Project 2
Numerical Methods for Continuous Systems

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Contents

1 Problem Statement and Strong Formulation

Consider the "extended" Stokes Equation

$$\begin{aligned} cu - \mu\Delta u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= g && \text{in } \Omega, \\ u(\mathbf{x}) &= u_D(\mathbf{x}) && \text{for } \mathbf{x} \in \partial\Omega \end{aligned} \tag{D}$$

for $\Omega = [0,1] \times [0,1]$, where $u(x,y) = \begin{bmatrix} u_1(x,y) & u_2(x,y) \end{bmatrix}^T$ is the fluid velocity vector, $p(x,y)$ is the pressure, μ is the dynamic viscosity of the fluid, c is a "mass" constant and u_D is for the Dirichlet boundary conditions.

1.1 Unit of Measure

The coordinates x have the measure of a length [L], the velocity u of length over time [L]/[T] and the dynamic viscosity μ of Pascal times a time (Pa [T]) while the pressure p of Pascal Pa.

2 Weak Formulation

The Weak Formulation of (D) is: find $u, p \in (H_0^1(\Omega))^2 \times L_0^2(\Omega) =: \mathcal{V} \times \mathcal{Q}$ such that boundary conditions hold and

$$\begin{aligned} c\int_\Omega u \cdot v \, d\Omega - \int_\Omega \mu\Delta u \cdot v \, d\Omega + \int_\Omega \nabla p \cdot v \, d\Omega &= \int_\Omega f \cdot v \, d\Omega && \text{for any } v \in \mathcal{V} \\ \int_\Omega q \operatorname{div} u \, d\Omega &= \int_\Omega gq \, d\Omega && \text{for any } q \in \mathcal{Q}. \end{aligned} \tag{W}$$

By Green's Lemma and writing $\nabla p \cdot v = \operatorname{div}(vp) - p \operatorname{div} v$, we obtain

$$\begin{aligned} c\int_\Omega u \cdot v \, d\Omega + \int_\Omega \nabla u : \nabla v \, d\Omega - \int_\Omega p \operatorname{div} v \, d\Omega &= \int_\Omega f \cdot v \, d\Omega && \text{for any } v \in \mathcal{V} \\ -\int_\Omega q \operatorname{div} u \, d\Omega &= -\int_\Omega gq \, d\Omega && \text{for any } q \in \mathcal{Q}, \end{aligned} \tag{W}$$

as the boundary terms vanishes because $v \in (H_0^1(\Omega))^2$. Finally, denoting

$$\begin{aligned} m(u,v) &= c\int_\Omega u \cdot v \, d\Omega & a(u,v) &= \int_\Omega \nabla u : \nabla v \, d\Omega & F(v) &= \int_\Omega f \cdot v \, d\Omega \\ b(v,q) &= -\int_\Omega q \operatorname{div} u \, d\Omega & G(q) &= -\int_\Omega gq \, d\Omega, \end{aligned}$$

the system of equations (W) reads: find $u, p \in \mathcal{V} \times \mathcal{Q}$ such that

$$\begin{aligned} m(u,v) + a(u,v) + b(v,p) &= F(v) && \text{for any } v \in \mathcal{V} \\ b(u,q) &= G(q) && \text{for any } q \in \mathcal{Q}. \end{aligned} \tag{W}$$

3 Weak Galerkin Formulation

Now, considering the triangulation $\mathcal{T}_h$ of Ω with nodes $\mathcal{N}_h = \{\mathbf{x}_1, \ldots, \mathbf{x}_{n_{\text{nodes}}}\}$ and elements $\mathcal{E}_h$, we can approximate the function space $H_0^1(\Omega)$ with

$$\widetilde{\mathcal{V}}_h = \operatorname{span}\{\phi_1, \ldots, \phi_{n_{\text{nodes}}}\},$$

where the ϕ_i are the chosen basis function. Then, we can approximate $\mathcal{V}$ with $e_1\widetilde{\mathcal{V}}_h \oplus e_2\widetilde{\mathcal{V}}_h$, where $e_1 = [1,0]^T$ and $e_2 = [0,1]^T$, so that

$$\mathcal{V}_h = \operatorname{span}\{\phi_1^{(1)}, \ldots, \phi_{n_{\text{nodes}}}^{(1)}, \phi_1^{(2)}, \ldots, \phi_{n_{\text{nodes}}}^{(2)}\},$$

where $\phi_i^{(j)} = e_j\phi_i$. Similarly, approximate $\mathcal{Q}$ with

$$\mathcal{Q}_h = \operatorname{span}\{\psi_1, \dots, \psi_{n_{\text{nodes}}}.\}$$

Then, as $\mathcal{V}_h \times \mathcal{Q}_h$ is a finite dimensional vector space, is enough to verify Equation (W) for the basis functions. This yields the Weak Galerkin Formulation: find $u_h, p_h \in \mathcal{V}_h \times \mathcal{Q}_h$ such that boundary conditions hold and

$$\begin{aligned} m(u_h, \phi_i^{(1)}) + a(u_h, \phi_i^{(1)}) + b(\phi_i^{(1)}, p_h) &= F(\phi_i^{(1)}) && \text{for } i = 1, \dots, n_{\text{nodes}} \\ m(u_h, \phi_i^{(2)}) + a(u_h, \phi_i^{(2)}) + b(\phi_i^{(2)}, p_h) &= F(\phi_i^{(2)}) && \text{for } i = 1, \dots, n_{\text{nodes}} \\ b(u_h, \psi_i) &= G(\psi_i) && \text{for } i = 1, \dots, n_{\text{nodes}}. \end{aligned} \tag{W_h}$$

Writing the approximate solution $u_h = \sum_{j=1}^{n_{\text{nodes}}} u_j^{(1)}\phi_j^{(1)} + \sum_{k=0}^{n_{\text{nodes}}} u_k^{(2)}\phi_k^{(2)}$ and $p_h = \sum_{l=0}^{n_{\text{nodes}}} p_l\psi_l$, the above can be stated as

$$\begin{aligned} \sum_j u_j^{(1)} m(\phi_j^{(1)}, \phi_i^{(1)}) + \sum_j u_j^{(1)} a(\phi_j^{(1)}, \phi_i^{(1)}) + \sum_l p_l b(\phi_i^{(1)}, \psi_l) &= F(\phi_i^{(1)}) && \text{for } i = 1, \dots, n_{\text{nodes}} \\ \sum_k u_k^{(2)} m(\phi_k^{(2)}, \phi_i^{(2)}) + \sum_k u_k^{(2)} a(\phi_k^{(2)}, \phi_i^{(1)}) + \sum_l p_l b(\phi_i^{(2)}, \psi_l) &= F(\phi_i^{(2)}) && \text{for } i = 1, \dots, n_{\text{nodes}} \\ \sum_j u_j^{(1)} b(\phi_j^{(1)}, \psi_i) + \sum_k u_k^{(2)} b(\phi_k^{(2)}, \psi_i) &= G(\psi_i) && \text{for } i = 1, \dots, n_{\text{nodes}}, \end{aligned} \tag{W_h}$$

as $m(\phi_i^{(1)}, \phi_j^{(2)}) = 0 = a(\phi_i^{(1)}, \phi_j^{(2)})$. In matrix form, by stacking the components of the solution as $\begin{bmatrix}\mathbf{u}^{(1)} & \mathbf{u}^{(2)} & \mathbf{p}\end{bmatrix}^T$, it reads

$$\begin{bmatrix} K & 0 & B_1^T \\ 0 & K & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{g} \end{bmatrix}, \tag{$\star$}$$

where $(\mathbf{f}_1)_i = F(\phi_i^{(1)})$, $(\mathbf{f}_2)_i = F(\phi_i^{(2)})$, $\mathbf{g}_i = G(\psi_i)$, $K_{ij} = M_{ij} + A_{ij}$, with $M_{ij} = m(\phi_j^{(1)}, \phi_i^{(1)}) = m(\phi_j^{(2)}, \phi_i^{(2)})$ and $A_{ij} = a(\phi_j^{(1)}, \phi_i^{(1)}) = a(\phi_j^{(2)}, \phi_i^{(2)})$, $(B_1)_{ij} = b(\phi_j^{(1)}, \psi_i)$ and $(B_2)_{ij} = b(\phi_j^{(2)}, \psi_i)$.

While the mass matrix M and the stiffness matrix A are as the velocity u was one dimensional, B_1 and B_2 are

$$(B_1)_{ij} = b(\phi_j^{(1)}, \psi_i) = -\int_\Omega \psi_i \operatorname{div} \phi_j^{(1)}\, d\Omega = -\int_\Omega \psi_i \operatorname{div}(e_1\phi_j)\, d\Omega = -\int_\Omega \psi_i \partial_x \phi_j\, d\Omega,$$

and

$$(B_2)_{ij} = b(\phi_j^{(2)}, \psi_i) = -\int_\Omega \psi_i \operatorname{div} \phi_j^{(2)}\, d\Omega = -\int_\Omega \psi_i \operatorname{div}(e_2\phi_j)\, d\Omega = -\int_\Omega \psi_i \partial_y \phi_j\, d\Omega.$$

4 Finite Elements and Assembly

4.1 P1 / P1

In $\mathcal{P}_1/\mathcal{P}_1$ we choose linear Lagrangian polynomial to discretize both H_0^1, where each component of the velocity lives, and L_0^2 for the pressure. Inside each element $e \in \mathcal{E}$ with vertices $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$ only the corresponding basis function ϕ_i, ϕ_j, ϕ_k are non zero. Their local expression is

$$\phi_l^{(e)}(x, y) = \frac{a_l + b_l x + c_l y}{2\Delta^{(e)}} \quad \text{for } l \in \{i, j, k\},$$

where $\Delta^{(e)}$ is the surface measure of the element,

$$\Delta^{(e)} = \frac{1}{2}\det\left(\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}\right),$$

and the coefficients $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are computed so that $\phi_l(\mathbf{x_m}) = \delta_{lm}$, i.e.,

$$
\begin{aligned}
a_i &= x_j y_k - x_k y_j & \qquad b_i &= y_j - y_k & \qquad c_i &= x_k - x_j \\
a_j &= x_k y_i - x_i y_k & b_j &= y_k - y_i & c_j &= x_i - x_k \\
a_k &= x_i y_j - x_j y_i & b_k &= y_i - y_j & c_k &= x_j - x_i.
\end{aligned}
$$

and then,

$$\nabla \phi_i^{(e)} = \frac{1}{2\Delta^{(e)}} \begin{bmatrix} b_i \\ c_i \end{bmatrix}$$

is constant.

Mass Matrix The local mass matrix $M^{(e)}$ for a 2d triangular element with linear Lagrangian basis functions is:

$$M^{(e)} = \frac{\Delta^{(e)}}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Stiffness Matrix: The local stiffness matrix is

$$A_{ij}^{(e)} = \mu \int_e \frac{1}{2\Delta^{(e)}} (b_i, c_i)^T \cdot \frac{1}{2\Delta^{(e)}} (b_j, c_j)^T \, d\Omega = \mu \frac{\Delta^{(e)}}{4\Delta^{(e)2}} (b_i b_j + c_i c_j) = \frac{\mu}{4\Delta^{(e)}} (b_i b_j + c_i c_j).$$

Compactly, if we collect $\mathbf{b} = \begin{bmatrix} b_i & b_j & b_k \end{bmatrix}$ and, likewise $\mathbf{c} = \begin{bmatrix} c_i & c_j & c_k \end{bmatrix}$, we can write

$$A^{(e)} = \frac{\mu}{4\Delta^{(e)}} (\mathbf{b}^T \mathbf{b} + \mathbf{c}^T \mathbf{c}).$$

B Matrices: Locally on the element $e \in \mathcal{E}$, the coefficients of B_1 are

$$(B_1)_{ij}^{(e)} = -\int_e \psi_i^{(e)} \partial_x \phi_j^{(e)} \, d\Omega = -\frac{1}{2\Delta^{(e)}} b_j \int_e \psi_i^{(e)} \, d\Omega = -\frac{1}{2\Delta^{(e)}} \frac{\Delta^{(e)}}{3} b_j = -\frac{b_j}{6}$$

and is independent of the row index i. Compactly,

$$B_1^{(e)} = -\frac{1}{6} \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \\ \mathbf{b} \end{bmatrix} \quad \text{and, similarly,} \quad B_2^{(e)} = -\frac{1}{6} \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \\ \mathbf{c} \end{bmatrix}.$$

Stabilization for P1/P1 space Since the P1/P1 space is not inf-sup stable, we need to add a stabilization term to have a well defined unique solution of the System ($\star$). The stabilized Stokes GLS formulation leads to the linear system

$$\begin{bmatrix} K & 0 & B_1^T \\ 0 & K & B_2^T \\ B_1 & B_2 & -\tau C \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{g} - \tau \, \mathbf{g}^{\text{stab}} \end{bmatrix},$$

where all the terms are as above and, additionaly, τ is an hyperparameter to tune the amount of additional stabililization and

$$\mathbf{g}_j^{\text{stab}} = \sum_{e \in \mathcal{E}_h} \int_e f \cdot \nabla \phi_j \, d\Omega \quad \text{and} \quad C_{ij} = \sum_{e \in \mathcal{E}_h} \int_e \nabla \phi_i \cdot \nabla \phi_j \, d\Omega,$$

where the ϕ_j are meant as basis function for the pressure space $\mathcal{Q}_h$. This term stems from adding in the Equation (W) the term $-\tau$ times a bilinear form

$$c((u_h, p_h), (v, q)) = \sum_{e \in \mathcal{E}_h} \int_e (-\mu \Delta u_h + \nabla p_h - f) \cdot (-\mu \Delta v + \nabla q) \, d\Omega$$

that is zero computed in the exact solution. As u_h and v in $\mathcal{V}_h$ are locally linear, the laplacians are null and the form depends actually only on p_h and q:

$$c((u_h, p_h), (v, q)) = \sum_{e \in \mathcal{E}_h} \int_e (\nabla p_h - f) \cdot \nabla q \, d\Omega = \sum_{e \in \mathcal{E}_h} \int_e \nabla p_h \cdot \nabla q \, d\Omega - \sum_{e \in \mathcal{E}_h} \int_e f \cdot \nabla q \, d\Omega = c(p_h, q).$$

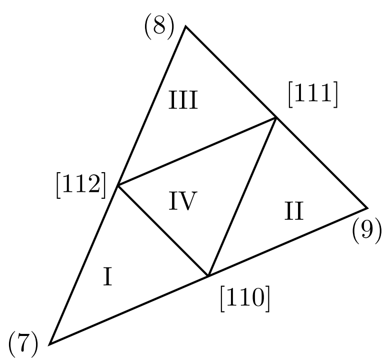
The elementwise sum makes it possible to tune τ specifically for each element. We prefer not to as the mesh provided are sufficiently regular. In this case the matrix C is exactly equal to the stiffness matrix A. Denoting $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$ the nodes of the element e, we compute the integral for the $\mathbf{g}^{\text{stab}}$ with the degree one quadrature rule

$$\int_e f \, d\Omega = \frac{\Delta^{(e)}}{3} \left(f(\mathbf{x}_i) + f(\mathbf{x}_j) + f(\mathbf{x}_k) \right),$$

that is of order $\mathcal{O}(h^2)$, enough for the desired order of convergence for $\mathcal{P}_1$ elements.

4.2 P1-iso-P2 / P1

For having an inf-sup stable space we need a functional space for the velocity sensibly richer than the functional space for the pressure. $\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$ fulfill this by using $\mathcal{P}_1$ elements on a finer triangulation $\mathcal{T}_h$ for the velocity and $\mathcal{P}_1$ elements on a coarse mesh $\mathcal{T}_{2h}$ for the pressure. In the implementation, we start from the coarse triangulation $\mathcal{T}_{2h}$ and refine it by adding for each coarse element the midpoints of each edge to the nodes of the fine mesh $\mathcal{N}_h$, if not already added, and add the four subtriangle children to the elements $\mathcal{E}_h$.



(a) Example of Triangulation Refinement.

Elements	Nodes index in $\mathcal{N}_{2h}$		
k	7	8	9

(b) Parent element in $\mathcal{E}_{2h}$

Elements	Nodes index in $\mathcal{N}_h$			Parent
I	7	110	112	1
II	8	111	110	2
III	9	112	111	3
IV	110	111	112	0

(c) Child elements in $\mathcal{E}_h$

We introduce a fourth component in each entry of the `element_h` matrix to record the local index (1, 2, or 3) of the coarse node from the parent triangle that is also present in the corresponding refined subtriangle. For the central subtriangle, which does not contain any of the original coarse nodes, we assign this value as 0. This additional information will later be used to assemble the B matrices.

Moreover, to avoid inserting the same midpoint multiple times into the refined node set $\mathcal{N}_h$, we use a sparse matrix `midpointEdgeIdx` of size $n_{\mathcal{N}_{2h}} \times n_{\mathcal{N}_{2h}}$. This matrix stores in position i, j, and j, i for symmetry, the global index in $\mathcal{N}_h$ of the midpoint associated with the edge, if present, of the coarse mesh linking the node i and j. When processing a new element, before computing and inserting a midpoint into $\mathcal{N}_h$, we check whether `midpointEdgeIdx(i,j)` is empty. If so, we compute the midpoint, assign it a new global index, and store it in both `midpointEdgeIdx(i,j)` and `midpointEdgeIdx(j,i)`. Else, we simply retrieve the existing global index from `midpointEdgeIdx(i,j)`.

This mechanism guarantees that each midpoint is added to $\mathcal{N}_h$ only once and that its index is consistently reused across all child elements sharing that edge.

4.2.1 Assembly

The assembly of the Mass and Stiffness matrices is standard as in $\mathcal{P}_1(\mathcal{T}_h)$. We process the topology parent element by parent element, child element by child element. In this way, we preserve parent-child relationship while going through all the elements. For the B matrices we exploit the parent local index we recorded in the last column of `element_h`. Indeed, locally in the child c_l of the element e, we have that

$$B_{1\,ij}^{(c_l)} = -\int_{c_l} \psi_i^{(e)} \partial_x \phi_j^{(c_l)}\, d\Omega = -\frac{1}{2\Delta^{(c_l)}} b_j \int_{c_l} \psi_i^{(e)}\, d\Omega,$$

where $\psi_1^{(e)}, \psi_2^{(e)}, \psi_3^{(e)}$ are the elemental basis function for the pressure space $\mathcal{P}_1(\mathcal{T}_{2h})$ and $\mathbf{b} = \begin{bmatrix} b_i & b_j & b_k \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} c_i & c_j & c_k \end{bmatrix}$ are such that the gradients of the elemental basis function for the component of the velocity $\mathcal{P}_1(\mathcal{T}_{2h})$ are

$$\begin{bmatrix} \nabla\phi_i & \nabla\phi_j & \nabla\phi_k \end{bmatrix} = \frac{1}{2\Delta^{(c_l)}} \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}.$$

By using the quadrature rule as before and denoting $\mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k$ the nodes of the child element, we can compute

$$\int_{c_l} \psi_i^{(e)}\, d\Omega = \frac{\Delta^{(e)}}{3} \left(\psi_i^{(e)}(\mathbf{y}_i) + \psi_i^{(e)}(\mathbf{y}_j) + \psi_i^{(e)}(\mathbf{y}_k) \right) = \begin{cases} \frac{\Delta^{(c_l)}}{3} \cdot (1 + 1/2 + 1/2) = \frac{2}{3}\Delta^{(c_l)} & \text{if } \texttt{parent\_idx}(c_l) = i \\ \frac{\Delta^{(c_l)}}{3} \cdot (0 + 1/2 + 0) \quad = \frac{\Delta^{(c_l)}}{6} & \text{if } \texttt{parent\_idx}(c_l) \neq i \\ \frac{\Delta^{(c_l)}}{3} \cdot (1/2 + 1/2 + 0) = \frac{\Delta^{(c_l)}}{3} & \text{if } \texttt{parent\_idx}(c_l) = 0. \end{cases}$$

In total,

$$B_{1\,ij}^{(c_l)} = -\frac{1}{2\Delta^{(c_l)}} b_j \cdot \begin{cases} \frac{2}{3}\Delta^{(c_l)} \\ \frac{\Delta^{(c_l)}}{6} \\ \frac{\Delta^{(c_l)}}{3} \end{cases} = -\begin{cases} \frac{b_j}{3} & \texttt{parent\_idx}(c_l) = i \\ \frac{b_j}{12} & \texttt{parent\_idx}(c_l) \neq i \\ \frac{b_j}{6} & \texttt{parent\_idx}(c_l) = 0, \end{cases}$$

and in matrix form,

$$B_{1\,ij}^{(c_l)} = \frac{1}{12}\, \texttt{diag}(\texttt{scale}) \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \\ \mathbf{b} \end{bmatrix} \quad \text{and similarly} \quad B_{2\,ij}^{(c_l)} = \frac{1}{12}\, \texttt{diag}(\texttt{scale}) \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \\ \mathbf{c} \end{bmatrix},$$

where the scaling vector `scale` is defined

$$\texttt{scale} = \begin{cases} [2\ 2\ 2] & \text{if } \texttt{parent\_idx}(c_l) = 0 \\ [s_i\ s_j\ s_k] & \text{otherwise} \end{cases} \quad \text{with } s_m = \begin{cases} 4 & \text{if } m = \texttt{parent\_idx}(c_l) \\ 1 & \text{otherwise.} \end{cases}$$

5 Convergence Analysis

To test the goodness of the methods, we compute the convergence error for four mesh refinements on a manufactured problem where the exact solution is known. That is Equation (D) with

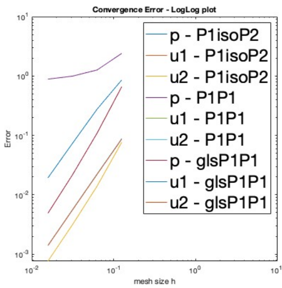
$$\begin{aligned} u_1(x, y) &= -\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y) \\ u_2(x, y) &= \sin(2\pi x)\cos(2\pi y) - \sin(2\pi y) \\ p(x, y) &= 2\pi(\cos(2\pi y) - \cos(2\pi x)) \\ f_1(x, y) &= cu_1(x, y) - 4\mu\pi^2 \sin(2\pi y)(2\cos(2\pi x) - 1) + 4\pi^2 \sin(2\pi x) \\ f_2(x, y) &= cu_2(x, y) + 4\mu\pi^2 \sin(2\pi x)(2\cos(2\pi y) - 1) - 4\pi^2 \sin(2\pi y) \end{aligned}$$

and boundary conditions of Dirichlet type given by the boundary values of u_1, u_2 and p. We set $c = 1$ and $\mu = 1$. Convergence error is computed as $\|u_{1,h} - u_1\|_2 = \left(\int_\Omega |u_{1,h}(\mathbf{x}) - u_1(\mathbf{x})|^2\, d\Omega\right)^{1/2}$ and accumulated numerically element by element.

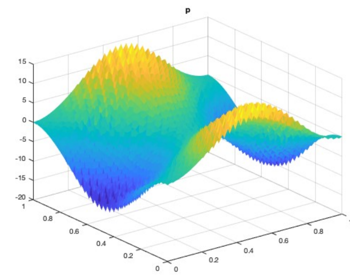
mesh	h	$\mathcal{P}_1/\mathcal{P}_1$			GLS $\mathcal{P}_1/\mathcal{P}_1$			$\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$		
-	-	u_1	u_2	p	u_1	u_2	p	u_1	u_2	p
1	0.125	8.84e-02	8.83e-02	2.40e+00	8.64e-02	8.64e-02	6.58e-01	7.64e-02	7.64e-02	8.57e-01
2	0.0625	2.25e-02	2.25e-02	1.26e+00	2.21e-02	2.21e-02	1.09e-01	1.40e-02	1.40e-02	2.77e-01
3	0.03125	5.65e-03	5.65e-03	9.94e-01	5.55e-03	5.55e-03	2.17e-02	3.14e-03	3.14e-03	7.36e-02
4	0.015625	1.41e-03	1.41e-03	8.85e-01	1.39e-03	1.39e-03	4.88e-03	7.60e-04	7.61e-04	1.92e-02

Table 1: Convergence error on four mesh refinements per method used. For GLS, $\tau = 0.0001$ has been used. For $\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$, the errors for the pressure are computed in the $\mathcal{T}_{2h}$ triangulation.

From Table 1 and Figure 2a we can notice that the convergence order is approximately of order 2 for both pressure and velocity and that the pressure solution for $\mathcal{P}_1/\mathcal{P}_1$ without stabilization is unstable and shows oscillations, as seen in Figure 2b.



(a) Convergence Error in log scale.



(b) Pressure solution without stabilization for mesh4. Oscillations and instability are evident.

Moreover, Table 2 shows for different mesh refinements the ratio

$$r_k = \frac{\mathtt{err}_{k-1}}{\mathtt{err}_k} \left(\frac{h_k}{h_{k-1}} \right)^2,$$

that approaches one when the order of the scheme is two, exactly as expected except for the pressure of the non stabilized $\mathcal{P}_1/\mathcal{P}_1$.

mesh	$\mathcal{P}_1/\mathcal{P}_1$			GLS $\mathcal{P}_1/\mathcal{P}_1$			$\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$		
-	u_1	u_2	p	u_1	u_2	p	u_1	u_2	p
1	1.0191	1.0191	2.1053	1.0212	1.0212	0.6641	0.7342	0.7344	1.2918
2	1.0040	1.0039	3.1475	1.0052	1.0052	0.7947	0.8969	0.8972	1.0644
3	0.9993	0.9994	3.5593	1.0012	1.0012	0.8994	0.9677	0.9678	1.0447

Table 2: Ratio r_k for mesh refinements for different schemes and functions.

In Figure 3 are shown the numerical solution with the mesh4 and $\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$ space.

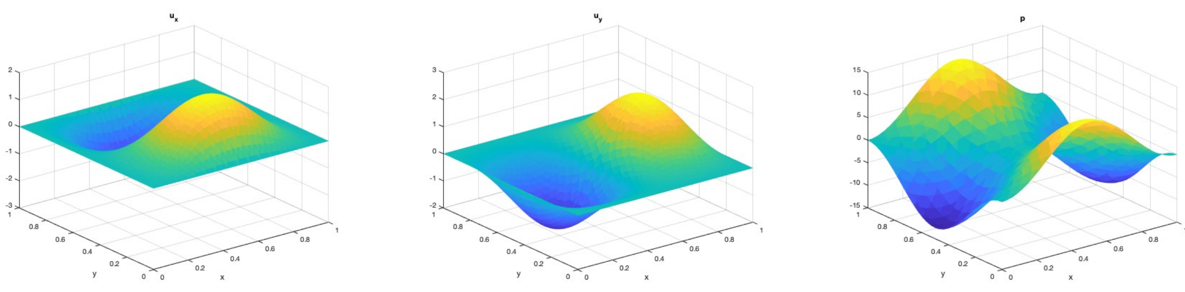


Figure 3: Numerical solution on the manifactured problem with $\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$ space. From left to right u_1, u_2 and p.

6 Numerical Solution to Lid-Driven Cavity Equation

We use the GLS $\mathcal{P}_1/\mathcal{P}_1$ the $\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$ elements to solve numerically the Lid-Driven Cavity Equation. The boundary conditions are illustrated in Figure 4 and the forcing terms f and g are set to 0. For the pressure, zero value is imposed in the origin as Dirichlet boundary condition.

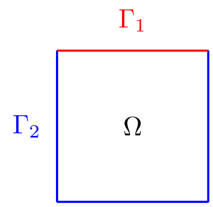


Figure 4: Domain of study for the Lid-Driven Cavity problem. The function $u = [1, 0]^T$ on Γ_1 and $[0, 0]^T$ on Γ_2.

Figure 5 includes the plot of the numerical solution found for the mesh4 with GLS $\mathcal{P}_1/\mathcal{P}_1$ and Figure 6 with $\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_1$. From a visual inspection, they look similar to the reference pictures provided in the Exercise Text.

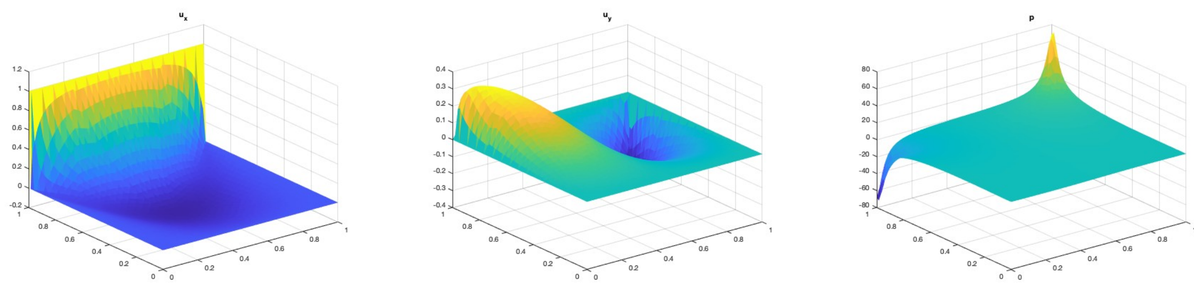


Figure 5: Numerical solution of the Lid-Driven Cavity Equation with GLS P1/P1 elements and $\tau = 0.0001$. From left to right u_1, u_2 and p.

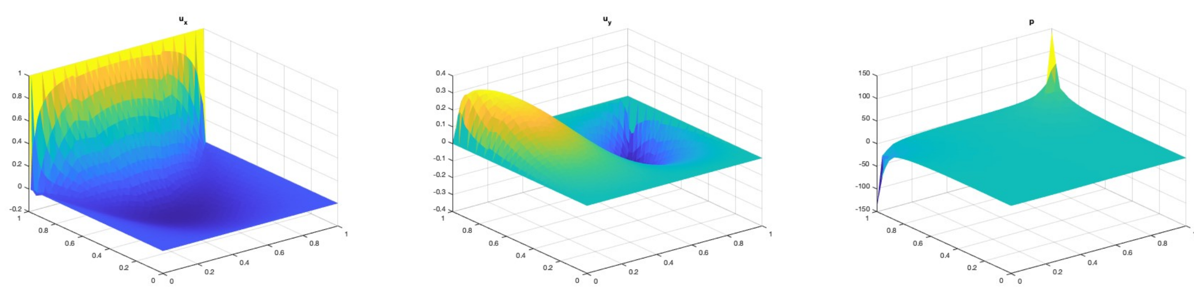


Figure 6: Numerical solution of the Lid-Driven Cavity Equation with P1-iso-P2/P1 elements. From left to right u_1, u_2 and p.