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Markov Reliability and Availability Analysis

Part II: Continuous-Time Discrete State Markov Processes

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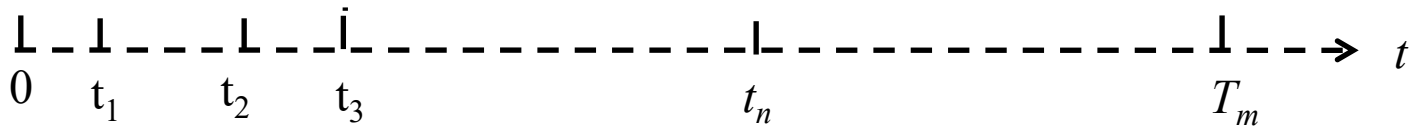
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Continuous Time Discrete State Markov Processes

- The **stochastic process** may be **observed** at:

- Discrete times

→ **DISCRETE-TIME DISCRETE-STATE MARKOV PROCESSES**

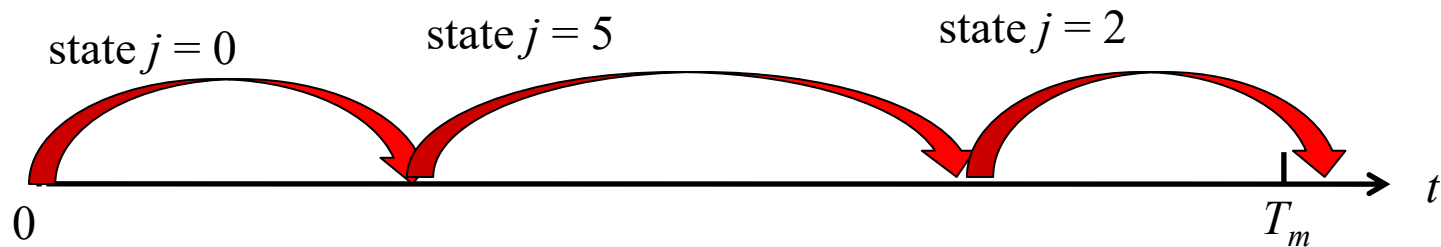


- Continuously

→ **CONTINUOUS-TIME DISCRETE-STATE MARKOV PROCESS**



- The stochastic process is **observed continuously** and **transitions** are assumed to **occur continuously in time**



- The random process of system transition between states in time is described by a **stochastic process** $\{X(t); t \geq 0\}$
- $X(t) :=$ system state at time t
 - $X(3.6) = 5$: the system is in state number 5 at time $t = 3.6$



OBJECTIVE:

Computing the probability that the system is in a given state
as a function of time, for all possible states

$$P[X(t) = j], t \in [0, T_m], j = 0, 1, \dots, N$$

Objective:

$$P[X(t) = j], t \in [0, T_m], j = 0, 1, \dots, N$$

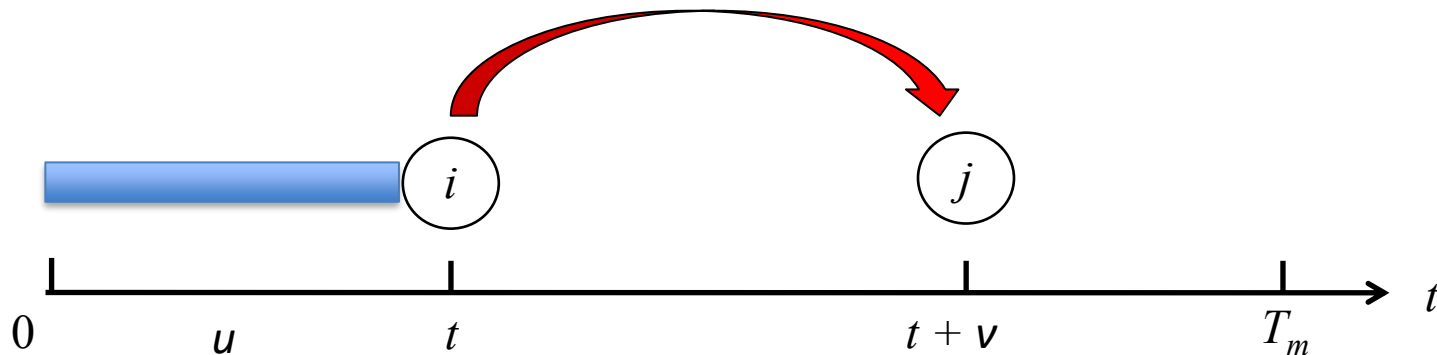


What do we need?

Transition Probabilities!

- **Transition probability** that the system moves to state j at time $t + v$ given that it is in state i at **current** time t and given **the previous system history**

$$P[X(t + v) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < t]$$
$$(i = 0, 1, \dots, N, j = 0, 1, \dots, N)$$



- **IN GENERAL STOCHASTIC PROCESSES:**

the **probability** of a **future** state of the system usually depends on its **entire life history**

$$P[X(t + \nu) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < t] \\ (i = 0, 1, \dots, N, j = 0, 1, \dots, N)$$

- **IN MARKOV PROCESSES:**

the **probability** of a **future** state of the system **only** depends on its **present state**

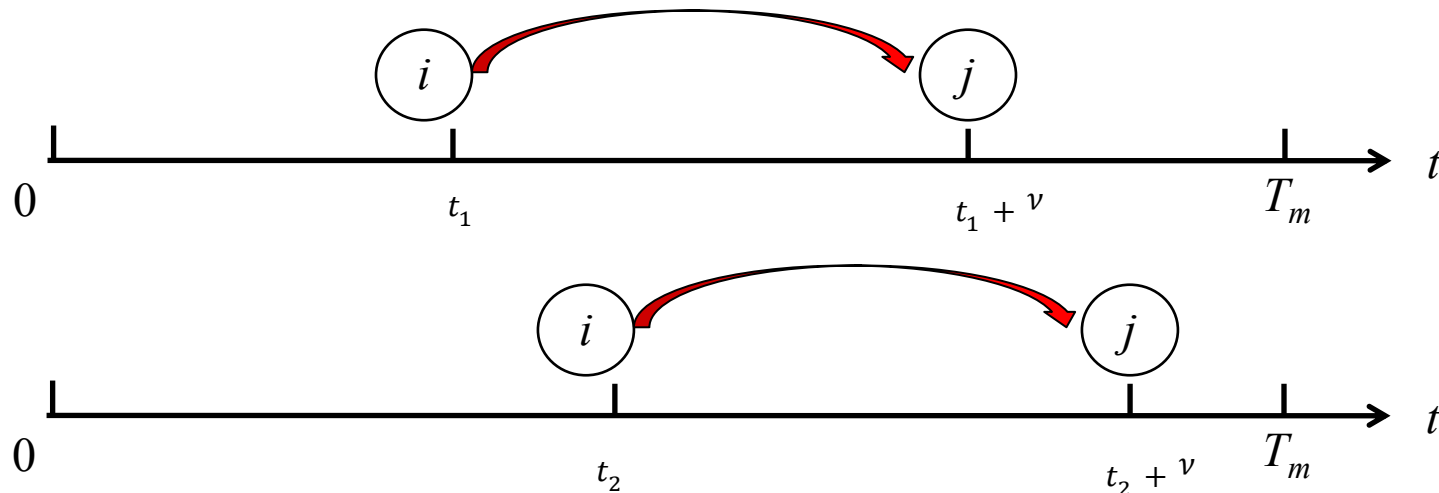
$$P[X(t + \nu) = j \mid X(t) = i, \cancel{X(u) = x(u)}, 0 \leq \cancel{u} < \cancel{t}] \\ = \\ P[X(t + \nu) = j \mid X(t) = i] \\ (i = 0, 1, \dots, N, j = 0, 1, \dots, N)$$

THE PROCESS HAS “NO MEMORY”

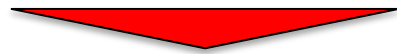
If the **transition probability** depends on the **interval ν** and **not** on the **individual times t and $t + \nu$**

- the transition probabilities are **stationary**
- the Markov process is **homogeneous** in time

$$p_{ij}(t, t + \nu) = P[X(t + \nu) = j \mid X(t) = i] = p_{ij}(\nu)$$



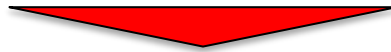
- Homogeneous process without memory



Transition time \rightarrow Exponential distribution

HYPOTHESIS:

- The time interval $\nu = dt$ is **small** such that **only one** event (i.e., one **stochastic transition**) can occur within it

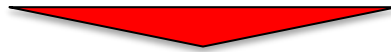


$$p_{ij}(dt) = P[X(t + dt) = j | X(t) = i] = 1 - e^{-\alpha_{ij} \cdot dt}$$

α_{ij} = **transition rate** from state i to state j

HYPOTHESIS:

- The time interval $\nu = dt$ is **small** such that **only one** event (i.e., one **stochastic transition**) can occur within it



$$p_{ij}(dt) = P[X(t + dt) = j | X(t) = i] = 1 - e^{-\alpha_{ij} \cdot dt}$$

= (Taylor 1st order expansion)

$$\alpha_{ij} \cdot dt + \theta(dt), \quad \lim_{dt \rightarrow 0} \frac{\theta(dt)}{dt} = 0$$

$$p_{ii}(dt) = 1 - \sum_{j \neq i} p_{ij}(dt) = 1 - dt \cdot \sum_{j \neq i} \alpha_{ij} + \theta(dt)$$

Discrete-time

$$p_{ij} = P[X(n+1) = j | X(n) = i]$$

$$\underline{P}(n+1) = \underline{P}(n) \cdot \underline{A}$$

$$\underline{A} = \begin{pmatrix} p_{00} & p_{01} & \dots & p_{0N} \\ p_{10} & p_{11} & \dots & p_{1N} \\ \dots & \dots & \dots & \dots \\ p_{N0} & p_{N1} & \dots & p_{NN} \end{pmatrix}$$

Continuous-time

$$\alpha_{ij} = P[X(t+dt) = j | X(t) = i]$$

$$\underline{P}(t+dt) = \underline{P}(t) \cdot \underline{A}$$

$$\underline{A} = \begin{pmatrix} 1 - dt \cdot \sum_{j=1}^N \alpha_{0j} & \alpha_{01} \cdot dt & \dots & \alpha_{0N} \cdot dt \\ \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{\substack{j=0 \\ j \neq 1}}^N \alpha_{1j} & \dots & \alpha_{1N} \cdot dt \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

$$[P_0(t+dt)P_1(t+dt)\dots P_N(t+dt)] = [P_0(t)P_1(t)\dots P_N(t)] \cdot \begin{pmatrix} 1 - dt \cdot \sum_{j=1}^N \alpha_{0j} & \alpha_{01} \cdot dt & \dots & \alpha_{0N} \cdot dt \\ \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{\substack{j=0 \\ j \neq 1}}^N \alpha_{1j} & \dots & \alpha_{1N} \cdot dt \\ \dots & \dots & \dots & \dots \end{pmatrix}$$




- First-equation:

$$P_0(t+dt) = \left[1 - dt \sum_{j=1}^N \alpha_{0j} \right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \dots + \alpha_{N0} P_N(t) dt$$

The conceptual model: the fundamental matrix equation (2)

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$$P_0(t + dt) = \left[1 - dt \sum_{j=1}^N \alpha_{0j} \right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \dots + \alpha_{N0} P_N(t) dt$$

 subtract $P_0(t)$ on both sides

$$P_0(t + dt) - P_0(t) = P_0(t) - P_0(t) - \sum_{j=1}^N \alpha_{0j} P_0(t) dt + \alpha_{10} P_1(t) dt + \dots + \alpha_{N0} P_N(t) dt$$

 divide by dt

$$\frac{P_0(t + dt) - P_0(t)}{dt} = - \sum_{j=1}^N \alpha_{0j} P_0(t) + \alpha_{10} P_1(t) + \dots + \alpha_{N0} P_N(t)$$

 let $dt \rightarrow 0$

$$\lim_{dt \rightarrow 0} \frac{P_0(t + dt) - P_0(t)}{dt} = \frac{dP_0}{dt} = - \sum_{j=1}^N \alpha_{0j} \cdot P_0(t) + \alpha_{10} \cdot P_1(t) + \dots + \alpha_{N0} \cdot P_N(t)$$

$$\frac{dP_0}{dt} = -\sum_{j=1}^N \alpha_{0j} P_0(t) + \alpha_{10} P_1(t) + \dots + \alpha_{N0} P_N(t) = [P_0(t), P_1(t), \dots, P_N(t)] \cdot \begin{bmatrix} -\sum_{j=1}^N \alpha_{0j} \\ \alpha_{10} \\ \dots \\ \alpha_{N0} \end{bmatrix}$$

- Extending to the other equations:

$$\frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{A}^*, \quad \underline{A}^* = \begin{pmatrix} -\sum_{j=1}^N \alpha_{0j} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & -\sum_{\substack{j=0 \\ j \neq 1}}^N \alpha_{1j} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

**TRANSITION RATE
MATRIX**

It will be indicated as A

System of $N+1$ linear, first-order differential equations in the unknown state probabilities

$$P_j(t), j = 0, 1, 2, \dots, N, t \geq 0$$

Example 1: one component/one repairman-Markov Diagram and transition rate matrix

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Consider a system made by one component which can be in two states: working or failed. Assume constant failure rate λ and constant repair rate μ . You are required to:

- Draw the Markov diagram
- Find the transition rate matrix, A

Example 2: system with N identical components and N repairman available

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Consider a system made by N identical components which can be in two states: working or failed. Assume constant failure rate λ , that N repairman are available and that the single component repair rate is constant and equal to μ . You are required to:

- Draw the Markov diagram
- Find the transition rate matrix, A

Example 3: system with N identical components and 1 repairman available

Consider a system made by N identical components which can be in two states: working or failed. Assume constant failure rate λ , that 1 repairman is available and that the component repair rate is constant and equal to μ . You are required to:

- Draw the Markov diagram
- Find the transition rate matrix

Solution to the Fundamental Equation

Solution to the fundamental equation of the Markov process continuous in time

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$$\left\{ \begin{array}{l} \frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{A} \\ \underline{P}(0) = \underline{C} \end{array} \right. \quad \text{where} \quad \underline{A} = \begin{pmatrix} -\sum_{j=1}^N \alpha_{0j} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & -\sum_{\substack{j=0 \\ j \neq 1}}^N \alpha_{1j} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

System of $N+1$ **linear, first-order differential equations** in the unknown state probabilities

$$P_j(t), j = 0, 1, 2, \dots, N, t \geq 0$$



USE LAPLACE TRANSFORM

Solution to the fundamental equation of the Markov process continuous in time: the Laplace Transform Method

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- Laplace Transform: $\tilde{P}_j(s) = L[P_j(t)] = \int_0^\infty e^{-st} P_j(t) dt, \quad j = 0, 1, \dots, N$
- First derivative: $L\left(\frac{dP_j(t)}{dt}\right) = s \cdot \tilde{P}_j(s) - P_j(0), \quad j = 0, 1, \dots, N$
- Apply the Laplace operator to $\frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{A}$

$$L\left[\frac{d\underline{P}(t)}{dt}\right] = L[\underline{P}(t) \cdot \underline{A}]$$

$$\text{First derivative} \leftarrow \boxed{s\tilde{\underline{P}}(s) - \underline{C}} = \boxed{\tilde{\underline{P}}(s) \cdot \underline{A}} \rightarrow \text{Linearity}$$

$$\tilde{\underline{P}}(s) = \underline{C} \cdot [s \cdot \underline{I} - \underline{A}]^{-1} \rightarrow \underline{P}(t) = \text{inverse transform of } \tilde{\underline{P}}(s)$$

Solution to the Fundamental Equation: Steady State Probabilities

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- At steady state $\underline{P}(t) = \underline{\Pi}$
 $\frac{d\underline{P}(t)}{dt} = 0 \quad \Rightarrow \quad \frac{d\underline{P}(t)}{dt} = \underline{P}(t) \cdot \underline{A} = \underline{\Pi} \cdot \underline{A} = 0$
- Solve the (linear) system:
$$\left\{ \begin{array}{l} \underline{\Pi} \cdot \underline{A} = 0 \\ \sum_{j=0}^N \Pi_j = 1 \end{array} \right.$$

Solution to the Fundamental Equation: Steady State Probabilities

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- At steady state: $\underline{P}(t) = \underline{\Pi}$

$$\frac{d\underline{P}(t)}{dt} = 0 \quad \Rightarrow \quad \frac{d\underline{P}(t)}{dt} = \underline{P}(t) \cdot \underline{A} = \underline{\Pi} \cdot \underline{A} = 0$$

- Solve the (linear) system:
$$\begin{cases} \underline{\Pi} \cdot \underline{A} = 0 \\ \sum_{j=0}^N \Pi_j = 1 \end{cases}$$

- It can be shown that:
$$\Pi_j = \frac{D_j}{\sum_{i=0}^N D_i} \quad j = 0, 1, 2, \dots, N$$

D_j = determinant of the **square matrix** obtained from \underline{A}
by **deleting** the j -th row and column

Example 4: one component/one repairman – Solution to the fundamental equation (1)

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Consider a system made by one component which can be in two states: working ('0') or Failed ('1'). Assume constant failure rate λ and constant repair rate μ and that the component is working at $t = 0$: $\underline{C} = [1 \ 0]$

- You are required to find the component steady state and instantaneous availability

Quantity of Interest

- **Unconditional** probability of **arriving in state j** in the next dt **departing from state i** at time t : $P[X(t + dt) = j, X(t) = i]$

$$P[X(t + dt) = j, X(t) = i] = P[X(t + dt) = j | X(t) = i] \cdot P[X(t) = i] = p_{ij}(dt)P_i(t)$$

- **Frequency of departure from state i to state j :**

$$v_{ij}^{dep}(t) = \lim_{dt \rightarrow 0} \frac{P[X(t + dt) = j, X(t) = i]}{dt} = \lim_{dt \rightarrow 0} \frac{p_{ij}(dt)P_i(t)}{dt} = \alpha_{ij}P_i(t)$$

- **Total frequency of departure from state i to any other state j :**

$$v_i^{dep}(t) = \sum_{\substack{j=0 \\ j \neq i}}^N v_{ij}^{dep}(t) = \sum_{\substack{j=0 \\ j \neq i}}^N \alpha_{ij} \cdot P_i(t) = P_i(t) \sum_{\substack{j=0 \\ j \neq i}}^N \alpha_{ij} = -\alpha_{ii} \cdot P_i(t)$$

- **Unconditional** probability of **arriving in state j** in the next dt **departing from state i** at time t : $P[X(t + dt) = j, X(t) = i]$

$$P[X(t + dt) = j, X(t) = i] = P[X(t + dt) = j | X(t) = i] \cdot P[X(t) = i] = p_{ij}(dt)P_i(t)$$

- **Frequency of departure from state i to state j :**

$$v_{ij}^{dep}(t) = \lim_{dt \rightarrow 0} \frac{P[X(t + dt) = j, X(t) = i]}{dt} = \lim_{dt \rightarrow 0} \frac{p_{ij}(dt)P_i(t)}{dt} = \alpha_{ij}P_i(t) \quad (\text{at steady state}) = v_{ij}^{dep} = \alpha_{ij} \cdot \Pi_i$$

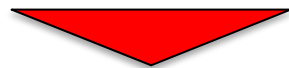
- **Total frequency of departure from state i to any other state j :**

$$v_i^{dep}(t) = \sum_{\substack{j=0 \\ j \neq i}}^N v_{ij}^{dep}(t) = \sum_{\substack{j=0 \\ j \neq i}}^N \alpha_{ij} \cdot P_i(t) = P_i(t) \sum_{\substack{j=0 \\ j \neq i}}^N \alpha_{ij} = -\alpha_{ii} \cdot P_i(t) \quad (\text{at steady state}) \quad v_i^{dep} = -\alpha_{ii} \cdot \Pi_i$$

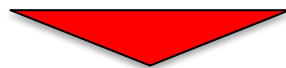
- **In analogy**, considering the **arrivals** to state i from any state k :

$$v_i^{arr}(t) = \sum_{\substack{k=0 \\ k \neq i}}^N \alpha_{ki} \cdot P_k(t)$$

$$v_i^{arr} = \sum_{\substack{k=0 \\ k \neq i}}^N \alpha_{ki} \cdot \Pi_k \quad (\text{at steady state})$$



$$\underline{\underline{\Pi}} \cdot \underline{\underline{A}} = 0 \quad \Rightarrow \quad \sum_{k=0}^N \alpha_{ki} \cdot \Pi_k = 0 \quad (i = 0, 1, 2, \dots, N)$$



$$-\alpha_{ii} \cdot \Pi_i = \sum_{\substack{k=0 \\ k \neq i}}^N \alpha_{ki} \cdot \Pi_k \quad (i = 0, 1, 2, \dots, N)$$

AT STEADY STATE:

frequency of departures from state i = frequency of arrivals to state i

- **SYSTEM FAILURE INTENSITY w_f :**
 - **Rate** at which **system failures** occur
 - **Expected number** of system **failures** per **unit of time**
 - **Rate of exiting a success state** to go into one of **fault**

- **SYSTEM FAILURE INTENSITY W_f :**
 - **Rate** at which **system failures** occur
 - **Expected number** of system **failures** per **unit of time**
 - **Rate of exiting a success state** to go into one of **fault**

$$W_f(t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \rightarrow F}$$

S = set of success states of the system

F = set of failure states of the system

$P_i(t)$ = probability of the system being in the functioning state i at time t

$\lambda_{i \rightarrow F}$ = conditional (transition) probability of leaving success state i towards failure states

- **SYSTEM REPAIR INTENSIT W_r :**
 - **Rate** at which **system repairs** occur
 - **Expected number** of system **repairs** per **unit of time**
 - **Rate of exiting a failed state** to go into one of **success**

$$W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \rightarrow S}$$

S = set of success states of the system

F = set of failure states of the system

$P_j(t)$ = probability of the system being in the failure state j at time t

$\mu_{j \rightarrow S}$ = conditional (transition) probability of leaving failure state j towards success states

Example 5: one component/one repairman

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Consider a system made by one component which can be in two states: working ('0') or Failed ('1'). Assume constant failure rate λ and constant repair rate μ and that the component is working at $t = 0$: $\underline{C} = [1 \ 0]$

- You are required to find the failure and repair intensities

- **Time of occupance of state i (sojourn time), T_i** = time spent in a state i
- Is the time (t) that the system has already been in state i influencing the time (s) the system will remain in state i ?


$$P(T_i > t + s | T_i > t) = P(X(t + u) = i, 0 \leq u \leq s | X(\tau) = i, 0 \leq \tau \leq t) =$$

$$= P(X(t + u) = i, 0 \leq u \leq s | X(t) = i) \text{ (by Markov property)}$$

$$= P(X(u) = i, 0 \leq u \leq s | X(0) = i) \text{ (by homogeneity)}$$

$$= P(T_i > s) \quad \textbf{Memoryless Property}$$

- The only distribution satisfying the memoryless property is the **Exponential distribution**


$$T_i \sim \text{Exp}$$

- System departure **rate from state i** (at steady state): $-\alpha_{ii}$

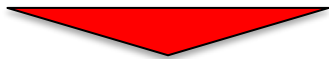



$$T_i \sim \text{Exp}(-\alpha_{ii})$$

- **Expected sojourn time l_i** : average time of occupancy of state i

$$l_i = \mathbb{E}\{T_i\} = \frac{1}{-\alpha_{ii}}$$

- Total frequency of departure at steady state: $v_i^{dep} = -\alpha_{ii} \cdot \Pi_i$
- Average time of occupancy of state: $l_i = \frac{1}{-\alpha_{ii}}$


$$v_i^{dep} = -\alpha_{ii} \cdot \Pi_i = \frac{\Pi_i}{l_i}$$

$$\Pi_i = v_i^{arr} \cdot l_i$$

$v_i^{dep} = v_i^{arr}$

The **mean** proportion of time Π_i that the system spends in state i is equal to the **total frequency of arrivals to state i** multiplied by the mean duration of one visit in state i

- **System instantaneous availability** at time t
= **sum** of the **probabilities** of being in a **success** state at time t

$$p(t) = \sum_{i \in S} P_i(t) = 1 - q(t) = 1 - \sum_{j \in F} P_j(t)$$



In the Laplace domain

$$\tilde{p}(s) = \sum_{i \in S} \tilde{P}_i(s) = \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s)$$

S = set of success states of the system

F = set of failure states of the system

- **TWO CASES:**

1) Non-Reparable Systems
→ No repairs allowed

2) Reparaible Systems
→ Repairs allowed

- No repairs allowed \Rightarrow **Reliability = Availability** $R(t) \equiv p(t) = 1 - q(t)$
- In the Laplace Domain: $\tilde{R}(s) = \sum_{i \in S} \tilde{P}_i(s) = \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s)$
- **Mean Time to Failure (MTTF):**

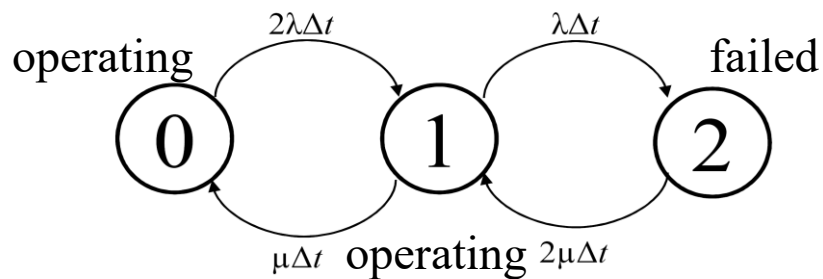
$$MTTF = \int_0^{\infty} R(t) dt = \left[\int_0^{\infty} R(t) e^{-st} dt \right]_{s=0} = \tilde{R}(0) = \sum_{i \in S} \tilde{P}_i(0) = \left[\frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s) \right]_{s=0}$$

- **TWO CASES:**

- 1) **Non-reparable systems**
→ **No repairs allowed**

- 2) **Reparable systems**
→ **Repairs allowed**

Parallel System of two identical components

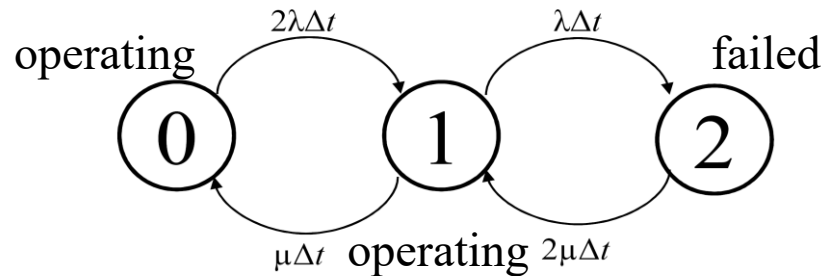


$$S = \{0, 1\}$$
$$F = \{2\}$$

$$R(t) = P(T > t) = P\{X(\tau) = 0 \text{ or } X(\tau) = 1, \forall \tau \in [0, t)\} = P_0^*(t) + P_1^*(t)$$

Can I transform the Markov Diagram in such a way that $R(t) = P_0^*(t) + P_1^*(t)$?

Parallel System of two identical components

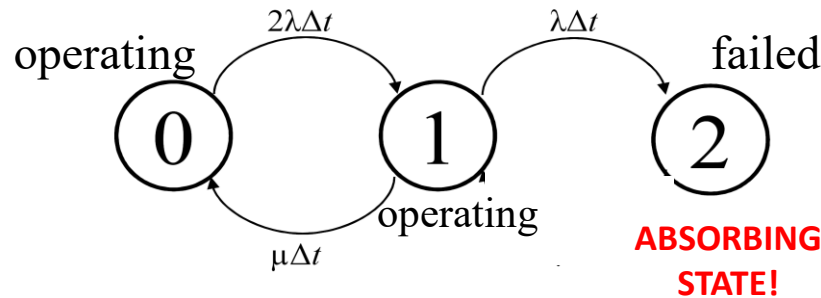


$$S = \{0, 1\}$$

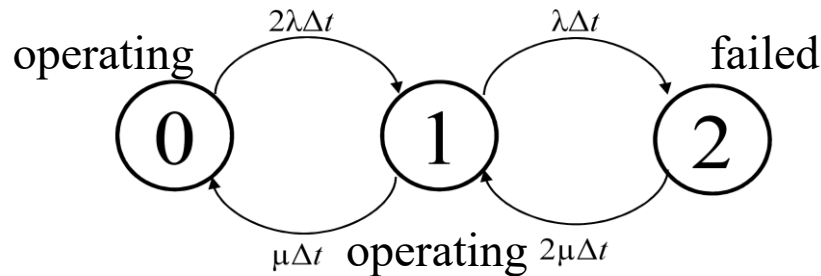
$$F = \{2\}$$

$$R(t) = P(T > t) = P\{X(\tau) = 0 \text{ or } X(\tau) = 1, \forall \tau \in [0, t)\} = P_0^*(t) + P_1^*(t)$$

Can I transform the Markov Diagram in such a way that $R(t) = P_0^*(t) + P_1^*(t)$?



1. Transform the failed states $j \in F$ into absorbing states (the system cannot be repaired \rightarrow it is not possible to escape from a failed state). **i.e., exclude all the failed states $j \in F$ from the transition rate matrix**



$$S = \{0, 1\}$$

$$F = \{2\}$$

$$\underline{\underline{A}} = \left(\begin{array}{cc|c} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu + \lambda) & \lambda \\ \hline 0 & 2\mu & -2\mu \end{array} \right) \xrightarrow{\text{red arrow}} \underline{\underline{A}}^* = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu + \lambda) & \lambda \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{red arrow}} \underline{\underline{A}}^* = \begin{bmatrix} -2\lambda & 2\lambda \\ \mu & -(\mu + \lambda) \end{bmatrix}$$

The new matrix $\underline{\underline{A}}^*$ contains the transition rates for transitions **only among the success states** $i \in S$

(the “reduced” system is virtually functioning continuously with no interruptions)

2. Solve the **reduced problem** of \underline{A}^* for the probabilities $P_i^*(t)$, $i \in S$ of being in these **(transient) safe states**

$$\frac{d\underline{P}^*(t)}{dt} = \underline{P}^*(t) \cdot \underline{A}^*$$



Reliability

$$R(t) = \sum_{i \in S} P_i^*(t)$$



Mean Time To Failure (MTTF)

$$MTTF = \int_0^{\infty} R(t) dt = \sum_{i \in S} \tilde{P}_i^*(0) = \tilde{R}(0)$$

NOTICE: in the reduced problem we have only transient states $\Rightarrow \Pi_i^* = P_i^*(\infty) = 0$

Consider a system made by 2 identical components in parallel. Each component can be in two states: working or failed. Assume constant failure rate λ , that 2 repairmen are available and that the single component repair rate is constant and equal to μ . You are required to:

- find the system reliability
- find the system MTTF

Consider a system made by 2 identical components in series. Each component can be in two states: working or failed. Assume constant failure rate λ , that 2 repairmen are available and that the single component repair rate is constant and equal to μ . You are required to:

- find the system reliability
- find the system MTTF