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A generalized Gronwall inequality and its application to a fractional differential equation

Haiping Ye [a,*], Jianming Gao [a], Yongsheng Ding [b]

[a] *Department of Applied Mathematics, Donghua University, Shanghai 201620, PR China*
[b] *College of Information Sciences and Technology, Donghua University, Shanghai 201620, PR China*

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Abstract

This paper presents a generalized Gronwall inequality with singularity. Using the inequality, we study the dependence of the solution on the order and the initial condition of a fractional differential equation.

Keywords: Integral inequality; Differential equation; Fractional order

1. Introduction

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations; cf. [1]. The celebrated Gronwall inequality known now as Gronwall–Bellman–Raid inequality provided explicit bounds on solutions of a class of linear integral inequalities. On the basis of various motivations, this inequality has been extended and used in various contexts [2–4].

Let us recall the standard Gronwall inequality which can be found in [10, p. 14].

Theorem A. *If*

$$x(t) \leqslant h(t) + \int_{t_0}^{t} k(s)x(s)\,ds, \quad t \in [t_0, T),$$

* Corresponding author.
E-mail addresses: hpye@dhu.edu.cn (H. Ye), ysding@dhu.edu.cn (Y. Ding).

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where all the functions involved are continuous on $[t_0, T)$, $T \leqslant +\infty$, *and* $k(t) \geqslant 0$, *then* $x(t)$ *satisfies*

$$x(t) \leqslant h(t) + \int_{t_0}^{t} h(s)k(s)\exp\left[\int_{s}^{t} k(u)\,du\right] ds, \quad t \in [t_0, T).$$

If, in addition, $h(t)$ *is nondecreasing, then*

$$x(t) \leqslant h(t)\exp\left(\int_{t_0}^{t} k(s)\,ds\right), \quad t \in [t_0, T).$$

However, sometimes we need a different form, to discuss the weakly singular Volterra integral equations encountered in fractional differential equations. In this paper we present a slight generalization of the Gronwall inequality which can be used in a fractional differential equation. Using the inequality, we study the dependence of the solution on the order and the initial condition for a fractional differential equations with Riemann–Liouville fractional derivatives.

2. An integral inequality

In this section, we wish to establish an integral inequality which can be used in a fractional differential equation. The proof is based on an iteration argument.

Theorem 1. *Suppose* $\beta > 0$, $a(t)$ *is a nonnegative function locally integrable on* $0 \leqslant t < T$ (*some* $T \leqslant +\infty$) *and* $g(t)$ *is a nonnegative, nondecreasing continuous function defined on* $0 \leqslant t < T$, $g(t) \leqslant M$ (*constant*), *and suppose* $u(t)$ *is nonnegative and locally integrable on* $0 \leqslant t < T$ *with*

$$u(t) \leqslant a(t) + g(t)\int_{0}^{t} (t-s)^{\beta-1}u(s)\,ds$$

on this interval. Then

$$u(t) \leqslant a(t) + \int_{0}^{t} \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right] ds, \quad 0 \leqslant t < T.$$

Proof. Let $B\phi(t) = g(t)\int_0^t (t-s)^{\beta-1}\phi(s)\,ds$, $t \geqslant 0$, for locally integrable functions ϕ. Then

$$u(t) \leqslant a(t) + Bu(t)$$

implies

$$u(t) \leqslant \sum_{k=0}^{n-1} B^k a(t) + B^n u(t).$$

Let us prove that

$$B^n u(t) \leqslant \int_{0}^{t} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}u(s)\,ds \tag{1}$$

and $B^n u(t) \to 0$ as $n \to +\infty$ for each t in $0 \leqslant t < T$.

We know this relation (1) is true for $n = 1$. Assume that it is true for some $n = k$. If $n = k + 1$, then the induction hypothesis implies

$$B^{k+1}u(t) = B\big(B^k u(t)\big) \leqslant g(t)\int_0^t (t-s)^{\beta-1}\left[\int_0^s \frac{(g(s)\Gamma(\beta))^k}{\Gamma(k\beta)}(s-\tau)^{k\beta-1}u(\tau)\,d\tau\right]ds.$$

Since $g(t)$ is nondecreasing, it follows that

$$B^{k+1}u(t) \leqslant \big(g(t)\big)^{k+1}\int_0^t (t-s)^{\beta-1}\left[\int_0^s \frac{(\Gamma(\beta))^k}{\Gamma(k\beta)}(s-\tau)^{k\beta-1}u(\tau)\,d\tau\right]ds.$$

By interchanging the order of integration, we have

$$\begin{aligned}B^{k+1}u(t) &\leqslant \big(g(t)\big)^{k+1}\int_0^t\left[\int_\tau^t \frac{(\Gamma(\beta))^k}{\Gamma(k\beta)}(t-s)^{\beta-1}(s-\tau)^{k\beta-1}\,ds\right]u(\tau)\,d\tau\\ &= \int_0^t \frac{(g(t)\Gamma(\beta))^{k+1}}{\Gamma((k+1)\beta)}(t-s)^{(k+1)\beta-1}u(s)\,ds,\end{aligned}$$

where the integral

$$\begin{aligned}\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{k\beta-1}\,ds &= (t-\tau)^{k\beta+\beta-1}\int_0^1 (1-z)^{\beta-1}z^{k\beta-1}\,dz\\ &= (t-\tau)^{(k+1)\beta-1}B(k\beta,\beta)\\ &= \frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)}(t-\tau)^{(k+1)\beta-1}\end{aligned}$$

is evaluated with the help of the substitution $s = \tau + z(t-\tau)$ and the definition of the beta function (cf. [6, pp. 6–7]).

The relation (1) is proved.

Since $B^n u(T) \leqslant \int_0^t \frac{(M\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}u(s)\,ds \to 0$ as $n \to +\infty$ for $t \in [0,T)$, the theorem is proved. □

For $g(t) \equiv b$ in the theorem we obtain the following inequality. This inequality can be found in [5, p. 188].

Corollary 1. [5] *Suppose $b \geqslant 0$, $\beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leqslant t < T$ (some $T \leqslant +\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leqslant t < T$ with*

$$u(t) \leqslant a(t) + b\int_0^t (t-s)^{\beta-1}u(s)\,ds$$

on this interval; *then*

$$u(t) \leqslant a(t) + \int_0^t\left[\sum_{n=1}^{\infty}\frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds, \quad 0 \leqslant t < T.$$

Corollary 2. *Under the hypothesis of Theorem* 1*, let* $a(t)$ *be a nondecreasing function on* $[0, T)$*. Then*

$$u(t) \leqslant a(t)E_\beta\big(g(t)\Gamma(\beta)t^\beta\big),$$

where E_β *is the Mittag-Leffler function defined by* $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

Proof. The hypotheses imply

$$u(t) \leqslant a(t)\left[1 + \int_0^t \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}\,ds\right] = a(t)\sum_{n=0}^{\infty} \frac{(g(t)\Gamma(\beta)t^\beta)^n}{\Gamma(n\beta+1)}$$
$$= a(t)E_\beta\big(g(t)\Gamma(\beta)t^\beta\big).$$

The proof is complete. □

3. Application to dependence of solution on parameters

In this section we will show that our main result is useful in investigating the dependence of the solution on the order and the initial condition to a certain fractional differential equation with Riemann–Liouville fractional derivatives.

It is frequently stated that the physical meaning of initial conditions expressed in terms of Riemann–Liouville fractional derivatives is unclear or even nonexistent. But in [9], N. Heymans and I. Podlubny demonstrate that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann–Liouville fractional derivatives and that it is possible to obtain initial values for such initial conditions by appropriate measurements or observations. Therefore, our discussion here is meaningful.

Let us consider the following initial value problem [6] in terms of the Riemann–Liouville fractional derivatives:

$$D^\alpha y(t) = f\big(t, y(t)\big), \tag{2}$$

$$D^{\alpha-1} y(t)|_{t=0} = \eta, \tag{3}$$

where $0 < \alpha < 1$, $0 \leqslant t < T \leqslant +\infty$, $f : [0, T) \times R \to R$ and D^α denotes Riemann–Liouville derivative operator.

Riemann–Liouville derivative and integral are defined below [6–8].

Definition 1. The fractional derivative of order $0 < \alpha < 1$ of a continuous function $f : R^+ \to R$ is given by

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t)\,dt$$

provided that the right side is pointwise defined on R^+.

Definition 2. The fractional primitive of order $\alpha > 0$ of a function $f : R^+ \to R$ is given by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt$$

provided the right side is pointwise defined on R^+.

The existence and uniqueness of the initial value problem (2)–(3) have been studied in the literature [6]. Also the dependence of a solution on initial conditions has been discussed in [6]. In this section we present the dependence of the solution on the order and the initial condition. We shall consider the solutions of two initial value problems with neighboring orders and neighboring initial values. It is important to note that here we are considering a question which does not arise in the solution of differential equations of integer order.

First, let us reduce the problem (2)–(3) to a fractional integral equation. We obtain

$$y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(\tau, y(\tau)\big)\, d\tau. \tag{4}$$

It is clear that Eq. (4) is equivalent to the initial value problem (2)–(3) (cf. [6, pp. 127–128]).

Theorem 2. *Let $\alpha > 0$ and $\delta > 0$ such that $0 < \alpha - \delta < \alpha \leqslant 1$. Let the function f be continuous and fulfill a Lipschitz condition with respect to the second variable*; *i.e.,*

$$\big|f(t,y) - f(t,z)\big| \leqslant L|y-z|$$

for a constant L independent of t, y, z in R. For $0 \leqslant t \leqslant h < T$, assume that y and z are the solutions of the initial value problems (2)–(3) *and*

$$D^{\alpha-\delta} z(t) = f\big(t, z(t)\big), \tag{5}$$

$$D^{\alpha-\delta-1} z(t)|_{t=0} = \bar{\eta}, \tag{6}$$

respectively. Then, for $0 < t \leqslant h$ the following holds:

$$\big|z(t) - y(t)\big| \leqslant A(t) + \int_0^t \left[\sum_{n=1}^{\infty} \left(\frac{L}{\Gamma(\alpha)} \Gamma(\alpha-\delta)\right)^n \frac{(t-s)^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))} A(s)\right] ds,$$

where

$$A(t) = \left|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)} t^{\alpha-\delta-1} - \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1}\right| + \left|\frac{t^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right| \cdot \|f\| + \left|\frac{t^{\alpha-\delta}}{\alpha-\delta}\left[\frac{1}{\Gamma(\alpha-\delta)} - \frac{1}{\Gamma(\alpha)}\right]\right| \cdot \|f\|$$

and

$$\|f\| = \max_{0 \leqslant t \leqslant h} \big|f(t,y)\big|.$$

Proof. The solutions of the initial value problem (2)–(3) and (5)–(6) are given by

$$y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(\tau, y(\tau)\big)\, d\tau$$

and

$$z(t) = \frac{\bar{\eta}}{\Gamma(\alpha-\delta)} t^{\alpha-\delta-1} + \frac{1}{\Gamma(\alpha-\delta)} \int_0^t (t-\tau)^{\alpha-\delta-1} f\big(\tau, z(\tau)\big)\, d\tau,$$

respectively. It follows that

$$
\begin{aligned}
&\big|z(t)-y(t)\big| \\
&\quad\leqslant \left|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}-\frac{\eta}{\Gamma(\alpha)}t^{\alpha-1}\right| \\
&\qquad+\left|\frac{1}{\Gamma(\alpha-\delta)}\int_0^t (t-\tau)^{\alpha-\delta-1} f\big(\tau,z(\tau)\big)\,d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-\delta-1} f\big(\tau,z(\tau)\big)\,d\tau\right| \\
&\qquad+\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-\delta-1} f\big(\tau,z(\tau)\big)\,d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-\delta-1} f\big(\tau,y(\tau)\big)\,d\tau\right| \\
&\qquad+\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-\delta-1} f\big(\tau,y(\tau)\big)\,d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f\big(\tau,y(\tau)\big)\,d\tau\right| \\
&\quad\leqslant A(t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-\delta-1}L\big|z(\tau)-y(\tau)\big|\,d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
A(t)={}&\left|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}-\frac{\eta}{\Gamma(\alpha)}t^{\alpha-1}\right|+\left|\frac{t^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right|\cdot\|f\| \\
&+\left|\frac{t^{\alpha-\delta}}{\alpha-\delta}\left[\frac{1}{\Gamma(\alpha-\delta)}-\frac{1}{\Gamma(\alpha)}\right]\right|\cdot\|f\|.
\end{aligned}
$$

An application of Theorem 1 yields

$$
\big|z(t)-y(t)\big|\leqslant A(t)+\int_0^t\left[\sum_{n=1}^{\infty}\left(\frac{L}{\Gamma(\alpha)}\Gamma(\alpha-\delta)\right)^n\frac{(t-s)^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))}A(s)\right]ds
$$

and the theorem is proved. □

Remark 1. It follows from Theorem 2 that for every ϵ between 0 and h small changes in order and initial condition cause only small changes of the solution in the closed interval $[\epsilon,h]$ (which does not contain zero).

Remark 2. Existence and uniqueness are granted in the autonomous case, i.e., when f depends only on y. See [7, Lemma 4.2].

A general theorem of existence and uniqueness for the nonautonomous case $f(t,y)$ can be found in [6, p. 127].

Corollary 3. [6] *Under the hypothesis of Theorem* 2*, if* $\delta=0$*, then*

$$
\big|z(t)-y(t)\big|\leqslant|\bar{\eta}-\eta|t^{\alpha-1}E_{\alpha,\alpha}\big(Lt^{\alpha}\big),
$$

for $0<t\leqslant h$*, where* $E_{\alpha,\alpha}$ *is the Mittag-Leffler function defined by* $E_{\alpha,\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\alpha)}$ $(\alpha>0)$.

Proof. If $\delta = 0$, then

$$A(t) = \left| \frac{t^{\alpha-1}}{\Gamma(\alpha)} (\bar{\eta} - \eta) \right|.$$

By Theorem 2, we obtain

$$\begin{aligned} \left|z(t) - y(t)\right| &\leqslant A(t) + \int_0^t \left[\sum_{n=1}^{\infty} L^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)} A(s) \right] ds = |\bar{\eta} - \eta| t^{\alpha-1} \sum_{n=0}^{\infty} \frac{(Lt^\alpha)^n}{\Gamma(\alpha n + \alpha)} \\ &= |\bar{\eta} - \eta| t^{\alpha-1} E_{\alpha,\alpha}\left(Lt^\alpha\right), \end{aligned}$$

for $0 < t \leqslant h$. The proof is complete. □

Corollary 3 can be found in [6, pp. 134–136].

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