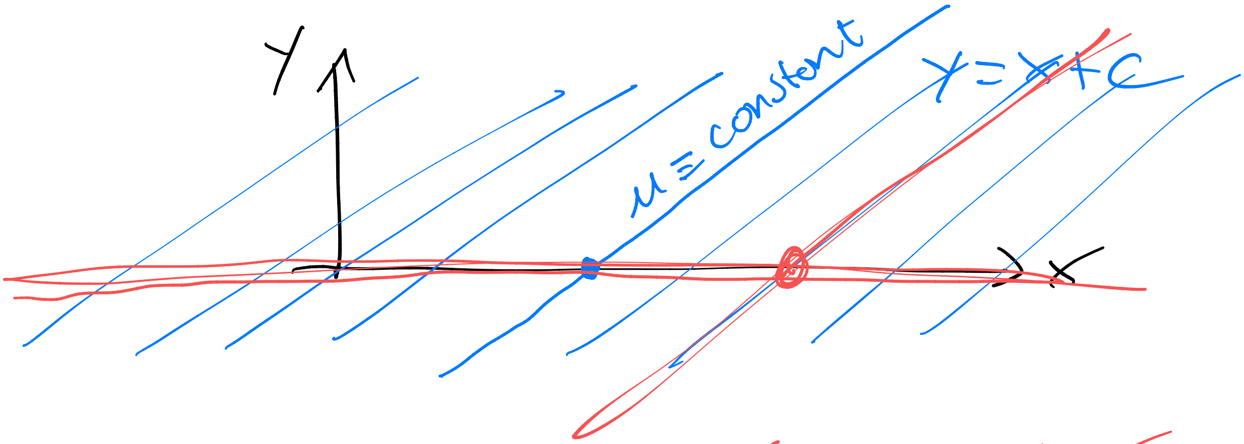
$$\begin{cases} \xi = x+y \\ \eta = y-x \end{cases} \qquad v(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$$

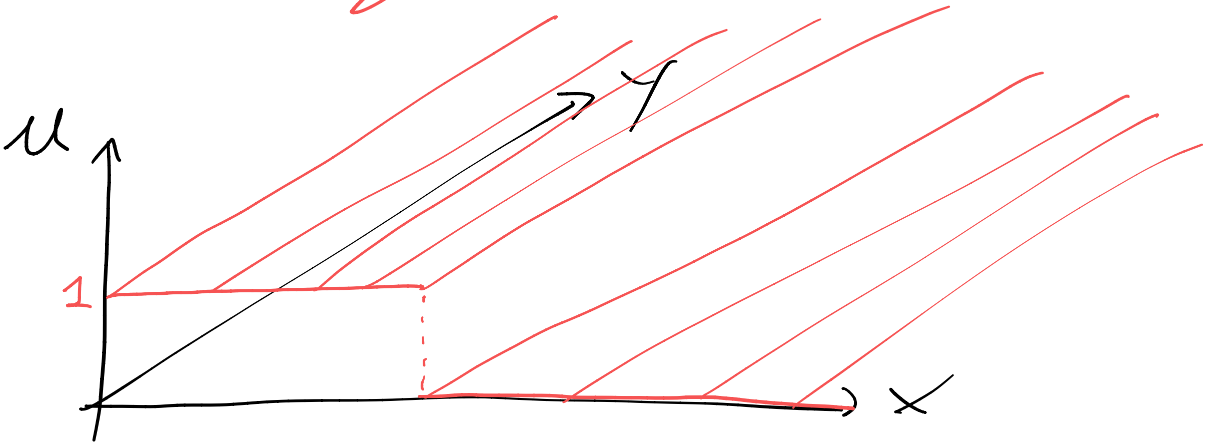
$$v_\xi = 0 \xrightarrow{\text{integrate}} v(\xi,\eta) = f(\eta)$$

$$u(x,y) = f(\eta(x,y)) = f(y-x)$$

u is constant $\Longleftarrow$ for $\overset{\|}{c}$

$y - x = c$ characteristics of the PDE





note that the "red function" satisfies the PDE. In particular this function is discontinuous across some characteristics

More in general, take $a, b, c, f : \mathbb{R}^2 \to \mathbb{R}$, $a, b \in C^1(\mathbb{R}^2)$

$\nexists (x,y) \in \mathbb{R}^2 : a(x,y) = b(x,y) = 0$

$$a(x,y)\, u_x + b(x,y)\, u_y + c(x,y)\, u = f(x,y) \quad \text{in } \overset{\Omega}{\overset{\shortparallel}{\mathbb{R}^2}}$$

As before can use method of charact.

- characteristic curves γ: $y = y(x)$ solution of

$$-a \frac{dy}{dx} + b = 0$$

(2) Second order linear PDEs in $\mathbb{R}^2$

$a, b, c, d, e, f, g : \Omega \to \mathbb{R}$, $\Omega = \mathbb{R}^2$

$$\boxed{a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} + d\,u_x + e\,u_y + f\,u = g}$$

class of 2^{nd} order linear PDEs

Def (Discriminant): $D = D(\underline{x}) = b^2 - ac$ is the DISCRIMINANT of the 2^{nd} order PDE. (x, y)

- $D > 0$ at $\underline{x}$, the PDE is hyperbolic at $\underline{x}$
- $D = 0$ " , " parabolic "
- $D < 0$ " , " elliptic "

The PDE is said hyperbolic, etc if it is such in the whole of Ω.

note: lower order terms are irrelevant for this characterisation of 2^{nd} order PDEs.

<u>examples</u>:
- hyperbolic: $u_{tt} + c\,u_{xx} = 0$ $\quad (c < 0)$ wave eq.
- parabolic: $u_t + a\,u_{xx} = 0$ $\quad (a < 0)$ heat (or diffusion) eq.
- elliptic: $\underbrace{u_{xx} + u_{yy}}_{\Delta u = \nabla \cdot (\nabla u)} = 0$ Laplace

"Physically" these examples reflect the general character of each class:

- elliptic eq. $\to$ time indep
- parabolic eq. $\to$ time-dep diffusion phenomena

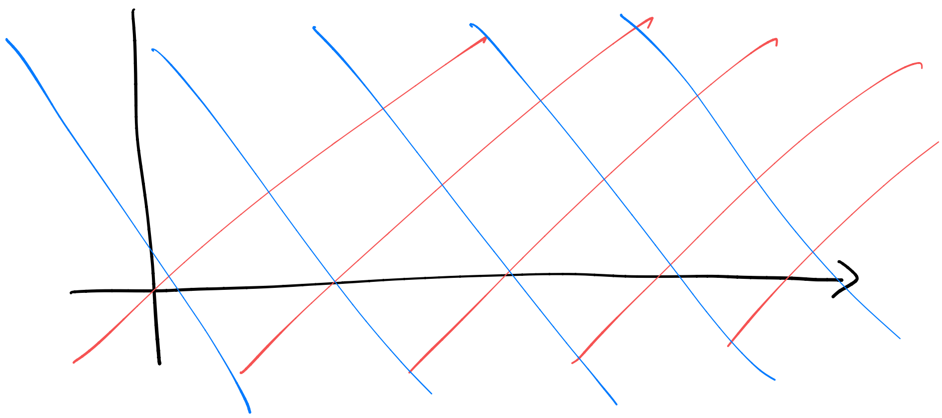
- hyperbolic eq. $\longrightarrow$ transport wave-like phenomena with finite speed propagation.

Theorem: (invariance of type): The sign D of a 2^{nd} order linear PDE is invariant under smooth non-singular transformation of coordinates (change of variables).

example: $u_{xx} - u_{yy} = 0$

$$\begin{cases} \xi = x+y \\ \eta = y-x \end{cases} \rightarrow v_{\xi\eta} = 0 \longrightarrow v(\xi,\eta) = F(\xi) + G(\eta) \quad \forall F, G \in C^2(\mathbb{R})$$

$$\longrightarrow u(x,y) = F(\underbrace{x+y}_{c_1 \in \mathbb{R}}) + G(\underbrace{y-x}_{c_2 \in \mathbb{R}})$$



these two families of characteristic curves are found as the solutions of ODE

$$\frac{dy}{dx} = \frac{b \pm \sqrt{D}}{a} = \pm 1 \qquad (a=1, b=0, c=-1)$$

In general, for $a u_{xx} + 2b u_{xy} + c u_{yy} + \dots = f$,

we get the CHARACTERISTIC EQ.

$$a\left(\frac{dy}{dx}\right)^2 - 2b\,\frac{dy}{dx} + c = 0$$

$\Rightarrow$ with roots

$$\frac{dy}{dx} = \frac{b \pm \sqrt{D}}{a}$$

- $D > 0$: 2 solutions $\Rightarrow$ 2 families of char eq.
- $D = 0$: 1 solution $\Rightarrow$ 1 family
- $D < 0$: 0 solution $\Rightarrow$ no characteristics !

Remark: all this still valid for semilinear PDEs!

Now, let's "fix the constants".

Def (Cauchy Problem): Let $F(x, u, \dots) = 0$ given PDE of order k.

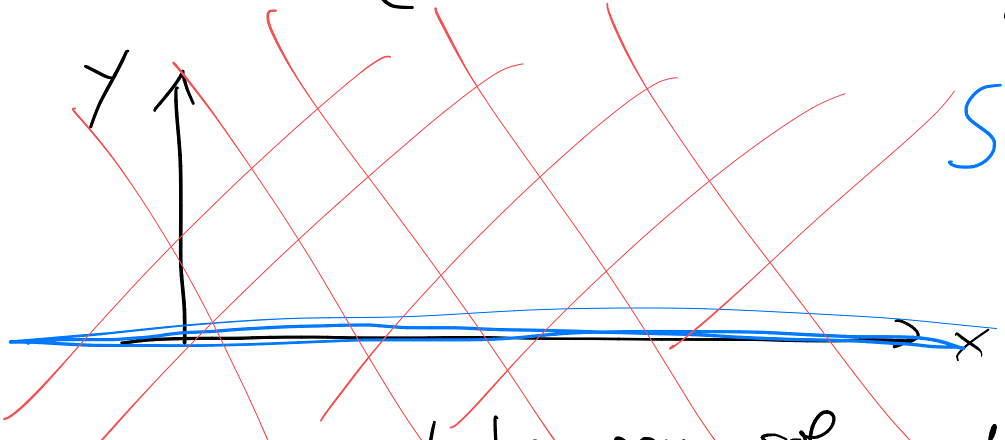
Let S be a surface in $\mathbb{R}^d$, $n = n(\underline{x})$ its normal,

and suppose that $\forall \underline{x} \in S$, the "INITIAL CONDITIONS"

$$u(\underline{x}) = f_0(\underline{x}),\ \frac{\partial u}{\partial \underline{n}}(\underline{x}) = f_1(\underline{x}),\ \dots,\ \frac{\partial^{k-1} u}{\partial \underline{n}^{k-1}}(\underline{x}) = f_{k-1}(\underline{x})$$

are given. The <u>Cauchy problem</u>: find the solutions of the PDE satisfying the I.C.

example: $\begin{cases} u_{xx} - u_{yy} = 0 \\ u(x,0) = \sin x \, ; \, u_y(x,0) = 0 \end{cases}$ Cauchy Pbm



$S = \{ (x,y) : y = 0 \}$

solution: take gen. sol. $u(x,y) = F(x+y) + G(y-x)$

apply the I. C., gives

$$u(x,y) = \frac{1}{2} \left(\sin(x+y) + \sin(x-y) \right)$$

$\exists!$ solution of the Cauchy problem!

Theorem (Cauchy-Kovalevskaya):

Given 2nd order linear PDE and an initial surface S with initial data f_i, and given $\underline{x_0} \in S$. Suppose that:

- S is analytic
- S is not a characteristic surface at $\underline{x_0}$
- data all analytic (coeffs, f_i)

Then, Cauchy problem has a solution in a neighbourhood of $\underline{x_0}$ (local result!)

which is analytic.

comments

☆ general but "limited" to analytic data

☆ gives a result about $\exists!$. Thus does not imply that the solution is "well behaved"

<u>Def</u>: A PDE problem is WELL POSED in the sense of Hadamard if

- it has a unique solution
- the solution depends continuously on the PDE coeffs and the problem's data

see provided example for Laplace eq. showing $\exists!$ of a Cauchy problem for which a small perturbation of data $\Rightarrow$ exponentially large change in solution!

$\rightsquigarrow$ Cauchy problem "wrong" for Laplace!

Why numerical methods?

- Theory of PDEs: study of well-posedness of PDE problems, rarely can write down the solutions

- Numerical methods look for approximate solutions that can be computed in practice!

(Quarteroni)

Let $P(\underset{\text{sol}}{u}, \underset{\text{data}}{g}) = 0$ be a PDE problem

$P_N(u_N, g_N) = 0$

N: dimension of discrete problem

Def: A numerical method $P_N(u_N, g_N)$ is

☆ CONVERGENT: if $\|u - u_N\| \xrightarrow{N \to \infty} 0$

$\|\cdot\|$: some norm

☆ CONSISTENT: if $P_N(u, g) \xrightarrow{N \to +\infty} 0$

STRONGLY CONSISTENT $P_N(u, g) = 0 \quad \forall N$

☆ STABLE: if small perturbations in

g_H results in small perturbation in resulting numerical solution

So :- $g_H \longrightarrow g_H + \delta g_H$

- solve $P_H(\underbrace{u_H + \delta u_H}, g_H + \delta g_H) = 0$

$\forall \epsilon > 0 \; \exists \delta_\epsilon > 0$: if $\| \delta g_H \| < \delta$

then $\| \delta u_H \| \leq \epsilon \quad \forall H$.

Theorem (Lax-Richtmyer equivalence):

Consistency + Stability $\Rightarrow$ convergent