Homework 1: Linkage Analysis

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1. Steady-state rate of 7450 parts in seven hours

2 Free-Body Diagrams

3 Statics and Dynamics Equations

3.1 Forces and Moments

3.1.1 Static Equilibrium

At static equilibrium, the net force and moment on each link is zero. Figures **??**, **??**, **??**, **??**, and **??** show free-body diagrams of each link, with an input torque τ_{in}, weights W, at center of masses s, and with artifact weight W_{art}.

Each link yields one vector force equations; link AB also yields one vector moment equation. All force equations have two components, and the moment equation has three components.

For link AB, with moments summed at point A:

$$R_{a,x} - R_{b,x} = 0 \qquad R_{a,y} - R_{b,y} - W_{ab} = 0 \qquad \vec{\tau}_{in} - \vec{r}_{s/a} \times \vec{W}_{ab} - \vec{r}_{b/a} \times \vec{R}_b$$

For link BC:

$$R_{b,x} - R_{c,x} = 0 \qquad R_{b,y} - R_{c,y} - W_{bc} = 0$$

For link CDE:

$$R_{d,x} + R_{c,x} - R_{e,x} = 0 \qquad R_{d,y} + R_{c,y} - R_{e,y} - W_{cde} = 0$$

For link EF:

$$R_{e,x} - R_{f,x} = 0 \qquad R_{e,y} - R_{f,y} - W_{ef} = 0$$

For link FG:

$$R_{f,x} - R_{g,x} = 0 \qquad R_{f,y} - R_{g,y} - W_{fg} - W_{\text{art}} = 0$$

3.1.2 Dynamic Equilibrium

3.2 Mass Calculations

All links were modeled in Onshape with material of Aluminum 7075. For all binary links, the centroid of the flat face is placed at the origin and the extrude is symmetric to ensure the center of mass also coincides with the origin. Link FG was extended by 1.843m beyond joint F and uses a similar rounded end with a pin for the gripper. For link CDE, joint D is placed at the origin and joint C at 0.98m distance. Link AB is shown as an example in Figure 1. The joint-to-joint center distance was varied to produce different links. The mass properties are computed with Onshape's "Mass and Section Properties" feature and are in Table 1.

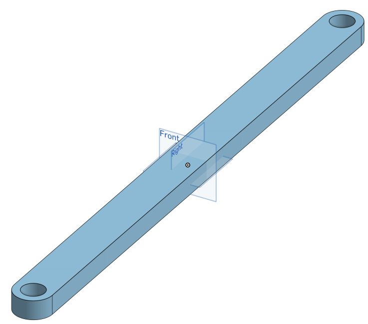

Figure 1: Link AB modeled in Onshape.

Link	Mass (kg)
AB	8.35
BC	20.2
CDE	34.85
EF	16.82
FG	62.11

Table 1: Link masses.

The mass moments of inertia are given by Onshape's "Mass and Section Properties" feature mentioned in Section 3.2, and are shown in Table 2. The MMI is about the axis formed by the intersection of the Front and Top planes in Figure 1.

Link	MMI (kg·m^2)
AB	0.266
BC	3.53
CDE	18.54
EF	2.05
FG	103

Table 2: Link mass moments of inertia.

3.3 Kinematics Equations

I developed my kinematic equations from two vector loops: A-B-C-D-A and D-E-F-G-D. Note that these represent two four-bar linkages in series, with ternary link DEC as the common link.

3.3.1 Position

For loop ABCDA, the position loop is:

$$\vec{r}_{b/a} + \vec{r}_{c/b} + \vec{r}_{d/c} + \vec{r}_{a/d} = 0$$

For loop DEFGD, the position loop is:

$$\vec{r}_{e/d} + \vec{r}_{f/e} + \vec{r}_{g/f} + \vec{r}_{g/d} = 0$$

3.3.2 Velocity

Differentiate the positions loop equations to derive the velocity loop equations:

$$\vec{v}_{b/a} + \vec{v}_{c/b} + \vec{v}_{d/c} + \vec{v}_{a/d} = 0$$

$$\vec{v}_{e/d} + \vec{v}_{f/e} + \vec{v}_{g/f} + \vec{v}_{g/d} = 0$$

The velocity of a joint j relative to i can be decomposed into translational and rotational components:

$$\vec{v}_{j/i} = \vec{v}_i + (\omega_{ij} \times \vec{r}_{j/i})$$

For a ground joint i, $\vec{v}_i$ is zero. For joints i and j of the same rigid link, their relative translational velocity is also zero. All joints and links in this linkage satisfy these conditions, so all translational velocity terms are zero for all joints. Therefore, the velocity loop equations for the loop equations reduce to the rotational components:

$$(\vec{\omega}_{ab} \times \vec{r}_{b/a}) + (\vec{\omega}_{bc} \times \vec{r}_{c/b}) + (\vec{\omega}_{cd} \times \vec{r}_{d/c}) + (\vec{\omega}_{da} \times \vec{r}_{a/d}) = 0$$

$$(\vec{\omega}_{de} \times \vec{r}_{e/d}) + (\vec{\omega}_{ef} \times \vec{r}_{f/e}) + (\vec{\omega}_{fg} \times \vec{r}_{g/f}) + (\vec{\omega}_{dg} \times \vec{r}_{g/d}) = 0$$

To solve the loop equations we require the steady-state crank angular velocity, ω_1. With a part per hour rate of 7450 parts per seven hours, we can compute:

$$\dot{p} = \frac{7450\text{part}}{7\text{hr}} = 0.29\frac{\text{part}}{\text{sec}}$$

$$p^{-1} = 3.38\frac{\text{s}}{\text{part}}$$

By inspection of the PMKS+ model, we see that the output link completes one cycle at the same rate as the input link. Thus:

$$\omega_1 = \omega_5$$

The output link delivers one part per revolution, so the output link angular velocity is:

$$\omega_5 = \frac{2\pi}{p^{-1}} \cdot 1 \text{ part}$$

$$\Rightarrow \omega_1 = 1.86\text{rad/s}$$

3.3.3 Acceleration

Differentiate the velocity loop equations to derive the acceleration loop equations. Begin by differentiating the general rotational velocity:

$$\frac{d}{dt}\left(\omega_{j/i} \times \vec{r}_{j/i}\right) = (\vec{\alpha}_{j/i} \times \vec{r}_{j/i}) + \omega_{j/i} \times (\omega_{j/i} \times \vec{r}_{j/i})$$

Written compactly, the acceleration loop equations are thus:

$$\Sigma(\vec{\alpha}_{j/i} \times \vec{r}_{j/i}) + \omega_{j/i} \times (\omega_{j/i} \times \vec{r}_{j/i}) \text{ for } i, j \cap \{(a, b), (b, c), (c, d), (d, a)\}$$

$$\Sigma(\vec{\alpha}_{j/i} \times \vec{r}_{j/i}) + \omega_{j/i} \times (\omega_{j/i} \times \vec{r}_{j/i}) \text{ for } i, j \cap \{(d, e), (e, f), (f, g), (g, d)\}$$

3.4 Accelerations at CMs

The acceleration at the center of mass of a link

4 Results

4.1 First Position

4.1.1 Joint Forces and Torques

4.1.2 Postion, Velocity, and Acceleration

4.1.3 Masses and Mass Moments of Inertia

4.2 Plots

4.3 Comparison with PMKS+

5 MATLAB Code

6 Discussion

7 References