8.1. The NLS equation

The Nonlinear Schödinger (NLS) equation

$$\partial_t u = \mathrm{i}\nu_1 \partial_x^2 u + \mathrm{i}\nu_2 |u|^2 u, \tag{8.1}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}$, $u = u(x,t) \in \mathbb{C}$, and coefficients $\nu_1, \nu_2 \in \mathbb{R}$, can be derived by multiple scaling analysis in order to describe the evolution of the envelope of a spatially and temporarily oscillating wave packet, as will be discussed in Chapter 11. By rescaling $u \mapsto \beta_1 u$, $x \mapsto \beta_2 x$, and $t \mapsto \beta_3 t$ with $\beta_j \in \mathbb{R}$ for $j \in \{1,2,3\}$ it can be transformed into

$$\partial_t u = -\mathrm{i}\partial_x^2 u + \alpha \mathrm{i} |u|^2 u, \qquad \alpha = \pm 1. \tag{8.2}$$

See Exercise 8.1. Two cases remain, namely $\alpha = -1$ and $\alpha = 1$. The case $\alpha = -1$ is called focusing and the case $\alpha = +1$ is called defocusing. In particular, the focusing NLS equation is widely used in nonlinear optics to describe the evolution and interaction of optical pulses. The NLS equation also plays a role in some theories about the occurrence of so called freak or rogue waves, cf. [**Osb10**]. The variant

$$\partial_t u = -\mathrm{i}\partial_x^2 u + \mathrm{i}Vu + \alpha \mathrm{i} |u|^2 u, \tag{8.3}$$

with $V = V(x)$ some potential, is called the Gross-Pitaevski equation and can be derived for the description of Bose-Einstein condensates, cf. [**Pel11**]. In summary the NLS equation is a widely used model in nonlinear physics.

The evolution of the NLS equation is not smoothing in the sense of the KPP or the Burgers equation. Hence, the phase spaces are not connected by smoothing. We can have global existence or stability in one space, but explosion or instability in another space.

8.1.1. Nonlinear oscillations and pulse solutions. The NLS equation consists of two parts, namely the dispersion part $-\mathrm{i}\partial_x^2 u$ and the nonlinear oscillation part $\alpha \mathrm{i} u |u|^2$.

Making the ansatz $u(x,t) = v(t) = r(t)\mathrm{e}^{\mathrm{i}\phi(t)}$ for x-independent solutions we find

$$\partial_t r = 0 \quad \text{and} \quad \partial_t \phi = \alpha r^2,$$

with the solutions $r(t) = r(0)$ and $\phi(t) = \phi(0) + \alpha r^2(0)t$, i.e., oscillations where the frequency increases with r. Next, we search for solutions of the form $u(x,t) = v(t)\mathrm{e}^{\mathrm{i}kx} = r(t)\mathrm{e}^{\mathrm{i}\phi(t)}\mathrm{e}^{\mathrm{i}kx}$. We find

$$\partial_t r = 0 \quad \text{and} \quad \partial_t \phi = k^2 + \alpha r^2,$$

with the solutions $r(t) = r(0)$ and $\phi(t) = \phi(0) + \omega(k,r)t$, where $\omega(k,r) = (k^2 + \alpha r^2)$. Thus, we have solutions which are periodic in time and space, namely

$$u(x,t) = u(x,t;k,r,\phi_0) = r\mathrm{e}^{(\mathrm{i}(kx+\phi_0+(k^2+\alpha r^2)t))}.$$

In the defocusing case, $\alpha = 1$, where always $\omega(k, r) = k^2 + \alpha r^2 > 0$, all periodic waves travel left, whereas in the focusing case, $\alpha = -1$, where $\omega(k, r) = k^2 + \alpha r^2$ can have either sign, the periodic waves can travel left or right. For illustration, we sketch some of these nonlinear oscillations in Figure 8.1. Interestingly, in the focussing case they are all unstable, which is known as Benjamin–Feir instability. See Exercise 8.23 where we consider the related stability question for spatially periodic solutions in the GL equation.

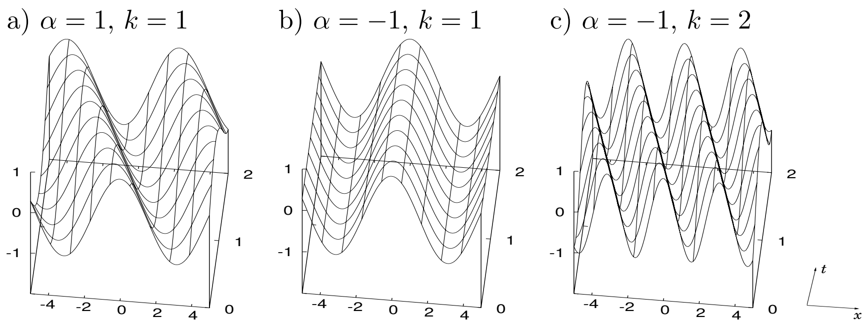

Figure 8.1. Nonlinear oscillations (real part) for the NLS equation, with $r = 1$. a) shows the defocusing case, where all waves travel left, while b), c) show the focusing case, where waves can travel left or right.

However, from the point of applications of the NLS equation, in particular due to its derivation for the description of modulations of electromagnetic waves, pulse solutions are more interesting. In order to find them, we make the ansatz

$$u(x, t) = B(x - ct)\mathrm{e}^{\mathrm{i}(qx - \omega t + \phi_0)},$$

with $B(\xi) \in \mathbb{R}$, which yields

$$-\mathrm{i}\omega B - cB' = -\mathrm{i}B'' + 2qB' + \mathrm{i}q^2 B + \alpha \mathrm{i} B^3.$$

Separating real and imaginary part gives the relations $c = -2q$ and

$$0 = B'' - (\omega + q^2)B - \alpha B^3. \tag{8.4}$$

This ODE can be discussed using the methods from §2.3.3, see in particular Remark 2.3.21. One finds that for $\omega + q^2 > 0$ and $\alpha = -1$ there exist solutions of (8.4) which are homoclinic to the origin. Figure 8.2 shows the phase portrait. In fact, there exist explicit formulas for homoclinic solutions of (8.4). Before computing these in Exercise 8.4, it is useful to have a look at the symmetries of the NLS equation in Exercise 8.3.

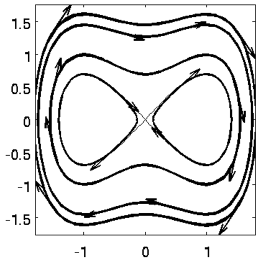

Figure 8.2. Phase portrait for (8.4), $\omega + q^2 = 1$ and $\alpha = -1$.

8.1.2. Dispersion. The linear Schrödinger equation

$$\partial_t u = -\mathrm{i}\partial_x^2 u \tag{8.5}$$

is a prototype example of a linear dispersive equation. Dispersion means that the phase velocity $c_p(k) = \omega(k)/k$ of harmonic waves $\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}$ depends in a non-trivial way on k. This has a number of consequences which we explain in the following. For (8.5) we have solutions $u(x,t) = \mathrm{e}^{\mathrm{i}(kx-\omega t)}$ where $\omega = -k^2$. Hence, $c_p(k) = k$ and the amplitude of the harmonic waves is conserved. This behavior is in contrast to the linear diffusion equation $\partial_t u = \partial_x^2 u$ where we have $u(x,t) = \mathrm{e}^{\mathrm{i}kx+\lambda t}$, with $\lambda = -k^2$, and thus all spatially harmonic waves are damped with some exponential rate, except for $k = 0$.

Like for the diffusion equation there exists an explicit solution formula which can be derived by Fourier transform and which at first looks very similar to the one of the diffusion equation.

Lemma 8.1.1. *The initial value problem for the linear Schrödinger equation* $\partial_t u = -\mathrm{i}\partial_x^2 u$, $u|_{t=0} = u_0$, *is solved by*

$$u(x,t) = \frac{1}{\sqrt{-4\pi \mathrm{i}t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}(x-y)^2}{4t}} u_0(y)\,\mathrm{d}y. \tag{8.6}$$

The integral exists for $u_0 \in L^1(\mathbb{R})$ *and extends to an isometry in* $L^2(\mathbb{R})$, *i.e.,* $\|u(\cdot,t)\|_{L^2} = \|u_0\|_{L^2}$.

Proof. Proceeding exactly as in Example 7.3.20 it remains to calculate $G(x,t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k^2 t}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k$. We have

$$\partial_x G(x,t) = \int_{\mathbb{R}} \mathrm{i}k\mathrm{e}^{\mathrm{i}k^2 t}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = -\frac{\mathrm{i}x}{2t} G(x,t).$$

This differential equation is solved by $G(x,t) = C\mathrm{e}^{-\mathrm{i}x^2/4t}$. The constant C can be computed through

$$C = G(0,t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k^2 t}\,\mathrm{d}k = \sqrt{\frac{1}{t}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y^2}\,\mathrm{d}y = \sqrt{\frac{\pi}{t}}\mathrm{e}^{\mathrm{i}\pi/4} = \sqrt{\frac{\pi}{-\mathrm{i}t}}.$$

The integral $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y^2}\,\mathrm{d}y$ exists as improper integral due to faster and faster oscillations of $\mathrm{e}^{\mathrm{i}y^2}$ for $|y| \to \infty$. The isometry follows immediately from

$$\|u(\cdot,t)\|_{L^2} = (2\pi)^{1/2}\|\widehat{u}(\cdot,t)\|_{L^2} = (2\pi)^{1/2}\|\widehat{u}(\cdot,0)\|_{L^2} = \|u_0\|_{L^2}. \qquad \square$$

Like for diffusion, from (8.6) we immediately obtain the estimate

$$\sup_{x\in\mathbb{R}} |u(x,t)| \leq \frac{C}{\sqrt{t}}\int_{-\infty}^{\infty} |u(x,0)|\,\mathrm{d}x, \tag{8.7}$$

i.e., solutions to spatially localized initial conditions decay uniformly towards zero with a rate $t^{-1/2}$. But there are major differences: by diffusion energy is lost, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\int |u(x,t)|^2\,\mathrm{d}x \leq 0$, while dispersion conserves energy, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\int |u(x,t)|^2\,\mathrm{d}x = 0$, but spreads it all over the real line.

Dispersion smoothes solutions locally in space. We can compute

$$\partial_x u(x,t) = \frac{1}{\sqrt{4\pi\mathrm{i}t}}\int_{-\infty}^{\infty} \frac{-2\mathrm{i}(x-y)}{4t}\mathrm{e}^{-\frac{\mathrm{i}(x-y)^2}{4t}}u(y,0)\,\mathrm{d}y < \infty$$

for all $x \in \mathbb{R}$ and all $t > 0$ if $\int_{-\infty}^{\infty} |y||u(y,0)|\,\mathrm{d}y < \infty$, i.e., $\partial_x u(x,t)$ can be computed point-wise, although the initial condition may only be continuous. More generally, if the moment $\int_{-\infty}^{\infty} |y|^n|u(y,0)|\,\mathrm{d}y$ is finite, then $\partial_x^n u(x,t)$ is finite for all $x \in \mathbb{R}$ and all $t > 0$.

The reason for this behavior is as follows: Consider initial conditions which are slow modulations in space of an underlying carrier wave, i.e., initial conditions of the form

$$u(x,0) = u_0(\varepsilon x)\mathrm{e}^{\mathrm{i}kx}$$

where $0 < \varepsilon \ll 1$ is a small parameter and u_0 a smooth spatially localized function. The right-hand side of the Schrödinger equation applied to this initial condition yields

$$\mathrm{i}\partial_x^2(u_0(\varepsilon x)\mathrm{e}^{\mathrm{i}kx}) = \mathrm{i}\mathrm{e}^{\mathrm{i}kx}\left(-k^2u_0(\varepsilon x) + \varepsilon 2\mathrm{i}ku_0'(\varepsilon x) + \varepsilon^2 u_0''(\varepsilon x)\right).$$

This motivates us to make the ansatz

$$u(x,t) = B\left(\varepsilon(x - c_g t), \varepsilon^2 t\right)\mathrm{e}^{\mathrm{i}k(x-c_p t)},$$

with constants c_p and c_g, and a function $B = B(\xi,\tau)$ with $\xi = \varepsilon(x - c_g t)$ and $\tau = \varepsilon^2 t$. Inserting this ansatz and computing the coefficients in front of ε^0, ε^1, and ε^2 shows that

$$-\mathrm{i}kc_p = -\mathrm{i}k^2, \qquad -c_g\partial_\xi B = -2k\partial_\xi B, \qquad \text{and} \qquad \partial_\tau B = \mathrm{i}\partial_\xi^2 B.$$

The constant $c_p = \omega(k)/k = k$ with frequency $\omega = k^2$ is called phase velocity. The constant $c_g = \omega'(k) = 2k$ is called the group velocity. This calculation shows that wave packets with carrier wave $\mathrm{e}^{\mathrm{i}kx}$ with k large are transported with large group velocity $c_g = 2k$ towards infinity. If the initial condition is spatially localized, then nothing can come from infinity. Therefore, at fixed $x \in \mathbb{R}$ only wave packets with low derivatives remain, and hence a local smoothing occurs.

On the other hand this also has the consequence that if the initial condition is not spatially localized, then packets with high derivatives can come from infinity. In fact, there is no local existence and uniqueness of solutions in $C^0_{b,\mathrm{unif}}(\mathbb{R},\mathbb{R})$.

Remark 8.1.2. For the transport equation and the linear wave equation we obtain $\omega(k) = ck$ with a constant $c \in \mathbb{R}$ and hence the group velocity $\omega'(k) = c$ is bounded and the transport equation and the linear wave equation can be solved in $C^0_{b,\mathrm{unif}}(\mathbb{R},\mathbb{R})$. ⌋

It turns out that $X = L^2(\mathbb{R},\mathbb{C})$ is a good choice for the handling of the linear Schrödinger equation as a dynamical system.

Lemma 8.1.3. *The curve of solutions $t \mapsto U(t,u_0)$ with $U(t,u_0)(x) = u(x,t)$ and $u(x,0) = u_0$ is continuous in X if $u_0 \in X$. Moreover, $U(t+s,u_0) = U(t,U(s,u_0))$.*

Proof. First of all we have the semigroup property

$$\begin{aligned} U(t+s,u_0) &= \mathcal{F}^{-1}(k \mapsto \mathrm{e}^{\mathrm{i}k^2(t+s)}\widehat{u}_0(k)) \\ &= \mathcal{F}^{-1}(k \mapsto \mathrm{e}^{\mathrm{i}k^2 t}(\mathrm{e}^{\mathrm{i}k^2 s}\widehat{u}_0(k))) = U(t,U(s,u_0)) \end{aligned}$$

such that is sufficient to prove continuity in $t = 0$. For all $\varepsilon > 0$ we have to find a $t_0 > 0$ such that for all $t \in (0,t_0)$ we have

$$\begin{aligned} \|U(t,u_0) - u_0\|_{L^2}^2 = &\int_{\mathbb{R}} |u(x,t) - u(x,0)|^2 \,\mathrm{d}x = 2\pi \int_{\mathbb{R}} |\widehat{u}(k,t) - \widehat{u}(k,0)|^2 \,\mathrm{d}k \\ =&2\pi \int_{\mathbb{R}} |(\mathrm{e}^{\mathrm{i}k^2 t} - 1)\widehat{u}(k,0)|^2 \,\mathrm{d}k = 2\pi \int_{|k|<L} \ldots \,\mathrm{d}k + 2\pi \int_{|k|\geq L} \ldots \,\mathrm{d}k < \varepsilon^2, \end{aligned}$$

where L is chosen so large that

$$2\pi \int_{|k|\geq L} \ldots \,\mathrm{d}k \leq 4\pi \int_{|k|\geq L} |\widehat{u}(k,0)|^2 \,\mathrm{d}k < \varepsilon^2/2.$$

For this L we find for the first integral

$$\begin{aligned} 2\pi \int_{|k|<L} \ldots \,\mathrm{d}k &\leq 2\pi \sup_{|k|<L} |\mathrm{e}^{\mathrm{i}k^2 t} - 1|^2 \int_{\mathbb{R}} \widehat{u}(k,0)|^2 \,\mathrm{d}k \\ &\leq 2\pi |\mathrm{e}^{\mathrm{i}L^2 t} - 1|^2 \int_{\mathbb{R}} |\widehat{u}(k,0)|^2 \,\mathrm{d}k < \varepsilon^2/2 \end{aligned}$$

if $t_0 > 0$ is sufficiently small. Therefore, we are done. □

The choice $X = L^2(\mathbb{R}, \mathbb{C})$ is only a good choice for the handling of the linear Schrödinger equation. For the NLS equation Sobolev spaces $H^m(\mathbb{R}, \mathbb{C})$ with $m > 1/2$ are more adequate.

8.1.3. Local existence and uniqueness, and the Hamiltonian. In the following we choose $X = H^m(\mathbb{R}, \mathbb{C})$ with $m > 1/2$ as phase space for the NLS equation.

Theorem 8.1.4. *Let $u_0 \in X$. Then there exist a $T_0 > 0$ and a unique local solution $u \in C([0, T_0], X)$ of the NLS equation with $u|_{t=0} = u_0$.*

Proof. According to Exercise 8.6 the operator $-\mathrm{i}\partial_x^2$ generates a C_0-semigroup $\mathrm{e}^{-\mathrm{i}t\partial_x^2} : X \to X$. Moreover, by Lemma 7.3.29 the nonlinearity $u \mapsto \mathrm{i}\alpha|u|^2 u$ is locally Lipschitz-continuous in X. The local existence thus follows by the variation of constant formula and the contraction mapping theorem, cf. the proof of Theorem 7.1.7. □

Remark 8.1.5. Like for the wave equation or the transport equation the solution also exists backwards in time, i.e., $u \in C([-T_0, T_0], X))$. ⌋

This local solution can be continued as long as the H^m-norm of the solution stays bounded. First we show that the L^2-norm is conserved along solutions. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u\overline{u}\,\mathrm{d}x = \int_{\mathbb{R}} (\partial_t u)\overline{u} + u(\partial_t \overline{u})\,\mathrm{d}x \\ &= 2\,\mathrm{Re}\int_{\mathbb{R}} \overline{u}\left(-\mathrm{i}(\partial_x^2 u - \alpha|u|^2 u)\right)\mathrm{d}x = 0. \end{aligned}$$

The estimate for the H^1-norm is related to the Hamiltonian structure of the NLS equation. It turns out that the NLS equation, like the KdV equation in the next section, is a completely integrable Hamiltonian system which can be solved explicitly. See [**DJ89**, §6] for an overview. Here, we only show that the NLS equation is a Hamiltonian system. With

$$H(u) = \int_{\mathbb{R}} \frac{1}{2}|\partial_x u(x)|^2 + \frac{1}{4}\alpha|u(x)|^4\,\mathrm{d}x$$

we find

$$\begin{aligned} \partial_u H[v] &= \lim_{\varepsilon\to 0} \varepsilon^{-1}(H(u+\varepsilon v) - H(u)) \\ &= \lim_{\varepsilon\to 0} \varepsilon^{-1}\int_{\mathbb{R}} |\partial_x(u+\varepsilon v)|^2/2 + \alpha|u+\varepsilon v|^4/4 - |\partial_x u|^2/2 - |u|^4/4\,\mathrm{d}x \\ &= \mathrm{Re}\int_{\mathbb{R}} (-\partial_x^2\overline{u} + \alpha\overline{u}|u|^2)v\,\mathrm{d}x \end{aligned}$$

which maps $v \in X = L^2(\mathbb{R}, \mathbb{C})$ linearly into $\mathbb{R}$, i.e., $\partial_u H$ is a linear map from X to $\mathbb{R}$ and hence an element of the dual space. According to the Riesz representation theorem [**Alt16**, Satz 4.1], in Hilbert spaces the dual space can be identified with X by defining a map (the canonical isomorphism)

$$\beta : \mathrm{Lin}(X, \mathbb{R}) \to X, \qquad \left(v \mapsto \langle u, v\rangle_{L^2} = \mathrm{Re} \int_{\mathbb{R}} u(x)\overline{v(x)}\,\mathrm{d}x\right) \mapsto u \tag{8.8}$$

and therefore $\beta\partial_u H = -\partial_x^2 u + \alpha u|u|^2$. We finally have

$$\partial_t u = -\mathrm{i}\partial_x^2 u + \alpha \mathrm{i} u|u|^2 = \mathrm{i}\beta\partial_u H(u) = J\beta\partial_u H(u)$$

where the operator $Ju = \mathrm{i}u$ is skew symmetric in X since

$$\langle Ju, v\rangle_{L^2} = \mathrm{Re} \int_{\mathbb{R}} \overline{\mathrm{i}u(x)} v(x)\,\mathrm{d}x = -\,\mathrm{Re} \int_{\mathbb{R}} \overline{u(x)}\mathrm{i}v(x)\,\mathrm{d}x = -\langle u, Jv\rangle_{L^2}.$$

The Hamiltonian is well defined on H^1 since $H \leq \|\partial_x u\|_{L^2}^2 + \|u\|_{C^0}^2\|u\|_{L^2}^2$. In the defocusing case the Hamiltonian is positive definite, i.e., $H(u) > 0$ for $u \neq 0$, whereas in the focusing case it is indefinite. Hence, in the defocusing case we get the H^1-estimate for free since then

$$\|u(t)\|_{H^1}^2 \leq H(t) + \|u(t)\|_{L^2}^2 = H(0) + \|u_0\|_{L^2}^2$$

for all $t \in \mathbb{R}$ and thus the solution exists globally with a uniform bound in $H^1(\mathbb{R})$, and we already proved half of the following theorem.

Theorem 8.1.6. *For $u_0 \in H^1(\mathbb{R})$ the local solution of the NLS equation exists globally in time and stays uniformly bounded in $H^1(\mathbb{R})$.*

Proof. It remains to consider the focusing case. We have $\|u(t)\|_{L^2}^2 = \|u(0)\|_{L^2}^2$ and $H(u(t)) = H(u(0))$ and thus

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2\,\mathrm{d}x = H(u(0)) + \frac{1}{4}\int_{\mathbb{R}} |u|^4\,\mathrm{d}x. \tag{8.9}$$

To estimate the last term we use the Gagliardo-Nirenberg estimate, cf. Lemma 6.3.10,

$$\int_{\mathbb{R}} |u|^{q+1}\,\mathrm{d}x \leq \left(\int_{\mathbb{R}} |u|^2\,\mathrm{d}x\right)^{(2(q+1)-d(q-1))/4} \left(\int_{\mathbb{R}} |\nabla u|^2\,\mathrm{d}x\right)^{d(q-1)/4}, \tag{8.10}$$

which for $q = 3$ and $d = 1$ yields

$$\int_{\mathbb{R}} |u|^4\,\mathrm{d}x \leq \left(\int_{\mathbb{R}} |u|^2\,\mathrm{d}x\right)^{3/2} \left(\int_{\mathbb{R}} |\partial_x u|^2\,\mathrm{d}x\right)^{1/2}. \tag{8.11}$$

Therefore, with $\phi(u) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2\,\mathrm{d}x$ we have

$$0 \leq \phi(u(t)) \leq H(u(0)) + \|u(0)\|_{L^2}^{3/2}\sqrt{\phi(u(t))}$$

since the L^2-norm and the Hamiltonian are conserved. This inequality immediately implies a uniform bound for $\phi(u(t))$ and so also the H^1-norm stays bounded. $\square$

Remark 8.1.7. Note that (8.10) depends on d. In fact, for $x \in \mathbb{R}^d$ with $d \geq 2$ the solutions of $\partial_t u = -\mathrm{i}\Delta u - \mathrm{i}|u|^2 u$ can blow up in finite time T, i.e., $\|u(t)\|_{H^1} \to \infty$ as $t \to T$, cf. [**SS99b, Fib15**]. $\rfloor$

As already said, we come back to the NLS equation in Part IV, while here we close with some bounds for oscillatory integrals which are used in the derivation of dispersive estimates.

8.1.4. The method of stationary phase. We explain a method which gives more insight into dispersion and which especially allows to compute decay rates like (8.7) from the Fourier representation $\widehat{u}(k,t) = \widehat{G}(k,t)\widehat{u}(k,0)$ of the solution u also in situations where in physical space no explicit representation formula for G such as (8.6) is known. Our simple approach is based on the subsequent Lemma of van der Corput, cf. [**Ste93**].

Lemma 8.1.8. (Lemma of van der Corput) *Let $\phi : \mathbb{R} \to \mathbb{R}$ in C^ℓ with $|\phi^{(\ell)}| \geq 1$ for all $\theta \in [a,b]$. For $\ell = 1$ assume the monotonicity of ϕ'. Then there exists a $C_\ell > 0$ such that*

$$\left| \int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta) d\theta} \right| \leq C_\ell t^{-1/\ell}.$$

Proof. For $\ell = 1$ we obtain

$$\begin{aligned} I(t) &= \int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta)}\,\mathrm{d}\theta = \int_a^b \frac{1}{\mathrm{i}t\phi'(\theta)} \left(\frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{e}^{\mathrm{i}t\phi(\theta)})\right) \mathrm{d}\theta \\ &= \frac{1}{\mathrm{i}t\phi'(\theta)} \mathrm{e}^{\mathrm{i}t\phi(\theta)}|_a^b - \frac{1}{\mathrm{i}t} \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\phi'(\theta)}\right) \mathrm{e}^{\mathrm{i}t\phi(\theta)} d\theta. \end{aligned}$$

This can be estimated by

$$\begin{aligned} |tI(t)| &\leq \frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(a)|} + \int_a^b \left| \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\phi'(\theta)} \right| \mathrm{d}\theta \\ &\leq \frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(a)|} + \frac{1}{|\phi'(b)^{-1} - \phi'(a)^{-1}|} \leq 4 \end{aligned}$$

where we used the monotonicity of ϕ'.

For all other ℓ we use induction. Assume that $|\phi^{(\ell+1)}(\theta)| \geq 1$ for all $\theta \in [a,b]$. Then there exists at most one point $\theta_0 \in [a,b]$ with $\phi^{(\ell)}(\theta_0) = 0$ and we have $|\phi^{(\ell)}(\theta)| \geq \delta$ for $|\theta - \theta_0| \geq \delta$. If this point does not exist we apply the Lemma after a possible rescaling for ℓ. Thus, we assume the existence of such a θ_0 and write $I(t) = I_1(t) + I_2(t)$ where $I_1(t)$ is the integral over $(a, \theta_0 - \delta) \cup (\theta_0 + \delta, b)$ and $I_2(t)$ over $(\theta_0 - \delta, \theta_0 + \delta)$. We find

$$|I_1(t)| \leq 2C_\ell (\delta t)^{-1/\ell}$$

since for $\theta \in (a, \theta_0 - \delta) \cup (\theta_0 + \delta, b)$ we have $|\phi^{(\ell)}(\theta)| \geq \delta$ such that the induction for ℓ can be applied. Moreover, we have

$$|I_2(t)| \leq |\int_{\theta_0 - \delta}^{\theta_0 + \delta} \mathrm{e}^{\mathrm{i}t\phi(\theta)} \, \mathrm{d}\theta| \leq 2\delta.$$

Choose $\delta = t^{-1/(\ell+1)}$. Since $(\delta t)^{1/\ell} = (t^{-1/(\ell+1)} t)^{1/\ell} = t^{-1/(\ell+1)}$ we have $I(t) \leq C_{\ell+1} t^{-1/(\ell+1)}$, with $C_{\ell+1} = 2(1 + C_\ell)$. □

Remark 8.1.9. Since no compactness argument has been used in the proof of Lemma 8.1.8 values $a = -\infty$ and $b = \infty$ are allowed if the integrals exist.

We use Lemma 8.1.8 and Remark 8.1.9 to confirm the dispersive estimate (8.7).

Example 8.1.10. The linear Schrödinger equation $\partial_t u = \mathrm{i}\partial_x^2 u$ with $t, x \in \mathbb{R}$, $u(x,t) \in \mathbb{C}$ is solved by $u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) u(y,0) \, \mathrm{d}y$ with

$$G(x,t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k^2 t} \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\phi(k)} \, \mathrm{d}k$$

such that $\phi(k) = -k^2 + kx/t$. The integral is estimated for every fixed $\xi = x/t$. Since $\phi''(k) = -2$ we thus have $\sup_{\xi \in \mathbb{R}} |G(\xi t, t)| \leq C t^{-1/2}$. From $\sup_{\xi \in \mathbb{R}} |G(\xi t, t)| = \sup_{x \in \mathbb{R}} |G(x,t)|$ it follows

$$\|u(\cdot,t)\|_{L^\infty} \leq C \|G(\cdot,t)\|_{L^\infty} \|u(\cdot,0)\|_{L^1} \leq C t^{-1/2} \|u(\cdot,0)\|_{L^1}$$

due to Lemma 7.3.19. ⌋

Example 8.1.11. The Airy equation $\partial_t u = \partial_x^3 u$ with $t, x, u(x,t) \in \mathbb{R}$ is solved by $u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) u(y,0) dy$ with

$$G(x,t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k^3 t} \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\phi(k)} \, \mathrm{d}k$$

such that $\phi(k) = -k^3 + kx/t$. Since $\phi'''(k) = -6$ we have like in Example 8.1.10 that $\|G(\cdot,t)\|_{L^\infty} \leq C t^{-1/3}$ and therefore

$$\|u(\cdot,t)\|_{L^\infty} \leq C \|G(\cdot,t)\|_{L^\infty} \|u(\cdot,0)\|_{L^1} \leq C t^{-1/3} \|u(\cdot,0)\|_{L^1}.$$

⌋

The handling of oscillatory integrals by the stationary phase method is a well developed theory, cf. [**Ste93**]. Even more complicated dispersive estimates can be obtained with this approach, cf. [**LP09**]. Such dispersive estimates can be transferred in so called Strichartz estimates. In case of the linear Schrödinger semigroup, for $2 \leq q \leq \infty$, $\frac{2}{q} + \frac{d}{p} = \frac{d}{2}$, and $1 \leq p < \infty$, we have for instance

$$\|e^{it\Delta/2}u_0\|_{L^q((0,t),L^p)} \leq C\|u_0\|_{L^2}.$$

Such estimates allow to prove local and global existence in lower regularity spaces, cf. [**Tao06**].

8.2. The KdV equation

The Korteweg-deVries (KdV) equation [**KdV95**]

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u, \tag{8.12}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $u = u(x,t) \in \mathbb{R}$, can be derived as a modulation equation from various models for the description of long unidirectional waves, such as long wave length surface water waves, see Chapter 12. It consists of two parts, namely the dispersive term $-\partial_x^3 u$ and the nonlinear transport term $6u\partial_x u$. By rescaling u, x, and t every other value for the coefficients can be obtained. The choice in (8.12) is the one which is most often used in the literature and is motivated by its derivation from the water wave problem. Our presentation follows in big parts the textbook [**DJ89**].

The linearized KdV equation or Airy equation is given by

$$\partial_t u = -\partial_x^3 u.$$

It possesses solutions $u(x,t) = \mathrm{e}^{ikx+\lambda t}$ with $\lambda = ik^3$, i.e., the amplitude of harmonic waves is preserved. Moreover, the total energy is conserved by the Airy equation since

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2(x,t)\,\mathrm{d}x = -2\int_{\mathbb{R}} u(x,t)\partial_x^3 u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} \partial_x(\partial_x u(x,t))^2\,\mathrm{d}x = 0.$$

The Airy equation shows dispersion, i.e., energy is spread over the real line. The estimate

$$\sup_{x\in\mathbb{R}} |u(x,t)| \leq Ct^{-1/3}\int_{\mathbb{R}} |u(y,0)|\,\mathrm{d}y,$$

can be established with the stationary phase method, cf. Example 8.1.11. The decay rate $t^{-1/3}$ is a consequence of $\lambda''(0) = 0$, but $\lambda'''(0) \neq 0$. Alternatively, it can be obtained by an explicit solution formula which can be build with the help of self-similar solutions of the form $u(x,t) = t^{-1/3}\widetilde{u}(xt^{-1/3})$ as fundamental solutions, cf. [**Rau91**].

The nonlinear transport term

$$\partial_t u = 6u\partial_x u$$

already appeared in the Burgers equation. We found that shocks may be created in finite time. Like in the Burgers equation, the linear semigroup, here generated by ∂_x^3, inhibits the creation of shocks in the full KdV equation.