CO-2 data analysis

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Task and dataset

Section 1

Preliminary analysis

Section 2

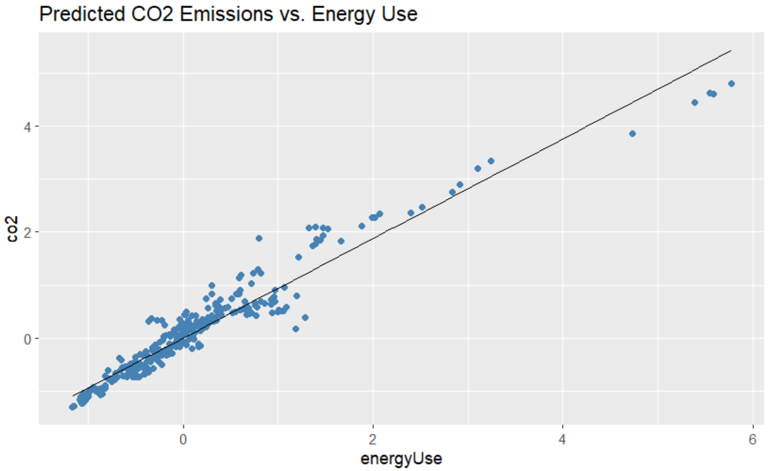
BAS analysis and feature selection

Section 3

- Before considering more complex models
- Split in train and test

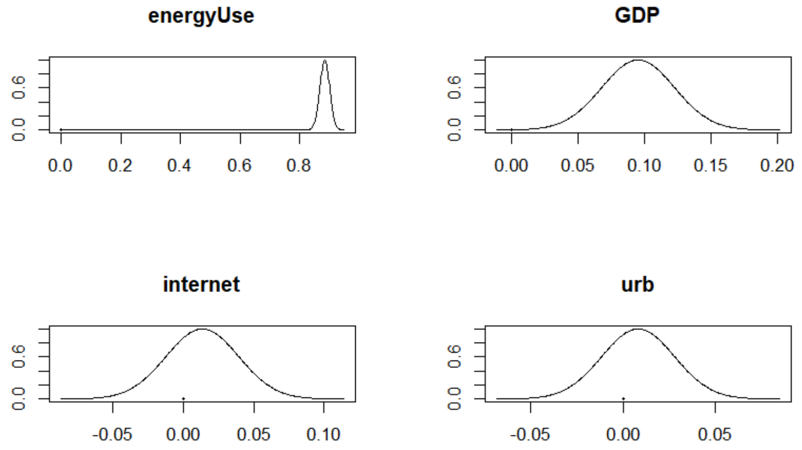
prediction using only energyUse

```
g-prior | alpha = n
[1] "Train Mean sum of squared error
is: 0.0666247331734019"
[1] "Test Mean sum of squared error
is: 0.0591558419947747"
```



prediction using all the covariates

```
g-prior | alpha = n
[1] "Train Mean sum of squared error
is: 0.0576266600005661"
[1] "Test Mean sum of squared error
is: 0.0543753086210353"
```



Jeffreys-Zellner-Siow (JZS) priors

```
prior="JZS" | alpha=1

Showing results only for HPM

[1] "Intercept" "energyUse" "GDP"

[1] "Train Mean sum of squared error is:
0.0577376077419626"
[1] "Test Mean sum of squared error is:
0.0541967365249909"
```

- Features selection
- Showing results only for HPM
- we decided not to use a BMA since we are interested in the best model and not in the average of the models

```
          P(B != 0 | Y) model 1 model 2 model 3 model 4 model 5
Intercept         1.000   1.000   1.000   1.000   1.000   1.000
energyUse         1.000   1.000   1.000   1.000   1.000   1.000
GDP               0.998   1.000   1.000   1.000   0.000   1.000
internet          0.029   0.000   1.000   0.000   1.000   1.000
urb               0.026   0.000   0.000   1.000   0.000   1.000
BF                   NA   1.000   0.028   0.027   0.002   0.001
PostProbs            NA   0.946   0.026   0.025   0.002   0.001
R2                   NA   0.943   0.943   0.943   0.940   0.943
dim                  NA   3.000   4.000   4.000   3.000   5.000
logmarg              NA 405.047 401.471 401.423 398.607 397.930

 Marginal Posterior Summaries of Coefficients:

 Using  HPM

 Based on the top  1 models
           post mean  post SD  post p(B != 0)
Intercept  0.02865    0.01416  1.00000
energyUse  0.88544    0.01624  1.00000
GDP        0.11161    0.01678  0.99841
internet   0.00000    0.00000  0.02884
urb        0.00000    0.00000  0.02607
```

Adding more covariates

- non-linear relationships
- GDP^2 and energyUse^2
- Feature selection

Jeffreys-Zellner-Siow (JZS) priors

```
prior="JZS" | alpha=1

Showing results only for HPM

[1] "Intercept" "energyUse" "GDP"

[1] "Train Mean sum of squared error is: 0.0577376077419626"

[1] "Test Mean sum of squared error is: 0.0541967365249909"
```

	P(B != 0 | Y)	model 1	model 2	model 3	model 4	model 5
Intercept	1.000	1.000	1.000	1.000	1.000	1.000
energyUse	1.000	1.000	1.000	1.000	1.000	1.000
GDP	0.438	0.000	1.000	0.000	1.000	0.000
internet	0.045	0.000	0.000	1.000	0.000	0.000
urb	0.037	0.000	0.000	0.000	1.000	0.000
energyUse2	1.000	1.000	1.000	1.000	1.000	1.000
GDP2	0.035	0.000	0.000	0.000	0.000	1.000
BF	NA	1.000	0.781	0.060	0.045	0.040
PostProbs	NA	0.499	0.390	0.030	0.022	0.020
R2	NA	0.948	0.949	0.948	0.949	0.948
dim	NA	3.000	4.000	4.000	5.000	4.000
logmarg	NA	417.241	416.993	414.426	414.138	414.021

```
Marginal Posterior Summaries of Coefficients:

Using  HPM

Based on the top  1 models
```

	post mean	post SD	post p(B != 0)
Intercept	0.028655	0.013570	1.000000
energyUse	1.078736	0.020586	1.000000
GDP	0.000000	0.000000	0.437515
internet	0.000000	0.000000	0.044818
urb	0.000000	0.000000	0.036636
energyUse2	-0.046170	0.005378	0.999997
GDP2	0.000000	0.000000	0.035117

```
[1] "Intercept"  "energyUse"  "energyUse2"
[1] "Train Mean sum of squared error is: 0.0530340808788469"
[1] "Test Mean sum of squared error is: 0.0431591495882435"
```

Conclusion of regression with BAS

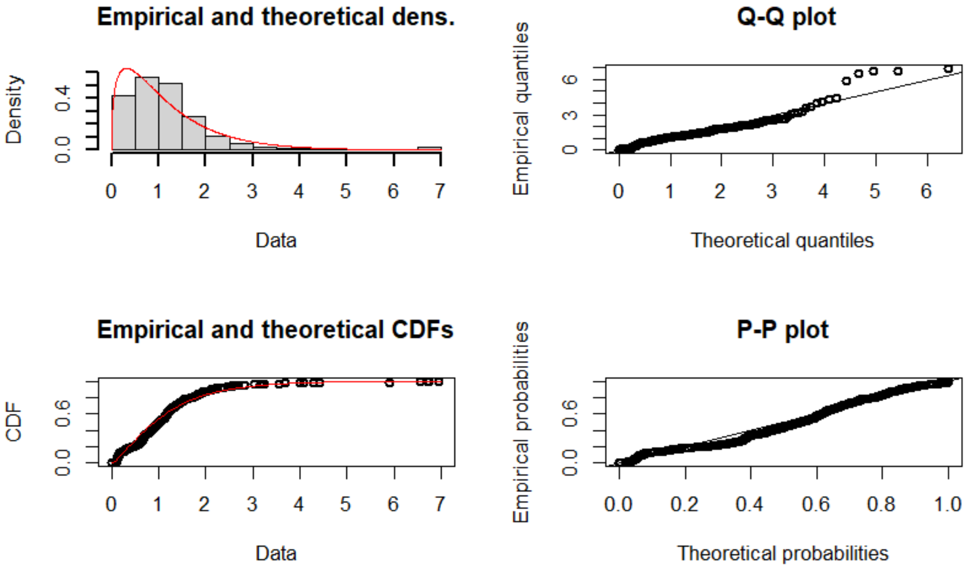
- If we avoid adding more covariates, the model with the JZS prior has the best performance and among the selected the one with only energyUse and GDP as covariates is the best. It has a lower BIC compared to the other subset of models. The model doesn't overfit on the test set and has a good performance. Lastly it shows a strong relashionship between co2 and GDP since the posterior probability is close to 1.
- If instead we add more covariates we can see that the model with the squared of energyUse and energy has a lower BIC compared to the other subset of models and get better overall performances. In this case the importance of GDP with co2 decays.

Models (JAGS)

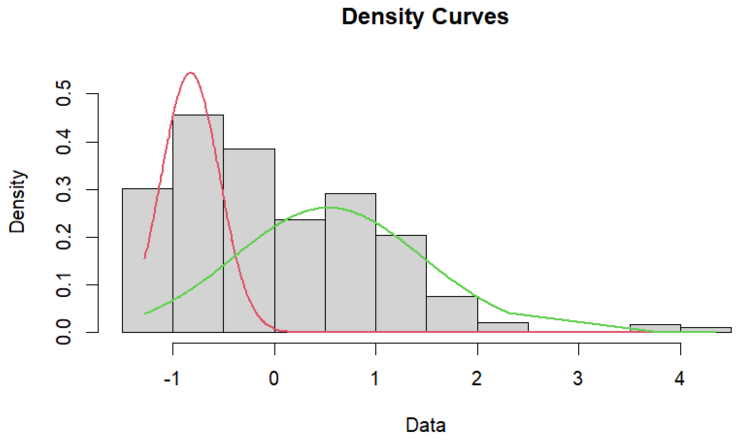
Section 4.0 - 4.1

- This part was only a trial
- Study in deep the distribution of the covariates
- Add this information into the model to see if the prediction improves
- Get the distribution shape (keeping in mind variables have been standardized and it doesn’t reflect the true distribution)

EnergyUse



GDP



JAGS model considering covariate distribution energyUse

$$y_i \sim \mathcal{N}(\beta_o + X_i\beta, \sigma_y^2)$$
$$X_1 \sim \mathcal{G}(alphaEnergy, betaEnergy)$$
$$\beta_0 \sim \mathcal{N}(0, 100)$$
$$\beta_i \sim \mathcal{N}(0, 100)$$
$$\sigma_y^{-2} \sim \mathcal{G}(0.01, 0.01)$$
$$alphaEnergy \sim \mathcal{G}(1, 1)$$
$$betaEnergy \sim \mathcal{G}(1, 1)$$

	Mean	SD	Naive SE	Time-series SE
alphaEnergy	1.35769	0.091352	1.055e-03	2.081e-03
beta[1]	0.90722	0.015266	1.763e-04	2.942e-04
beta[2]	0.08923	0.024638	2.845e-04	5.072e-04
beta[3]	0.01262	0.023156	2.674e-04	4.540e-04
beta[4]	0.01061	0.017120	1.977e-04	2.286e-04
beta0	-1.05792	0.021904	2.529e-04	4.091e-04
betaEnergy	1.16403	0.093768	1.083e-03	2.114e-03
var.y	0.05773	0.004384	5.062e-05	5.093e-05

2. Quantiles for each variable:

	2.5%	25%	50%	75%	97.5%
alphaEnergy	1.18667	1.293468	1.35418	1.41789	1.54281
beta[1]	0.87681	0.897028	0.90730	0.91765	0.93662
beta[2]	0.04128	0.072276	0.08916	0.10571	0.13721
beta[3]	-0.03205	-0.002955	0.01253	0.02858	0.05688
beta[4]	-0.02246	-0.001153	0.01080	0.02199	0.04451
beta0	-1.10080	-1.072313	-1.05802	-1.04332	-1.01375
betaEnergy	0.98633	1.100396	1.15981	1.22482	1.35676
var.y	0.04979	0.054642	0.05755	0.06053	0.06698

JAGS model considering covariate distribution energyUse a

Likelihood:

$$y_i \sim N(\mu_i, \sigma_y^2),$$
$$\mu_i = \beta_0 + \sum_{j=1}^{p} x_{ij}\beta_j,$$
$$x_{i1} \sim \text{Gamma}(\alpha_{\text{Energy}}, \beta_{\text{Energy}}), \quad \text{for energyUse covariate}$$
$$z_i \sim \text{Bernoulli}(p_{\text{GDP}}),$$
$$x_{i2} \sim N(\mu_1 z_i + \mu_2(1-z_i), \tau_3 z_i + \tau_4(1-z_i)), \quad \text{for GDP cova}$$

Priors:

$$\beta_j \sim N(0, 100), \quad \text{for } j = 1, \dots, p$$
$$\beta_0 \sim N(0, 100),$$
$$\sigma_y^{-2} \sim \text{Gamma}(0.01, 0.01), \quad \text{where } \sigma_y^2 = \frac{1}{\sigma_y^{-2}},$$
$$\alpha_{\text{Energy}} = 2,$$
$$\beta_{\text{Energy}} = 2,$$
$$\mu_1 \sim N(-0.8, 100),$$
$$\mu_2 \sim N(0.5, 100),$$
$$\tau_3 \sim \text{Gamma}(0.01, 0.01),$$
$$\tau_4 \sim \text{Gamma}(0.01, 0.01),$$
$$p_{\text{GDP}} \sim \text{Beta}(1, 1).$$

	Mean	SD	Naive SE	Time-series SE
beta[1]	0.90737	0.015500	0.0001790	0.0003201
beta[2]	0.08868	0.024702	0.0002852	0.0005059
beta[3]	0.01345	0.023294	0.0002690	0.0004602
beta[4]	0.01030	0.017080	0.0001972	0.0002306
beta0	-1.05835	0.022125	0.0002555	0.0004334
mu1	-0.83426	0.051364	0.0005931	0.0019826
mu2	0.51722	0.101250	0.0011691	0.0031127
pGDP	0.38227	0.052767	0.0006093	0.0019452
var.y	0.05769	0.004278	0.0000494	0.0000493

2. Quantiles for each variable:

	2.5%	25%	50%	75%	97.5%
beta[1]	0.87672	0.896838	0.90732	0.91766	0.93752
beta[2]	0.04132	0.071764	0.08839	0.10520	0.13775
beta[3]	-0.03286	-0.002063	0.01378	0.02894	0.05911
beta[4]	-0.02324	-0.001317	0.01024	0.02186	0.04405
beta0	-1.10182	-1.073324	-1.05842	-1.04350	-1.01585
mu1	-0.94126	-0.868485	-0.83167	-0.79826	-0.73958
mu2	0.32106	0.448521	0.51691	0.58484	0.71749
pGDP	0.27716	0.345979	0.38372	0.42015	0.48108
var.y	0.04990	0.054692	0.05748	0.06043	0.06678

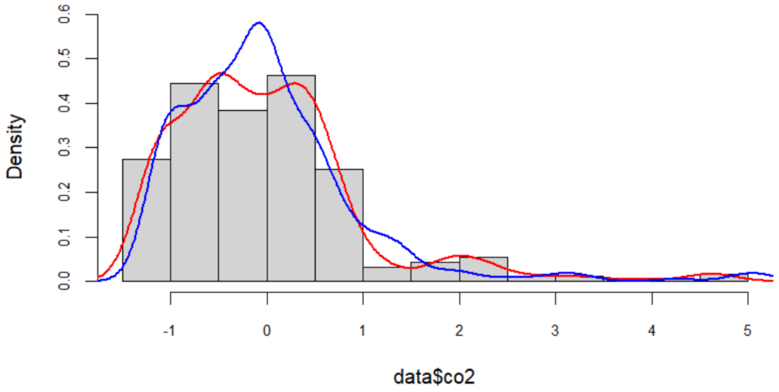
JAGS model no threshold on GDP

$$y_i \sim \mathcal{N}(\beta_o + X_i\beta, \sigma_y^2)$$
$$\beta_0 \sim \mathcal{N}(100)$$
$$\beta_i \sim \mathcal{N}(0, 100)$$
$$\sigma_y^{-2} \sim \mathcal{G}(0.01, 0.01)$$

	Mean	SD	Naive SE	Time-series SE
beta[1]	0.90697	0.015372	1.775e-04	3.116e-04
beta[2]	0.08965	0.024734	2.856e-04	5.249e-04
beta[3]	0.01254	0.023072	2.664e-04	4.495e-04
beta[4]	0.01069	0.016864	1.947e-04	2.284e-04
beta0	-1.05752	0.021857	2.524e-04	4.280e-04
var.y	0.05763	0.004394	5.074e-05	5.074e-05

2. Quantiles for each variable:

	2.5%	25%	50%	75%	97.5%
beta[1]	0.87713	0.8966404	0.90687	0.91714	0.93736
beta[2]	0.04167	0.0729711	0.08969	0.10657	0.13861
beta[3]	-0.03235	-0.0029236	0.01244	0.02774	0.05806
beta[4]	-0.02303	-0.0006827	0.01049	0.02227	0.04367
beta0	-1.10026	-1.0717167	-1.05756	-1.04309	-1.01424
var.y	0.04962	0.0545871	0.05744	0.06042	0.06692



- The prediction of previous models doesn't improve but it allowed us to extract the parameters of GDP and energyUse

Bayesian LASSO prior

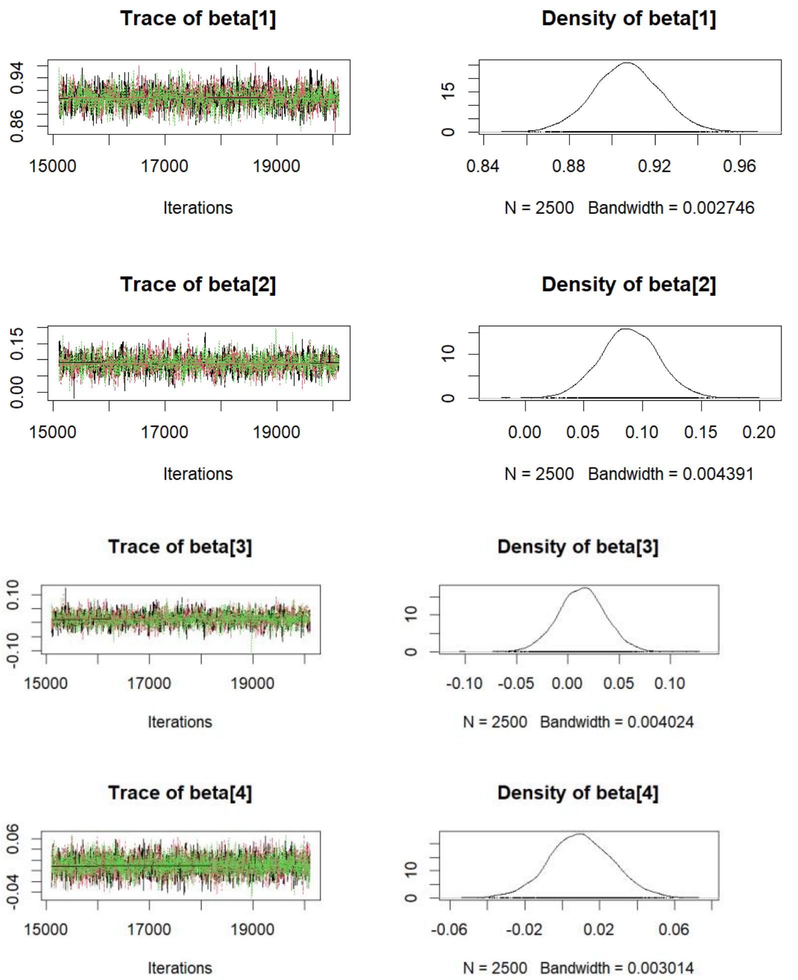
We use now lasso regression to select again the most important covariates in the model, and see if the model without some covariates can improve without degrading too much.

$$
\begin{aligned}
y_i &\sim N(\beta_0 + X_i\beta, \sigma_y^2)\\
\beta_0 &\sim N(0, 100)\\
\beta_i &\sim \mathrm{DE}(0, \sigma_b^2\sigma^2)\\
\sigma_y^2 &\sim \mathrm{IG}(0.01, 0.01)\\
\sigma_b^2 &\sim \mathrm{IG}(0.01, 0.01)
\end{aligned}
$$

	Mean	SD	Naive SE	Time-series SE
beta[1]	0.90688	0.01543	0.0001782	0.0003694
beta[2]	0.08862	0.02468	0.0002849	0.0005653
beta[3]	0.01353	0.02295	0.0002650	0.0005160
beta[4]	0.01014	0.01694	0.0001956	0.0002715
beta0	-1.05745	0.02201	0.0002541	0.0004957

2. Quantiles for each variable:

	2.5%	25%	50%	75%	97.5%
beta[1]	0.87623	0.896427	0.906878	0.91717	0.93735
beta[2]	0.03993	0.072306	0.088606	0.10543	0.13740
beta[3]	-0.03169	-0.001674	0.013657	0.02863	0.05913
beta[4]	-0.02284	-0.001261	0.009799	0.02150	0.04427
beta0	-1.10094	-1.072499	-1.057443	-1.04231	-1.01483



Threshold

Section 4.2.1

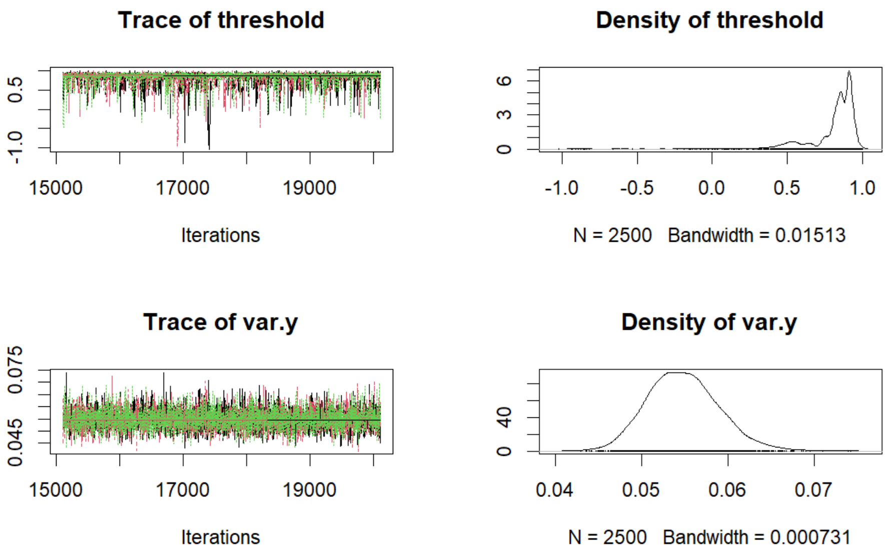
- For this analysis different models have been deployed

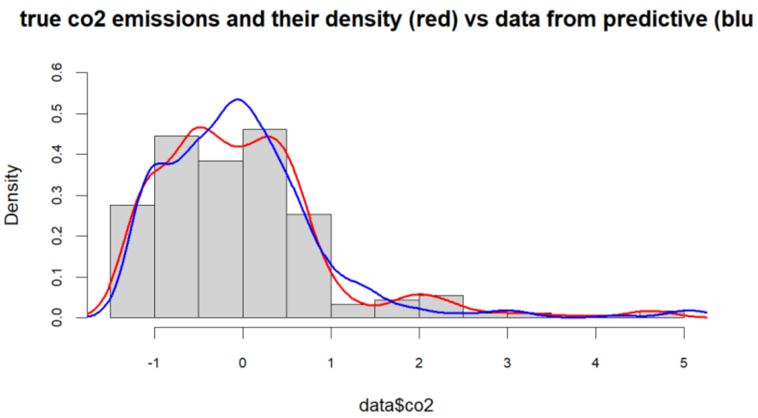
Find the threshold in GDP considering other variables

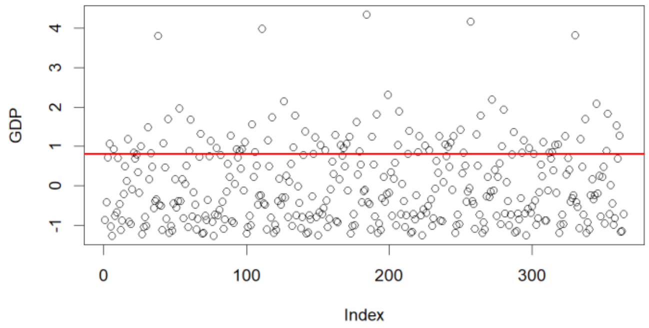
The model has a form:

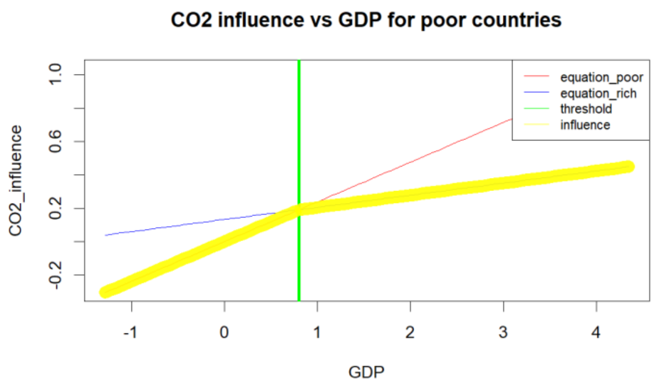
$$y_i \sim \mathcal{N}(\mu_i, \sigma_y^2)$$
$$\mu_i = \beta_0 + \sum_{j=1}^{p-1} x_{ij}\beta_j + betapoor \cdot \mathrm{GDP}_i \cdot I(\mathrm{GDP}_i < \mathrm{threshold}) + betarich \cdot \mathrm{GDP}_i \cdot I(\mathrm{GDP}_i >= \mathrm{threshold})$$
$$\beta_0 \sim \mathcal{N}(100)$$
$$\beta_i \sim \mathcal{N}(0, 100)$$
$$betapoor \sim \mathcal{N}(0, 100)$$
$$betarich \sim \mathcal{N}(0, 100)$$
$${\sigma_y}^{-2} \sim \mathcal{G}(0.01, 0.01)$$
$$\mathrm{threshold} \sim \mathcal{U}(-2, 1)$$

```
              Mean        SD  Naive SE Time-series SE
beta[1]    0.89524 0.015315 1.768e-04      3.484e-04
beta[2]   -0.03050 0.025109 2.899e-04      5.835e-04
beta[3]   -0.00564 0.017090 1.973e-04      2.486e-04
beta0     -0.99153 0.028691 3.313e-04      7.761e-04
beta_poor  0.23855 0.044293 5.114e-04      1.211e-03
beta_rich  0.07283 0.024845 2.869e-04      4.626e-04
threshold  0.80185 0.194603 2.247e-03      5.389e-03
var.y      0.05478 0.004112 4.748e-05      5.029e-05

          2. Quantiles for each variable:

                2.5%      25%       50%       75%
                      97.5%
   beta[1]   0.86507  0.88506  0.895469  0.905597
                      0.92454
   beta[2]  -0.07941 -0.04786 -0.030390 -0.013649
                      0.01849
   beta[3]  -0.03925 -0.01729 -0.005745  0.006217
                      0.02823
  beta0     -1.04490 -1.01105 -0.993111 -0.973394 -
                      0.93136
```









- The coefficient beta_poor for poor countries is higher than the coefficient beta_rich for rich countries. This means that the relationship between CO_2 emissions and GDP is stronger for poor countries than for rich countries.
- However because of all the parameters that are free we think this model is too general and the threshold refers for countries with very high GDP (40000)
- In this we use a single intercept: the assumption is the baseline CO_2 emissions level (intercept) is the same across all GDP levels, but the rate of change (slope) with respect to GDP changes at the threshold. Instead all the intercepts are summarized in the first parameter

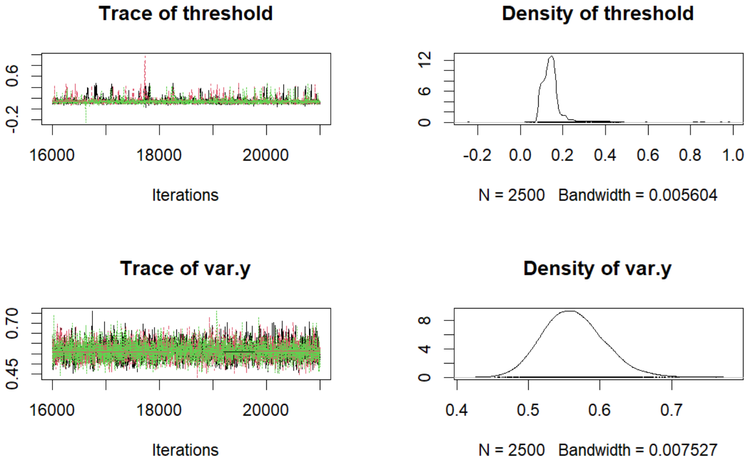
Find the threshold in GDP considering only GDP

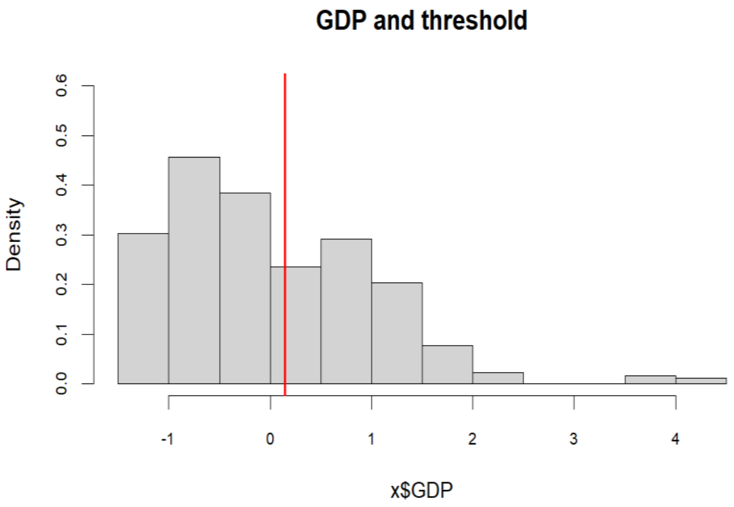
$$y_i \sim \mathcal{N}(\mu_i, \sigma_y^2)$$
$$\mu_i = (betapoor \cdot \mathrm{GDP}_i + betapoor_0) \cdot I(\mathrm{GDP}_i < \mathrm{threshold}) + (betarich \cdot \mathrm{GDP}_i + betarich_0) \cdot I(\mathrm{GDP}_i >= \mathrm{threshold})$$
$$betapoor \sim \mathcal{N}(0, 100)$$
$$betarich \sim \mathcal{N}(0, 100)$$
$$betapoor_0 \sim \mathcal{N}(0, 100)$$
$$betarich_0 \sim \mathcal{N}(0, 100)$$
$$\sigma_y^{-2} \sim \mathcal{G}(0.01, 0.01)$$
$$\mathrm{threshold} \sim \mathcal{U}(-1.5, 1)$$

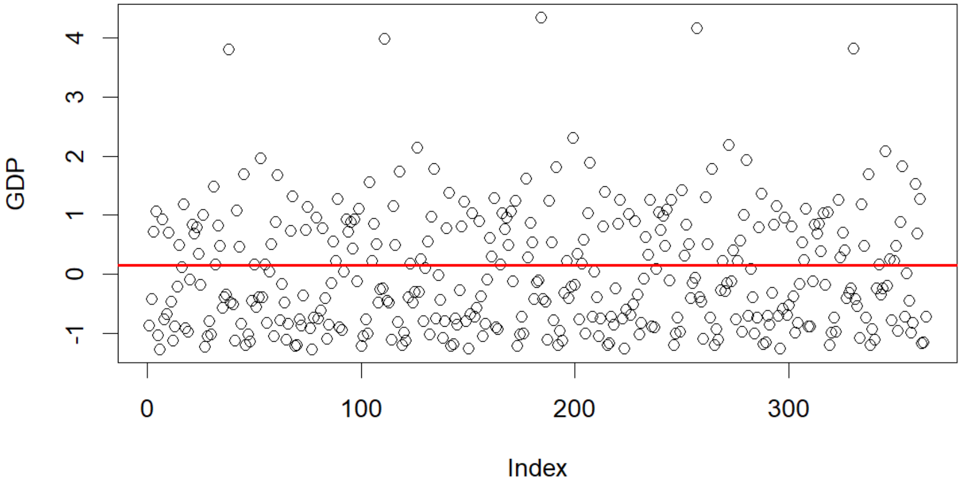
	Mean	SD	Naive SE	Time-series SE
beta_poor	1.41552	0.14797	0.0017086	0.003881
beta_poor_0	0.58793	0.11624	0.0013422	0.002984
beta_rich	0.53322	0.08458	0.0009766	0.001601
beta_rich_0	0.01984	0.10842	0.0012519	0.002093
threshold	0.14651	0.05753	0.0006643	0.001518
var.y	0.56299	0.04235	0.0004890	0.000489

2. Quantiles for each variable:

	2.5%	25%	50%	75%	97.5%
beta_poor	1.11997	1.31635	1.41820	1.5160	1.6991
beta_poor_0	0.35660	0.51157	0.59076	0.6674	0.8118
beta_rich	0.36989	0.47665	0.53184	0.5893	0.7027
beta_rich_0	-0.20049	-0.04954	0.02132	0.0933	0.2267
threshold	0.08693	0.11585	0.13877	0.1581	0.3309
var.y	0.48722	0.53313	0.56090	0.5898	0.6537







- This model is more robust threshold at 30000 (standardized variables)

$$y_i \sim \mathcal{N}(\mu_i, \sigma_y^2)$$

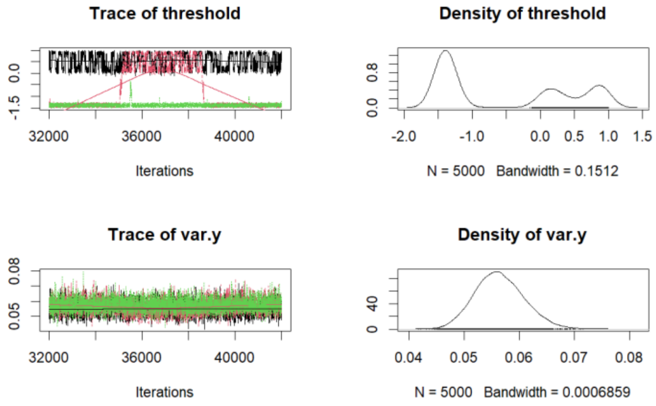
$$\mu_i = \beta_0 + \sum_{j=1}^{p-1} x_{ij}\beta_j + (betapoor \cdot \mathrm{GDP}_i + betapoor_0) \cdot I(\mathrm{GDP}_i < \mathrm{threshold}) + (betarich \cdot \mathrm{GDP}_i + betarich_0) \cdot I(\mathrm{GDP}_i >= \mathrm{threshold})$$

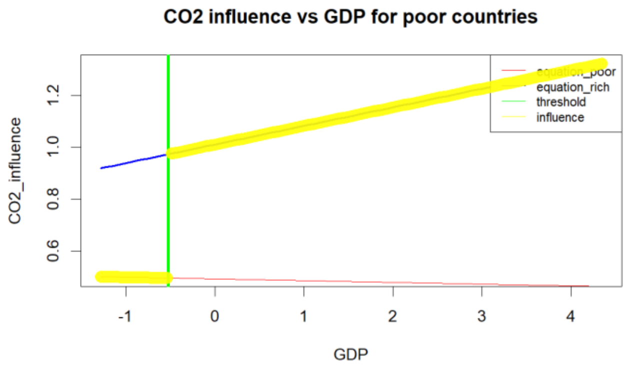
$$\beta_0 \sim \mathcal{N}(100)$$
$$\beta_i \sim \mathcal{N}(0, 100)$$
$$betapoor \sim \mathcal{N}(0, 100)$$
$$betarich \sim \mathcal{N}(0, 100)$$
$$betapoor_0 \sim \mathcal{N}(0, 100)$$
$$betarich_0 \sim \mathcal{N}(0, 100)$$
$$\sigma_y^{-2} \sim \mathcal{G}(0.01, 0.01)$$
$$\mathrm{threshold} \sim \mathcal{U}(-1.5, 1)$$

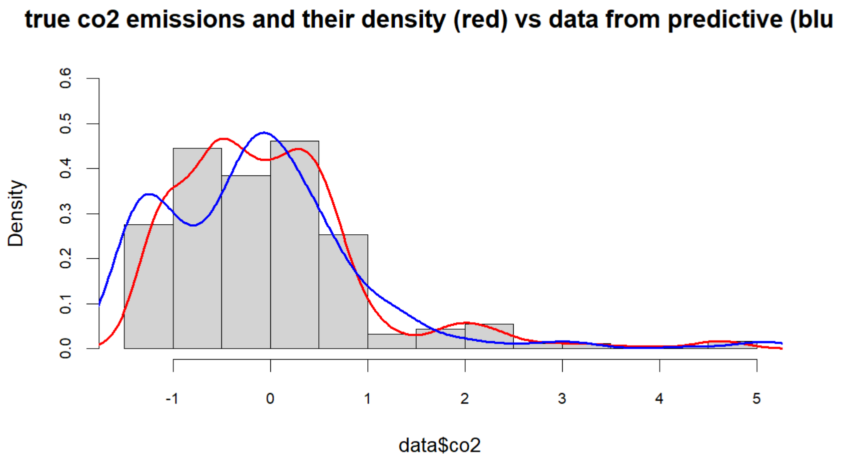
	Mean	SD	Naive SE	Time-series SE
beta[1]	0.902602	0.016226	1.325e-04	3.037e-04
beta[2]	-0.008552	0.034486	2.816e-04	2.376e-03
beta[3]	0.003648	0.018780	1.533e-04	5.034e-04
beta0	-2.017844	1.212530	9.900e-03	1.380e-01
beta_poor	-0.006683	7.514279	6.135e-02	6.063e-02
beta_poor_0	0.492876	7.327803	5.983e-02	1.181e-01
beta_rich	0.071660	0.038503	3.144e-04	1.310e-03
beta_rich_0	1.010556	1.218549	9.949e-03	1.245e-01
threshold	-0.528305	0.975763	7.967e-03	1.471e-01
var.y	0.056470	0.004435	3.621e-05	7.808e-05

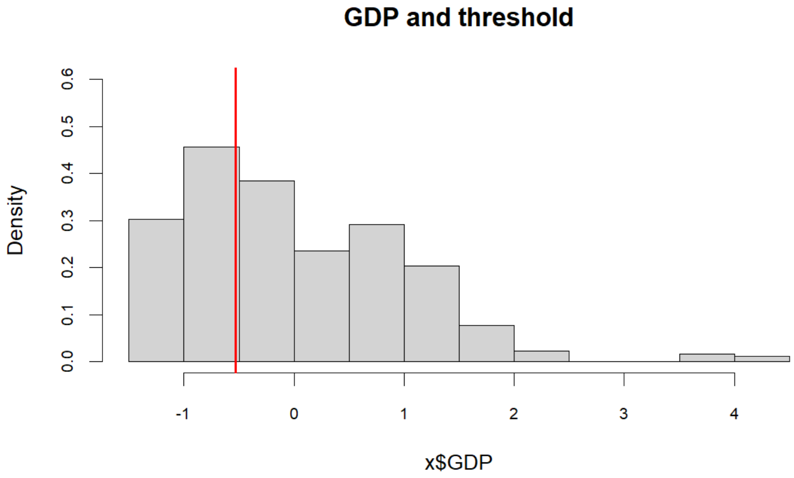
2. Quantiles for each variable:

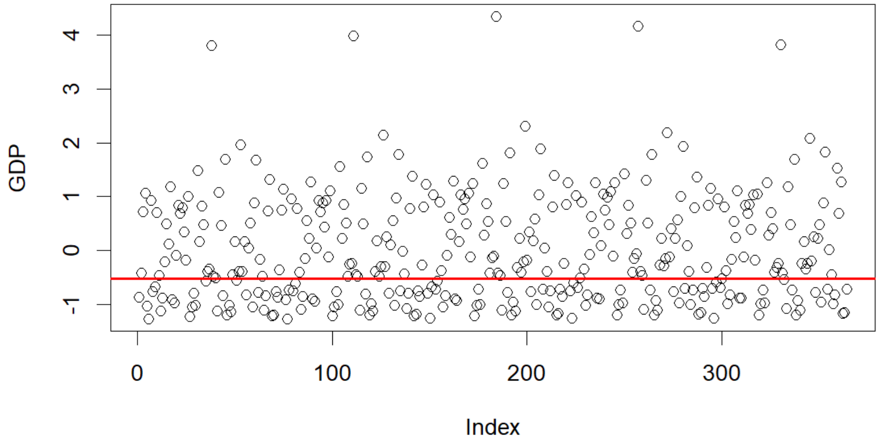
	2.5%	25%	50%	75%	97.5%
beta[1]	0.87124	0.891503	0.902701	0.91368	0.93448
beta[2]	-0.07634	-0.033781	-0.005914	0.01701	0.05251
beta[3]	-0.03301	-0.009233	0.003680	0.01625	0.04005
beta0	-4.51276	-3.152240	-1.477989	-1.10852	-0.35529
beta_poor	-17.40161	-1.182368	0.222070	0.85561	17.11251
beta_poor_0	-16.75105	-0.877324	0.581135	2.68696	16.51105
beta_rich	-0.01222	0.047735	0.076237	0.09879	0.13790
beta_rich_0	-0.70674	0.099705	0.499394	2.13322	3.45284
threshold	-1.48994	-1.400170	-1.296270	0.40722	0.92799
var.y	0.04834	0.053384	0.056238	0.05932	0.06562




CO2 influence vs GDP for poor countries
equation_poor
equation_rich
threshold
influence
CO2_influence
GDP


true co2 emissions and their density (red) vs data from predictive (blu
Density
data$co2


GDP and threshold
Density
x$GDP


GDP
Index

Section 4.2.1: Threshold

To answer the thesis of a threshold in the GDP above which the GDP does not influence the CO_2 production we start by putting a **uniform prior** on the threshold.

Define the linear predictor, only for μ :
$\mu[i] = \langle X[i,], \beta \rangle$

Normal model:
$\text{co2percap}[i] \sim \mathcal{N}(\mu[i], \beta_\gamma)$

Indicator variable:
$\text{indicator}[i] = (\text{unGDP}[i] \geq \text{threshold})$

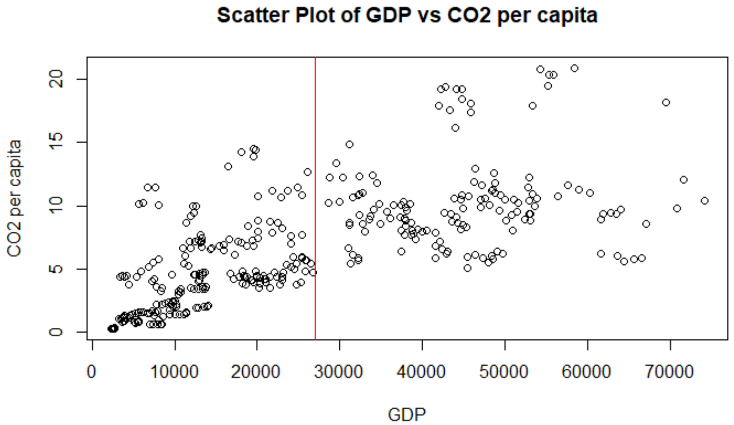
Covariate matrix $X[i,]$:
$X[i,1] = (1 - \text{indicator}[i])$
$X[i,2] = \text{indicator}[i]$
$X[i,3] = \text{GDP}[i] \cdot (1 - \text{indicator}[i])$
$X[i,4] = \text{GDP}[i] \cdot \text{indicator}[i]$

Prior for the variance β_γ :
$\beta_\gamma \sim \text{Gamma}(0.001, 0.001)$

Uninformative priors for β parameters:
$\beta[1] \sim \mathcal{N}(0, 0.01)$
$\beta[2] \sim \mathcal{N}(0, 0.01)$
$\beta[3] \sim \mathcal{N}(0, 0.01)$

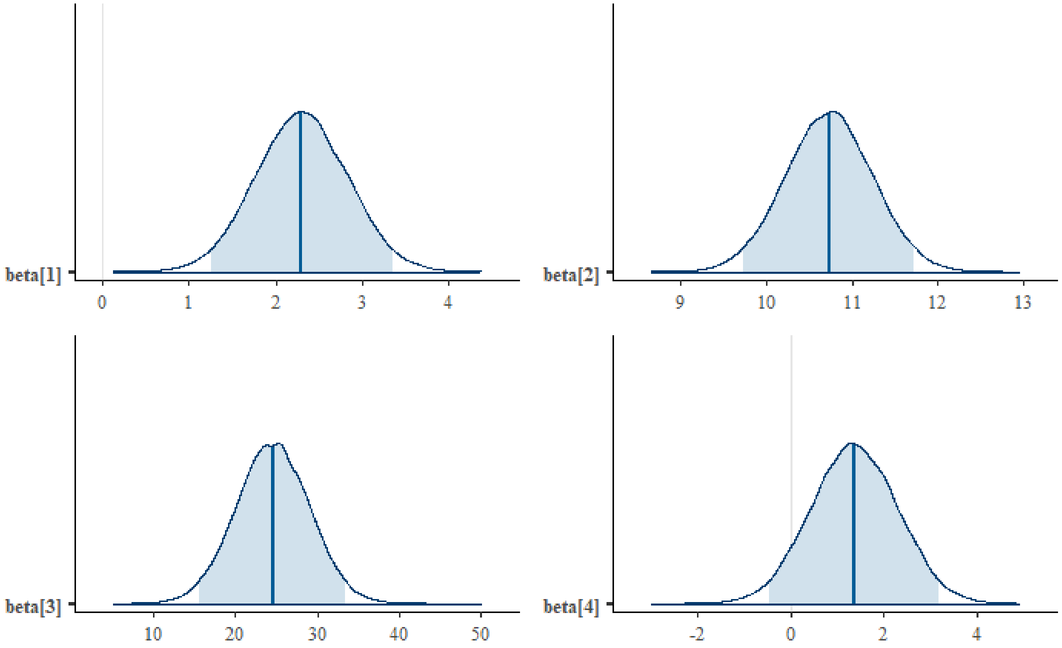
Informative prior:
$\beta[4] \sim \mathcal{N}(0, 1)$

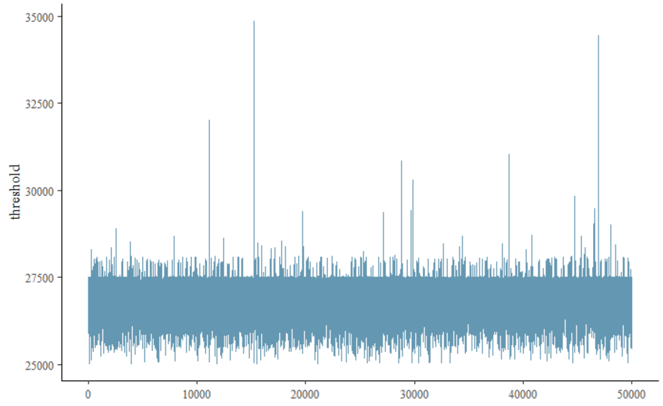
Prior for threshold:
$\text{threshold} \sim \text{Uniform}(25000, 35000)$

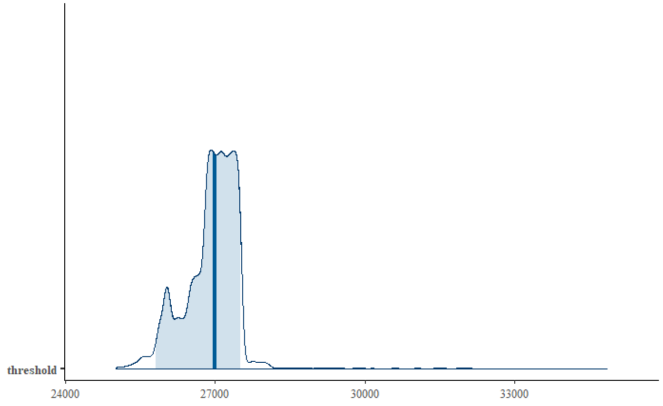


The threshold has been found around **27000**. This will have to be confirmed by a **broader** model.

Section 4.2.1: Threshold







Threshold 97.5% credible interval: 25800-27510

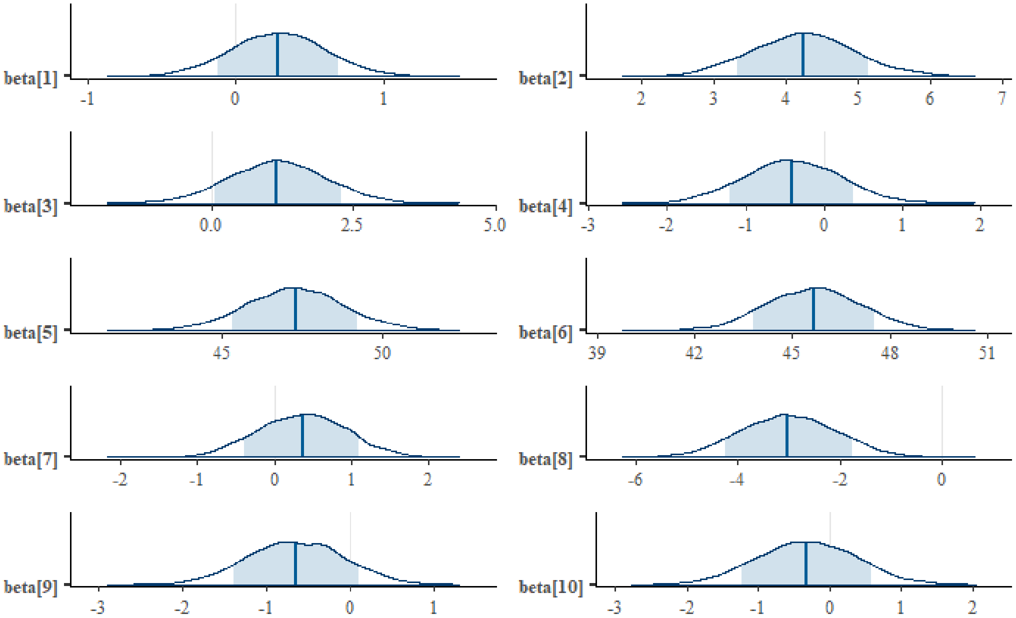
Normal model with threshold

Section 4.2.2

Section 4.2.2: Normal model with threshold

We now add all more relevant covariates in a Normal model, keeping a flat prior, to find out if our thesis hold.

$$
\begin{aligned}
\mu_i &\leftarrow \text{inprod}(X_i, \beta) \\
\text{co2percap}_i &\sim \mathcal{N}(\mu_i, \beta_\gamma) \\
\text{indicator}_i &\leftarrow (\text{unGDP}_i \geq \text{threshold}) \\
X_{i,1} &\leftarrow (1 - \text{indicator}_i) \\
X_{i,2} &\leftarrow \text{indicator}_i \\
X_{i,3} &\leftarrow \text{GDP}_i \cdot (1 - \text{indicator}_i) \\
X_{i,4} &\leftarrow \text{GDP}_i \cdot \text{indicator}_i \\
X_{i,5} &\leftarrow \text{EnergyUse}_i \cdot (1 - \text{indicator}_i) \\
X_{i,6} &\leftarrow \text{EnergyUse}_i \cdot \text{indicator}_i \\
X_{i,7} &\leftarrow \text{urb}_i \cdot (1 - \text{indicator}_i) \\
X_{i,8} &\leftarrow \text{urb}_i \cdot \text{indicator}_i \\
X_{i,9} &\leftarrow \text{internet}_i \cdot (1 - \text{indicator}_i) \\
X_{i,10} &\leftarrow \text{internet}_i \cdot \text{indicator}_i
\end{aligned}
$$



Section 4.2.2: Normal model with threshold

It should be noticed we **separated** the two dataset completely, without assuming that the other covariates would have remained the same below and above the threshold. We did this since we noticed how in our more general model **this was already the case**, without the need of assuming it. This is a stronger way of demonstrating our thesis, since we don’t need to leave out a level of complexity to let emerge the difference in the GDP parameter below and above.

Indeed, if we used the assumption of a single parameter for the remaining covariate in the two groups, we could have **missed** how **another covariate** was explaining better the variance in the observation with respect to the GDP parameter, making the difference in the GDP parameter near zero.

Normal model without threshold

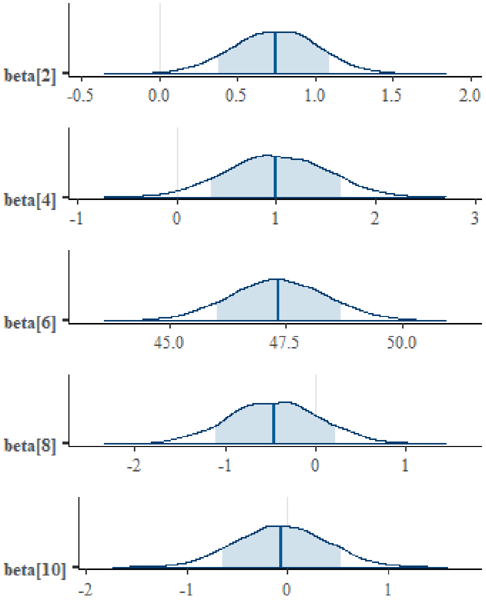
Section 4.2.3

Section 4.2.3: Normal model without threshold

To consider the same model **without** a threshold it is enough to move the threshold to 1000, in order to not create two groups, but just one.

$$
\begin{aligned}
\mu_i &\leftarrow \mathrm{inprod}(X_i, \beta) \\
\mathrm{co2percap}_i &\sim \mathcal{N}(\mu_i, \beta_\gamma)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{indicator}[i] &= 1 \\
X[i,2] &\leftarrow \mathrm{indicator}[i] \\
X[i,4] &\leftarrow \mathrm{GDP}[i] \cdot \mathrm{indicator}[i] \\
X[i,6] &\leftarrow \mathrm{EnergyUse}[i] \cdot \mathrm{indicator}[i] \\
X[i,8] &\leftarrow \mathrm{urb}[i] \cdot \mathrm{indicator}[i] \\
X[i,10] &\leftarrow \mathrm{internet}[i] \cdot \mathrm{indicator}[i]
\end{aligned}
$$



Gamma model with threshold

Section 4.2.4

Section 4.2.4: Gamma model with threshold

In order to establish the **robustness** of the Normal model, we tried also an analogous Gamma model.

$$\mu_i \leftarrow \langle X[i,], \beta \rangle$$

$$\mathrm{co2percap}[i] \sim \mathrm{Gamma}(\mu_i + 0.0001, \beta_\gamma)$$

$$\mathrm{indicator}[i] \leftarrow (\mathrm{unGDP}[i] \geq \mathrm{threshold})$$

$$
\begin{aligned}
X[i,1] &\leftarrow (1 - \mathrm{indicator}[i]) \\
X[i,2] &\leftarrow \mathrm{indicator}[i] \\
X[i,3] &\leftarrow \mathrm{GDP}[i] \cdot (1 - \mathrm{indicator}[i]) \\
X[i,4] &\leftarrow \mathrm{GDP}[i] \cdot \mathrm{indicator}[i] \\
X[i,5] &\leftarrow \mathrm{EnergyUse}[i] \cdot (1 - \mathrm{indicator}[i]) \\
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X[i,7] &\leftarrow \mathrm{urb}[i] \cdot (1 - \mathrm{indicator}[i]) \\
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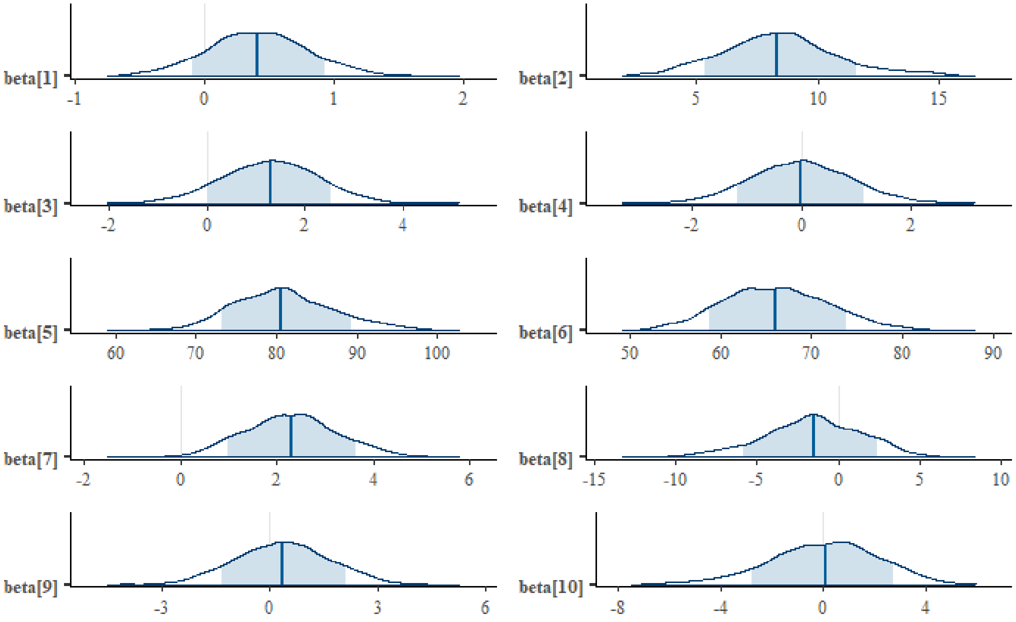
X[i,9] &\leftarrow \mathrm{internet}[i] \cdot (1 - \mathrm{indicator}[i]) \\
X[i,10] &\leftarrow \mathrm{internet}[i] \cdot \mathrm{indicator}[i]
\end{aligned}
$$

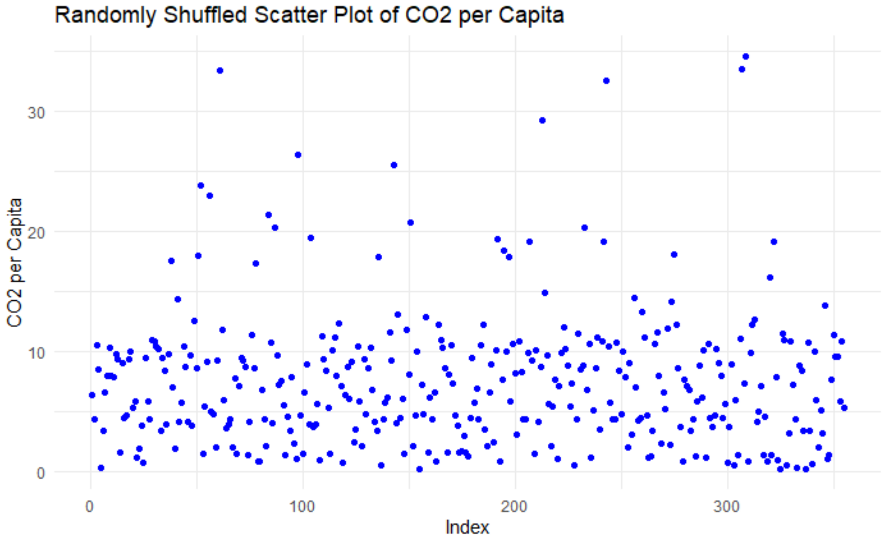


Time series model

Section 4.3

Section 4.3: Time series model

Let’s consider a Bayesian AR(1) model, where the current observation in the series is based on the **immediately preceding observation**, adjusted by a stochastic term. We insert also the **main covariates**.



The resulting time series have **much more stable** trajectory, demonstrating the potentially great effectiveness of this data organization in the case of this dataset.

Section 4.3: Time series model

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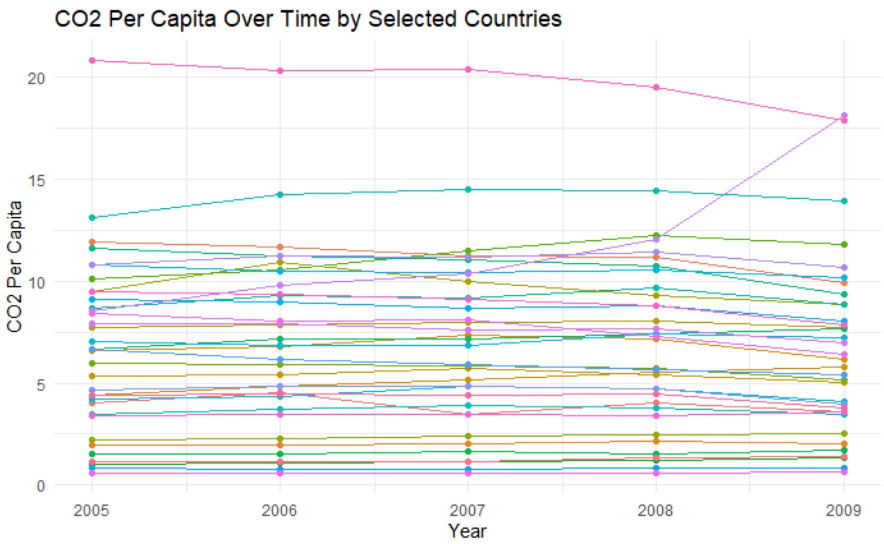


The resulting time series have **much more stable** trajectory, demonstrating the potentially great effectiveness of this data organization in the case of this dataset.

Section 4.3: Time series model

The structure is simply composed by a Normal model, with a linear combination of the covariates to compute the mean and flat priors for the standard deviation and the parameters.

The covariates used are:
Ar
EnergyUse
EnergyUse-difference
GDP
GDP-difference

$$Y[i] \sim N(\mu[i], \tau)$$
$$\mu[i] \leftarrow \alpha \cdot Y[i-1] + m_1 \cdot x_1[i-1] + m_2 \cdot x_2[i-1] + m_6 \cdot (x_1[i] - x_1[i-1]) + m_7 \cdot (x_2[i] - x_2[i-1])$$

$$\alpha \sim \mathrm{Uniform}(-1.5, 1.5)$$
$$\tau \sim \mathrm{Gamma}(0.1, 10)$$
$$m_1 \sim N(0.0, 1.0 \times 10^{-4})$$
$$m_2 \sim N(0.0, 1.0 \times 10^{-4})$$
$$m_6 \sim N(0.0, 1.0 \times 10^{-4})$$
$$m_7 \sim N(0.0, 1.0 \times 10^{-4})$$

Section 4.3: Time series model

The results of the model are:

In particular a distinction was made in the period pre crisis and during the crisis. The results were:

Section 4.3: Time series model

Noticeably all these models allow to have a satisfactory explanation, with a small MSE. The In-sample and Out-of-sample prediction results were:

Section 4.3: Time series model

The last distinction in the dataset that was tried on the AR(1) model was between countries above and under the GDP threshold:

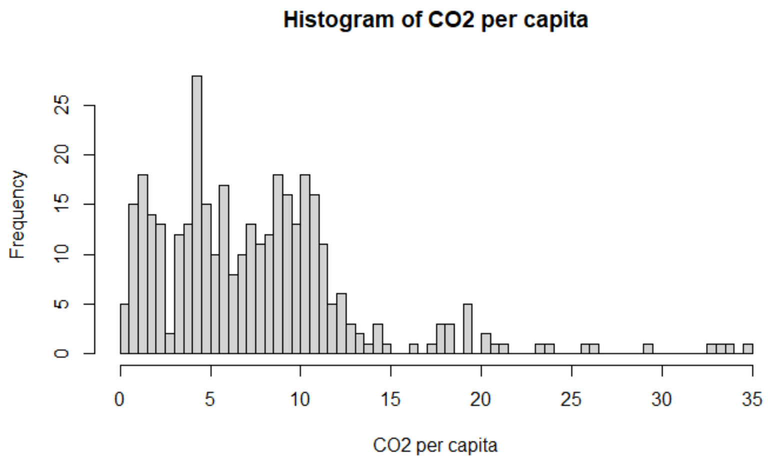
Clustering

Section 5

Section 5: Clustering

One-dimension mixture model, CO2 per capita clustered in 3 Normal curves.

$$
\begin{aligned}
&\#\ \text{Likelihood}\\
&z_i \sim \text{Categorical}(w)\\
&y_i \sim N(\mu_{z_i}, \sigma^2_{z_i})\\
\\
&\#\ \text{Prior}\\
&\mu_i \sim N(0, 0.01)\\
&\sigma^2_i \sim \text{Gamma}(0.01, 0.01)\\
\\
&w \sim \text{Dirichlet}(a)
\end{aligned}
$$



The resulting clusters have mean 1.25, 7.24 and 18.87. The largest is the the **middle** one (82%) containing most variability, followed by the **first** one of the less polluting countries (13%), finally the **third** group contains some “super-polluters” (5%).

Section 5: Clustering

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\\
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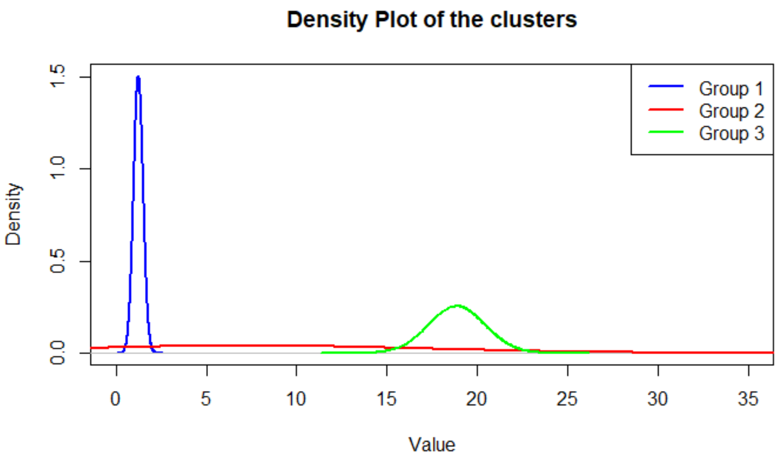
&\mu_i \sim N(0, 0.01) \\
&\sigma^2_i \sim \text{Gamma}(0.01, 0.01) \\
\\
&w \sim \text{Dirichlet}(a)
\end{aligned}
$$



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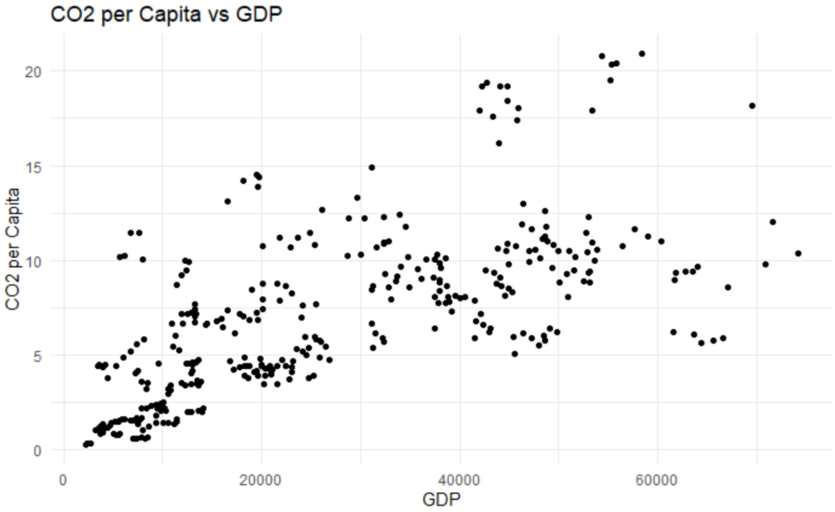
Section 5: Clustering

Two-dimension mixture model, CO2 per capita clustered in 2 Normal bivariate curves.

$$
\begin{aligned}
&z_i \sim \text{Bernoulli}(w) \\
&y_{i,1:2} \sim \text{Multivariate Normal}\left(\mu_{z_i+1,1:2}, \sigma_{z_i+1,1:2,1:2}\right) \\
\\
&\mu_{1,1} \sim N(0, 0.1) \\
&\mu_{1,2} \sim N(0, 0.1) \\
&\mu_{2,1} \sim N(0, 0.1) \\
&\mu_{2,2} \sim N(0, 0.1) \\
&w \sim \text{Beta}(1, 1) \\
\\
&\sigma^2_{1,1,1} \sim \text{Gamma}(1, 1) \\
&\sigma_{1,1,2} \sim N(0, 0.1) \\
&\sigma_{1,2,1} = \sigma_{1,1,2} \\
&\sigma^2_{1,2,2} \sim \text{Gamma}(1, 1) \\
\\
&\sigma^2_{2,1,1} \sim \text{Gamma}(1, 1) \\
&\sigma_{2,1,2} \sim N(0, 0.1) \\
&\sigma_{2,2,1} = \sigma_{2,1,2} \\
&\sigma^2_{2,2,2} \sim \text{Gamma}(1, 1)
\end{aligned}
$$



The two groups found have average GDP around 23000 and 40000, with the **first** group accounting for more than three times the point in the second group. The **second** group is also responsible for an average CO2 per capita production of 12.5, way higher than the 5.5 of the first group.

Section 5: Clustering

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z_i &\sim \text{Bernoulli}(w) \\
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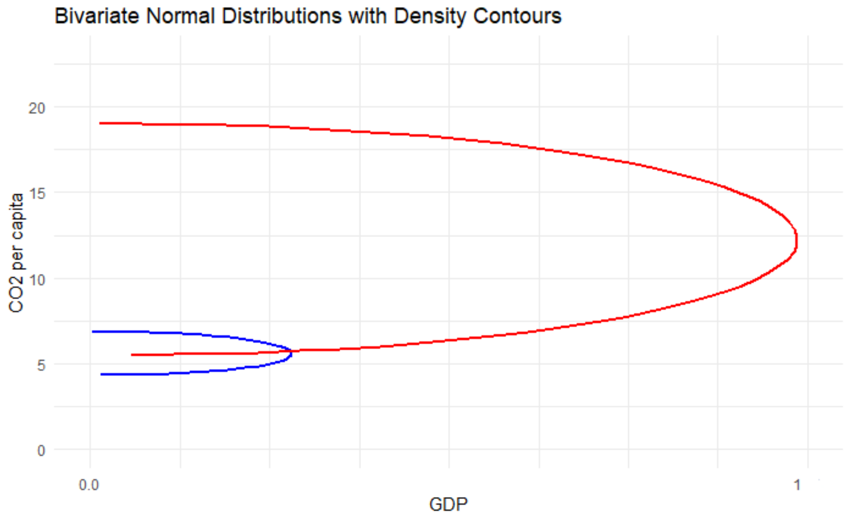
\\
\mu_{1,1} &\sim N(0, 0.1) \\
\mu_{1,2} &\sim N(0, 0.1) \\
\mu_{2,1} &\sim N(0, 0.1) \\
\mu_{2,2} &\sim N(0, 0.1) \\
w &\sim \text{Beta}(1, 1) \\
\\
\sigma^2_{1,1,1} &\sim \text{Gamma}(1, 1) \\
\sigma_{1,1,2} &\sim N(0, 0.1) \\
\sigma_{1,2,1} &= \sigma_{1,1,2} \\
\sigma^2_{1,2,2} &\sim \text{Gamma}(1, 1) \\
\\
\sigma^2_{2,1,1} &\sim \text{Gamma}(1, 1) \\
\sigma_{2,1,2} &\sim N(0, 0.1) \\
\sigma_{2,2,1} &= \sigma_{2,1,2} \\
\sigma^2_{2,2,2} &\sim \text{Gamma}(1, 1)
\end{aligned}
$$



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