How to Convert Volatility Surface from Dependent on Delta to Dependent on Moneyness and vice-versa

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Abstract

This article is aimed at providing a general method to convert a volatility surface of vanilla European call options from dependent on delta to dependent on moneyness. A simple algorithm, based on the fixed point algorithm, is described, with a focus on Balck's model for options on futures. Some empirical advice are also given, in order to improve the algorithm behaviour.

1 Procedure

We suppose to have a volatility surface dependent on delta, so we suppose to have $n \times k$ volatility points $\tilde{\sigma}_{ij}$ defined for delta $\tilde{\Delta}_i$ with $i = 1, \dots, n$ and for time-to-maturity T_j with $j = 1 \dots k$. Moreover we suppose to have an interpolation function σ such that $\sigma(\tilde{\Delta}_i, T_j) = \tilde{\sigma}_{ij}$. This function can be whatever we prefer to interpolate smiles of our volatility function (linear interpolation, splines...). We define moneynesses $m_1, m_2, \dots, m_n$ of our choice. We know that there exists a function $\Delta = F(m, T, \sigma)$ which, given a time to maturity T, a maturity m and a volatility σ, returns the associated Δ. Function F can be easily derived from the option delta formula (for both Black and Scholes's model and Black's model). By substituting the interpolation function in σ we obtain that

$$\Delta = F(m, T, \sigma(T, \Delta)) \tag{1}$$

For each pair m_i, T_j we look for δ_{ij} such that the following holds

$$\delta_{ij} = F(m_i, T_j, \sigma(\delta_{ij}, T_j))$$

Finding δ_{ij} that solves this equation, means solving a fixed point problem $f(x) = x$ or a zero-finding problem $f(x) - x = 0$ where $f(x) = F(m_i, T_j, \sigma(x, T_j))$. Once the problem has been solved for all pairs m_i, T_j, we can define the values of the new volatility surface as $\rho_{ij} = \sigma(\delta_{ij}, T_j)$. Therefore the new volatility surface is

$$(m_i, T_j) \to \rho_{ij}$$

Once ρ_{ij} has been calculated, we have a surface in 2 dimensions dependent on moneyness.

1.1 The case of Black's model

In this section we treat the case of options on futures, and thus the case of the Black's model. The delta of an option in Black's model is

$$\Delta = e^{-rT}\Phi(d_1)$$

with

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt, \quad d_1 = \frac{\ln \frac{S}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

where r is the interest rate and T the maturity of the option, S the price of the underlying future, K the strike of the option and σ the volatility of S. We define the moneyness $m = \frac{S}{K}$ and, from delta formula we obtain that

$$\Delta(m, T, \sigma) = e^{-rT}\Phi(d_1(m, T, \sigma)) \quad (2)$$

We suppose to have a volatility surface depending on delta, i.e. an interpolation function σ such that $\sigma(\Delta_i, T_j) = \tilde{\sigma}_{ij}$, and we suppose we have to convert the surface to be dependent on moneynesses $m_1, m_2, \ldots, m_n$. For each pair (m_i, T_j) we have to find δ_{ij} such that

$$\delta_{ij} = e^{-rT}\Phi(d_1(m_i, T_j, \sigma(\delta_{ij}, T_j))$$

Now the volatility surface, in $(m_i, T_j) \to \rho_{ij}$ where $\rho_{ij} = \sigma(\Delta_i, T_j)$. This defines the grid of points of the volatility surface, but obviously, an other interpolation funcion must be defined also for theese point. This means defining a function $\rho(m, T)$ such that $\rho(m_i, T_j) = \rho_{ij}$. Once again all the choices among the interpolation algorithm are possible.

1.2 Empirical considerations

As explained, the problem is reduced to solve a fixed point problem. One of the most popular and simple algorithm for the solution of the fixed point problem is the *fixed point algorithm.* If works as follows: let's suppose we have to find x such that $f(x) = x$ where f is a continuous function. Chosen an initial point x_0 we define

$$x_{n+1} = f(x_n)$$

It is obvious that if x_n has limit, the limit is a fixed point for f. This simple algorithm is called fixed point algorithm. Under the assumption that f is Lipschitz continuous in an interval I, with Lipschitz constant less that 1, the fixed point algorithm converges to the unique fixed point of f, given $x_0 \in I$. Unfortunately, we can easily realise that the function in (1), even though σ is constant, is not Lipschitz continuous with Lipschitz constant less that 1. Despite the convergence is not guaranteed for (1), the algorithm often converges with real data volatility in few iterations. This is due to the fact that (1) is locally Lipschitz continuous if the volatility surface does not have strong variations in $\tilde{\sigma}_{ij}$. Therefore this algorithm is strongly recommended. Some other empirical considerations:

- **How to choose** x_0: It is easy to see that (2) is less than e^{-rT} since Φ is a cumulative density function and thus $\Phi < 1$. Therefore (2) has a fixed point in $(0, e^{-rT})$ and we can set $x_0 = e^{-rT}$
- **What can we do if the algorithm does not converge**: in case the fixed point algorithm does not converge, we can reformulate our problem as a zeros-finding problem. We define $g(x) = f(x) - x$ and we can look for a zero of g with any of the zero-finding algorithm, like bisection (recommended for its generality), secant method, Newton method... These methods can be implemented in the interval $(0, e^{-rT})$
- **Smile interpolation**: It is recommended to use a linear interpolation as interpolation function σ, because of its simple and predictable behaviour, in both volatility surface dependent on delta and on moneyness
- **How to invert the procedure**: Function F can be *inverted* respect to m, in the sense that there exists H such that $m = H(\Delta, T, \sigma)$. We can repeat the same procedure described in this article by using H and by inverting the role of m and Δ: we obtain a procedure to convert a volatility surface dependent on moneyness to one dependent on delta.