PREY-PREDATOR

Research Project

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Our analysis is based on the research paper

"Dynamics in a time-discrete food-chain model with strong pressure on preys"

By Alsedà, Vidiella, Solé, Lázaro and Sardanyés

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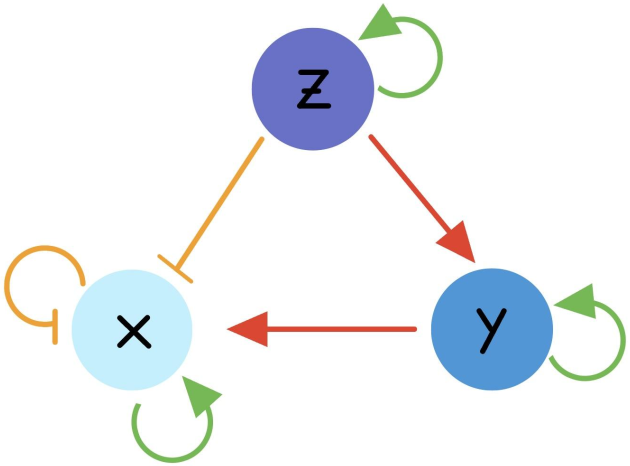
Food-chain scheme

We consider a discrete-time dynamical system that models the population dynamics of three interacting species x, y and z, with non-overlapping generations. x represents the prey, y represents the predator that preys on x, and z represents the top-predator, which preys on y and also interferes with the reproduction of x. Additionally, x also interferes with its own growth.



Mathematical model

The ecosystem can be described by the discrete-time system below.

x, *y* and *z* are the normalised population densities.

μ, *β* and *γ* are positive constants representing the reproduction rates of each population.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \mu x_n(1 - x_n - y_n - z_n) \\ \beta y_n(x_n - z_n) \\ \gamma y_n z_n \end{pmatrix} = T \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

Domain

In order to have a system well-defined we define the domain U

$$\mathbb{U} = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0 \ \wedge \ x + y + z \leq 1\}$$

Moreover, we restrict the set of parameters to study important phenomena such as cascade extinctions and periodic orbits.

$$Q = \{(\mu, \beta, \gamma) \in (0, 4] \times [2.5, 5] \times [5, 9.4]\}$$

Our goal is to investigate the conditions that lead to extinction. Clearly it is represented from the point (0,0,0), that can be reached in just one iteration from the points (1, 0, 0), {(0, y, 0) ∈ U} and {(0, 0, z) ∈ U}. Although extinction can occur also whenever the orbits move outside of the domain U.

Definition of the invariant set

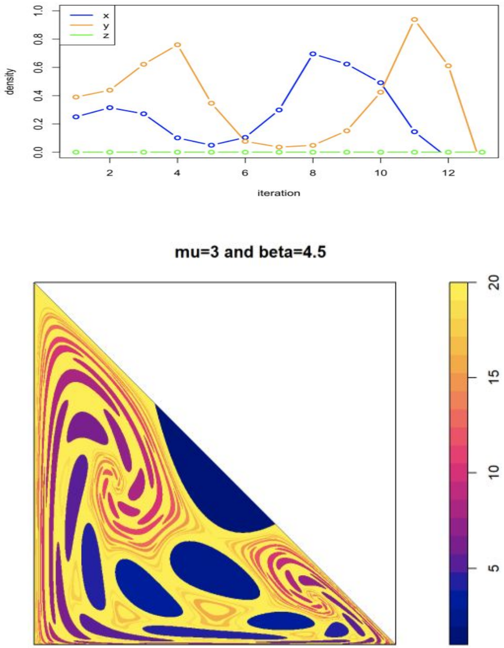
We want to define the invariant set S as the set of points whose orbits will always remain inside the domain U. However, to determine this set studying the system over all the domain U is very complex, so without loss of information we reduce the domain to

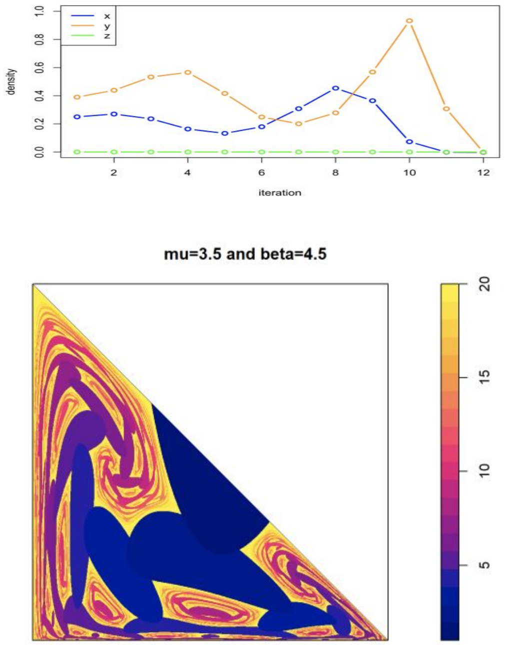
$$\mathcal{E} = \{(x, 0, z) \in \mathbb{U}\} \cup \{(x, y, z) \in \mathbb{U} : y > 0 \wedge x \geq z\}$$

It can be proven that S ⊂ ε ⊂ U, so all the phenomena we want to study in S are guaranteed to occur inside ε.

Complexity of the region S

To illustrate the complexity of studying the region S, we provide an example showing the amount of time it takes for a given system, with an initial condition in the plane *z = 0*, to exceed the carrying capacity and go extinct.





Computation of the fixed points

In order to compute the fixed points, we need to solve the following system:

$$
\begin{aligned}
x &= \mu x(1 - x - y - z) \\
y &= \beta y(x - z) \\
z &= \gamma y z
\end{aligned}
$$

We obtain:

- $P_1 = (0, 0, 0)$
- $P_2 = (\frac{\mu - 1}{\mu}, 0, 0)$
- $P_3 = (\frac{1}{\beta}, 1 - \frac{1}{\mu} - \frac{1}{\beta}, 0)$
- $P_4 = (\frac{1}{2}(1 - \frac{1}{\mu} + \frac{1}{\beta} - \frac{1}{\gamma}), \frac{1}{\gamma}, \frac{1}{2}(1 - \frac{1}{\mu} - \frac{1}{\beta} - \frac{1}{\gamma}))$
- $P_5 = (0, \frac{1}{\gamma}, -\frac{1}{\beta})$

To analyse the stability we exploit the Jacobian matrix of the system

$$P_1 = (0, 0, 0)$$

This fixed point always exists and it represents the extinction of all the species.

- ***$0 < \mu < 1$*** → locally asymptotically stable sink node
- ***$\mu > 1$*** → saddle with an unstable invariant manifold of dimension 1

$$P_2 = (\frac{\mu - 1}{\mu}, 0, 0)$$

P_2 is contained in the domain for *$\mu \geq 1$.* If it is reached the two predator species go extinct and the dynamic is only governed by the logistic map of the prey *x.*

- ***$1 < \mu < \beta/(\beta-1)$***→ locally asymptotically stable sink node
- ***$\beta/(\beta-1) < \mu < 3$*** → saddle with an unstable invariant manifold of dimension 1
- ***$3 < \mu \leq 4$*** → a saddle with an unstable invariant manifold of dimension 2

$$P_3 = (\frac{1}{\beta}, 1 - \frac{1}{\mu} - \frac{1}{\beta}, 0)$$

P_3 is contained in the domain for $\beta \leq 5$ and $1/\mu + 1/\beta \leq 1$. The biological meaning of this fixed point corresponds to the extinction of the top-predator and the survival of the prey and the predator.

However, this point can be reached only when the initial conditions of the system are $(x, y, z) = (x_0, y_0, 0)$, this means in an initial situation without the top-predator. Indeed, by construction of the system, the survival of z is highly linked to the predator y.

$$P_4 = (\frac{1}{2}(1 - \frac{1}{\mu} + \frac{1}{\beta} - \frac{1}{\gamma}), \frac{1}{\gamma}, \frac{1}{2}(1 - \frac{1}{\mu} - \frac{1}{\beta} - \frac{1}{\gamma}))$$

This fixed point exists for $1/\mu + 1/\beta + 1/\gamma \leq 1$.

Local bifurcations

We identify different regions of the parameters inside which the behaviour of the fixed points does not change, so that we can define the biological evolution of the system inside those zone

Zone A → $0 < \mu < 1$

There exist only one fixed point P_1, which is asymptotically stable, so in this region all the species go to extinction.

Zone B → $1 < \mu < \beta/(\beta-1)$

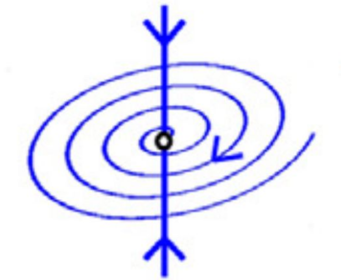
It contains two fixed points P_1 and P_2, the first one is a saddle, while the second one is asymptotically stable, so in this region only the prey survives, while the other two species go to extinction.

Zone C → $\beta/(\beta-1) < \mu < 2\beta(\beta - 1 - \sqrt{\beta(\beta - 2)})$

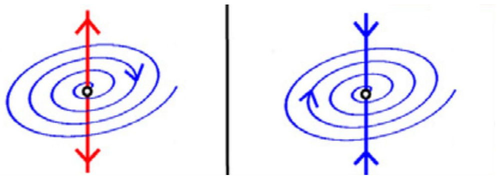
There are three fixed points P_1, P_2 and P_3. P_1 and P_2 are saddle points (from now on they don't change their behaviour) , while P_3 is asymptotically stable, so the top-predator goes to extinction in this region.

Zone D → . $2\beta(\beta - 1 - \sqrt{\beta(\beta - 2)}) < \mu < \beta\gamma/((\beta-1)\gamma-\beta)$

It contains three fixed points P_1, P_2 and P_3. P_3 is an asymptotically stable spiral-node sink, so the top-predator goes to extinction in this region, while the prey and the predator achieve an equilibrium of coexistence via damped oscillations.

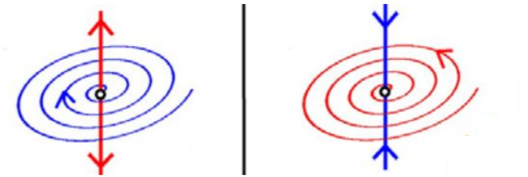


Zone E → $\beta\gamma/((\beta-1)\gamma-\beta) < \mu < \min\{3, \psi(\beta, \gamma)\}$



Zone H → $\psi(\beta, \gamma) \geq 3$ and $3 < \mu < \psi(\beta, \gamma)$

All four fixed points exist. P_3 is an unstable spiral-sink node-source and P_4 is asymptotically stable sink of spiral-node. This means that the three species achieve a statistic coexistence equilibrium.



Zone F → $\psi(\beta, \gamma) < \mu < \min\{3, \beta/(\beta-2)\}$ and $\psi(\beta, \gamma) < 3$

Zone I → $2.5 < \beta < 3$ and $\max\{3, \psi(\beta, \gamma)\} < \mu < \min\{4, \beta/(\beta-2)\}$

All four fixed points exist. P_3 is an unstable spiral-sink node-source and P_4 is unstable spiral-source node-sink. So we expect a fluctuating coexistence of all the species.

Zone G → $3 < \beta \leq 5$ and $\beta/(\beta-2) < \mu < 3$

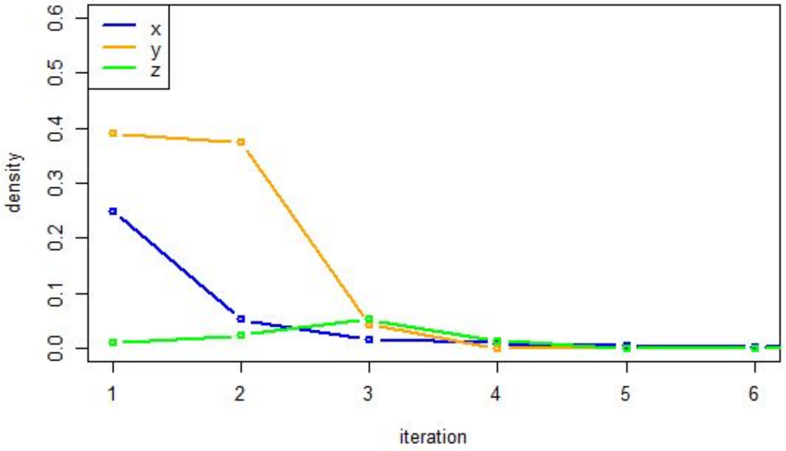
Zone J → $8/5 < \beta \leq 5$ and $\max\{3, \beta/(\beta-2)\} < \mu < 4$

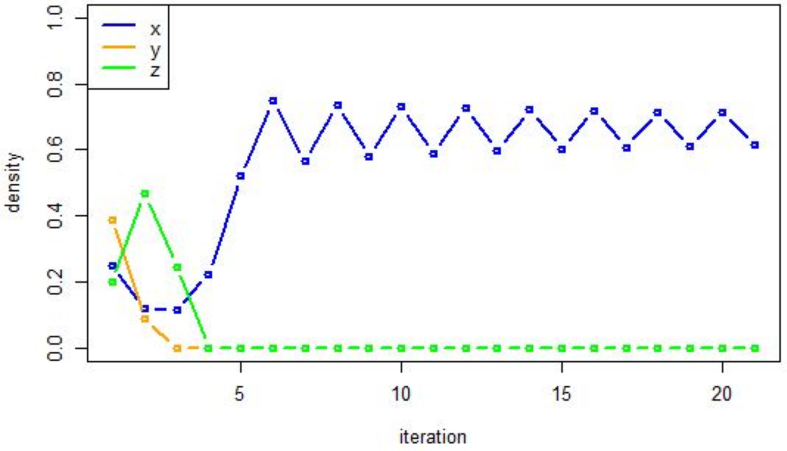
They contain all the four fixed points. P_3 is an unstable spiral-node source and P_4 is unstable spiral-source node-sink. This means that the three species coexist.

Global dynamics

One phenomenon is common among all the zones, above mentioned:

- If at a certain point the **prey species goes extinct** ($x_n = 0$), then we will witness a **cascade extinction** of all the three species.
- If the **predator species goes extinct** ($y_n = 0$) then the system will reduce to a 1-species **logistic map**, governed by μ.

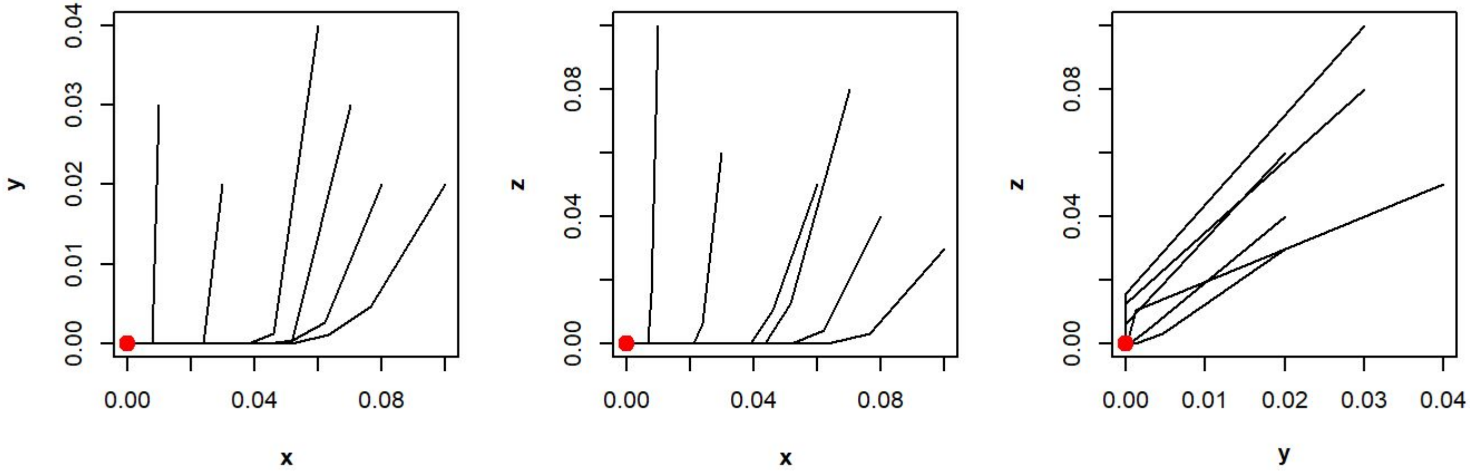




Zone A

Zone A is characterised by convergent orbits toward P_1, leading to the extinction of all the species. starting from all the valid starting conditions.

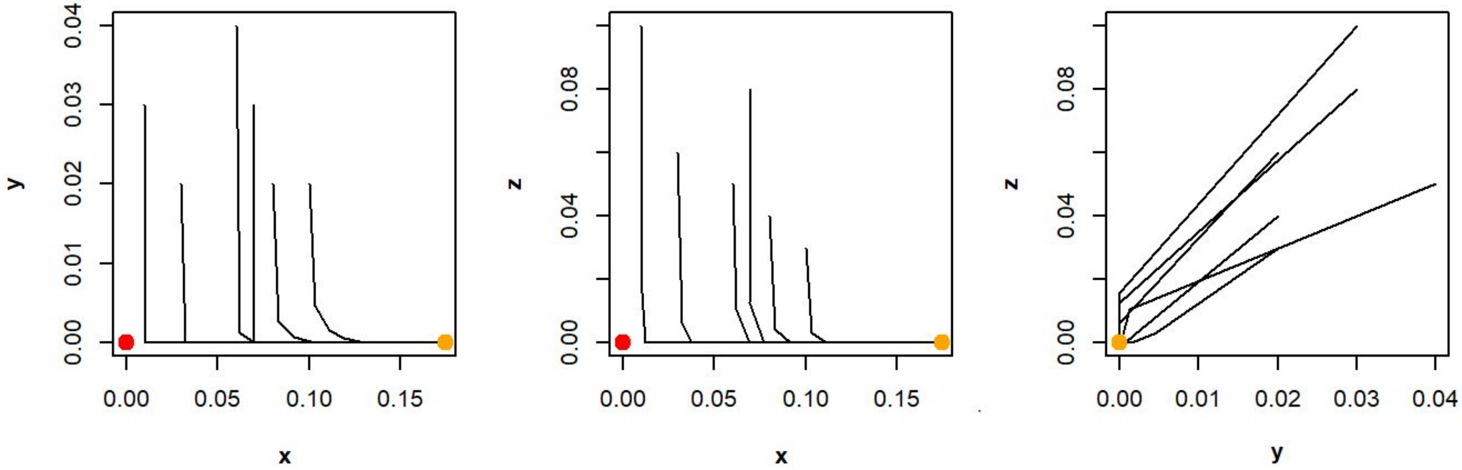
$$\lim_{n\to\infty} T^n(x,y,z) = (0,0,0) = P_1,\ \forall\,(x,y,z) \in S$$



Zone B

Zone B is characterised by convergent orbits toward P_2, leading to the extinction of all the species. starting from all the valid starting conditions.

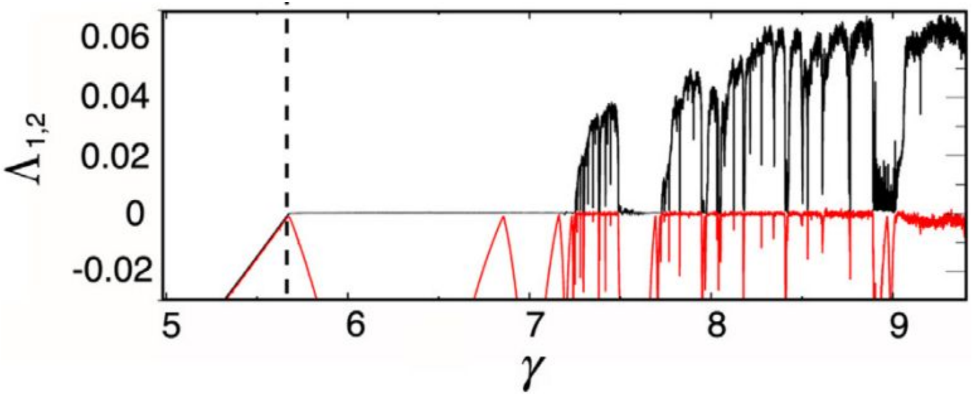
$$\lim_{n\to\infty} T^n(x,y,z) = (1-\frac{1}{\mu},0,0) = P_2$$

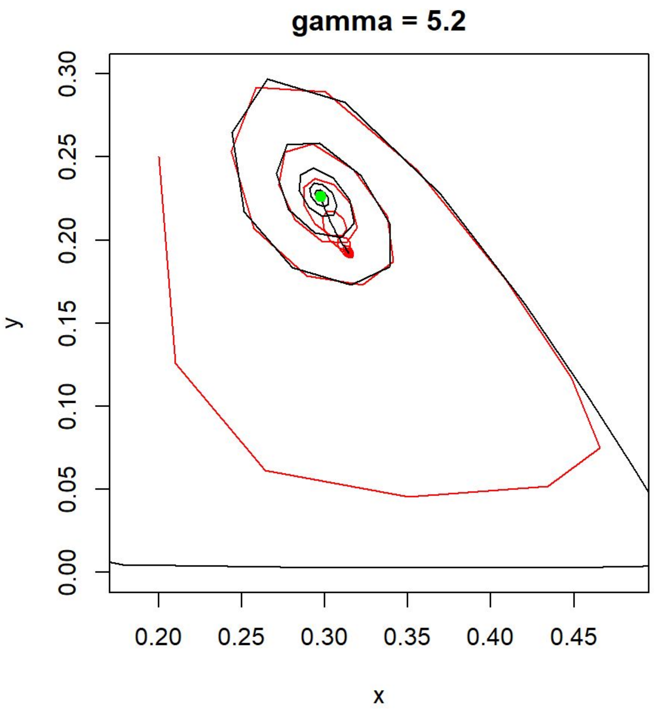


Chaotic behavior

To better analyze the system, the **Lyapunov exponents** Λ_i should be estimated, through a computational method.
This approach, when performed, allows to identify the points where **bifurcations** take place and also when the system becomes **chaotic** ($\Lambda>0$).

Given these premises, we'll study the system in the regions suggested by this procedure.



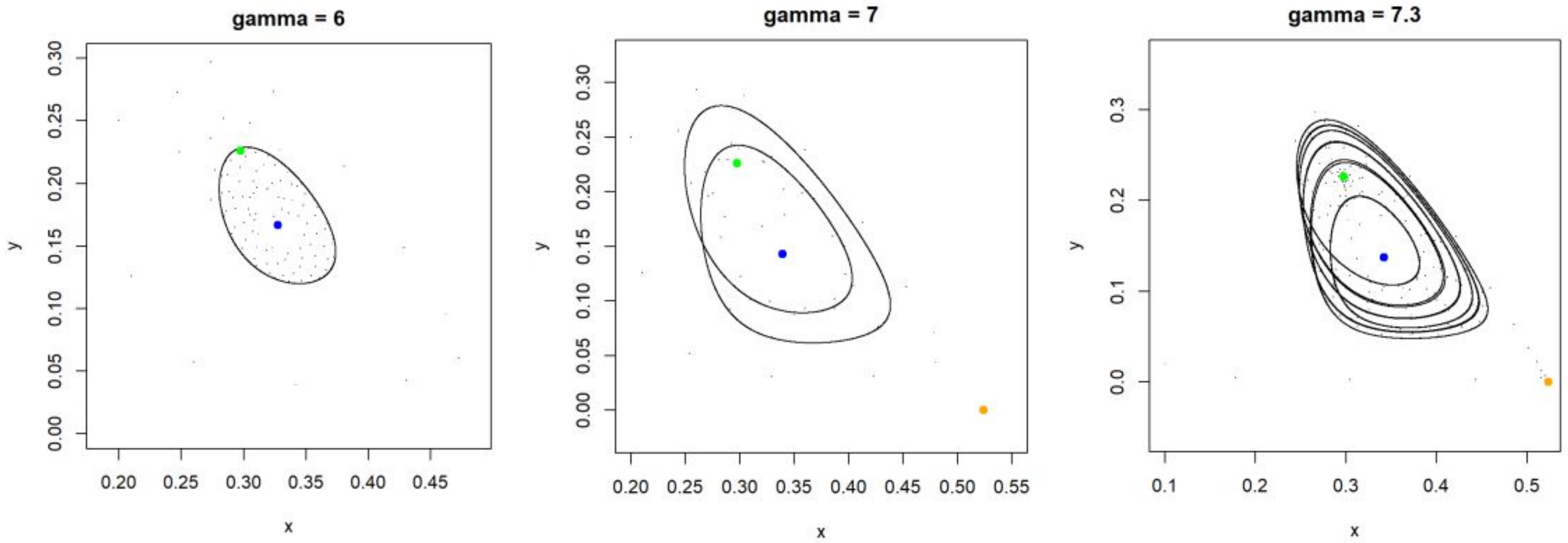


Convergent orbits *are obtained for $5 < \gamma < 5.673$*

P_4 is the unique stable fixed point. In this static equilibrium, reached after damped oscillations, in which the three species coexist.

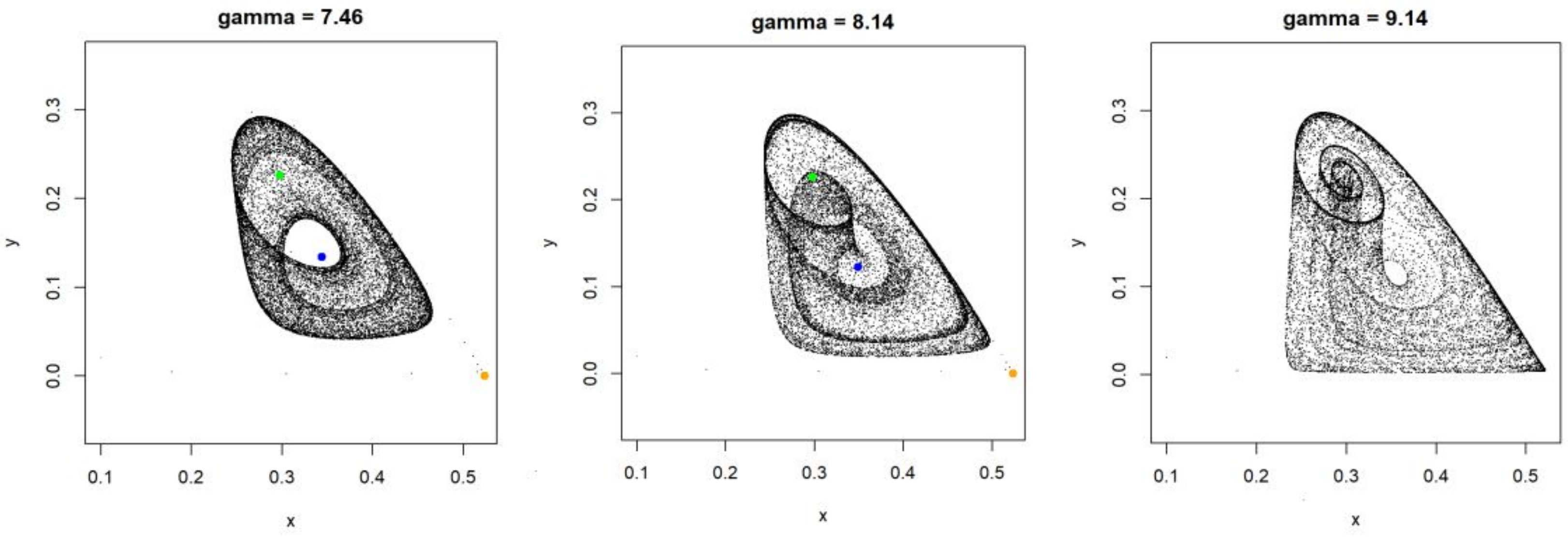
For $5.673 < \gamma < 7.25$ we observe **closed invariant curves**

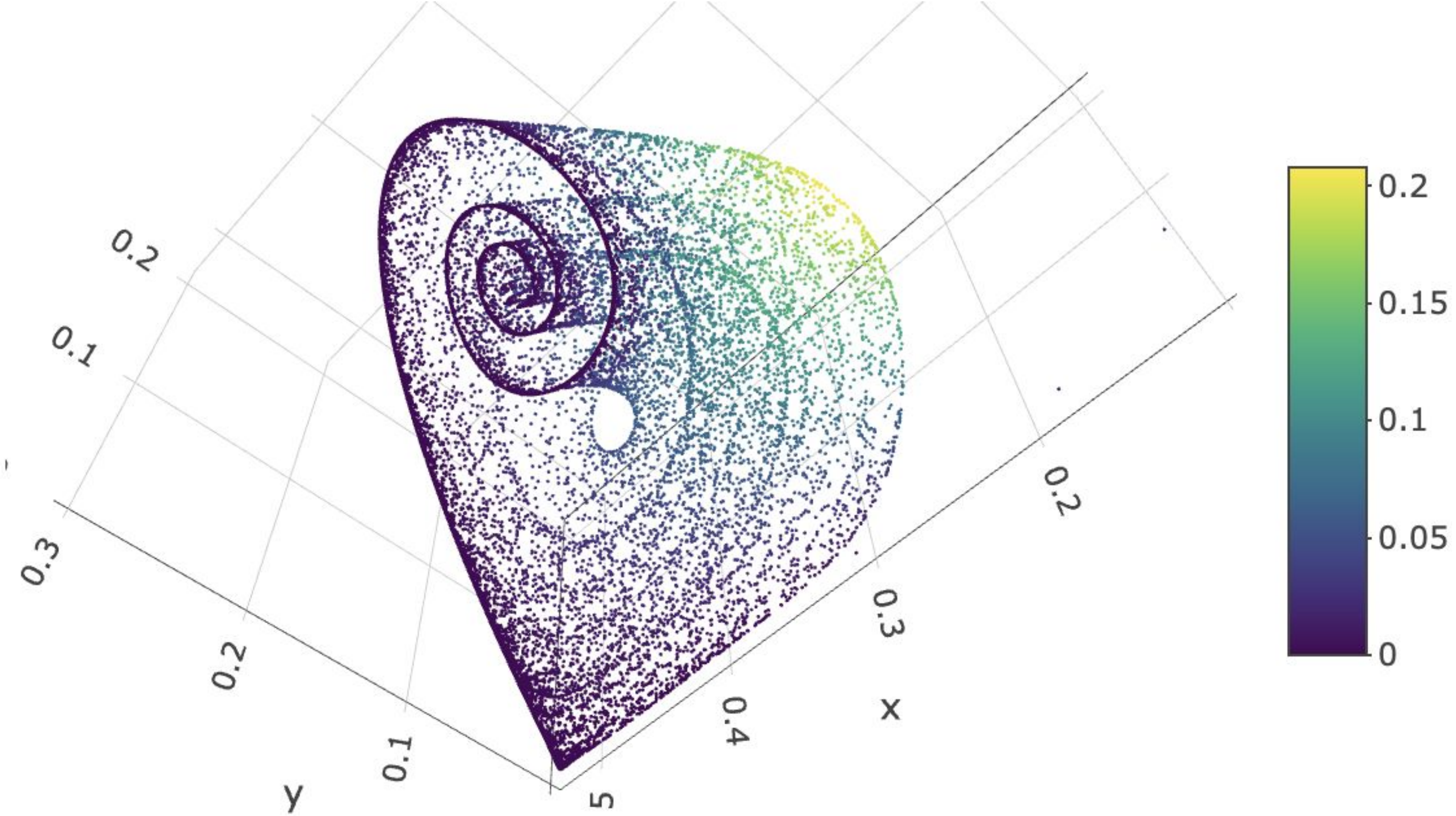
We are in *Zone F*, so all fixed points are unstable. The orbits are periodic with a period which increases as γ increases.



For $7.25 < \gamma < 9.4$ we identify a **route to chaos**

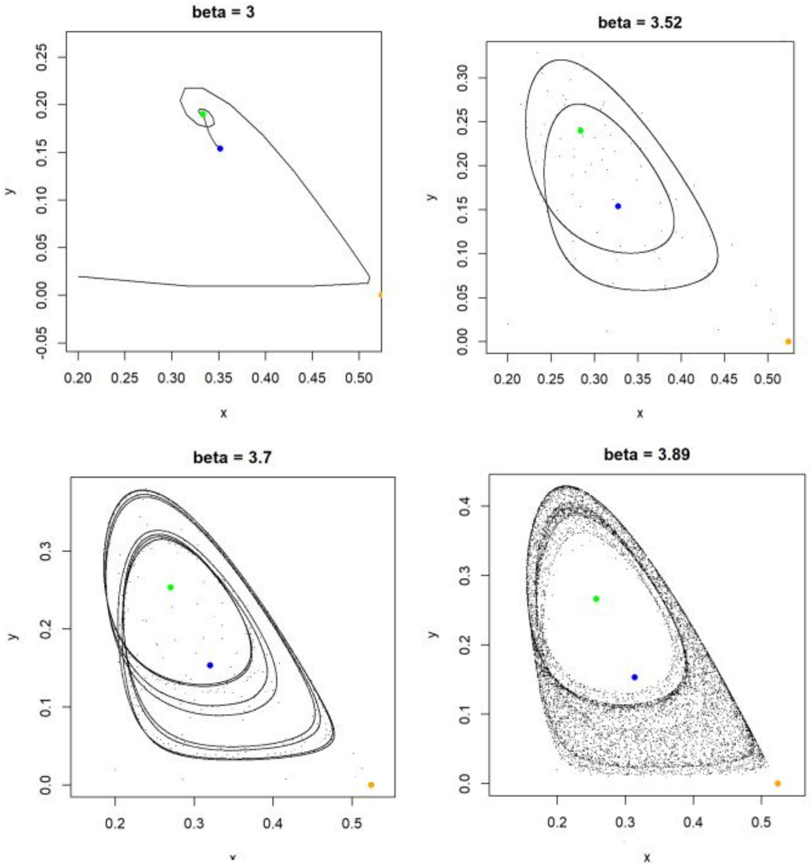
Indeed, as γ increases, the periodic orbits are denser and denser and we can see a sequence of period-doublings, making the period tend to infinite.




0.2
0.1
0.3
0.2
0.1
y
5
0.4
0.3
0.2
x
0.2
0.15
0.1
0.05
0

We can observe that a similar zone-transition occurs as β increases:

- for $2.5 \leq \beta \leq 2.7$ (i.e. zone D) the top predator z goes to extinction, while the prey x and predator y achieve a static equilibrium
- for $2.7 \leq \beta \leq 3.18$ (i.e. zone E) the three species coexist
- for $3.18 \leq \beta \leq 5$ (i.e Zone F and G) all of the fixed points are unstable and thus periodic dynamics can occur.



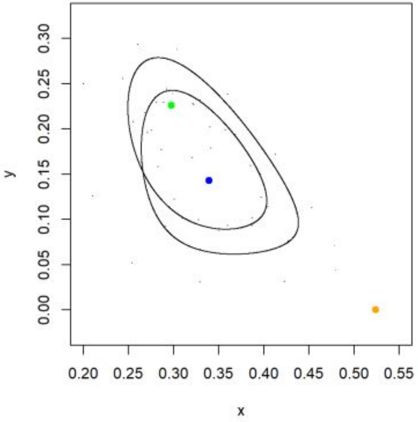
Local maxima approach

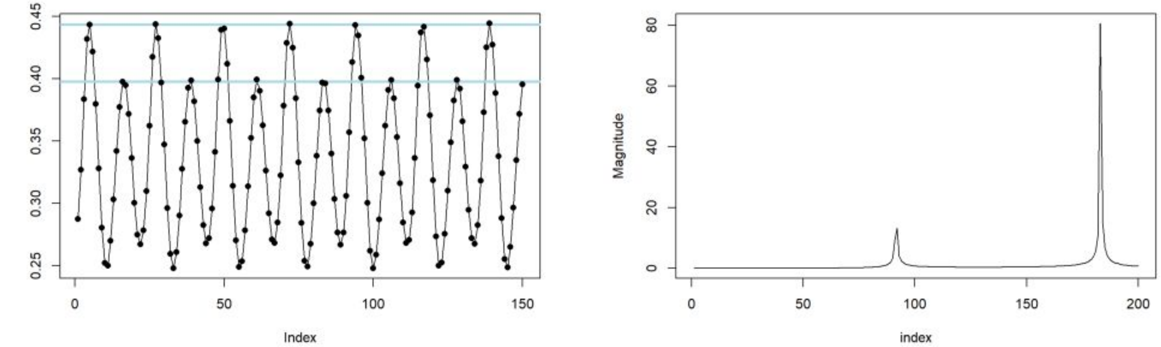
In order to characterize the route to chaos

We plot the **local maxima** of the periodic orbits

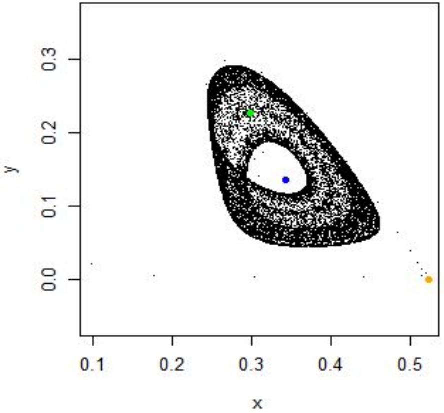
We plot the coefficients of the **Fourier transform** of the time-series

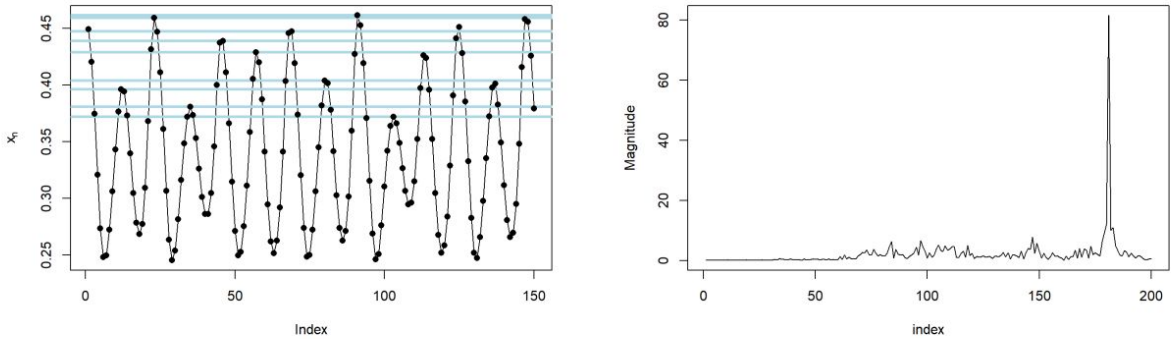
$\gamma = 7.1$



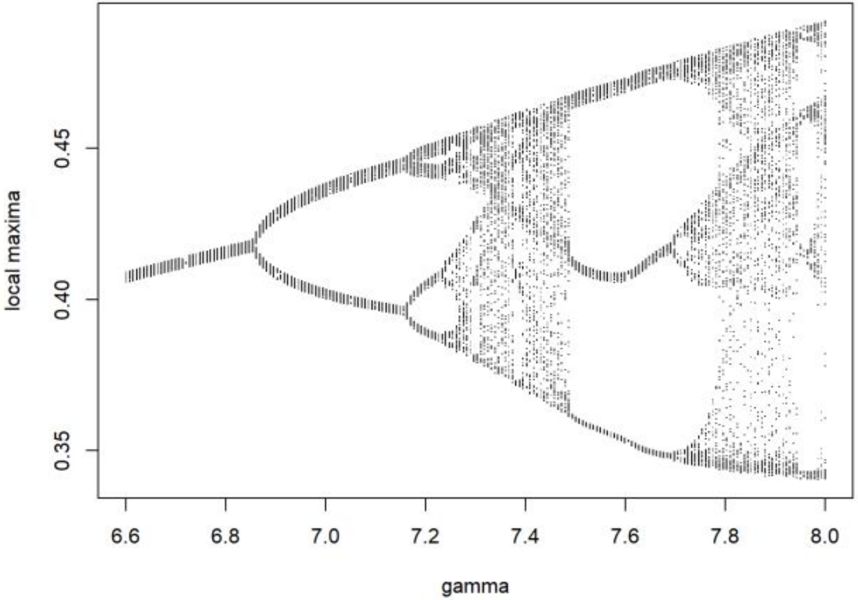


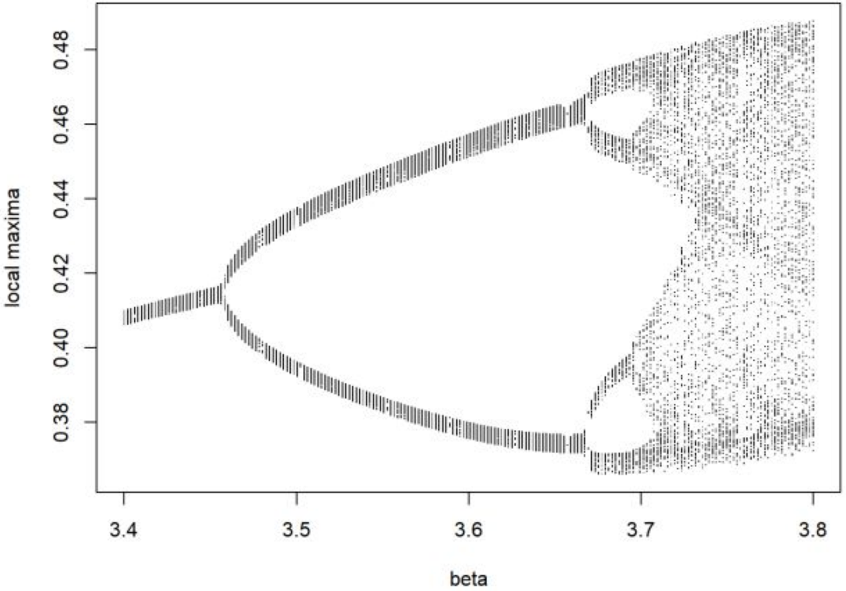
$\gamma = 7.4$





Finally, this approach allows us to use the estimated local maxima of the time-series to plot a **bifurcation diagram** of the system with respect to γ and to β.





Here, after a supercritical Neimark-Sacker bifurcation, we can observe the phenomenon of **period doubling,** which leads to **chaos.**

Analysing the system's equilibria and chaotic behaviours, we identify different phenomena.

extinction of the top predator

all-species extinctions

coexistence of the three species

extinction of both predators

Very easy food-chain scheme, with only three species interacting, leads us to understand how complex nature is and how easily the equilibrium of an ecosystem could be perturbed.

Thanks for your attention!