

Graph theory

Introduction and basic algorithms

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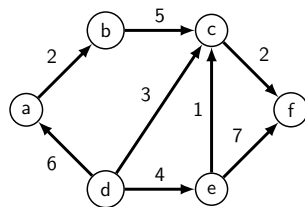
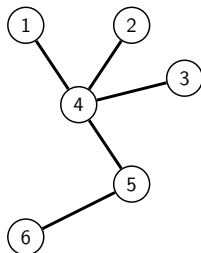
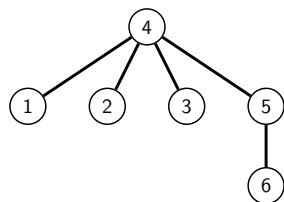
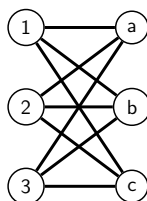
Table of contents

- 1 Introduction
 - Examples
 - Definitions
 - Kinds of graph
 - Graph representation
- 2 Graph traversals
 - BFS
 - DFS
- 3 Problems
 - MST
 - Connected components
 - Single source shortest path
 - Conclusion

Table of contents

- 1 Introduction
 - Examples
 - Definitions
 - Kinds of graph
 - Graph representation
- 2 Graph traversals
 - BFS
 - DFS
- 3 Problems
 - MST
 - Connected components
 - Single source shortest path
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Examples



Why graph theory?

- A huge number of problems can be converted in a graph problem
- Graph theory is fun $\backslash(^-^)/$

We will study problems in abstract form. Their application can be found in the most diverse areas.

Typical graphs problems

- Given the description of a city, find the shortest path between locations A and B or determine that it is impossible to reach B from A .
- On an electronic board, choose a set of connections whose sum in length is minimal and which allows to pass through all points of interest.
- Given dependency relationships, find a suitable order to install some packages (or attend some classes) or determine that it is impossible.

Definition: Graph

Graph $G = (V, E)$

A graph is defined as a pair of sets:

- V is a set of **vertexes/nodes**
- E is a set of **edges**

Definition: Vertexes and Edges

Vertex

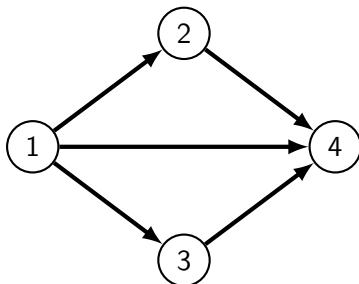
- Vertexes are also called *nodes*
- Vertexes are denoted with labels

Edge

- Each edge is defined by a pair of vertexes
- An edge *connects* the vertexes that define it
- In some cases, the vertexes can be the same

Example

- $G = (V, E)$
- $V = \{1, 2, 3, 4\}$
- $E = \{(1, 2), (1, 3), (1, 4), (2, 4), (3, 4)\}$



Definition: Paths and Cycles

Path

A path of length n in a graph $G = (V, E)$ is a sequence $v_0, \dots, v_n \in V$ such that $(v_{i-1}, v_i) \in E \ \forall \ 1 \leq i \leq r$.

A path is *simple* if all v_i differ.

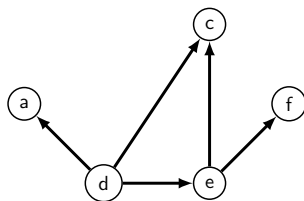
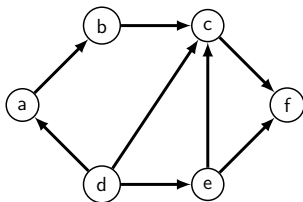
Cycle

A cycle is a path in which the first and the last node are the same.

Definition: Subgraph

Subgraph

A graph $G' = (V', E')$ is subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.



Graph size

Definitions

- $n = |V|$: number of nodes
- $m = |E|$: number of edges

Relationships between n and m

- In an undirected graph, $m \leq \frac{n(n-1)}{2} = O(n^2)$
- In a directed graph, $m \leq n^2 - n = O(n^2)$

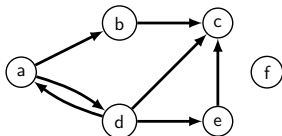
Directed and Undirected graphs

Directed graph $G = (V, E)$

- E is a set of *ordered* pairs (u, v) of nodes

$V = \{ a, b, c, d, e, f \}$

$E = \{ (a, b), (a, d), (b, c), (d, a), (d, c), (d, e), (e, c) \}$

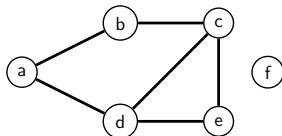


Undirected graph $G = (V, E)$

- E is a set of *unordered* pairs $[u, v]$ of nodes

$V = \{ a, b, c, d, e, f \}$

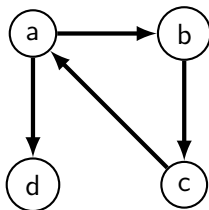
$E = \{ [a, b], [a, d], [b, c], [c, d], [c, e], [e, d] \}$



Cyclic and Acyclic graphs

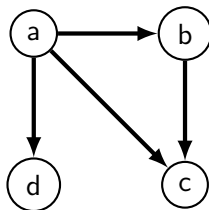
Cyclic graph

- Contains cycles



Acyclic graph

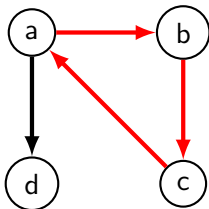
- Contains no cycles



Cyclic and Acyclic graphs

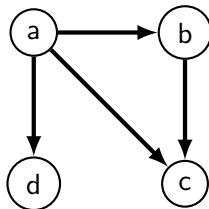
Cyclic graph

- Contains cycles



Acyclic graph

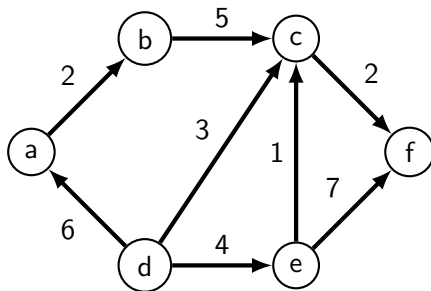
- Contains no cycles



Weighted graphs

Weighted graph

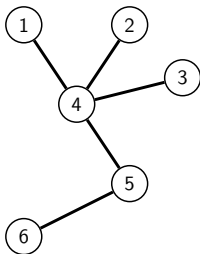
- Each edge is assigned a *weight*
- Weight typically shows cost of traversing



Trees

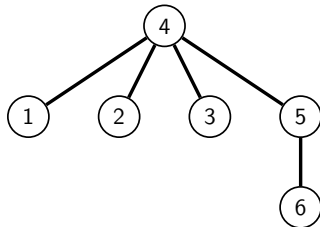
Tree

- *Connected* graph with $m = n - 1$



Rooted tree

- *Connected* graph with $m = n - 1$ in which some special node is designed as root



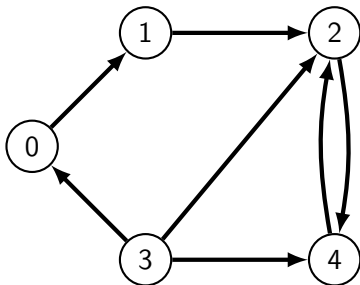
Representations

Two possible "classic" implementations

- Adjacency matrix
- Adjacency list

Adjacency matrix

$$m_{uv} = \begin{cases} 1, & \text{if } (u, v) \in E \\ 0, & \text{if } (u, v) \notin E \end{cases}$$



	0	1	2	3	4
0	0	1	0	0	0
1	0	0	1	0	0
2	0	0	0	0	1
3	1	0	1	0	1
4	0	0	1	0	0

Adjacency list

$$G.adj(u) = \{v \mid (u, v) \in E\}$$

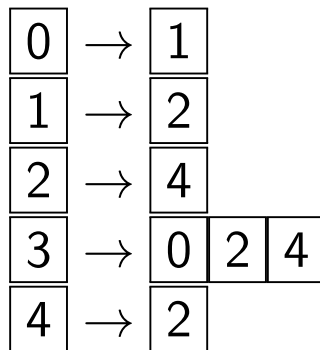
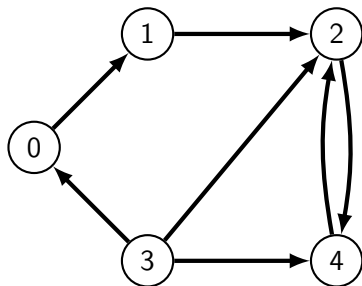


Table of contents

- 1 Introduction
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Breadth-first search

Problem definition

Given a graph $G = (V, E)$ and a vertex $r \in V$ (root), visit exactly once all the vertexes of the graph that can be reached from r .

Breadth-first search (BFS)

Traverse the graph by visiting the nodes by levels: first nodes at distance 1 from the source, then distance 2, etc.

- Application: shortest distances

Breadth-first search

```
def bfs(G, r):  
    Q = deque()  
    Q.append(r)  
    visited = {r}  
    while len(Q) > 0:  
        u = Q.popleft()  
        for v in G.adj(u):  
            if not v in visited:  
                visited.add(v)  
                Q.append(v)
```

Depth-first search

Problem definition

Given a graph $G = (V, E)$ and a vertex $r \in V$ (root), visit exactly once all the vertexes of the graph that can be reached from r .

Depth-first search (DFS)

Traverse the graph by visiting all the nodes that can be reached by a node, and all the nodes that can be reached by those nodes, etc.

- Application: topological sort
- Application: cycle detection
- Application: connected components
- Application: strongly connected components

Depth-first search: iterative

```
def dfs(G, r):  
    stack = [r]  
    visited = {r}  
    while len(stack) > 0:  
        u = stack.pop()  
        for v in G.adj(u):  
            if not v in visited:  
                visited.add(v)  
                stack.append(v)
```

Depth-first search: recursive

```
def dfs(G, u, visited):  
    visited.add(u)  
    for v in G.adj(u):  
        if not v in visited:  
            dfs(G, v, visited)
```

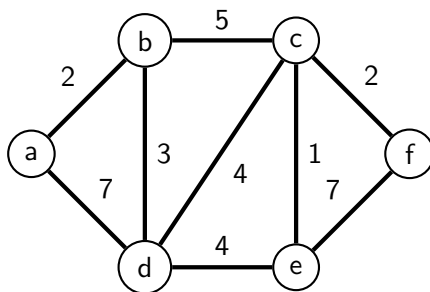
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 - DFS
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Minimum Spanning Tree (MST)

Problem definition

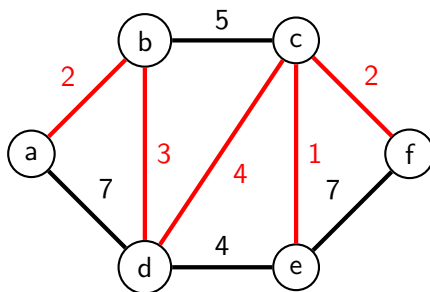
Given a connected weighted graph $G = (V, E)$, find $S \subseteq E$ such that (V, S) is still connected and S has the minimum total edge weight.



Minimum Spanning Tree (MST)

Problem definition

Given a connected weighted graph $G = (V, E)$, find $S \subseteq E$ such that (V, S) is still connected and S has the minimum total edge weight.



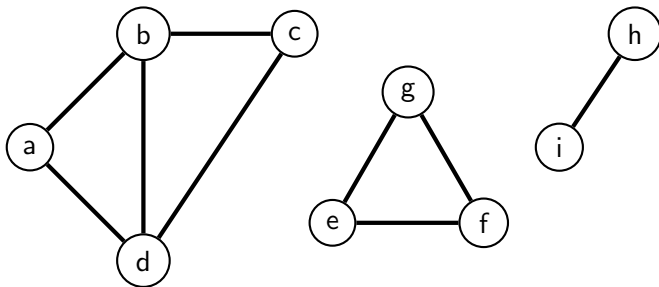
Kruskal algorithm for MST

```
def mst(E):  
    E.sort() # sort for increasing weight  
    ans = []  
    for edge in E:  
        if not Cycle(ans, edge):  
            ans.append(edge)  
  
    return ans
```

Connected Components

Problem definition

Given an undirected graph $G = (V, E)$, find the number of connected components.



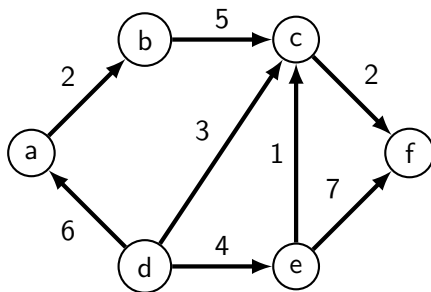
Connected Components

```
def connectedComponents(G):  
    visited = {}  
    count = 0  
    for u in G.V:  
        if not u in visited:  
            count += 1  
            dfs(G, u, visited)  
  
    return count
```


Single source shortest path

Problem definition

Given a weighted graph $G = (V, E)$ and a source node s , find the distance of every node from s . If there is not a path from s to a node v , $\text{dist}[v]$ should be $+\infty$.



Single source shortest path

```
def shortestPath(G, s):  
    Q = deque()  
    dist = [math.inf] * G.n  
    dist[s] = 0  
    Q.append(s)  
    while len(Q) > 0:  
        u = Q.popleft()  
        for (v,w) in G.adj(u):  
            if dist[v] > dist[u] + w:  
                dist[v] = dist[u] + w  
                Q.append(v)  
  
    return dist
```

More problems

- Given an undirected graph, find the minimum number of edges to add if you want to make it connected.
- Given a weighted graph and three nodes a, b, c , find a minimal length path which goes from a to c passing through b .
- Given a directed graph, find a permutation of V such that $\forall (u, v) \in E$ u comes before v , or tell it is impossible.
- Given a set of words S , find a minimal length string which contains each element of S as a substring.
- Given some dominoes, tell whether you can put them in a line such that the domino condition is satisfied.

Any questions?

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- 2 Graph traversals
 - BFS
 - DFS
- 3 Problems
 - MST
 - Connected components
 - Single source shortest path
 - Conclusion