University PSL-Dauphine

Mathematics department

Lorenzo Magnino and David Nassauer

Montecarlo Method and Finite Differences in Finance

Evaluation of Asian Options

Abstract:

In this project we study and implement different methods for the evaluation of an Asian Option. After introducing briefly the Asian Options we present three different Monte-Carlo schemes based on three different way to approximate the integral average of the stock price (methods are presented in B. Lapeyre and E. Temam. 2001). We analyse their convergence and their efficiency. Eventually we present a PDE approach, based on L.C.G. Rogers and Z. Shi (1995) ([4]).

1 Introduction

Asian options are a type of exotic financial derivative whose payoff is determined by the average price of the underlying asset over a specified period. Unlike traditional options, which depend on the spot price of the underlying asset at a specific point in time, Asian options take into account the average behavior of the asset over a predetermined time frame.
The pricing of Asian options is complex due to the averaging nature of their payoffs and the stochastic processes that govern the underlying asset's movements. Unlike European or American options, which often have closed-form solutions, Asian options require more sophisticated methodologies.
Monte Carlo methods, in particular, provide a flexible and powerful tool for pricing them.Monte Carlo methods are computational techniques that use random sampling to approximate numerical solutions for problems that may be deterministic but difficult to solve analytically.They are commonly used for pricing complex financial instruments when analytical solutions are not feasible.Monte Carlo estimates become more accurate as the number of simulated paths increases, making them a valuable tool for a wide range of financial derivatives.
Another method for evaluating them is through the use of PDE formalism. This approach involves formulating a partial differential equation that governs the dynamics of the Asian option's value over time and underlying asset price. For instance, the Black-Scholes partial differential equation (PDE) can be modified to consider the averaging mechanism that is inherent in Asian options. The resulting PDE offers a complete depiction of the option's evolution, taking into account the impact of volatility, interest rates, and the averaging period. We use the finite difference scheme to approximate them.

In section 2 we describe the mathematical and financial framework. In section 3 we present the three different methods that we have used in this paper. In section 4 we give a variance reduction technique in order to reduce the error of the method (link with the variance of our target variable). In section 5 we analyse the convergence of the method underline their different. In section 6 we present two different methods to evaluate an Asian Option with PDEs. In section 7 we present out results and finally in section 8 we give our conclusions.

2 Mathematical Framework

We work in the classical framework of Black-Scholes model in which the prices of the risky asset and the no-risky asset are given by:

$$dS_t = S_t(\mu dt + \sigma dB_t)$$
$$dS_t^0 = rS_t^0 dt$$

where μ and σ represent the drift and the volatility of the process We can then adopt the neutral risk probability approach in which we introduced a new probability measure (using Girsanov procedure) and then the risky asset satisfies the equation

$$dS_t = S_t(rdt + \sigma dW_t)$$

where r is the interest rate and W_t is the Wiener process obtained by the transformation $W_t = B_t + \frac{\mu - r}{\sigma}$ and it is a Brownian motion under the neutral risky probability measure mentioned before.
The solution of that SDE is a geometric Brownian motion:

$$S_t = S_{T_0} exp(\sigma W_t - \frac{\sigma^2}{2} t + rt).$$

S_{T_0} is the initial stock price (without loss of generality we are going to consider $T_0 = 0$). We're going consider Asian call with fixed strike whose payoff depends of the mean of the price for the risky asset on a given period. Thus its price at maturity T can be written as:

$$V_t(S, A) = e^{-r(T-t)} \mathbb{E}((A_S(t) - K)_+)$$

where

$$A_S(t) = \frac{1}{t} \int_0^t S_u du$$

.

One can also consider other similar types of functions inside the expected value in order to manage call/put with fixed/floating strike.

3 Numerical Schemes

In order to compute a price using Monte-Carlo method to simulated the expected value that appear in the price of an Asian Option we have to be able to approximate the integral $Y_T = \int_0^T S_u du$ within it. We present three methods to do it and we're going to analyse their efficiency and their rate of convergence. What is important here is that the price of the stock S_t can be exactly simulated using the simulation of a Brownian motion (with the numpy's module np.random.randn) In order to present the schemes mentioned we are going to define a time-mesh $times := t_k = kT/N = kh, k = 0, ..., N$

First scheme: Standard Scheme. This scheme is based on Riemann sums:

$$Y_T^{r,N} = h \sum_{k=0}^{N-1} S_{t_k}. \tag{3.1}$$

Then we can use this scheme inside our Monte-Carlo simulation. We take M number of drawings and we obtained the following approximation of the price at maturity of a fixed strike Asian call.

$$V_T(S, A) \approx \frac{e^{-rT}}{M} \sum_{j=1}^{M} \left(\frac{h}{T} \sum_{k=0}^{N-1} S_{t_k} - K \right)_+$$

As for every kind of Monte Carlo methods the time complexity of this algorithm is $O(\frac{1}{NM})$. Since we used to type of approximations we have two types of errors: one in refer to the Monte Carlo approximation while the other is the time step error. We're going to study those errors in the section 5 when we study the convergence of the methods.

Second scheme: Higher accuracy scheme 1. We know that in L^2 the closest variable to $g(X)$ is $\mathbb{E}(g(X)|X)$. Then in our case we can consider:

$$\mathbb{E}\left(\left(\frac{1}{T}\int_0^T S_u du - K\right)_+ |B_h\right)$$

where B_h is the $\sigma-$algebra generated by the S_{t_k} (i.e $B_h = \sigma(S_{t_k}, k = 0, ...N)$). Unfortunately we cannot compute explicitly this conditional expectation but we can work with

$$\left(\mathbb{E}\left(\frac{1}{T}\int_0^T S_u du \big| B_h\right) - K\right)_+ = \left(\frac{1}{T}\int_0^T \mathbb{E}\left(S_u|B_h\right) du - K\right)_+$$

since the conditional law of W_u (the only random part in S_u) with respect to B_h for $u \in [t_k, t_{k+1}$ is given by

$$\L(W_u|W_{t_k} = x, W_{t_{k+1}} = y) = \left(\frac{t_{k+1} - u}{h}x + \frac{u - t_k}{h}y, \frac{(t_{k+1} - u)(u - t_k}{h}\right) \tag{3.2}$$

It can be noticed that from Jensen inequality this quantity is less than the first conditional expectation, but it is a good approximation of the average Y_T. So now we can compute:

$$\mathbb{E}\left(\frac{1}{T}\int_0^T S_u du \big| B_h\right) = \frac{1}{T}\sum_{k=0}^{N-1}\int_{tk}^{t_{k+1}} e^{\sigma \frac{u-t_k}{h}(W_{t_{k+1}} - W_{t_k}) - \frac{\sigma^2}{w}\frac{(u-t_k)^2}{h} + ru} e^{\sigma W_{t_k} - \frac{\sigma^2}{2}t_k} du$$

Using a Taylor expansion with h small we arrive at the proper scheme:

$$Y_T^{e,N} = \frac{h}{T}\sum_{k=0}^{N-1} S_{t_k}\left(1 + \frac{rh}{2} + \sigma\frac{W_{t_{k+1}} - W_{t_k}}{2}\right). \tag{3.3}$$

It can be seen (B.Lapeyre, E.Teman 2001) that this scheme is equivalent to the well known trapezoidal method since the L^2 error between these two scheme is $O(1/n^3)$ against the rate of convergence of the High accuracy scheme 1 that is $O(1/n)$.

Third scheme: Higher accuracy scheme 2. This last scheme is similar to the previous one but process looking at the fact that the Brownian Motion W is a Gaussian process and so the integral between 0 and T of this process has a normal density with respect to the Lebesgue measure on $\mathbb{R}$. Following the computations made in the lectures notes of B.Bouchard the mean and the variance of this new process derived from:

$$E\left[\int_{t_i}^{t_{i+1}} W_t\, dt \middle| W_{t_i} = x, W_{t_{i+1}} = y\right] = \int_{t_i}^{t_{i+1}} E\left[W_t \middle| W_{t_i} = x, W_{t_{i+1}} = y\right] dt = h\frac{(x+y)}{2}$$

and

$$\begin{aligned}\mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} W_t dt\right)^2 |W_{t_i} = x, W_{t_{i+1}} = y\right] &= 2\int_{t_i}^{t_{i+1}}\int_{t_i}^{t} E\left[W_t W_u \middle| W_{t_i} = x, W_{t_{i+1}} = y\right] du\, dt \\ &= -\frac{1}{12}(t_i - t_{i+1})^2(t_i - t_{i+1} - 3(x+y)^2)\end{aligned}$$

Hence we can compute the integral as:

$$Y_T = \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\left(h + \frac{rh^2}{2} + \sigma\int_{t_k}^{t_{k+1}}(W_u - W_{t_k})du\right) \tag{3.4}$$

4 Variance Reduction: Control Variate MC Estimator

Recall that the accuracy of a MC estimator $Y_n = \frac{1}{n}\sum_{i=0}^{n} Y_i$ is proportional to $\sqrt{Var(Y_n)} = \sqrt{\frac{Var(Y)}{n}}$. So the goal is to reduce the variance of Y to improve the estimate. The method proposed is the so called Control Variate: find a real-valued random variable X such that $\mathbb{E}[X]$ is known and a constant $b \in \mathbb{R}$ such that

$$Var(Y - b(X - \mathbb{E}[X])) \ll Var(Y)$$

and use the estimator

$$Y_n(b) = Y_n + b(X_n - \mathbb{E}[X]).$$

It can be shown that this estimator is unbiased and consistent (by LLN) of the expected value of Y. Since the computational cost of $Y_n(b)$ is higher than Y_n than we need to know the optimal parameter b, i.e such b that minimize the variance of the new estimator. It holds:

$$b^* = -\frac{Cov(X,Y)}{Var(X)}$$

and

$$Var(Y(b^*)) = Var(Y)(1 - \rho_{XY}^2)$$

where ρ_{XY} stands for the correlation between X and Y. When b^* is not known explicitly we need to estimate by:

$$b_n^* = \frac{\sum_{i=1}^{n}(X_i - \bar{X_n})(Y_i - \bar{Y_n})}{\sum_{i=1}^{n}(X_i - \bar{X_n})^2}.$$

Eventually we need to find a random variable strong correlated with Y such that $\mathbb{E}[X]$. In our work following the method by Kemma and Vorst (1990) we approximate Y_T by $exp(1/T\int_0^T log(S_u)du$. Since r and σ are not too large they are similar. This is convenient since the variable $Z' = 1/T\int_0^T log(S_u)du$ has a normal law and we can compute explicitly:

$$\mathbb{E}(e^{-rT}(exp(Z') - K)_+).$$

Hence we can take as a control variable

$$Z = e^{-rT}\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}+\frac{\sigma}{2}\int_0^T W_u du} - K\right)_+$$

In particular:

$$E[Z] = e^{\frac{1}{2}(r-\frac{1}{6}\sigma^2)T} S_0 N(d) - KN\left(d - \sigma\sqrt{\frac{1}{3}T}\right)$$

where N is the cumulative standard normal distribution function and d can be written as:

$$d = \frac{log(S_0/K) + 1/2(r + \frac{1}{6}\sigma^2)T}{\sigma\sqrt{\frac{1}{3}T}}$$

For each scheme we obtained the following approximations for the control variable

$$Z^r = e^{-rT}\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}+\frac{\sigma}{2}\sum_0^{N-1} hW_{t_k}du} - K\right)_+$$

$$Z^e = e^{-rT}\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}+\frac{\sigma}{2}\sum_0^{N-1} \frac{h}{2}(W_{t_k}+W_{t_{k+1}})du} - K\right)_+$$

$$Z^p = e^{-rT}\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}+\frac{\sigma}{2}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} W_u du} - K\right)_+$$

5 Numerical Analysis

There are different ways to quantify the accuracy of a discretization scheme. Here we are going to consider to types of errors: the strong error and the weak error.

5.1 The Strong Error

The Strong error measures how far the discretized path is from the original path in terms of L^p-norm (we are going to consider L^2-norm). In general one consider:

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t - X_t^m|^p\right)^{\frac{1}{p}}$$

For our schemes it holds:

$$\textbf{Riemann:} \quad \mathbb{E}\left(\sup_{t\in[0,T]} |Y_t - Y_t^r|^2\right)^{\frac{1}{2}} \leq \frac{K_1(T)}{n}$$

$$\textbf{Second Scheme:} \quad \mathbb{E}\left(\sup_{t\in[0,T]} |Y_t - Y_t^e|^2\right)^{\frac{1}{2}} \leq \frac{K_2(T)}{n}$$

$$\textbf{Third Scheme:} \quad \mathbb{E}\left(\sup_{t\in[0,T]} |Y_t - Y_t^p|^2\right)^{\frac{1}{2}} \leq \frac{K_3(T)}{n^{3/2}}$$

(The explicit computations and the exact rates of convergence can be found in 'B. Lapeyre and E. Temam (2001)')

From these computations it turns out that the third scheme is the most efficient while there is no difference between the first and the second one. Now we are going to introduce another type of error that analyse the distributions of the schemes.

5.2 The Weak Error

In most of financial frameworks we are more interested in the weak error which measure how far the distribution of the discretized process is far from the original process (details computations on Temam, E. (2001)) It holds:

$$|\mathbb{E}[Y_T^r] - \mathbb{E}[Y_T]| \leq \frac{1}{n}$$

$$|\mathbb{E}[Y_T^e] - \mathbb{E}[Y_T]| \leq \frac{1}{n^{3/2}}$$

$$|\mathbb{E}[Y_T^p] - \mathbb{E}[Y_T]| \leq \frac{1}{n^{3/2}}$$

It can be seen now that the first two methods, for which the strong error is the same, differ in the weak error and, in particular the second one is more efficient. Looking at the numerical result present in section 7 this difference is shown by the experiments.

6 PDE approach

6.1 Martingale approach

One way to approach the evaluation of an Asian Option is by the formalism of the PDE. We follow the transformation presented by Rogers, L.C.G., and Shi, Z. (1995). We define:

$$\phi(t,x) := \mathbb{E}\left[\left(\int_t^T S_u \mu(du) - x\right)_+ |S_t = 1\right]$$

where is the usual geometric Brownian motion based on Black-Scholes Model and in our framework we have μ uniform on [0,T]. Then we can develop the martingale:

$$\begin{aligned} M_t &= \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^T S_u \mu(du) - K\right)_+ |F_t\right]\right. && (6.1) \\ &= S_t \mathbb{E}\left[\left(\frac{1}{T-t}\int_t^T \frac{S_u}{S_t} du - \frac{K - \frac{1}{t}\int_0^t S_u du}{S_t}\right)_+ |F_t\right] && (6.2) \\ &= S_t \phi(t, \xi_t) && (6.3) \end{aligned}$$

where

$$\xi_t := \frac{K - \frac{1}{t}\int_0^t S_u du}{S_t}$$

Now by Ito's formula we have:

$$d\xi_t = -\frac{1}{T}dt + \xi_t(-\sigma dB_t - r dt + \sigma^2 dt),$$

and using again Ito's formula on 6.3:

$$
\begin{aligned}
dM &= \phi dS + S(\dot{\phi}dt + \phi' d\xi + \frac{1}{2}\phi'' d[\xi]) + dSd\xi \\
&\doteq S[r\phi + \dot{\phi} - (\frac{1}{T} + r\xi)\phi' + \frac{1}{2}\sigma^2\xi^2\phi'']dt
\end{aligned}
$$

Now consider $f(t,x) := e^{-r(T-t)}\phi(t,x)$ and we find that f solves

$$\dot{f} + Gf = 0$$

where G is the operator

$$G := \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{x^2} - (1/T + rx)\frac{\partial}{\partial x}$$

For a fixed strike Asian Option we can set the following boundary condition:

$$f(T,x) = x^-$$

and then we have that the price at maturity T, fixed price K, and initial price S_0 is

$$V(S,T,K) = e^{-rT} S_0 \phi(0, KS_0^{-1})$$

It is also important to notice that for $x <= 0$ we have

$$\phi(t,x) = r^{-1}(e^{r(T-t)} - 1) - x$$

In order to approximate this PDE we use a Finite difference method: we set a time-space mesh $\{(t_n, x_i)\}$ using an equi-distance partition of [0,T] and $[S_0, S_{max}]$ (where we set and S_max in order to have a bounded domain). We're gonna consider f_i^n as the approximation of f in the points of the mesh. Then we can write:

- **Central approximation for the operator $\boldsymbol{G}$**: using Taylor expansion we can approximate the first and the second derivative as

$$
\begin{aligned}
\frac{\partial f}{\partial x} f(t_n, x_i) &\approx \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x} \\
\frac{\partial^2 f}{\partial^2 x^2} f(t_n, x_i) &\approx \frac{f_{i+1}^n + f_{i-1}^n - 2f_i^n}{(\Delta x)^2}.
\end{aligned}
$$

  So the operator can be approximate with

$$
\begin{aligned}
G_{\Delta x} \approx & \left(\frac{\sigma^2 x^2}{2(\Delta x)^2} - \left(\frac{1}{T} + rx\right)\frac{1}{2\Delta x}\right) f_{i+1}^n + \\
& + \left(\frac{\sigma^2 x^2}{2(\Delta x)^2} + \left(\frac{1}{T} + rx\right)\frac{1}{2\Delta x}\right) f_{i-1}^n - \frac{\sigma^2 x^2}{(\Delta x)^2} f_i^n
\end{aligned}
$$

Then for the approximation of the time derivative of f we can use an implicit scheme or an explicit scheme, changing the side of the approximation:

- **Explicit scheme**:
$$\partial_t f(t_n, x_i) \approx \frac{f_i^n - f_i^{n-1}}{\Delta t}$$
So using the previous computation we can approximate the whole PDE as
$$f^{n-1} = (Id + \Delta t G_{\Delta x}) f^n$$
So at every step we need to do a mtrix moltiplication.

- **Implicit scheme**:
$$\partial_t f(t_n, x_i) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$
In this case it turns out:
$$f^{n+1} = (Id + \Delta t G_{\Delta x}) f^n$$
and so, in this case, at each step we need to solve s linear system.

These schemes are both consistent of order (1,2) i.e
$$P_{\Delta t, \Delta x} = \partial_t f(t, x) + Gf(t, x) + O(\Delta t, (\Delta x)^2)$$

The main difference stands in the stability. We say that a scheme is stable if $\exists C \geq 0$ such that
$$|f^n|_\infty \leq |f^m|_\infty \quad \text{for all } n = 0, ..., m-1$$
for any terminal condition u_m. The explicit scheme so is stable under the CFL condition :
$$\textbf{CFL:} \quad \bar{\sigma}^2 \Delta t \leq (\Delta x)^2$$
where $\bar{\sigma} = \sigma * x$. Implicit scheme are usually unconditionally stable since they do not present conditions like before but on the other hand, certain implicit methods may exhibit instability near specific parameter values, such as near zero. The stability of implicit methods can be affected by the stiffness of the problem.

6.2 Splitting Method

Another method to evaluate Asian Option through the PDE approach is to use a splitting method: the idea is to treat different parts of an equation or of a complex problem separately, one after the other in the right order. For our particular aim can be explained in the following way: in order to solve numerically
$$\frac{\partial u}{\partial t} + F_1([u]) + F_2([u]) = 0 \quad in\ \mathbb{R}^N x(0, \Delta t)$$
where we use the notation [u] standing for (u, Du, Du^2) we apply successively the schemes
$$u^{1/2} = u^0 + \Delta F_1([u^0])$$

and

$$u^1 = u^{1/2} + \Delta t F_2([u^{1/2}]).$$

Notice that the approximation stands with order $O(\Delta t)^2)$ In our framework, to compute an Asian Option one has to consider the solutions u of the equation

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru - \frac{S}{T}\frac{\partial u}{\partial Z} = 0 \quad \text{in} \quad \mathbb{R}^+ \,\mathrm{x}\, \mathbb{R}^+ \,\mathrm{x}\, (0, T),$$

with the terminal data

$$u(S, Z, T) = (Z - S)^+ \quad \text{in } \mathbb{R}^+.$$

The price of the Asian Option at time $0 \leq t \leq T$ is given by $u\left(S, \frac{1}{T}\int_0^t S_u du, t\right)$.

We can split our problem with 3 variables into one heat equation problem and one a transport equation problem, solved successively as mentioned before:

$$\textbf{Heat Equation:} \qquad -\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = 0$$

with boundary conditions:

$$u(S, T) = (S - K)^+, \qquad u(0, t) = 0$$

$$\textbf{Transport Equation:} \qquad -\frac{\partial v}{\partial t} + S\frac{\partial v}{\partial S} = 0$$

with boundary conditions:

$$v(S, T) = max(S, v_{max})$$

We can solve these problems with a Finite Difference scheme as mentioned before.

7 Numerical implementations and results

7.1 Monte Carlo

We implemented all three schemes and tested them on the parameters $M = 100000$ (i.e. 100000 sample paths), $T = 1$, $S_0 = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.2$, and variable N (time discretization). The code was run on a 2023 M2 MacBook air in a standard Jupyter Notebook environment.

As we can see from figures 1, 2 and 3, scheme 1 performs much worse than the other 2 methods. Between the latter, they're both very similar in terms of (weak) convergence, which confirms what we expected.

Scheme	Time steps	Estimator	Confidence Interval	Time (s)
1	10	6.790	[6.362, 6.459]	0.988
	50	6.926	[6.873, 6.978]	1.13
	100	7.015	[6.963, 7.068]	1.24
	1000	7.074	[7.021, 7.127]	3.78
2	10	7.020	[6.967, 7.072]	1.37
	50	7.049	[6.996, 7.102]	1.50
	100	7.077	[7.024, 7.131]	1.63
	1000	7.080	[7.027, 7.133]	4.46
3	10	7.050	[6.997, 7.103]	1.57
	50	7.066	[7.013, 7.119]	1.76
	100	7.070	[7.017, 7.124]	1.97
	1000	7.033	[6.980, 7.086]	5.90

Table 1: Comparing the 3 schemes, without variance reduction. The number of Monte Carlo iterations was 100000.

Scheme	Time steps	Estimator	Confidence Interval	Time (s)
1C	10	6.791	[6.789, 6.792]	1.23
	50	6.993	[6.992, 6.995]	1.31
	100	7.018	[7.016, 7.019]	1.43
	1000	7.040	[7.039, 7.042]	4.02
2C	10	7.042	[7.040, 7.043]	1.57
	50	7.042	[7.041, 7.044]	1.70
	100	7.042	[7.041, 7.044]	1.83
	1000	7.043	[7.041, 7.044]	4.47
3C	10	7.039	[7.038, 7.041]	1.77
	50	7.043	[7.041, 7.044]	1.94
	100	7.041	[7.040, 7.043]	2.23
	1000	7.041	[7.041, 7.044]	6.15

Table 2: Comparing the 3 schemes, using variance reduction. The number of Monte Carlo iterations was 100000.

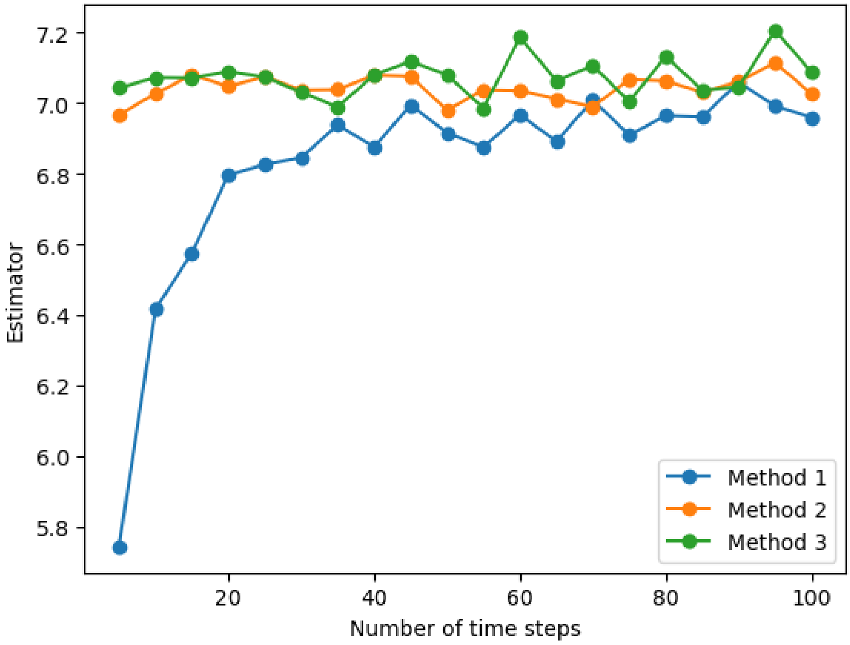

Figure 1: Convergence in the no variance reduction case.

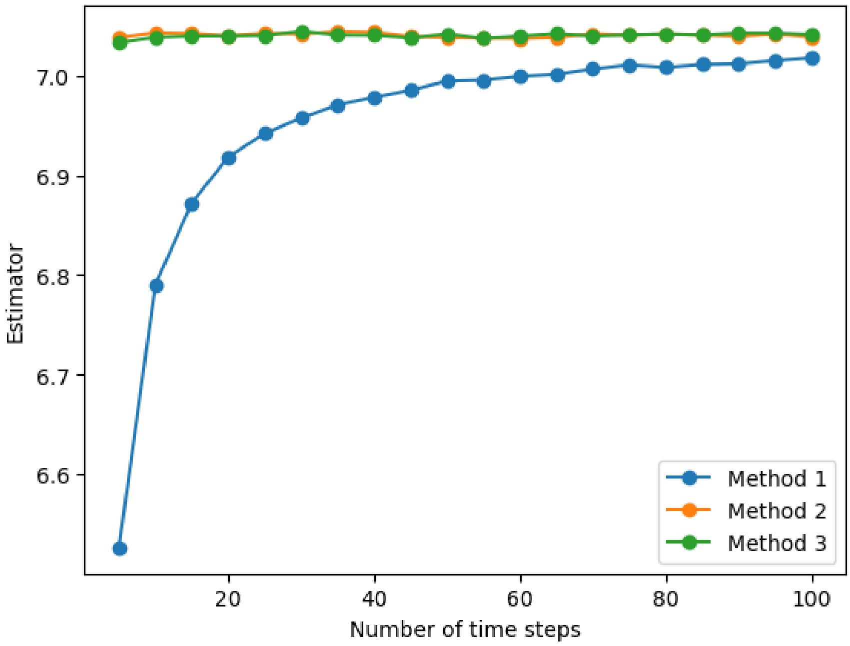

Figure 2: Convergence in the variance reduction case.

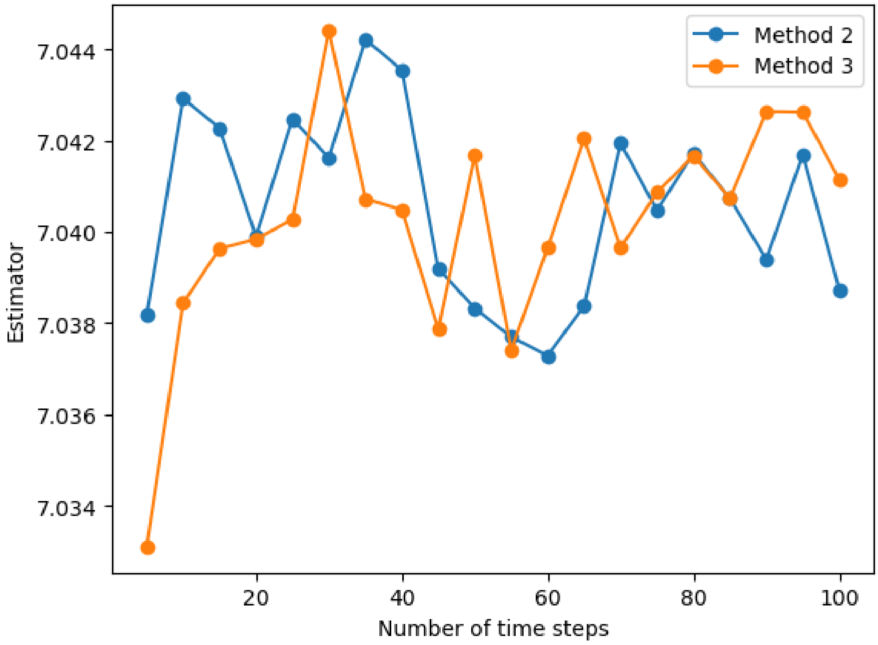

Figure 3: Zoom of the previous graph, excluding scheme 1.

7.2 PDE

We implemented the PDE method referenced in [4], and test both an explicit and implicit finite difference method on the following parameters: $m = 100$, $l = 100$ (grid discretization), $\alpha = 0$, $\beta = 2$ (strike price boundaries), $T = 1$, $S_0 = 100$, $K = 100$, $r = 0.02$, $\sigma = 0.05$.

The results are as follow:

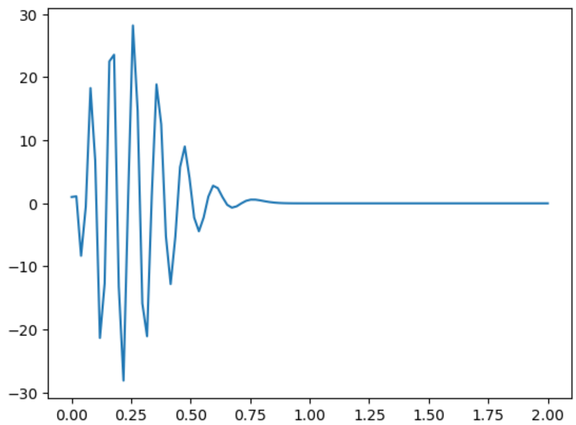

Figure 4: Explicit finite difference method.

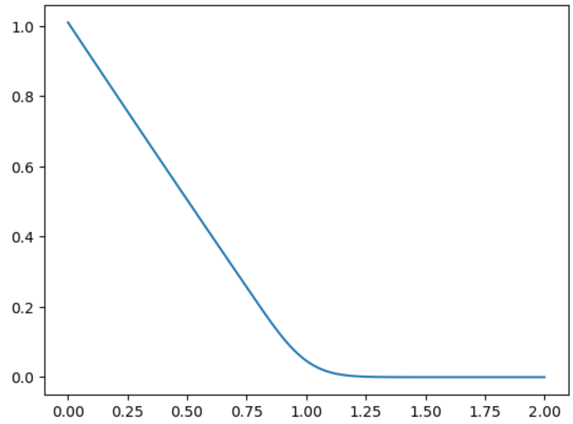

Figure 5: Implicit finite difference method.

As we can see, the explicit method is extremely unstable (some degree of instability is expected).

However, both methods produce the wrong output when using the parameters of the Monte Carlo section. This could be due to bad implementation on our part, or due to the fact that the boundary conditions are not exact.

8 Conclusion

We have studied and implemented three schemes to approximate the integral within the payoff of an Asian Option. The first is based on Riemann sums, and while much simpler than the others, its convergence is also much slower. We can also see that variance reduction schemes are not too much slower than the standard ones. This makes them a much more attractive option when computing the price of an asian option - we can get errors of the order 0.01 in less than 10 seconds, while using the same amount of sample paths in the Monte Carlo simulation.

Those result underline the fact that even if the strong error of the first and second scheme is the same they are really different in the convergence and the second one seem closer to the last one. This is explain well by the weak error. Indeed the last two schemes has the weak error of order $1/n^{3/2}$ while the first one only $1/n$.

We could improve on these schemes by considering even higher order approximations of the Talor expansions, for example.

We also implemented the first method described in the PDE approach. It can be seen from the numerical results how the explicit scheme is unstable near zero. This is because CFL condition is not satisfies near zero and so we obtain an oscillatory behaviour of the solution. The implicit scheme should give the right solution far from zero since also in that case we have a small unstability near zero since we are working with $\bar{\sigma} = \sigma * x$. But our results gives a wrong output, maybe for a wrong implementation of the scheme.

This shows us that Asian Options are quite difficult to manage since they are quite unstable near zero but the implicit method eventually work well and it is better than the explicit one.

A further work could be try to approximate the sensitivities of the Asian Options: since they have not a closed analytical form it is not possible to find an analytical form for the sensitive like the Delta or the Gamma. Although in financial situations it's really important to know how the price behave when one of the parameter change so it is necessary to have a clear view of the behave of the sensitive.

Another further work could be improve the accuracy and stability in approximating PDE. One way to do it is to use the Quadratic Upstream Interpolation for Convective Kinematics that use trigonometric function ([8], [9])

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