Network Dynamics and Learning: Homework 1

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***Abstract*—In this report has been explained how I solved the homework tasks and the results I obtained, with some brief theoretical recall when needed. In this homework I have exchanged ideas and compared the results with my colleague Jasmine Guglielmi.**

I. Exercise

The first exercise is about a simple 4-node graph represented in Fig. 1 with capacities c_e.

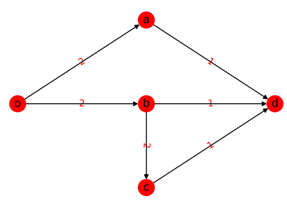

Fig. 1. Initial network with capacities

A. Question

In the first question is asked which is the minimum capacity to be removed in order to let not feasible flow exist between source node o and sink node d.

For the Network resilience interpretation of the Max-Flow Min-Cut theorem, the minimum total capacity to be removed from a the network to make d not reachable from o is equals to the min-cut capacity $c^*_{o,d}$.
The Min-Cut capacity between o and d is 3.
The 2 node sets obtained performing the cut are $U = \{o, a, b, c\}$ and $U^C = \{d\}$, so in order to disconnect the 2 nodes the capacity should be removed from the nodes passing through it.
In fact removing 1 unit of capacity from (a, d), (b, d) and (c, d), nodes o and d becomes not connected, as shown in Fig. 2.

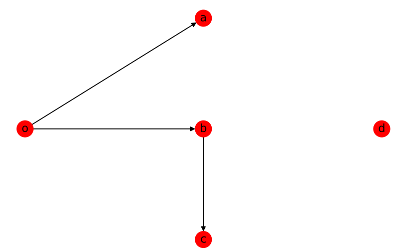

Fig. 2. Network after disconnecting o and d

B. Question

In this question is asked which is the maximum capacity that can be remove not affecting the maximum throughput between o and d.

Removing capacity from an edge passing through a cut with capacity corresponding to the Min-Cut one, Max-Flow will decrease and consequently throughput will be affected. According to that, the only edges of which capacity can be decreased are the ones not passing through any Min-Cut.
In this graph there are 2 min cuts, which are:
$U = \{o, a, b, c\}$, $U^C = \{d\}$ and $U = \{o, a\}$, $U^C = \{b, c, d\}$, so decreasing by 1 the capacity of edges (o, a) and (b, c) the throughput is not affected.
After performing that, any edge in the network pass through at lest one Min-Cut, so not any other edge capacity can be decreased without decreasing throughput.
In conclusion, the maximum total capacity that can be removed without affecting the throughput between o and d is 2.

C. Question

In this question is asked how $x > 0$ units of capacity can be distributed over the network in order to maximize throughput between o and d, and to plot the throughput as a function of x.

Throughput can be maximized, if the graph is imbalanced, simply adding capacity in order to balance it. If the graph is already balanced, the way is to add capacity to a random edge chosen between the ones belonging to any shortest path in terms of number of edges between o and d.
Form the plot in Fig. 3, can be observed that throughput can increase by one every adding of 2 units of capacity.

This can be explained by the fact that the shortest path in terms of number of edge between o and d has length 2. In fact it can also be observed that following the explained procedure edges (b, c) and (c, d) are never incremented after the first balancing, because they do not belong to any shortest

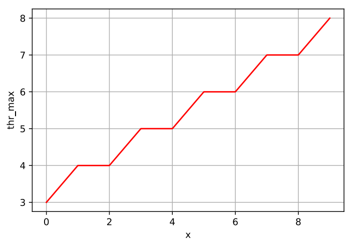

Fig. 3. Plot showing the maximum throughput respect to the extra units of capacity

path.

II. Exercise

In this exercise there is a set of people $p1, p2, p3, p4$ and a set of books $b1, b2, b3, b4$.
Each person is interested in 1 or more books, and this is explained by edges in Fig. 4.

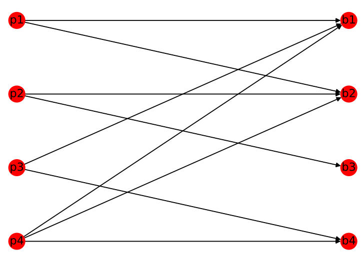

Fig. 4. Initial bipartite graph representing people on the left and books on the right

A. Question

In the first question is asked to find a perfect matching if present using the Max-Flow.

In this case, in order to compute max flow between people and books, origin o and destination d should be added, respectively connected to all people and books, as shown in Fig. 5.

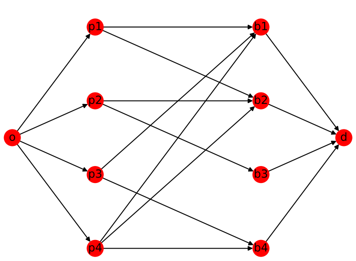

Fig. 5. Network with origin o and destination d added

If to each person can be assigned a book of interest and each book is assigned to a person interested to it,a perfect matching exists, and this happens if Max-Flow between o and d is equals to the number of people and books.
In this case Max-Flow between o and d is 4, so at least 1 perfect matching exists.
A possible perfect matching is:
$p_1 \rightarrow b_2,\ p_2 \rightarrow b_3,\ p_3 \rightarrow b_1,\ p_4 \rightarrow b_4$

B. Question

Assuming that there are multiple copies of books, respectively 2,3,2,2, and that each person can take an arbitrary number of different books, is asked to find the maximum number of different books that can be assigned in total.

In order to model this problem, the capacity of edges between o and people is infinite, because there's no a limit to the number of different books that can be assigned to a person, the edges between people and books have all capacity 1 because each person can take only one copy of each book, and between books and origin the capacity have been set according to the number of copies of each book, as shown in Fig. 6.

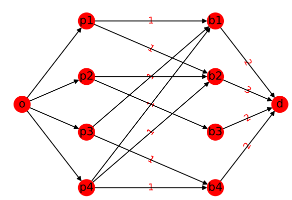

Fig. 6. Network with new capacities

Given this modelling, the max flow between o and d is 8, and it means that 8 over 9 books can be assigned respecting the given constraints.

C. Question

Being possible for the library to sell one copy of book s and buy a copy of another book b, is asked to find the s and b maximizing the number of assigned books.

It has been found that buying one copy of $b1$ and selling one copy of book $b3$ the Max-Flow increases to 9 and all books can be assigned.
This can be explained by the fact that only one person is interested in $b3$ and there are 2 copies of it, and on the other side 3 people are interested in $b1$ and there are only 2 copies of it.

III. Exercise

In the last exercise should be modeled using a graph a simplified map of Los Angeles highways.

This network can be constructed with the support of files containing: - B: node-link incidence matrix;
- c: capacities for each link;
- l: travel times for each link;
- f: flows for each link.
The obtained network is shown in Fig. 7.

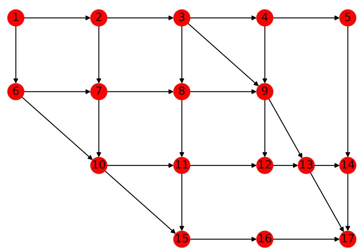

Fig. 7. Los Angeles highways network

A. Question

In the first question is asked to find the shortest path in terms of traveling time between origin 1 and destination 17.

This can be easily computed using networkx built-in functions.
The shortest path in terms of traveling time is $[1, 2, 3, 9, 13, 17]$, as shown in Fig. 8.

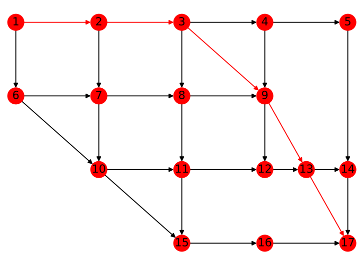

Fig. 8. Shortest path in terms of traveling time

B. Question

In this question is asked to compute the maximum flow between origin 1 and destination 17.

After adding capacities to edges in the correct way, it can be simply done by built-in networkx functions:
$max - flow = 22448$

C. Question

In this question is asked to compute the external inflow ν satisfying the flow balance constraint.

$$Bf = \nu = [16806, 8570, 19448, 4957, -746, 4768, 413, \\ -2, -5671, 1169, -5, -7131, -380, -7412, -7810, \\ -3430, -23544]$$

D. Question

Assuming that the exogenous inflow is a zero-filled vector except for $\nu_1 = 16806$ and $\nu_{17} = -\nu_1$, is asked to compute the social optimum f^* respect to the the delays on edges $\tau_e(f_e)$.

To do this, the cost function:

$$\sum_{e \in \mathcal{E}} f_e \tau_e(f_e) = \sum_{e \in \mathcal{E}} \frac{f_e l_e}{1 - \frac{f_e}{c_e}} = \sum_{e \in \mathcal{E}} (\frac{l_e c_e}{1 - \frac{f_e}{c_e}} - l_e c_e) \quad (1)$$

should be minimized, and it has been done with the support of cxvpy, setting f^* as variable and imposing the flow balance, nonnegativity and capacity constraints.
The total travel time at the social optimum is 25943.622, obtained with the flow vector:

$$f^* = [6.64219910e+03, 6.05893789e+03, \\ 3.13232779e+03, 3.13232589e+03, 1.01638009e+04, \\ 4.63831664e+03, 3.00634073e+03, 2.54263460e+03, \\ 3.13154448e+03, 5.83261212e+02, 1.45164550e-02, \\ 2.92659559e+03, 1.89781986e-03, 3.13232589e+03, \\ 5.52548426e+03, 2.85427264e+03, 4.88644874e+03, \\ 2.21523712e+03, 4.63720641e+02, 2.33768761e+03, \\ 3.31799129e+03, 5.65567890e+03, 2.37310712e+03, \\ 1.99567283e-03, 6.41411626e+03, 5.50543301e+03, \\ 4.88645073e+03, 4.88645073e+03]$$

E. Question

In this question is asked to compute the Wardrop Equilibrium $f^{(0)}$.

To do this, the cost function:

$$\sum_{e \in \mathcal{E}} \int_0^{f_e} \tau_e(s)\, ds \quad (2)$$

should be minimized.
In this case, given the delay function $\tau_e(f_e)$, the cost function to minimize is:

$$\sum_{e \in \mathcal{E}} \int_0^{f_e} \frac{l_e}{1 - \frac{s}{c_e}} ds = \sum_{e \in \mathcal{E}} -c_e l_e \log{(1 - \frac{f_e}{c_e})} \quad (3)$$

with $f^{(0)}$ respecting the flow constraints as in the previous point.
Applying these operations, the total delay at the Wardrop Equilibrium is 26292.963874629393 and the flow vector obtained is:

$$f^{(0)} = [6.71564895e+03, 6.71564803e+03, \\ 2.36740801e+03, 2.36740792e+03, 1.00903510e+04, \\ 4.64539489e+03, 2.80384316e+03, 2.28356194e+03, \\ 3.41848003e+03, 9.22328268e-04, 1.76829408e+02, \\ 4.17141061e+03, 8.92024178e-05, 2.36740792e+03, \\ 5.44495611e+03, 2.35317044e+03, 4.93333832e+03,$$

$1.84155266e+03, 6.97110629e+02, 3.03649261e+03,$
$3.05028094e+03, 6.08677356e+03, 2.58651143e+03,$
$1.24029072e-04, 6.91874216e+03, 4.95391934e+03,$
$4.93333845e+03, 4.93333845e+03]$

Given tolls ω defined as:

$$\omega_e = f_e^* \tau_e'(f_e^*) \tag{4}$$

is asked to found the new Wardrop Equilibrium $f^{(\omega)}$.

Being the $\tau_e'(f_e^*) = \frac{l_e c_e}{(c_e - f_e^*)^2}$ and f^* the social optimum, the cost function to minimize is:

$$\sum_{e \in \mathcal{E}} (\int_0^{f_e} \tau_e(s) ds + \omega_e f_e) = \sum_{e \in \mathcal{E}} -c_e l_e \log{(1 - f_e c_e)} + \sum_{e \in \mathcal{E}} f_e \omega_e \tag{5}$$

with $f^{(0)}$ respecting the flow constraints as in the previous point.
The new Wardrop Equilibrium $f^{(\omega)}$ is:

$f^{(\omega)} = [6.70985110e+03, 6.70985086e+03,$
$2.40098316e+03, 2.40098314e+03, 1.00961489e+04,$
$4.64702364e+03, 2.80354413e+03, 2.27990795e+03,$
$3.40793915e+03, 2.35827155e-04, 1.60447652e+02,$
$4.14842005e+03, 1.77521811e-05, 2.40098314e+03,$
$5.44912522e+03, 2.36499067e+03, 4.92761429e+03,$
$1.84347974e+03, 6.84083836e+02, 3.02038887e+03,$
$3.04907448e+03, 6.06946335e+03, 2.57570513e+03,$
$2.47813497e-05, 6.90169737e+03, 4.97668827e+03,$
$4.92761431e+03, 4.92761431e+03]$

with a total delay of 26271.429108843167. It can be observed that:

$$PoA(\omega) = \frac{total\ delay\ at\ Wardrop\ Equilibrium\ with\ \omega}{total delay at Social Optimum} \tag{6}$$

is $1.01347 \simeq 1$, and that means that this is a good toll design that keeps the system very close to the social optimum.

F. Question

Being the new cost of the system $\psi_e(f_e) = f_e(\tau_e(f_e) - l_e)$, it has been asked to compute the new social optimum and construct tolls ω^* such that the Wardrop Equilibrium coincides with f^*. In order to compute the new Social optimum, the cost function:

$$\sum_{e \in \mathcal{E}} f_e(\tau_e(f_e) - l_e) \tag{7}$$

should be minimized respecting the flow constraints as in the previous point.
The obtained flow vector is:

$f^* = [6.65329658e+03, 5.77466230e+03,$
$3.41971657e+03, 3.41971062e+03, 1.01527034e+04,$
$4.64278036e+03, 3.10584008e+03, 2.66218478e+03,$
$3.00907935e+03, 8.78634280e+02, 7.42401749e-03,$
$2.35493830e+03, 5.94907576e-03, 3.41971062e+03,$
$5.50992306e+03, 3.04369256e+03, 4.88180506e+03,$
$2.41557456e+03, 4.43662730e+02, 2.00804968e+03,$
$3.48735309e+03, 5.49540277e+03, 2.20377848e+03,$
$2.20338871e-03, 6.30070364e+03, 5.62348910e+03,$
$4.88180726e+03, 4.88180726e+03]$

with a total delay of 1002795968.2529671.

In order to construct tolls ω^* as requested, the function:

$$\sum_{e \in \mathcal{E}} (\int_0^{f_e} \psi_e(s) ds + \omega_e f_e) \tag{8}$$

where f is the social optimum, should be minimized.
The obtained tolls are:

$\omega^* = [-5.87275093e-16, -4.87717810e-16,$
$3.12056271e-17, 3.12390244e-17, -8.12982074e-16,$
$-3.03961469e-16, 1.59827547e-16, 3.93267366e-16,$
$2.04877686e-16, 3.71065951e-15, 5.39301448e-10,$
$6.06501589e-16, 6.67533797e-10, 3.12386131e-17,$
$-4.51559991e-16, 1.88402577e-16, -3.49859506e-16,$
$5.60153831e-16, 8.56513165e-15, 9.25679538e-16,$
$6.52682972e-18, -4.49472639e-16, 7.33228033e-16,$
$1.74839837e-09, -5.50717233e-16, -4.67453391e-16,$
$-3.49860181e-16, -3.49860181e-16]$

The new Wardrop equilibrium with constructed tolls is:

$f^{(\omega^*)} = [6.78171987e+03, 6.00489786e+03,$
$3.26626180e+03, 3.26625112e+03, 1.00242801e+04,$
$4.66556319e+03, 3.02889528e+03, 2.64103860e+03,$
$3.03881965e+03, 7.76822009e+02, 2.40314706e+02,$
$2.49832135e+03, 1.06787614e-02, 3.26625112e+03,$
$5.35871687e+03, 2.88151848e+03, 4.89068831e+03,$
$2.41348992e + 03, 6.28171384e + 02, 2.10055098e +$
$03, 3.50968837e + 03, 5.61023936e + 032.26259356e +$
$031.49121230e - 036.38646544e + 035.52884468e +$
$034.89068980e + 034.89068980e + 03]$

with a total delay of 1003639627.5199927

$PoA(\omega^*)$ is very close to 1, so the result is verified.