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Exact solutions of a two-dimensional nonlinear Schrödinger equation

Aly R. Seadawy

Mathematics Department, Faculty of Science, Beni-Suef University, Egypt

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ABSTRACT

In the present study, we converted the resulting nonlinear equation for the evolution of weakly nonlinear hydrodynamic disturbances on a static cosmological background with self-focusing in a two-dimensional nonlinear Schrödinger (NLS) equation. Applying the function transformation method, the NLS equation was transformed to an ordinary differential equation, which depended only on one function ξ and can be solved. The general solution of the latter equation in ζ leads to a general solution of NLS equation. A new set of exact solutions for the two-dimensional NLS equation is obtained.

1. Introduction

The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model that describes many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, plasma physics, heat pulses in solids and various other nonlinear instability phenomena [1–4].

The evolution of weakly nonlinear hydrodynamic disturbances on a static cosmological background, paying special attention to nonlinear modulational instabilities, solitons and self-focusing, is studied [5]. The two-dimensional cubic NLS equation is found to govern the long-term evolution of the envelope of weakly nonlinear, nearly plane-symmetric, almost monochromatic acoustic waves. Nonlinear modulational instability may even arise within a range of wave-numbers for which both modes of the linearized theory are stable, leading to the possibility of soliton solution formation [5–7].

The NLS equation has been used to explain a variety of effects in the propagation of optical pulses. As is well known, the balance between the self-phase modulation and group velocity dispersion leads to the so-called soliton solutions for the NLS equation [8,9]. Solitary wave solutions have been known to exist in a variety of nonlinear and dispersive media for many years. In the context of optical communications, a pulse propagating in an optical fiber with Kerr-law nonlinearity can form an envelope soliton [10]. This behavior of the pulse propagation offers the potential for understanding pulse transmission over very long distances. Just as the balance between self-phase-modulation and group-velocity dispersion can lead to the formation of temporal solitons in single-mode fibers, diffraction and self-focusing can compensate each other, and it is also possible to have an analogous spatial soliton [9–13]. The importance of studying optical solitons comes from the fact that they have potential applications in optical transmission and all-optical processing. Since analytical solutions are known for only a few cases, investigations into the properties of solutions are normally performed numerically using such approaches. However, it is often desirable to have an analytical model describing the dynamics of pulse propagation in a fiber [14–16].

This paper is organized as follows. An introduction is given in Section 1. In Section 2, the basic equations governing the problem together with the multiple scale method are sketched, and the self-focusing nonlinear NLS equation is briefly reported. The exact solutions of the two-dimensional NLS equation are obtained in Section 3.

2. Problem formulation

Consider that the system to be studied consists of mass conservation and Euler equations for each fluid constituent, together with a Poisson equation for the modified potential, whose source is the sum of the density contrasts of the ordinary

E-mail address: aly742001@yahoo.com.

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and dark matter. Suppose that the rotation of each velocity field vanishes and the functions $n(x,y,t)$, $N(x,y,t)$, $u(x,y,t)$, $v(x,y,t)$, $U(x,y,t)$, $V(x,y,t)$, and $\phi(x,y,t)$ satisfy [5]

$$\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}+\frac{\partial (nv)}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+c^2\frac{\partial (\log n)}{\partial x}+\frac{\partial \phi}{\partial x}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+c^2\frac{\partial (\log n)}{\partial y}+\frac{\partial \phi}{\partial y}=0, \tag{3}$$

$$\frac{\partial N}{\partial t}+\frac{\partial (NU)}{\partial x}+\frac{\partial (NV)}{\partial y}=0, \tag{4}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+C^2\frac{\partial (\log N)}{\partial x}+\frac{\partial \phi}{\partial x}=0, \tag{5}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+C^2\frac{\partial (\log N)}{\partial y}+\frac{\partial \phi}{\partial y}=0, \tag{6}$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=(n-n_0)+(N-N_0), \tag{7}$$

where the quantities c^2 and C^2 are constants representing typical propagation velocities in these units and are proportional to the temperatures of the fluids. The two fluids are coupled only by gravitation.

In a weakly nonlinear system, the nonlinear effects can become comparable with the linear ones only if they accumulate over long times or distances. The strength of the nonlinearity thus introduces new length and time scales into the problem. Kates and Kaup [5] assumed that the physical quantities of interest can be developed in asymptotic expansions depending on the coordinates and the small amplitude parameter ϵ in a prescribed way. The ϵ-dependence can be written as

$$n\simeq n_0+\epsilon n_1+\epsilon^2 n_2+\epsilon^3 n_3+\cdots,\qquad N\simeq N_0+\epsilon N_1+\epsilon^2 N_2+\epsilon^3 N_3+\cdots, \tag{8}$$

$$u\simeq \epsilon u_1+\epsilon^2 u_2+\epsilon^3 u_3+\cdots,\qquad U\simeq \epsilon U_1+\epsilon^2 U_2+\epsilon^3 U_3+\cdots, \tag{9}$$

$$v\simeq \epsilon^2 v_2+\epsilon^3 v_3+\epsilon^4 v_4+\cdots,\qquad V\simeq \epsilon^2 V_2+\epsilon^3 V_3+\epsilon^4 V_4+\cdots, \tag{10}$$

$$\phi\simeq \epsilon\phi_1+\epsilon^2\phi_2+\epsilon^3\phi_3+\cdots. \tag{11}$$

Kates and Kaup [5] deduced the two-dimensional NLS equation for acoustic disturbances in the form

$$i\frac{\partial F}{\partial \tau}+D\frac{\partial^2 F}{\partial \zeta^2}+E\frac{\partial^2 F}{\partial \eta^2}+2W|F|^2F=0, \tag{12}$$

and

$$E=-\frac{1+D_2}{2D_2},\qquad W=\frac{12D_5+6D_4-3D_3-4D_2D_3-D_2^2-4D_3^2}{24D_2},$$
$$D=\frac{D_2-4D_3-3D_2^2}{2D_2^3},\qquad D_m=\frac{n_0\omega^{2m-2}}{(\omega^2-c^2k^2)^m}+\frac{N_0\omega^{2m-2}}{(\omega^2-C^2k^2)^m}, \tag{13}$$

where $m=1,2,\ldots,5$, and k and ω are the wavenumber and frequency.

3. Exact solutions of the NLS equation

Now we shall find classes of solutions of the two-dimensional NLS equation, by applying the function transformation method. Suppose that the solution of Eq. (12) is in the form

$$F(\tau,\zeta,\eta)=f(\xi)\exp(i\theta),\quad \theta=\alpha\zeta+\beta\eta+\gamma\tau,\quad \xi=\zeta+\eta-2(\alpha D+\beta E)\tau, \tag{14}$$

where α, β and γ are constants and $f(\xi)$ is a real function. Substituting (14) into (12), we obtain the ordinary differential equation as

$$(D+E)\frac{d^2f(\xi)}{d\xi^2}-(\gamma+\alpha^2D+\beta^2E)f(\xi)+2Wf(\xi)^3=0, \tag{15}$$

which can be written as

$$\frac{d^2f(\xi)}{d\xi^2} = \left(\frac{\gamma}{D+E} + \alpha^2 + \beta^2\right) f(\xi) - \frac{2W}{D+E} f(\xi)^3. \tag{16}$$

If the right hand side is expressed in the form of a gradient, Eq. (16) becomes

$$\frac{d^2f(\xi)}{d\xi^2} = \frac{d}{df}\left[\frac{1}{2}\left(\frac{\gamma}{D+E} + \alpha^2 + \beta^2\right) f(\xi)^2 - \frac{W}{2(D+E)} f(\xi)^4\right]. \tag{17}$$

Taking into account that

$$\frac{d^2f(\xi)}{d\xi^2} = \frac{d}{df}\left[\frac{1}{2}\left(\frac{df(\xi)}{d\xi}\right)^2\right], \tag{18}$$

Eq. (16) can be written as

$$\frac{df}{\sqrt{\left(\frac{\gamma}{D+E} + \alpha^2 + \beta^2\right) f^2 - \frac{W}{D+E} f^4 + A}} = d\xi, \tag{19}$$

where A is an arbitrary constant of integration.

Now, different values for A give also different analytic solutions of Eq. (19). We give the following cases.

Case (I) In order to get localized solutions we will take $A = 0$ and $\frac{W}{D+E} < 0$ in Eq. (19). Thus, the following bright solitary wave solution for the ordinary differential equation (16) can be obtained as

$$f(\xi) = \frac{4\sqrt{\lambda}\ \exp\left(\sqrt{\lambda}(\xi + \xi_0)\right)}{4\mu\ \exp\left(2\sqrt{\lambda}(\xi + \xi_0)\right) - 1}, \tag{20}$$

and

$$f(\xi) = \frac{4\sqrt{\lambda}\ \exp\left(\sqrt{\lambda}(\xi + \xi_0)\right)}{4\mu\ \exp\left(2\sqrt{\lambda}\xi_0\right) - \exp\left(2\sqrt{\lambda}\xi\right)}, \tag{21}$$

where ξ_0 is a constant of integration, $\lambda = \frac{\gamma}{D+E} + \alpha^2 + \beta^2$ and $\mu = \frac{W}{D+E}$. Then the solution of the two-dimensional NLS equation (12) is in the form

$$F(\tau, \zeta, \eta) = \frac{4\sqrt{\lambda} \exp\left(\sqrt{\lambda}(\zeta + \eta - 2(\alpha D + \beta E)\tau + \xi_0)\right)}{4\mu\ \exp\left(2\sqrt{\lambda}(\zeta + \eta - 2(\alpha D + \beta E)\tau + \xi_0)\right) - 1} \cdot \exp(i(\alpha\zeta + \beta\eta + \gamma\tau)), \tag{22}$$

and

$$F(\tau, \zeta, \eta) = \frac{4\sqrt{\lambda} \exp\left(\sqrt{\lambda}(\zeta + \eta - 2(\alpha D + \beta E + \xi_0))\right)}{4\mu\ \exp\left(2\sqrt{\lambda}\xi_0\right) - \exp\left(2\sqrt{\lambda}(\zeta + \eta - 2(\alpha D + \beta E))\right)} \cdot \exp(i(\alpha\zeta + \beta\eta + \gamma\tau)). \tag{23}$$

Case (II) (a) If $A = 0$, $\xi_0 = 0$, $\frac{\gamma}{D+E} + \alpha^2 + \beta^2 > 0$ and $\frac{W}{D+E} > 0$ in Eq. (19), the following bell-shaped solitary wave solution of Eq. (16) can be obtained:

$$f(\xi) = \sqrt{\frac{\lambda}{\mu}}\, \text{sech}\left(\sqrt{\lambda}\xi\right). \tag{24}$$

The solution of the two-dimensional NLS equation (12) becomes

$$F(\tau, \zeta, \eta) = \sqrt{\frac{\lambda}{\mu}}\, \text{sech}\left(\sqrt{\lambda}(\zeta + \eta - 2(\alpha D + \beta E)\tau)\right) \cdot \exp(i(\alpha\zeta + \beta\eta + \gamma\tau)). \tag{25}$$

(b) If $A = 0$, $\xi_0 = 0$, $\frac{\gamma}{D+E} + \alpha^2 + \beta^2 < 0$ and $\frac{W}{D+E} > 0$ in Eq. (19), the following triangular solution of Eq. (16) can be obtained:

$$f(\xi) = \sqrt{-\frac{\lambda}{\mu}}\, \sec\left(\sqrt{-\lambda}\xi\right). \tag{26}$$

The two-dimensional NLS equation (12) has the solution

$$F(\tau,\zeta,\eta)=\sqrt{-\frac{\lambda}{\mu}}\sec\left(\sqrt{-\lambda}(\zeta+\eta-2(\alpha D+\beta E)\tau)\right)\cdot\exp(i(\alpha\zeta+\beta\eta+\gamma\tau)). \tag{27}$$

Case (III) (a) If $A=\frac{-4(\gamma+\alpha^2+\beta^2)^2}{W(D+E)}$, $\xi_0=0$, $\frac{\gamma}{D+E}+\alpha^2+\beta^2<0$ and $\frac{W}{D+E}>0$ in Eq. (19), the following kink-shaped solitary wave solution of Eq. (16) can be found:

$$f(\xi)=\sqrt{-\frac{\lambda}{\mu}}\tanh\left(\sqrt{-\lambda}\xi\right). \tag{28}$$

Then the solution of the two-dimensional NLS equation (12) becomes

$$F(\tau,\zeta,\eta)=\sqrt{-\frac{\lambda}{\mu}}\tanh\left(\sqrt{-\lambda}(\zeta+\eta-2(\alpha D+\beta E)\tau)\right)\cdot\exp(i(\alpha\zeta+\beta\eta+\gamma\tau)). \tag{29}$$

(b) If $A=\frac{-4(\gamma+\alpha^2+\beta^2)^2}{W(D+E)}$, $\xi_0=0$, $\frac{\gamma}{D+E}+\alpha^2+\beta^2>0$ and $\frac{W}{D+E}>0$ in Eq. (19), the following triangular solution for the ordinary differential equation (16) can be found:

$$f(\xi)=\sqrt{\frac{\lambda}{\mu}}\tan\left(\sqrt{\lambda}\xi\right). \tag{30}$$

Then the solution of the two-dimensional NLS equation (12) becomes

$$F(\tau,\zeta,\eta)=\sqrt{\frac{\lambda}{\mu}}\tan\left(\sqrt{\lambda}(\zeta+\eta-2(\alpha D+\beta E)\tau)\right)\cdot\exp(i(\alpha\zeta+\beta\eta+\gamma\tau)). \tag{31}$$

Case (IV) Eq. (16) admits three Jacobi elliptic function solutions as

(a) If $A=\frac{\lambda^2m^2(m^2-1)}{\mu(2m^2-1)^2}$, $\frac{\gamma}{D+E}+\alpha^2+\beta^2>0$ and $\frac{W}{D+E}<0$ in Eq. (19), the following solution for Eq. (16) can be obtained:

$$f(\xi)=\sqrt{\frac{-\lambda m^2}{\mu(2m^2-1)}}cn\left(\sqrt{\frac{\lambda}{2m^2-1}}\xi\right). \tag{32}$$

The two-dimensional NLS equation (12) has the solution

$$F(\tau,\zeta,\eta)=\sqrt{\frac{-\lambda m^2}{\mu(2m^2-1)}}cn\left(\sqrt{\frac{\lambda}{2m^2-1}}(\zeta+\eta-2(\alpha D+\beta E)\tau)\right)\cdot\exp(i(\alpha\zeta+\beta\eta+\gamma\tau)). \tag{33}$$

(b) If $A=\frac{\lambda^2(1-m^2)}{\mu(2-m^2)^2}$, $\frac{\gamma}{D+E}+\alpha^2+\beta^2>0$ and $\frac{W}{D+E}<0$ in Eq. (19), the following solution for Eq. (16) can be obtained:

$$f(\xi)=\sqrt{\frac{-\lambda}{\mu(2-m^2)}}dn\left(\sqrt{\frac{\lambda}{2-m^2}}\xi\right). \tag{34}$$

Then the solution of the two-dimensional NLS equation (12) becomes

$$F(\tau,\zeta,\eta)=\sqrt{\frac{-\lambda}{\mu(2-m^2)}}dn\left(\sqrt{\frac{\lambda}{2-m^2}}(\zeta+\eta-2(\alpha D+\beta E)\tau)\right)\cdot\exp(i(\alpha\zeta+\beta\eta+\gamma\tau)). \tag{35}$$

(c) If $A=\frac{\lambda^2m^2}{\mu(m^2+1)^2}$, $\frac{\gamma}{D+E}+\alpha^2+\beta^2<0$ and $\frac{W}{D+E}>0$ in Eq. (19), the following solution for Eq. (16) can be found:

$$f(\xi)=\sqrt{\frac{-\lambda m^2}{\mu(m^2+1)}}sn\left(\sqrt{-\frac{\lambda}{m^2+1}}\xi\right). \tag{36}$$

Then the solution of the two-dimensional NLS equation (12) becomes

$$F(\tau,\zeta,\eta)=\sqrt{\frac{-\lambda m^2}{\mu(m^2+1)}}sn\left(\sqrt{-\frac{\lambda}{m^2+1}}(\zeta+\eta-2(\alpha D+\beta E)\tau)\right)\cdot\exp(i(\alpha\zeta+\beta\eta+\gamma\tau)). \tag{37}$$

The only restrictive condition in the above solutions is $\alpha^2 + \beta^2 > -\frac{\gamma}{D+E}$, which corresponds to singular solutions. The four following particular cases arise.

Case (1) $\gamma > 0, D > -E$.

Case (2) $\gamma < 0, D < -E$.

Case (3) $\gamma > 0, E < -D, \alpha^2 + \beta^2 > -\frac{\gamma}{D+E}$.

Case (4) $\gamma < 0, E > -D, \alpha^2 + \beta^2 > -\frac{\gamma}{D+E}$.

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