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Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations

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Abstract. The authors recently derived a family of nonlinear Schrödinger equations on $\mathbb{R}^3$ from fundamental considerations of generalized symmetry: $i\hbar\partial_t\psi = -(\hbar^2/2m)\nabla^2\psi + F[\psi,\bar{\psi}]\psi + i\hbar D\{\nabla^2\psi + (|\nabla\psi|^2/|\psi|^2)\psi\}$, where F is an arbitrary real functional and D a real, continuous quantum number. These equations, descriptive of a quantum mechanical current that includes a diffusive term, correspond to unitary representations of the group $\mathit{Diff}(M)$ parametrized by D, where $M = \mathbb{R}^3$ is the physical space. In the present paper we explore the most natural ansatz for F, which is labelled by five real coefficients. We discuss the invariance properties, describe the stationary states and some non-stationary solutions, and determine the extra, dissipative terms that occur in the Ehrenfest theorem. We identify an interesting, Galilean-invariant subfamily whose properties we investigate, including the case where the dissipative terms vanish.

1. Introduction

Considerations of generalized symmetry in quantum mechanics have led to a partial classification and interpretation of the unitary representations of a certain infinite-dimensional group and to the corresponding self-adjoint representations of its Lie algebra. The group is the natural semidirect product $G(M)$ of the additive group $\mathcal{A}(M)$ of smooth, real-valued scalar functions on the manifold M, with the group $\mathit{Diff}(M)$ of diffeomorphisms of M under composition (here all functions and diffeomorphisms are trivial at infinity). Its Lie algebra $\mathcal{S}(M)$, regarded as the natural (kinematical) *symmetry algebra* of M, is the semidirect sum of the Abelian Lie algebra of functions $\mathcal{A}(M)$, with the algebra $\mathit{Vect}(M)$ of vector fields on M (trivial at infinity), equipped with the usual Lie bracket.

This approach to quantum theory began with a (singular) local algebra of gauge-invariant currents for $M = \mathbb{R}^n$, proposed by Dashen and Sharp [1]; the corresponding non-singular current algebra and the semidirect product group $G(M)$ were derived and represented by Goldin [2, 3]. For further applications, see [4–9]. Doebner, Tolar and their collaborators quantized physical systems localized on smooth, topologically non-trivial manifolds M [10–14]. In the resulting 'quantum Borel kinematics', the result is a self-adjoint representation of an algebra $\mathcal{P}(M)$, consisting of smooth, real-valued functions on M under pointwise operations coupled to $\mathit{Vect}(M)$. But even for $M = \mathbb{R}^3$ (a single, spinless point particle), the irreducible representations of $\mathcal{P}(\mathbb{R}^3)$ or alternatively $\mathcal{S}(\mathbb{R}^3)$ via essentially self-adjoint operators are not unique. Under appropriate conditions they are parametrized (up to unitary equivalence) by a new real, continuous quantum number D [11, 15, 16].

From the assumption of local probability conservation in these representations, we derived recently a family of nonlinear Schrödinger equations, depending on D, governing the time evolution of the wavefunction [17]. For more detail concerning the representations and their interpretation, see [18–21]. The time-dependent quantum-mechanical probability density and current, obtained as expectation values, are (in the Schrödinger picture)

$$\begin{aligned} \rho(x,t) &= \langle \rho_{\rm op}(x)\rangle = \overline{\psi(x,t)}\psi(x,t) \\ j^D(x,t) &= \langle J^D_{\rm op}(x)\rangle = (\hbar/2im)(\bar\psi\nabla\psi - \psi\nabla\bar\psi) - D\nabla(\bar\psi\psi). \end{aligned} \tag{1}$$

Probability conservation is expressed locally by

$$\partial_t\rho = -\nabla\cdot j^D = -\nabla\cdot j^{D=0} + D\nabla^2\rho \tag{2}$$

giving us a Fokker–Planck type of evolution equation, where the quantum number D is a (quantum) diffusion coefficient. Combining (1) with (2) gives the nonlinear Schrödinger equation in the form presented in [17],

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + F[\psi]\psi + i\hbar D\left\{\nabla^2\psi + \frac{|\nabla\psi|^2}{|\psi|^2}\psi\right\} \tag{3}$$

where $F[\psi]$ is an arbitary real functional of ψ, $\bar\psi$ and their derivatives; see below for a more direct formulation of (3). We interpret the right-hand side of (3) as $H[\psi]$ so that $i\hbar\partial_t\psi = H[\psi]$ on an appropriate domain, giving us the most general nonlinear 'Hamiltonian' consistent with (2). Previously we considered some properties of (3) in the cases where $F \equiv 0$, or where $F = V(x)$ is just an external potential. In this paper we consider more general choices of F.

Before proceeding further, we remark that as a consequence of the nonlinearity of the time evolution it is not possible to write '$\rho_{\rm op}(x,t)$' and '$J^D_{\rm op}(x,t)$' as linear operators in a Heisenberg picture. The Schrödinger picture, in which the wavefunctions evolve and the operators are fixed, is essential. This corrects a tacit, erroneous assumption in earlier papers [16–21] but does not affect our conclusions.

2. A parametrized family of equations

Since it is the imaginary part $I[\psi]$ of the nonlinear functional multiplying ψ that is constrained, it is natural to rewrite (3) in the form

$$i\hbar\partial_t\psi = H_0\psi + +i\,I[\psi]\psi + R[\psi]\psi \qquad \text{with} \qquad I[\psi] = \tfrac{1}{2}\hbar D\frac{\nabla^2(\bar\psi\psi)}{\bar\psi\psi} = \tfrac{1}{2}\hbar D\frac{\nabla^2\rho}{\rho} \tag{4}$$

where the linear term $H_0 = -(\hbar^2/2m)\nabla^2 + V(x)$; thus R is the real part of the nonlinear functional multiplying ψ. Our theory to this point gives no further information about $R[\psi]$. It is reasonable to assume then that $R[\psi]$ is of a form similar to $I[\psi]$; i.e. (i) it is proportional to $\hbar D'$, where D' has the dimensions of diffusion coefficient (later we shall make the natural choice $D' = D$); (ii) it is complex homogeneous of order 0, so that $R[\alpha\psi] = R[\psi]$; (iii) it is a rational function, with derivatives no higher than second order occurring in the numerator only; and (iv) it is invariant under the three-dimensional

Euclidean group $E(3)$. Under these conditions we derive a family of nonlinear Schrödinger equations parametrized by five real coefficients together with D' [20, 21]:

$$R[\psi] = \hbar D' \left\{ \lambda_1 \operatorname{Re} \frac{\nabla^2 \psi}{\psi} + \lambda_2 \operatorname{Im} \frac{\nabla^2 \psi}{\psi} + \lambda_3 \operatorname{Re} \frac{(\nabla\psi)^2}{\psi^2} + \lambda_4 \operatorname{Im} \frac{(\nabla\psi)^2}{\psi^2} + \lambda_5 \frac{|\nabla\psi|^2}{|\psi|^2} \right\}. \quad (5)$$

Note that we can keep non-zero terms in $R[\psi]\psi$ by permitting $D' \neq 0$ when $D = 0$, or (formally) by setting $D' = D$ and letting the λ_i be proportional to $1/D$ before passing to the limit as $D \to 0$. Thus we also obtain an interesting family of real nonlinear Schrödinger equations from the considerations here.

Alternatively, it is instructive to express R directly in terms of the physical quantities ρ and $j^{D=0}$. Then we obtain

$$\begin{aligned} R[\psi] &= \hbar D' \left\{ c_1 \frac{m}{\hbar} \frac{\nabla \cdot j^{D=0}}{\rho} + c_2 \frac{\nabla^2 \rho}{\rho} + c_3 \frac{m^2}{\hbar^2} \frac{(j^{D=0})^2}{\rho^2} + c_4 \frac{m}{\hbar} \frac{j^{D=0} \cdot \nabla\rho}{\rho^2} + c_5 \frac{(\nabla\rho)^2}{\rho^2} \right\} \\ &= \hbar D' \sum_{i=1}^{5} c_i R_i[\psi] \end{aligned} \quad (6)$$

where $c_1 = \lambda_2$, $c_2 = \lambda_1/2$, $c_3 = \lambda_5 - \lambda_1 - \lambda_3$, $c_4 = \lambda_4$ and $c_5 = (\lambda_5 - \lambda_1 + \lambda_3)/4$. Conversely, we also have $\lambda_1 = 2c_2$, $\lambda_2 = c_1$, $\lambda_3 = 2c_5 - \frac{1}{2}c_3$, $\lambda_4 = c_4$ and $\lambda_5 = 2c_2 + \frac{1}{2}c_3 + 2c_5$. Using the parametrization with the c_i and setting $D' = D$, we write the parametrized family of equations that results as $\mathcal{F}_D(c_1, \ldots, c_5)$. The equation first derived in [17] with $F[\psi] \equiv 0$ corresponds to the choices $\lambda_2 = -1$, $\lambda_j = 0$ $(j \neq 2)$; equivalently, it is the member $\mathcal{F}_D(-1, 0, 0, 0, 0)$ of our family.

We remark that the 'Hamiltonian' $H[\psi] = H_0\psi + \mathrm{i}I[\psi]\psi + R[\psi]\psi$, as a nonlinear operator on some domain ϑ_0, is not essentially self-adjoint; assuming it to be so would give a quantization (i.e. a representation of $\mathcal{S}(\boldsymbol{R}^3)$) with $D = 0$. Thus H is not a physical observable in the sense of axiomatic quantum theory. Furthermore, the connection between the energy and the formal Hamiltonian, viewed as the generator of the one-dimensional group of time translations, can differ in the linear and the nonlinear cases. We discuss this point further in the next section and indicate one approach to controlling the situation.

The meaning of energy as an observable is just one of several conceptual difficulties faced when one considers the possibility of a nonlinear evolution equation in quantum mechanics. Another unresolved question is how to interpret the possibility of using long-range quantum correlations to send instantaneous signals [22–25], or to communicate among different quantum mechanical worlds in a 'multiple-world' interpretation [26]. These issues have been raised in connection with other proposed nonlinear modifications of the Schrödinger equation [27–29], and in our view are far from being fully understood.

3. Invariance properties of the equations

Because of the form of H_0, $I[\psi]$ and $R[\psi]$, the family $\mathcal{F}_D$ is rotation-invariant if $V = V(|\boldsymbol{x}|)$ and $E(3)$-invariant if $V \equiv 0$. It is interesting and very instructive to inquire about Galilean invariance in the free-particle case [30–32]. Let $G_E^0(3)$ denote the subgroup generated by a central extension of $E(3)$ together with the Galilei boosts, parametrized by $(R, \boldsymbol{a}, \boldsymbol{v})$, where R is a rigid rotation in three dimensions, $\boldsymbol{a}$ is a translation and $\boldsymbol{v}$ is a velocity tranformation; the subscript E indicates the central extension. The full Galilean group $G_E(3)$ is the semidirect product $G_E^0(3) \otimes_s \boldsymbol{R}_t$, where $\boldsymbol{R}_t$ denotes the one-dimensional time translations paramaterized by t. We use this decomposition because $G_E^0(3)$ will enter via a *linear* representation in the Hilbert space, labelled as usual by the mass m (from representing the central element) and having a spin 0; while the representation of $\boldsymbol{R}_t$ is *nonlinear* because of the nonlinearity of H.

The transformation properties of ψ under Galilean boosts are given by $\psi \to \psi'$, where $\psi'(\boldsymbol{x},t) = \exp\,[-\frac{\mathrm{i}}{\hbar}(m\boldsymbol{v}\cdot\boldsymbol{x} + \frac{1}{2}mv^2t)]\,\psi(\boldsymbol{x}+\boldsymbol{v}t,t)$, or equivalently

$$\psi'(\boldsymbol{x}',t') = \exp\left[-\frac{\mathrm{i}}{\hbar}(m\boldsymbol{v}\cdot\boldsymbol{x} - \frac{1}{2}mv^2t)\right]\psi(\boldsymbol{x},t) \tag{7}$$

where $(\boldsymbol{x}',t') = (\boldsymbol{x}-\boldsymbol{v}t,t)$. We now determine the conditions under which (5) and (6) are G_E^0-invariant; i.e. we find the values of the coefficients for which if $\psi(\boldsymbol{x},t)$ is a solution, then $\psi'(\boldsymbol{x},t)$ is also a solution. A straightforward calculation yields the necessary and sufficient conditions,

$$\lambda_2 + \lambda_4 = 0 \qquad \text{and} \qquad \lambda_1 + \lambda_3 = \lambda_5 \tag{8}$$

or equivalently,

$$c_1 + c_4 = 0 \qquad \text{and} \qquad c_3 = 0\,.$$

The invariance of just a part of our family of equations, a subfamily that we label $\mathcal{F}_D^{\mathrm{Gal}}$, is perhaps surprising. We see that the Galilean invariance of the Fokker–Planck equation is broken at the level of the underlying Schrödinger equation. However, the nonlinear term, despite resembling a nonlinear potential (quantum potential), does not entirely spoil the invariance. It is notable that the 'generic' choice $\mathcal{F}_D(0,0,0,0,0)$, with $R \equiv 0$, is invariant.

Thus we can choose R such that invariance under a unitary representation U of $G_E^0(3)$-invariance is maintained. But the representation T of $\boldsymbol{R}_t$, defined at least for a neighbourhood of $t = 0$, is nonlinear; its differential is H. To decide whether U and T combine to give a representation $G_E(3)$ (at least in the neighbourhood of the unit element), we interpret the desired representation as a mapping from the Lie algebra $\mathcal{G}_E(3)$ of $G_E(3)$, to vector fields on $\mathcal{H}$ (regarded as a Hilbert manifold) whose domain of definition is ϑ_0. It is the Lie bracket of the resulting vector fields that must respect the bracket in $\mathcal{G}_E(3)$. Because of the nonlinearity, the Lie bracket does not correspond to a commutator between operators, but to a more involved expression. It is this interpretation that justifies our statement that the members of our family of equations subject to (8) are Galilean invariant. This point of view suggests to us that the physically observable expectation value of the energy be identified with the quantity $\langle \mathrm{i}\hbar\partial_t\rangle = \int \bar{\psi} H(\psi)\,\mathrm{d}^3x$.

A generalized Schrödinger equation in one-dimensional space was discussed recently by Malomed and Stenflo [33], extending earlier work of Stenflo, Yu and Shukla [34]. We note the connection between their equation, motivated by a nonlinear dispersion relation for waves on surfaces between plasmas, and our work. Rewriting [33], equation (7) in our notation, we have

$$\mathrm{i}\partial_t\psi = -\partial_x^2\psi - 2p|\psi|^2\psi + \left\{\overline{C}\frac{(\partial_x\psi)^2}{\psi^2} + C\frac{(\partial_x\overline{\psi})^2}{\overline{\psi}^2} - 2C\frac{\partial_x^2\overline{\psi}}{\overline{\psi}} - 2C\frac{|\partial_x\psi|^2}{|\psi|^2}\right\}\psi \tag{9}$$

where p is a real number and C is introduced as an arbitrary complex parameter. This family of equations intersects the unidimensional version of ours when we take $\hbar = 1$, $m = \frac{1}{2}$,

$V \equiv 0$, $p = 0$ and $C = -\mathrm{i}D/2$; this is the special case of $c_1 = \lambda_2 = 1$, $c_4 = \lambda_4 = -1$ and $c_1 = c_3 = c_5 = 0$ (respectively, $\lambda_1 = \lambda_3 = \lambda_5 = 0$). Thus we obtain $\mathcal{F}_D(1, 0, 0, -1, 0)$ with $V \equiv 0$, which is Galilean-invariant. The discussion in [33], particularly the remark that (9) can be derived from Hamiltonian density $H(x) = |\partial_x\psi|^2 + 2\,\mathrm{Re}[C(\psi/\bar{\psi})(\partial_x\bar{\psi})^2]$, thus pertains to this subfamily. Results on the region in the space of coefficients that give modulational stability for plane-wave solutions to our family of equations (see below) have been obtained by Goldin, Malomed and Stenflo [35] using methods analogous to [33].

4. The Ehrenfest relations

Equation (4) for the nonlinear time evolution lets us calculate the time dependence of the expectation value $\langle A\rangle$ for any observable in the Schrödinger representation. In fact, for $\psi \in \vartheta_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi, A\psi) = \frac{\mathrm{i}}{\hbar}\{(\psi, H_0A\psi) - (\psi, AH_0\psi)\} + \frac{\mathrm{i}}{\hbar}\{(\psi, R[\psi]A\psi) - (\psi, AR[\psi]\psi)\} + \frac{1}{\hbar}\{(\psi, I[\psi]A\psi) + (\psi, AI[\psi]\psi)\}. \tag{10}$$

When A is the position operator $\boldsymbol{x}_{\mathrm{op}}$, we have $R[\psi]\boldsymbol{x}_{\mathrm{op}}\psi - \boldsymbol{x}_{\mathrm{op}}R[\psi]\psi \equiv 0$, while $I[\psi]\boldsymbol{x}_{\mathrm{op}}\psi + \boldsymbol{x}_{\mathrm{op}}I[\psi]\psi = 2\boldsymbol{x}I[\psi]\psi$ and $(\psi, \boldsymbol{x}I[\psi]\psi) = 0$; thus $\langle\boldsymbol{x}_{\mathrm{op}}\rangle$ continues (as in the linear case) to behave classically:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{x}_{\mathrm{op}}\rangle = \frac{\mathrm{i}}{\hbar}\langle[H_0, \boldsymbol{x}]\rangle = \frac{1}{m}\langle\boldsymbol{p}_{\mathrm{op}}\rangle\,. \tag{11}$$

This is expected from the way in which the time dependence was introduced. When A is the momentum operator $\boldsymbol{p}_{\mathrm{op}} = -\mathrm{i}\hbar\nabla$, we calculate

$$(\psi, I[\psi]\,\boldsymbol{p}_{\mathrm{op}}\psi) + (\psi, \boldsymbol{p}_{\mathrm{op}}I[\psi]\psi) = Dm\int\frac{\nabla^2\rho}{\rho}\,\boldsymbol{j}^{D=0}\mathrm{d}^3x \tag{12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{p}_{\mathrm{op}}\rangle = -\langle\nabla V\rangle + Dm\int\frac{\nabla^2\rho}{\rho}\,\boldsymbol{j}^{D=0}\mathrm{d}^3x - \int\rho\nabla R[\psi]\,\mathrm{d}^3x\,. \tag{13}$$

With $R[\psi]$ given by (6), the second term of (13) (stemming from the imaginary nonlinear potential $\mathrm{i}I[\psi]$) can be rewritten:

$$Dm\int\frac{\nabla^2\rho}{\rho}\,\boldsymbol{j}^{D=0}\mathrm{d}^3x = \hbar D\int\rho\,\nabla(R_1[\psi] - R_4[\psi])\mathrm{d}^3x \tag{14}$$

where our calculation makes use of the fact that for $i, k = 1, 2, 3$, with $\boldsymbol{j}^{D=0} = (j_k)$, the quantity $(\partial/\partial x_i)(j_k/\rho)$ is symmetric under exchange of the indices i and k. Furthermore, from the fact that $\int\nabla(\nabla^2\rho/\rho)\mathrm{d}^3x = 0$, we derive

$$2\,(\psi, \nabla R_2[\psi]\psi) = (\psi, \nabla R_5[\psi]\psi)\,. \tag{15}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{p}_{\mathrm{op}}\rangle = -\langle \nabla V\rangle - \hbar D \int \rho\, \nabla[(c_1 - 1)R_1 + (c_2 + 2c_5)R_2 + c_3 R_3 + (c_4 + 1)R_4]\, \mathrm{d}^3x\,. \tag{16}$$

We see that sufficient conditions for the extra, 'dissipative' terms in (22) to vanish are given by

$$c_1 = 1 \qquad c_2 + 2c_5 = 0 \qquad c_3 = 0 \qquad c_4 = -1\,. \tag{17}$$

The corresponding one-parameter family $\mathcal{F}_D^{\mathrm{Ehr}} = \mathcal{F}_D(1, c_2, 0, -1, -c_2/2)$, $c_2 \in \boldsymbol{R}$, is automatically Galilean invariant from (8). For these equations, the time dependences of $\langle \boldsymbol{x}_{\mathrm{op}}\rangle$ and $\langle \boldsymbol{p}_{\mathrm{op}}\rangle$ are independent of D and have the usual classical limits given by the Ehrenfest theorem.

For the equation $\mathcal{F}_D(-1, 0, 0, 0, 0)$ studied in [17], we have from (16)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{p}_{\mathrm{op}}\rangle &= -\langle \nabla V\rangle - m \int \bar{\psi}\left(\frac{\boldsymbol{j}^{D=0}}{\rho}\right)\left(\frac{-D\nabla^2\rho}{\rho}\right)\psi\, \mathrm{d}^3x \\ &\quad + m \int \bar{\psi}\left(\frac{-D\nabla\rho}{\rho}\right)\left(\frac{\nabla\cdot \boldsymbol{j}^{D=0}}{\rho}\right)\psi\, \mathrm{d}^3x\,. \end{aligned} \tag{18}$$

The first term can be understood as arising from friction due to the velocity density $\boldsymbol{j}^{D=0}/\rho$ and the second term as due to the diffusive velocity density $-D\nabla\rho/\rho$; the terms cancel when $\boldsymbol{j}^{D=0}$ is directly proportional to $\nabla\rho$. This corrects typographical errors in equation (6) of [17].

5. On the solutions to the family of equations

Next we construct and examine some solutions to our family of equations, including stationary and time-dependent solutions.

When $V(\boldsymbol{x}) \equiv 0$, we have plane-wave solutions

$$\psi_{\boldsymbol{k}}(\boldsymbol{x}, t) = \exp \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) \tag{19}$$

with a dispersion relation $\omega = (\hbar/2m)k^2 + (1/\hbar)R[\psi]$; since for a plane wave $\boldsymbol{j}^{D=0} = \hbar\boldsymbol{k}/m$, we obtain

$$\omega = \left(\frac{\hbar}{2m} + Dc_3\right)k^2\,. \tag{20}$$

This dispersion relation physically distinguishes c_3 from the other coefficients.

As in the linear $D = 0$ case, there are solutions to (9) in which the space and time coordinates separate,

$$\psi_E(\boldsymbol{x}, t) = \psi_E(\boldsymbol{x}) \exp\left(-\mathrm{i}\frac{E}{\hbar}t\right) \qquad H[\psi_E(\boldsymbol{x})] = E\psi_E(\boldsymbol{x}) \tag{21}$$

with $E \in \boldsymbol{R}$, where ψ_E is square integrable and depends on $V(\boldsymbol{x})$, $c_1, \ldots, c_5$ and D. When such solutions exist we call them stationary states, because of the fact that

$\partial_t\rho = \partial_t[\overline{\psi_E(x,t)}\psi_E(x,t)] = 0$. Note that defining 'stationarity' via $\partial_t\rho = 0$ gives us $\nabla \cdot j^D(x,t) = 0$; i.e. $j^D(x,t)$ is a divergenceless vector field density that is otherwise arbitrary. To find solutions of the form of (21) to our family of nonlinear Schrödinger equations, we construct *affiliated linear* Schrödinger equations, in the sense that solutions to the latter yield solutions to the former via a nonlinear transformation of (21). In this way, we actually obtain eigenfunctions. Such methods are applicable to certain types of nonlinear partial differential equations, e.g. to Ricatti equations.

Let us write (omitting an index to indicate possible degeneracy in E),

$$\psi_E(x) = f_E(x)\exp(\mathrm{i}h_E(x)) \tag{22}$$

where $f_E(x)$ and $h_E(x)$ are real and $f_E(x) \geqslant 0$. Making the generic assumption $j^D(x,t) = 0$ to ensure stationarity now implies

$$-\frac{\hbar}{m}f_E^2(x)\nabla h_E(x) + D\nabla f_E^2(x) = 0\,. \tag{23}$$

Hence

$$h_E(x) = \Gamma \log f_E^2(x) \tag{24}$$

where $\Gamma = Dm/\hbar$ is the dimensionless constant introduced in [17]. From (21)–(24), we obtain after a straightforward calculation the following result. *Stationary solutions to the family of equations $\mathcal{F}_D$ given by (4)–(6), satisfying $H[\psi_E] = E\psi_E$, are obtained from solutions to an affiliated linear Schrödinger equation*

$$-\frac{\hbar^2}{2m^*}\nabla^2\phi_E(x) + V(x)\phi_E(x) = E\phi_E(x) \tag{25}$$

with the same potential $V(x)$ as in $\mathcal{F}_D$, but with a shifted 'effective mass' $m^* = \beta m$; where $\beta \in \boldsymbol{R}$ is given by

$$\beta = \beta(\Gamma, c_1, \ldots, c_5) = \frac{1 - 4\Gamma^2 - 8\Gamma(c_2 + c_5) - 8\Gamma^2(c_1 + c_4) - 8\Gamma^3 c_3}{(1 - 4\Gamma c_2 - 4\Gamma^2 c_1)^2}\,. \tag{26}$$

The stationary solutions of $\mathcal{F}_D$ are now given by

$$\psi_E(x) = \phi_E(x)^\alpha \exp\left[\mathrm{i}\Gamma\alpha\log\phi_E(x)^2\right] \tag{27}$$

with

$$\alpha = \alpha(\Gamma, c_1, \ldots, c_5) = \frac{1}{\beta\,(1 - 4\Gamma c_2 - 4\Gamma^2 c_1)}\,. \tag{28}$$

In addition $\psi_E(x)$ is square integrable if $[\phi_E(x)]^\alpha$ is square integrable and has the same degeneracy as $\phi_E(x)$. Note, however, that a set of degenerate eigenfunctions will not in general span a degeneracy subspace of eigenfunctions, because of the nonlinearity in the equation. It is desirable that $\beta > 0$, because otherwise the 'effective mass' m^* becomes negative (or, alternatively, the potential changes its sign). For any Γ, there are values of $c_1, \ldots, c_5$ fulfilling these conditions, suggesting that a variety of dissipative processes can be described by R. If $R \equiv 0$, then $\alpha = 1/(1 - 4\Gamma^2)$ and $\beta = 1 - 4\Gamma^2$.

The cases $\alpha \equiv 1$ (for all Γ) and $\beta \equiv 1$ (for all Γ) are of special interest. The case $\alpha \equiv 1$ is specified by the subfamily

$$\mathcal{F}_D^{\alpha\equiv 1}: \qquad c_1 + 2c_4 = -1 \qquad c_2 + 2c_5 = 0 \qquad c_3 = 0 \tag{29}$$

and the case $\beta \equiv 1$ by the subfamily

$$\mathcal{F}_D^{\beta\equiv 1}: \qquad c_1 = 0 \qquad 2\,(c_2^2 + c_4) = -1 \qquad c_3 = c_5 = 0\,. \tag{30}$$

These families intersect in the equation $\mathcal{F}_D(0,0,0,-1/2,0)$. The subfamily $\mathcal{F}_D^{\mathrm{Ehr}}$ given by (17) is contained in $\mathcal{F}_D^{\alpha\equiv 1}$; in fact, $\mathcal{F}_D^{\mathrm{Ehr}}$ is just the (one-parameter) intersection of the Galilean-invariant subfamily with $\mathcal{F}_D^{\alpha\equiv 1}$:

$$\mathcal{F}_D^{\alpha\equiv 1} \cap \mathcal{F}_D^{\mathrm{Gal}} = \mathcal{F}_D^{\mathrm{Ehr}}\,. \tag{31}$$

Furthermore, $\mathcal{F}_D^{\beta\equiv 1} \cap \mathcal{F}_D^{\mathrm{Gal}}$ is empty. As discussed in [17], $\mathcal{F}_D(-1,0,0,0,0)$ gives $\alpha \equiv 1$, $\beta = 1/(1+4\Gamma^2)$.

The structure of these stationary solutions is clearest when we set $\alpha = 1$, so that (27) gives

$$\psi_E(\boldsymbol{x},t) = \phi_E(\boldsymbol{x}) \exp\left[-\mathrm{i}\frac{E}{\hbar}t\right] \exp\left[\mathrm{i}\Gamma \,\log\, \phi_E(\boldsymbol{x})^2\right]. \tag{32}$$

We see that, because of the nonlinearity of (4), the bound-state solutions $\phi_E(\boldsymbol{x})$ of the affiliated linear Schrödinger equation with shifted mass m^* and potential $V(\boldsymbol{x})$ acquire a phase that depends on $\boldsymbol{x}$ through $\phi_E(\boldsymbol{x})$. A similar situation holds for $\alpha \neq 1$. The transition from the nonlinear equation to a related linear one is 'mediated' partly through this $\boldsymbol{x}$-dependent phase.

Some non-stationary solutions to our family of equations are known. Noting the role of the phase, Goldin [18] wrote a one-dimensional Gaussian solution for $\mathcal{F}_D(-1,0,0,0,0)$, with $V \equiv 0$. Further Gaussian solutions have been constructed and studied by Dodonov, Mizrahi, Nattermann, Scherer, Ushveridze and the authors [36–39], in the cases $V \equiv 0$ and $V(x)$ proportional to x^2. Among the results are soliton solutions in the free, Galilean-invariant case. For the harmonic oscillator potential it is shown in [38] that for certain members of our family of equations Gaussian solutions approach asymptotically in time the ground state of the affiliated linear Schrödinger equation. Thus the ground state behaves like an attractor of a flow in $\mathcal{H}$ (understood as a Hilbert manifold).

6. Summary and Conclusions

A new family of nonlinear Schrödinger equations, that includes a diffusion term, originates from fundamental considerations of local symmetry in quantum mechanics. Natural assumptions on the form of an arbitrary, real nonlinear functional multiplying ψ, allowing for the usual linear potential $V(\boldsymbol{x})$, lead to a five-parameter family $\mathcal{F}_D$ of evolution equations, which are candidates for describing dissipative and diffusion processes in quantum mechanics.

The family has some very appealing properties. As $D \to 0$ it goes over smoothly into the usual Schrödinger equation. For $V = V(\boldsymbol{x})$, $\mathcal{F}_D$ is rotation invariant; for $V \equiv 0$ it is Euclidean-invariant; and for certain choices of the five parameters it is even invariant under a central extension of the Galileo group in which the time-translation subgroup is represented nonlinearly. An 'arrow of time' associated with the sign of D is introduced into ordinary, nonrelativistic quantum mechanics in the simplest possible way. The time derivative of $\langle x_{\rm op}\rangle$ is as in ordinary quantum mechanics, while the time derivative of $\langle p_{\rm op}\rangle$ *has additional terms proportional to D indicating dissipation or friction.*

Because our equation is homogeneous, it generalizes to a hierarchy of N-particle evolution equations that respect the separation property; i.e. initially uncorrelated subsystems, in the absence of interactions, remain uncorrelated [27, 40].

We have found square-integrable stationary solutions. Bound states of an affiliated linear Schrödinger equation, with the same potential as in the linear case but with a shifted effective mass m^*, give solutions of the nonlinear equation through a spatially dependent phase proportional to D. Extending this technique, time-dependent Gaussian solutions are also found.

If such a nonlinear evolution equation proves acceptable, it has possible experimentally testable consequences. For example, the effective mass can be measured in precision quantum mechanical experiments, e.g. in two-level systems, while observation of a non-stationary particle's 'path' together with the Ehrenfest relation leads in principle to an independent measurement of the mass m. Such experiments can give an upper bound on Γ.

The group-theoretical justification for this nonlinear modification of quantum mechanics seems to suggest it as a 'minimal' nonlinear generalization of the Schrödinger equation. Such a model needs considerably more mathematical and physical investigation, and most importantly a convincing physical interpretation (which we do not yet have) for the sources of dissipation and the origin of the 'arrow of time' that it describes.

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