NSPDE/ANA 2022 - Lecture 10

Last lecture:

- The generalised Galerkin method
- Advection-dominated Advection-diffusion problems

Today:
- Strongly consistent FEM for advection-dominated problems
- Intro to Parabolic problems

References:
- Quarteroni
- Larsson-Thomee

Seen last week: for

$$\begin{cases} -a u'' + b u' = 0 & \text{in } \Omega = (0,1) \\ u(0) = 0 ; u(1) = 1 \end{cases}$$

exact solution: $u(x) = \dfrac{e^{b/a x} - 1}{e^{b/a} - 1}$ monotone!

FEM/FD solution: $U_j = \dfrac{1 - \left(\frac{1+Pe_h}{1-Pe_h}\right)^j}{1 - \left(\frac{1+Pe_h}{1-Pe_h}\right)^M} \quad \forall j$

oscillatory if $Pe_h > 1$

$Pe_h := \dfrac{bh}{2a}$ mesh Péclet number

(similar issues for reaction-dominated reaction-diffusion problems, see eg Quarteroni)

Towards a cure of numerical "instability"

generic equation of our FEM/FD (k=1) scheme

$$-a\,\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}+b\,\frac{u_{i+1}-u_{i-1}}{2h}=0$$

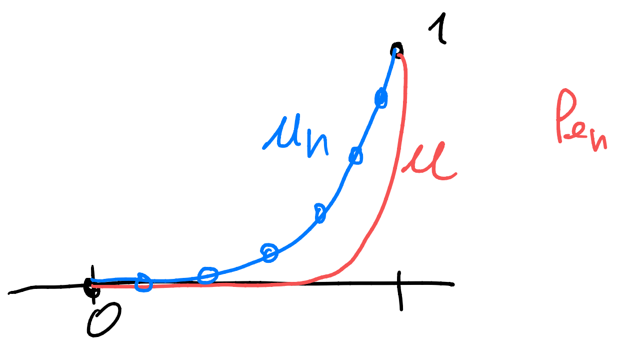
$$-\underbrace{a}_{\text{small}}\,u''+b\,u'=0$$

alternatively, use UPWIND difference for the advection term: $(b>0)$

$$-a\,\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}+b\,\frac{U_j-U_{j-1}}{h}=0$$

solution:

$$U_j=\frac{(1+2\,\mathrm{Pe}_h)^j-1}{(1+2\,\mathrm{Pe}_h)^N-1}\qquad \text{monotone?}$$



upwind solution is over diffusive.

$$b\,\frac{U_j - U_{j-1}}{h} = b\,\frac{U_j - \frac{1}{2}U_{j-1} - \frac{1}{2}U_{j-1} + \underbrace{\frac{1}{2}U_{j+1} - \frac{1}{2}U_{j+1}}_{=0}}{h}$$

$$= b\,\underbrace{\frac{U_{j+1} - U_{j-1}}{2h}}_{\text{central FD}} - \frac{bh}{2}\,\underbrace{\frac{U_{j+1} - 2U + U_{j-1}}{h^2}}_{\text{central FD for } \frac{bh}{2} u''}$$

upwind for $-a u'' + b u' = 0$ written as CD

for $-\underbrace{\left(a + \frac{bh}{2}\right)}_{a_h} u'' + b u' = 0$

$$Pe_h = \frac{bh}{2a_h} = \frac{Pe}{1 + Pe} < 1$$

UPWIND $\equiv$ numerical / artificial diffusion / viscosity

Adding viscosity/diffusion $\Rightarrow$ reduction in consistency

Perhaps accuracy can be raised by optimizing the amount of artificial diffusion:

In general, define

$$a_h := a\left(1+\phi(Pe_h)\right)$$

for example:

- upwind : $\phi(t)=t$ $\longrightarrow$ $O(h)$ scheme
- Scharfetter-Gummel : $\phi(t) = t-1+B(2t)$

$$B(t) = \begin{cases} \dfrac{t}{e^t-1} & \text{if } t>0 \\ 1 & \text{if } t=0 \end{cases}$$

Bernoulli function $\longrightarrow$ $O(h^2)$ scheme

<u>analysis</u> of the artificial diffusion scheme

we have $\quad a_h(u,v) = \int_\Omega a_h\, u' v'$, so

by Strang

$$\|u-u_h\|_{H^1} \le \inf_{v_h\in V_h} \left\{ \left(1+\frac{\gamma}{\alpha_{0,h}}\right)\|u-v_h\|_{H^1} + \frac{1}{\alpha_{0h}} \sup_{\substack{w_h\in V_h\\ w_h\neq 0}} \frac{|a(v_h,w_h)-a_h(v_h,w_h)|}{\|w_h\|_{H^1}} \right\}$$

(quarteroni)

$$\le C\,\frac{h^k}{a(1+\phi(Pe))}\|u\|_{H^{k+1}} + \frac{\phi(Pe)}{1+\phi(Pe)}\|u\|_{1,\Omega}$$

for FEM of degree k

for $k=1$, upwind $\mathcal{O}(h)$

SG $\mathcal{O}(h^2)$ (sore for $k>1$)

What about multi-dim problems?

Can we have a strongly consistent method

(inconsistency term also of $\mathcal{O}(k)$)

model problem: $-a\Delta u + b\cdot\nabla u = f$

$u_h \in V_h : \underset{\text{PDE + BC}}{A(u_h, v_h)}$ $= \ell(v_h)$

$+\sum_T \frac{h}{b}\int_T \underbrace{(-a\Delta u + b\cdot\nabla u - f)}_{\text{for strong consistency}}(b\cdot\nabla v)$

$(\alpha(h)\nabla u$ artificial diffusion scheme $\nabla v)$

streamline-diffusion scheme

stabilizing term

$\Rightarrow$ streamline-diffusion method

[Hughes-Brooks 1979]

$$B_h(u_h, v_h) := \sum_T \frac{\delta h_T}{b} \int_T (-a\,\Delta u_h + b \cdot \nabla u_h) \cdot (b \cdot \nabla v_h)$$

find $u_h \in V_h$:

$$A(u_h, v_h) + B_h(u_h, v_h) = \underbrace{\ell(v_h) + \sum_T \int_T \frac{h}{b} f\,(b \cdot \nabla v_h)}_{\int f v_h} \quad \forall v_h \in V_h$$

another interpretation: Petrov-Galerkin scheme (≡ test space ≠ trial space)

$$\sigma_T := \frac{\delta h_T}{b}$$ streamline-diffusion parameter

Analysis of SD - FEM

note :

- H^1 norm is not the right norm as we do not have stabilisation in such norm

- instead : SD - norm :

$$\|v\|_{SD} := \left(a\, |v|_1^2 + \sum_{T\in\mathcal{T}_h} \tau_T \|b\cdot\nabla v\|^2_{L^2(T)} \right)^{1/2}$$

Lemma : (stability) If $0 < \tau_T < \frac{h_T^2}{2a\,\mu_I}$, $\mu_I > 0$ constant (from $\|\Delta v_h\|^2_{0,T} \le \mu_I h_T^{-2} |v|^2_{1,T}$),

then $\mathcal{A}(v_h, v_h) + B_h(v_h, v_h) \geq \frac{1}{2} \|v_h\|^2_{SD}$

Theorem : If $\tau_h = \begin{cases} \delta_0 h_T & \text{if } Pe_h > 1 \text{ (conv-dom)} \\ \delta_1 \frac{h_T^2}{a} & \text{if } Pe_h \le 1 \text{ (diff. dominated)} \end{cases}$

for $\delta_0, \delta_1 > 0$, then

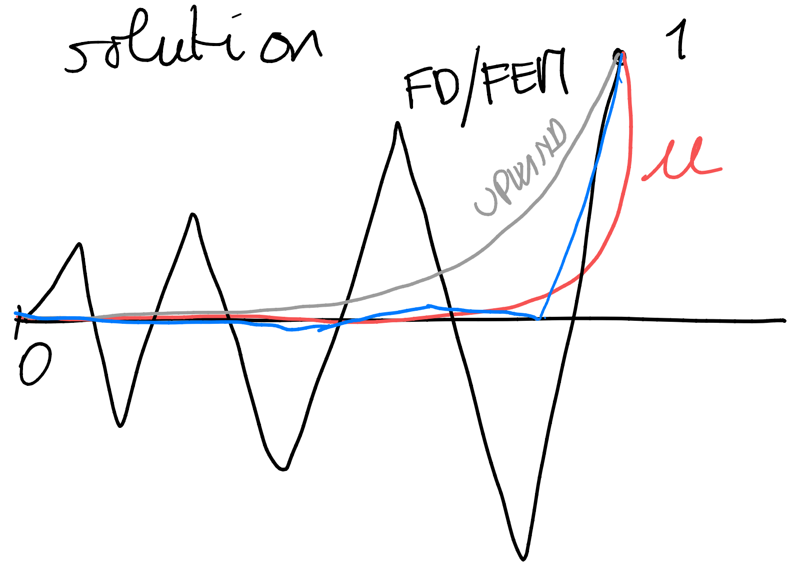
$$\|u-u_h\|_{SD} \le c\,(a^{1/2} + h^{1/2})\, h^k\, |u|_{k+1}$$

$$\left(|u-u_h\|^2_{H^1} + \underline{\sum_T \tau_T \|b\cdot\nabla(u-u_h)\|^2} \right)^{1/2}$$

$\Rightarrow$ diffusion dom case $\quad \mathcal{O}(h^k)$ for H^1 norm

$\mathcal{O}(h^{1/2})$

$\quad\;\;$ convection dom. case $\quad \mathcal{O}(h^k)$ for $\|b\cdot\nabla u\|$ norm typical solution



SD solution is $O(h^k)$ accurate but not monotone.

PARABOLIC PROBLEMS

In general, a parabolic equation in $\mathbb{R}^d$,

$$u_t + \mathcal{L} u = f$$

with $\mathcal{L}$ linear elliptic 2^{nd} order operator

$\left(\frac{\partial}{\partial t} + \mathcal{L} \right.$ = "parabolic op." $\left. \right)$

(Larson-Thomee) example: $\Omega \subset \mathbb{R}^d$ bounded

$$(P_0) \begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}^+ \quad \text{(heat equation)} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \quad \text{(boundary cond)} \\ u(\cdot, 0) = u_0 & \text{in } \Omega \quad \text{(initial cond.)} \end{cases}$$

with $u_0 \in L^2(\Omega)$.

Proposition: $\exists$ a countable set of smooth <u>eigenfunctions</u> $\{\varphi_i\}_{i=1}^{+\infty}$ and <u>eigenvalues</u> $\{\lambda_i\}_{i=1}^{+\infty}$ of the Laplace operator, i.e.

$$\begin{cases} -\Delta \varphi_i = \lambda_i \varphi_i & \text{in } \Omega \\ \varphi_i = 0 & \text{on } \partial\Omega. \end{cases}$$

and $\{\varphi_i\}_{i=1}^{+\infty}$ is a basis of $L^2(\Omega)$.

Moreover, $0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_i \le \cdots$, and $\lim_{i\to\infty} \lambda_i = +\infty$

and $\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j = \lambda_i \delta_{ij}$

→ we it in separation of variables: seek sol. (P_0) in the form:

$$u(\underline{x}, t) = \sum_{i=1}^{+\infty} \hat{u}_i(t)\, \varphi_i(\underline{x})$$

↑ $\hat{u}_i : \mathbb{R}^+ \to \mathbb{R}$ to be determined

Inserting such u in (P_0) gives:

$$\sum_{i=1}^{+\infty} \left(\hat{u}_i'(t) + \lambda_i \hat{u}_i(t)\right) \underbrace{\varphi_i(\underline{x})}_{\text{basis for } L^2} = 0 \qquad \text{in } \Omega \times \mathbb{R}^+$$

$$\Rightarrow \quad \hat{u}_i'(t) + \lambda_i \hat{u}_i(t) = 0 \qquad \forall i, \; t \in \mathbb{R}^+$$

with sol. $\hat{u}_i(t) = \hat{u}_i(0)\, e^{-\lambda_i t}$

const. of int. to be determined using the initial cond:

$$= u_0(x) = u(\underline{x}, 0) = \sum_{i=1}^{+\infty} \hat{u}_i(0)\, \varphi_i(\underline{x})$$

$$\overset{L^2}{=} \sum_{i=1}^{+\infty} u_0^i\, \varphi_i(\underline{x}) \qquad \exists\, u_0^i \quad i = 1, \dots, +\infty$$

$$\Rightarrow u(\underline{x}, t) = \sum_i u_0^i \underbrace{e^{-\lambda_i t}}_{\text{exponential decay in time}} \varphi_i(\underline{x})$$

$$\|u(\cdot, t)\|_{L^2(\Omega)} \le C\, \|u_0\|_{L^2(\Omega)} \qquad \text{(proof Parseval relation)}$$

$$\Rightarrow u(\cdot, t) \in L^2(\Omega)$$

For nonhomogeneous problem: $I = [0, T]$

$$(P)\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times I \\ u = 0 & \text{on } \partial\Omega \times I \\ u(\underline{x}, 0) = u_0 & \text{in } \Omega \end{cases}$$

Denote by $E(t)u_0$ the solution operator of the homog. problem (P_0).

Then

$$u(\cdot,t)\left(=u(t)\right) = E(t)u_0 + \int_0^t E(t-s)f(s)\,ds$$

Duhamel Principle

(proof in LT)

and

$$\|u(t)\|_0 \leq \|u_0\|_0 + \int_0^t \|f(s)\|\,ds$$

stability

($\Rightarrow$ uniqueness + cont. dependence on data follows)

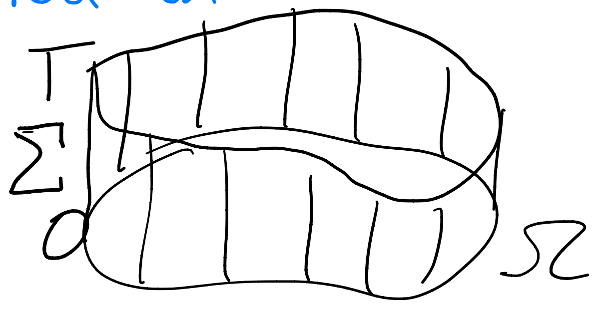
Moreover we have maximum principle:

$u_t - \Delta u \leq 0$ smooth in $\Omega \times (0,T]$, Then

the max. of u is attained in

$\Omega \times \{0\} \cup \Sigma$

$\Sigma = \partial\Omega \times (0,T]$



$$\|u\|_{C(\bar{\Omega}\times\bar{I})} \le \max\left\{\|u\|_{C(\Sigma)}, \|u_0\|_{C(\bar{\Omega})}\right\} + C\,\|f\|_{C(\bar{\Omega}\times\bar{I})}$$

Weak formulation

for
$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega\times I \\ u = 0 & \text{in } \Sigma \\ u(\underline{x},0) = u_0 & \text{in } \Omega \end{cases}$$

$\forall t \quad v \in H_0^1(\Omega)$ test and integrate:

$$\underbrace{\frac{d}{dt}\int_\Omega u\, v}_{\frac{d}{dt}(u,v)} + \underbrace{\int_\Omega \nabla u\cdot\nabla v}_{a(u,v)} = \int_\Omega f\, v \qquad \forall v\in H_0^1(\Omega)$$

$\forall t$

$$\frac{d}{dt}(u,v) + a(u,v) = (f,v) \qquad \forall v\in H_0^1(\Omega)$$

$$u(\cdot,t) = u(t) : I \longrightarrow V$$
$$t \longrightarrow u(t)$$

right setting:

$$u \in L^2(I; H_0^1(\Omega)) \cap C^0(\bar{I}; L^2(\Omega))$$

where,

$u \in L^2(I; H_0^1(\Omega))$ if $u = u(t) \in H_0^1(\Omega)$ and

$$\int_I \|u(t)\|^2_{H_0^1} dt < \infty$$

$u \in C^0(\bar{I}; L^2(\Omega))$ if $\forall t \in \bar{I}$,

$$\lim_{s \to t} \|u(s) - u(t)\|_{L^2(\Omega)} = 0$$

Well-posedness

Consider the problem: find $u \in L^2(I; H_0^1) \cap C^0(\bar{I}; L^2)$:

$$(P_W) \begin{cases} \frac{d}{dt}(u(t), v) + a(u(t), v) = F(t, v) & \forall v \in V \\ u(0) = u_0 \in L^2(\Omega) \end{cases}$$

with $a(u, v) = (\alpha \nabla u, \nabla v) - (\underline{b} u, \nabla v) + (c u, v)$

is well-posed? yes if:

- a is continuous
- if a is coercive

Actually, the coercivity requirement can be relaxed somehow. It is enough to require the validity of

e.g. H_0^1

Def: $\mathcal{A}$ is weakly coercive in V if

$\exists \alpha_0 > 0, \ \lambda \geq 0 \ : \ \forall v \in V$

$$\mathcal{A}(v,v) + \lambda \|v\|^2_{L^2(\Omega)} \geq \alpha_0 \|v\|_1^2$$

(Gårding inequality)

(see Quarteroni)

otherwise, in case of Gårding look here $(V = H_0^1)$

Theorem: if $\mathcal{A}$ is cont. and coercive in $V \times V$, $f \in L^2(I; L^2(\Omega))$, $u_0 \in L^2(\Omega)$. Then $\exists!\ u \in L^2(I; V) \cap C^0(\bar{I}; L^2)$ sol. of (P_W) and

$$u_t \in L^2(I; V')$$

$$\underbrace{\max_{t \in I} \|u(t)\|_0^2}_{\text{norm of}} + \underbrace{\int_I \|u(t)\|_V^2}_{\text{norm of}} \leq \|u_0\|^2_{L^2(\Omega)} + \frac{1}{\alpha_0} \int_I \|f(t)\|^2$$

"energy estimate"

that is, the problem is well-posed ?