

NSPDE/ANA 2022 - Lecture 7

Last lecture:

- Other boundary conditions
- General elliptic problems
- Symmetric case: Dirichlet principle
- General case: Lax Milgram lemma
- The method of Galerkin

Today:

- The Finite Element Method (FEM) in 1D
- Error analysis
- Conditioning of the FE matrix
- FEM in more dimensions

References:

- Quarteroni
- Larsson & Thomée

FEM

1. FEM two points bvp

$$\begin{cases} \mathcal{L}u = -(\alpha u')' + cu = f & \Omega = (0, 1) \\ u(0) = 0 = u(1) \end{cases}$$

$f \in L^2(\Omega)$, $\alpha(x) > 0$, $c \geq 0$

weak prob: Find $u \in H_0^1(\Omega)$:

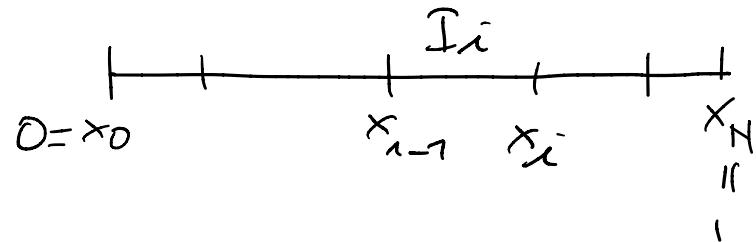
$$A(u, v) := \int_{\Omega} \alpha u' v' + \int_{\Omega} c u v = \int_{\Omega} f v = l(v) \quad \forall v \in H_0^1(\Omega)$$

FE SPACES.

- Fix partition of Ω , $I_i = (x_{i-1}, x_i)$

$$x_i = x_{i-1} + h_i$$

$$x_0 = 0$$



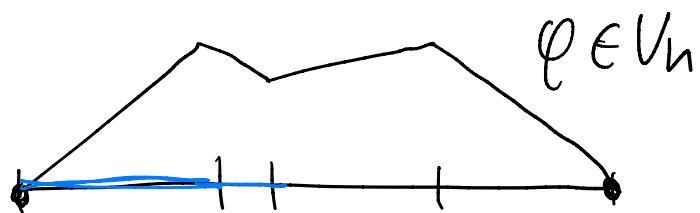
$$h = \max_i h_i$$

- Fix finite dim. space

$$V_h = V_h^k = \left\{ v \in C_0(\bar{\Omega}) : v|_{I_i} \in P^k(I_i) : i=1, \dots, N \right\}$$

$$h \in \mathbb{N}$$

example : $k=1$



Notice $V_h \subset V$.

FE method (\mathcal{Z}^0 -conforming) :

$$\text{Find } u_h \in V_h : \mathcal{A}(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h$$

FE in algebraic form

fix a basis $\{q_j\}_{j=1}^m$ of V_h , then

$$u_h = \sum_{j=1}^m v_j q_j$$

Hence, testing discrete prb. with test fun.,

Find $\{v_j\}_{j=1}^m$ such that

$$\sum_{j=1}^m v_j \left(\underbrace{\int_{\Omega} \varphi_j' \varphi_i'}_{S_{ij}} + \underbrace{\int_{\Omega} \varphi_j \varphi_i}_{M_{ij}} \right) = l(\varphi_i) \quad \forall i = 1, \dots, m$$

with:

$$A := S + M$$

$$F = (l(\varphi_i))$$

$$\Leftrightarrow \boxed{AU = F}$$

The "assembly" of A and F requires the eval. of integrals. Practical + efficient assembly is achieved by selecting "good" basis (local support):

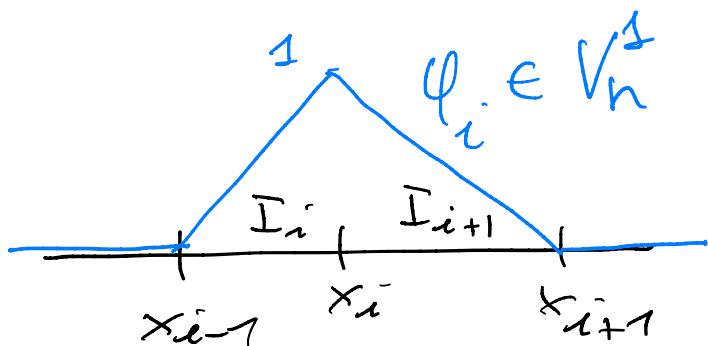
① $k=1$ (linear FEM)

φ_i select :

$$i = 1, \dots, n-1$$

(internal nodes)

gives basis



$$\varphi_i(x) = \begin{cases} 0 & I_j \quad j < i \\ \frac{x - x_{i-1}}{h_i} & I_i \\ \frac{x_{i+1} - x}{h_{i+1}} & I_{i+1} \\ 0 & I_j \quad j > i \end{cases}$$

then

$$\text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) = \emptyset \quad |i-j| > 1$$

"hat function"

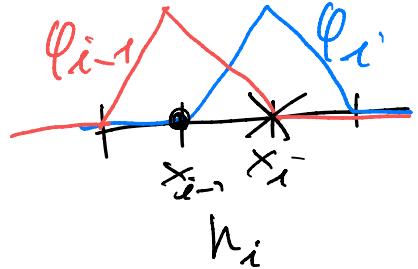
example: $\alpha = 1$

$$\int_0^1 \varphi_i'(x) \varphi_{i+1}'(x) dx$$

$$= \int_{x_{i-1}}^{x_i} \varphi_i'(x) \varphi_{i+1}'(x) dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right) \left(-\frac{1}{h_i}\right) dx = -\frac{1}{h_i^2} h_i = -\frac{1}{h_i}$$

$$= \int_0^1 \varphi_i'(x) \varphi_{i+1}'(x) dx = \frac{2}{h_i}$$



$$(h_i = h) \quad S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

\equiv FD matrix apart from scaling $1/h$ instead of $1/h^2$

Right-hand side is
discretised differently:

- FD $F_i = f(x_i)$

- FEM $F_i = \int_0^i f(x) \varphi_i(x) dx$
 $= \int_{x_{i-1}}^{x_{i+1}} f(x) \varphi_i(x) dx \sim h$

Question: how we compute $\int f \varphi$
 (similarly $\int (\alpha \varphi_i' \varphi_j' + b \varphi_i \varphi_j)$)

Answer: use appropriate quadrature formulas

Exercise: If $\int f \varphi_i$ approximated by trapezoidal rule, then also rhs of FEM = rhs of FD

fact: In 1D for $\alpha=1$, ($c=0$), if $\int f \varphi_i$ computed exactly, then
 $u_h(x_i) = u(x_i)$!
 (nodally exact)

Assembly: (of A, S, F)

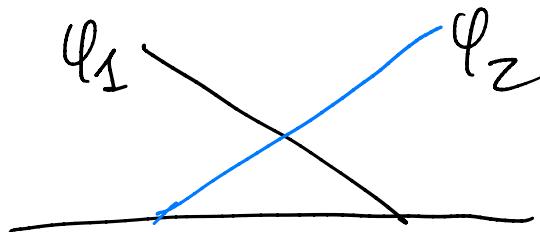
Assembly of system is by loop over the elements (intervals)

for $i = 1 : N-1$
 $A_{I,i}$ local matrix $\begin{pmatrix} \int_{I_i} (\varphi_{i-1}')^2 & \int_{I_{i-1}} \varphi_{i-1}' \varphi_i' \\ \int_I \varphi_i' \varphi_{i-1}' & \int_I (\varphi_i')^2 \end{pmatrix}$

distribute terms in A right places

For the computation of A_{I_i} use quadrature rules on ref. elemnt

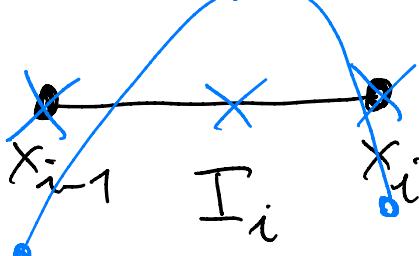
$$I = (0, 1)$$



use affine map from $I \rightarrow I_i$

$$\textcircled{2} \quad h > 1$$

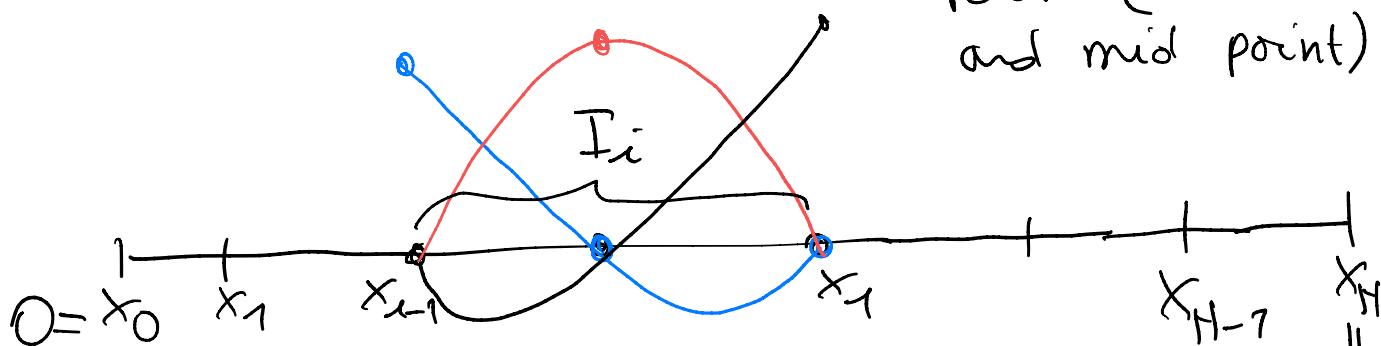
$$\bullet k = 2$$



fix "local" basis
of 3 quadratic
functions,

$$\varphi_l(x_m) = \sum_{l,m}$$

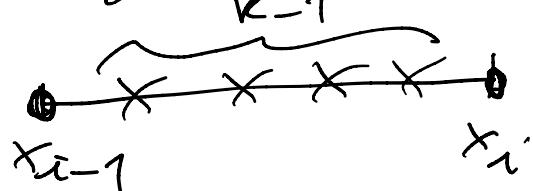
where $l = 1, 2, 3$
 x_m are the three
points (end points
and mid point)



$$\dim V_h^2 = (H-1) + N$$

$$\bullet k \text{ general}$$

$$\dim V_h^k = (H-1) + N(k-1)$$



Analysis of FEM:

use abstract analysis results: cont const

$$\textcircled{1} \text{ Lemma of Léa: } \|u - u_h\|_V \leq \frac{\kappa}{d_0} \inf_{v_h \in V_h} \|u - v_h\|_V$$

coerc. const

$$V = H_0^1(\Omega)$$

$$\|v\|_V^2 = \int_0^1 (v')^2 dx \quad \left(\text{or } + \int_0^1 v^2 dx \right)$$

$$+ \|v\|_{L^2}^2$$

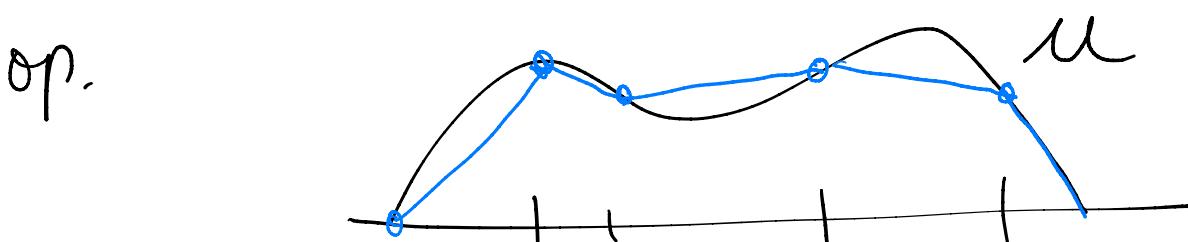
$$\|v\|_S^2 = \int_0^1 a(v')^2 dx + \int_0^1 c v^2 dx$$

$$\textcircled{2} \quad \inf_{v_h \in V_h} \|u - v_h\|_V^2 \leq \|u - I_h u\|_V^2$$

for some "special" choice $I_h u \in V_h$

$$= \sum_{i=1}^{N-1} \|u - I_h u\|_{I_i}^2$$

For instance, I_h can be the interpolation op.



assuming $u \in C^0(\Omega)$.

Def (FEM interpolant): $I_h : \mathbb{Z}^0 \rightarrow V_h$

as: $\forall v \in \mathbb{Z}^0$; $I_h v \in \mathbb{Z}^0 \cap V_h$:

$$I_h v|_{I_i} \in P^k(I_i)$$

Lagrange interpolant

Theorem (LT): $\left(\|v - I_h v\|_{L^2(I_i)} \leq C h_i^k |v|_{H^{k+1}(I_i)} \right)$

(see Helton)

$$\|(v - I_h v)'\|_{L^2(I_i)} \leq C h_i^k |v|_{H^{k+1}(I_i)}$$

ex: $k=1$

$$\begin{array}{ll} L^2 & O(h_i^2) \\ H^1 & O(h_i) \end{array}$$

from which "global" interp. error bounds also follow

Theorem (A priori error bound for FEM):

If $u \in H^{k+1}(\Omega)$, then $\left(\sum_{i=1}^{N-1} h_i \|u\|_{H^{k+1}(I_i)}^2 \right)^{1/2}$

$$\|(u - u_h)'\|_{L^2(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

$$\|u - u_h\|_{L^2(\Omega)} \leq C h^{k+1} |u|_{H^{k+1}(\Omega)}$$

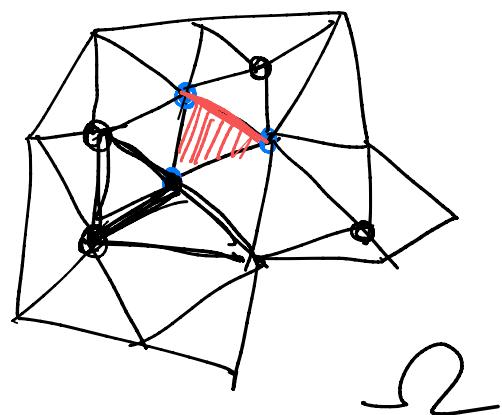
Moreover, if $u \in H^s(\Omega)$ $1 \leq s \leq k+1$,

$$\| (u - u_h)' \|_{L^2(\Omega)} \leq C h^5 \| u \|_{H^5(\Omega)}$$

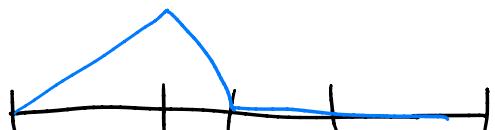
FEM in more dimensions

FE spaces (\mathbb{R}^2 for example)

Ω polygon



$d=1$



direct
functions defined
by value at 3 points?

- Let \mathcal{T}_h a triangulation of Ω

$\mathcal{T}_h = \{T\}$, T triangle

$\bar{\Omega} = \bigcup T$ + \mathcal{T}_h is "regular"

$\curvearrowright T_i \cap \bar{T}_j = \begin{cases} \emptyset & \text{1 vertex} \\ \text{1 edge} & \end{cases}$

- $V_h^k = \{v \in \mathcal{C}^0(\bar{\Omega}) : v|_T \in P^k(T), \forall T \in \mathcal{T}_h\}$

Proposition : $\nabla \in H^1(\Sigma) \Leftrightarrow \left\{ \begin{array}{l} \nabla_{T_i} \in H^1(T) \quad \forall T \in \mathcal{T}_n \\ \text{if } F = T_i \cap T_j \text{ then} \\ (\nabla|_{T_1})_F = (\nabla|_{T_2})_F \end{array} \right.$

Given Σ and \mathcal{T}_n
regular triangulation