Parameterized Algorithms for Matrix Completion with Radius Constraints

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Applications

Best@1: find the best item over a set

- Question Answering
- Recommendation System
- Machine Translation
- Web Search

Techniques Employed

- ML sorting algorithm and select the first element
- ML pairwise classifier and work on the induced graph

Warm Up

T **is a DAG**

The Champion beats everyone $\implies$ we have a $O(n)$ algorithm!

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General Case: Lower Bound

We need to unfold at least $\Omega\left(n^2\right)$ arcs $\implies$ Brute force is optimal

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Exploit the tournament structure $\longrightarrow$ **parametrized complexity**

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Parametrize with ℓ

the number of matches lost by the champion

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Proof

$$M = \begin{pmatrix} 0 & * & * & * & * & * \\ * & 0 & * & * & * & * \\ * & * & 0 & * & * & * \\ * & * & * & 0 & * & * \\ * & * & * & * & 0 & * \\ * & * & * & * & * & 0 \end{pmatrix}$$

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The **optimality certificate** consists of $\Omega\left(\ell n\right)$ entries of M

Can we Match the Lower Bound $\Omega(\ell n)$?

$$\begin{pmatrix} 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & * & 1 & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & * & 1 & 0 & 0 & * \\ 0 & 0 & * & 1 & 0 & * \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}$$

Can we Match the Lower Bound $\Omega(\ell n)$?

Find the optimality certificate

guessing zeros randomly row by row

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Las Vegas Algorithm

Worst Case: $\Theta(\ell n \log(n))$ w.h.p.

Bound Unmatched!

Algorithm

Kill Threshold α

$\alpha/2 \leq \ell < \alpha$

Kill Until

$2\alpha + 1$ survive

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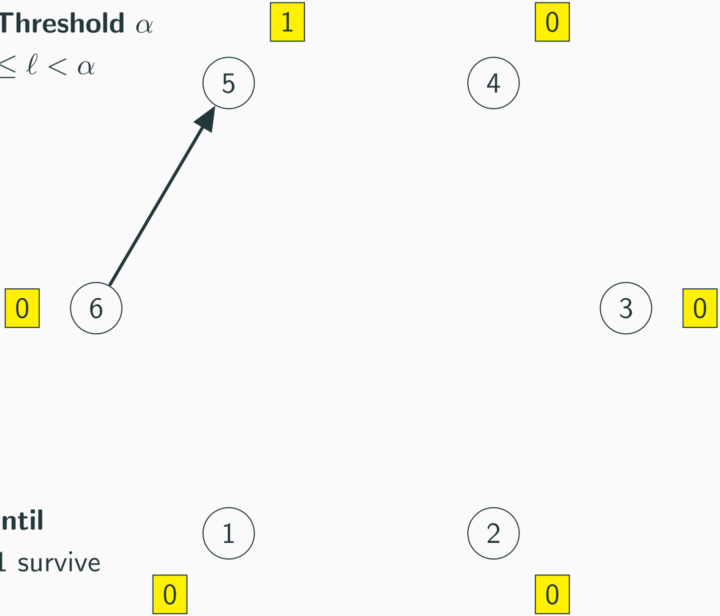

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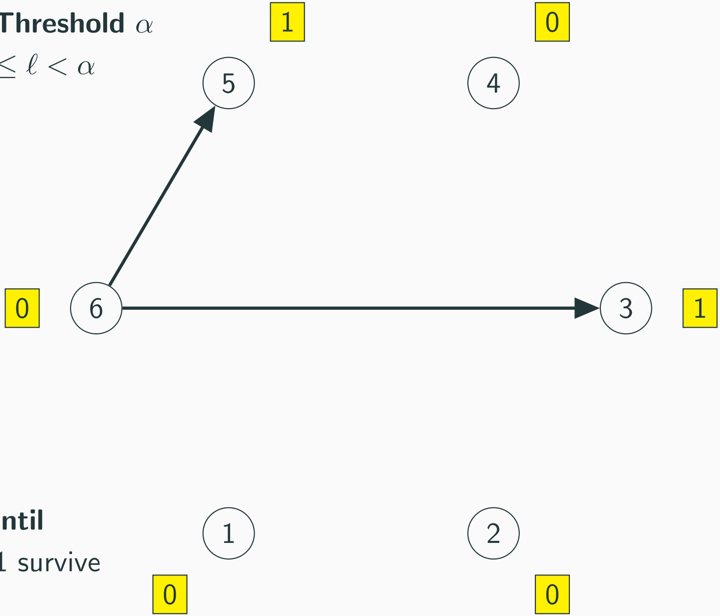

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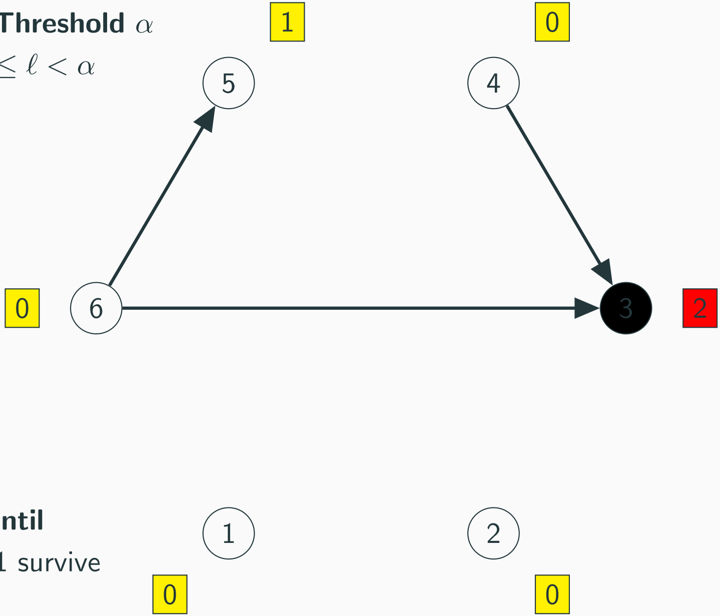

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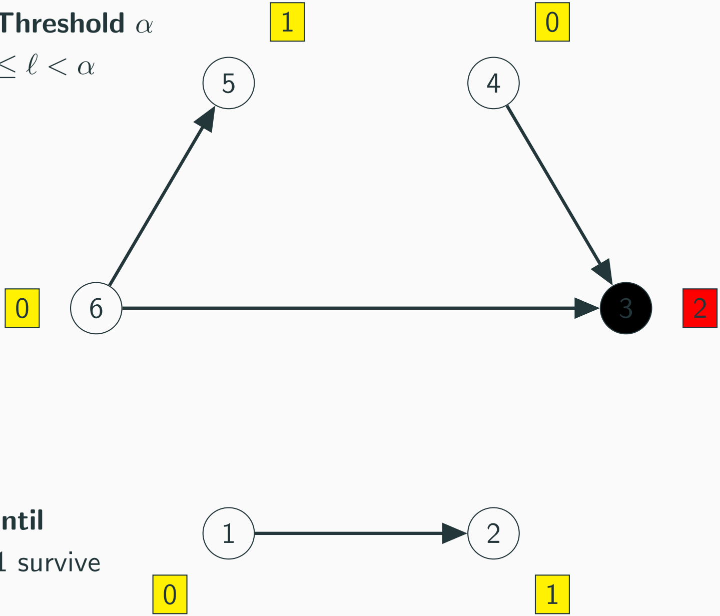

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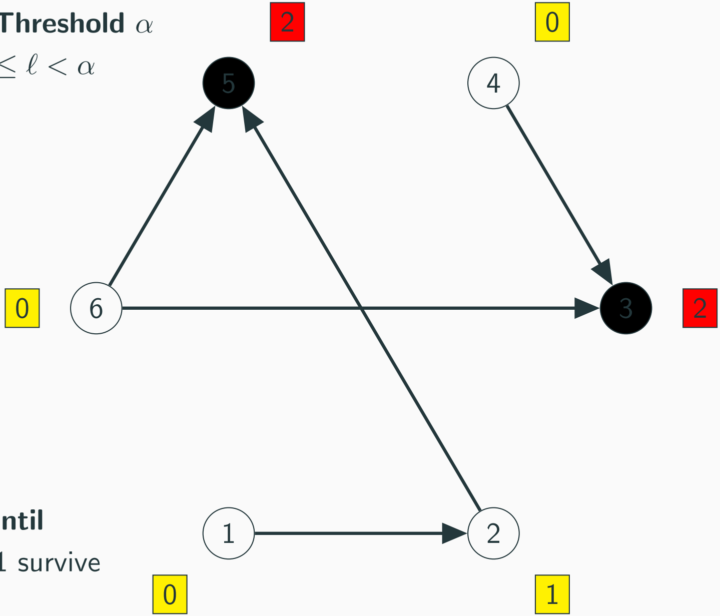

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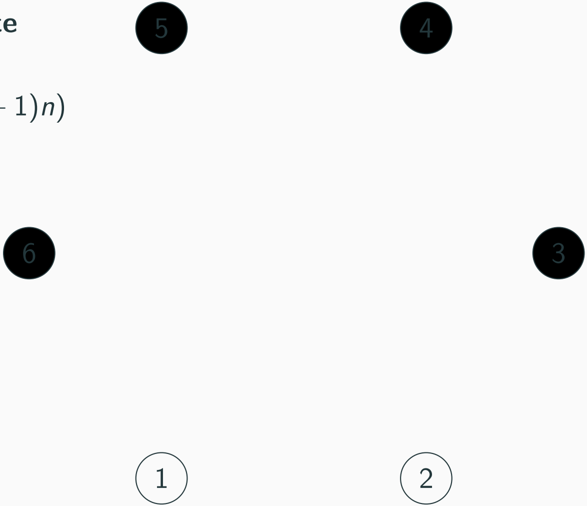

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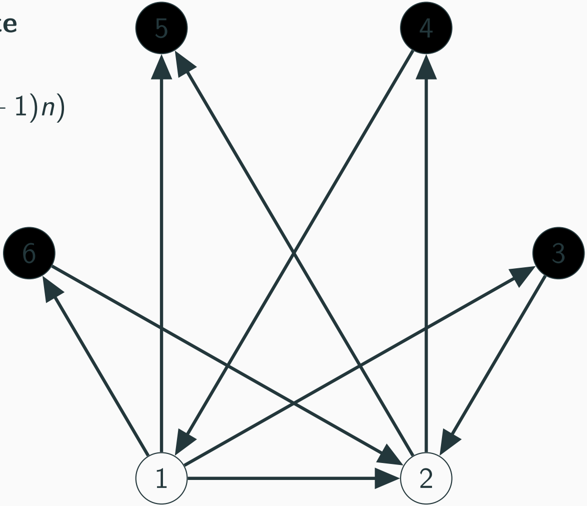

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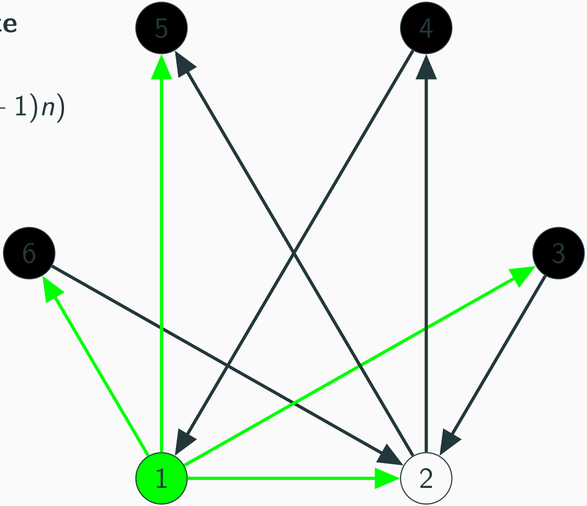

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- Every arc unfolded increase a counter by one
- Every counter is always $\leq \alpha$

$\implies$ The first phase takes $O(\alpha n)$

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- We start the second phase with $2\alpha + 1$ survivals
- Every survival plays $n - 1$ games

$\implies$ The second phase takes $O(\alpha n)$

$$\alpha/2 < \ell \implies \text{Complexity: } O(\ell n)$$

Correctness

We assumed to know α such that $\alpha/2 \leq \ell < \alpha$

- We don't know ℓ in advance!
- Perform an exponential search over α
- If the candidate has lost $< \alpha$ games, then it is the champion

Complexity is Preserved

$$\sum_{2^i \leq \ell} 2^i n = O(\ell n)$$

Correctness

Can we keep killing vertices until $2\alpha + 1$ survive?

- Consider a sub-tournament of $2\alpha + 1$ alive vertices
- Suppose every vertex loses $< \alpha$ games
- Double counting total losses gives a contradiction

Open Problems

- Is there any better parametrization?
- Is there a Monte Carlo algorithm that runs faster?
- Waiting for further advancements in ML pairwise classifiers...

Thank You!