Network Dynamics and Learning: Homework 2

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***Abstract*—In this report has been explained how I solved the Homework 2 tasks and the results I obtained, with some brief theoretical recall when needed.**

I. Exercise

The first problem consists of studying a single particle performing a continuous-time random walk over the network represented by Fig. 1. with transition matrix Λ.

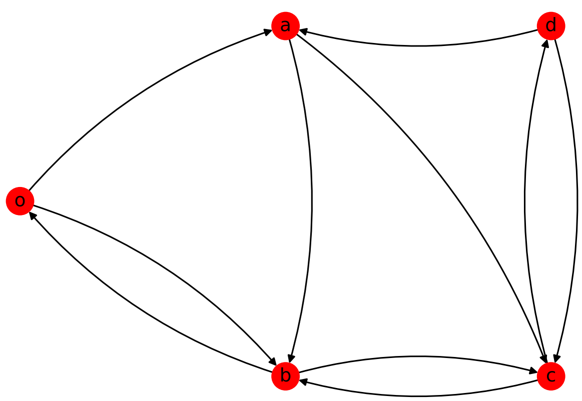

Fig. 1. Initial network

$$\Lambda = \begin{array}{c} \left(\begin{array}{ccccc} 0 & 2/5 & 1/5 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 & 1/3 & 0 \end{array}\right) \end{array} \begin{array}{c} o \\ a \\ b \\ c \\ d \end{array}$$

A. Question

In this task is required to compute according to simulations the return time of the particle for the node a, intended as the the average time for the particle to leave and reach back the node.

To simulate that, each node i in the network is equipped with its own Poisson clock with rate:
$\omega_i = \sum_j \Lambda_{ij}$.
The time between 2 consecutive ticks of a Poisson Clock with rate r denoted as t_{next} is computed as:
$t_{next} = -\frac{ln(u)}{r}$
where u is a random variable with uniform distribution $u \in U(0, 1)$.
When a particle is in the node i and the clock of that node ticks, it jumps to a neighbor j with probability: $P_{ij} = \frac{\Lambda_{ij}}{\omega_i}$

Over 10000 Random Walks, the average return time for node a is 6.7963.

B. Question

In this task is required to compare the results of the previous simulation with the theoretical return-time $\mathbb{E}_a[T_a^+]$.

The Theoretical return time for a node is defined as

$$\mathbb{E}[T_i^+] = \frac{1}{\bar{\pi}_i \omega_i} \tag{1}$$

where $\bar{\pi}$ is the stationary distribution of the continuous time random walk and is defined as:

$$\bar{\pi} = Q' \bar{\pi} \tag{2}$$

where

$$Q_{ij} = \frac{\Lambda_{ij}}{\omega_*} \qquad Q_{ii} = 1 - \sum_{i \neq j} Q_{ij} \tag{3}$$

According to this computation, I obtained:
$\mathbb{E}_a[T_a^+] = 6.7500$
so the value obtained through the simulation is coherent with it.

C. Question

In this task is required to compute according to simulations the return time of the particle from node o to node d, intended as the the average time for the particle to leave and reach back the node.

The simulation is performed in the same way as in point (a) and over 1000 Random Walks, the average hitting time from node o to node d is 8.4487.

D. Question

In this task is required to compare the results of the previous simulation with the theoretical hitting-time $\mathbb{E}_o[T_d]$.

The theoretical hitting times $\mathbb{E}_i[T_S])_{i \in R}$ for the set S and for all nodes $i \in R = \mathcal{V} \setminus S$ can be computed as:

$$\mathbb{E}_i[T_S] = \frac{1}{\omega_i} + \sum_j P_{ij} \mathbb{E}_j[T_S] \quad \text{for } i \notin S \tag{4}$$

in this case $S = \{d\}$ and $i = 0$.
According to this computation, I obtained:
$\mathbb{E}_o[T_d] = 8.7857$
so the value obtained through the simulation is coherent with it.

E. Question

In this task is asked to interpret Λ as the weight matrix of a graph G and simulate the French-DeGroot dynamics on G with an arbitrary initial condition $x(0)$.

In French-DeGroot dynamics $x(t+1) = Px(t)$ where P is the normalized weight matrix of G.
Being G is strongly connected and aperiodic:

$$\lim_{t \to +\infty} x(t) = \alpha \mathbf{1} \tag{5}$$

, where $\alpha = \pi' x(0)$ consensus and π invariant distribution of P for every initial condition $x(0)$.
The simulation has been performed with initial condition $x(0) = [2.5, 6, 3, 7.2, 9.8]$ and after 50 iterations the reached consensus is $\alpha = 5.7348$.

F. Question

In this task is asked to consider every component of the initial condition a $i.i.d$ random variable with variance σ^2, and compare the variance of the consensus $var(\alpha)$ with σ^2.

Using uniform distribution: $x_i(0) = \xi_i \sim U(0, 10)$, $\sigma^2 = \frac{10^2}{12} = 8.3333$
After 50 iterations of the French-DeGroot dynamics: $var(\alpha)_{simul} = 1.6806$.
Being $\alpha = \pi' x(0)$, $var(\alpha) = var(x_i(0)) \sum_i \pi_i^2 = 1.7801$.

As expected the simulated value is close to the theoretical one, and the consensus has a smaller variance of the initial state.

G. Question

In this task is asked to remove edges (d, a) and (d, c) form G and to describe the asymptotic behaviour of the dynamic on the obtained graph.
Considering every component of the initial condition a $i.i.d$ random variable with variance σ^2, is also asked to compare the variance of the consensus $var(\alpha)$ with σ^2.

The obtained graph G_g is represented by Fig. 2.

Starting from the same initial condition of point F, after 50 iterations of the French-DeGroot dynamics: $var(\alpha)_{simul} = 8.5133$, which is close to σ^2.
Having its condensation graph a sink, the invariant distribution centrality is 1 for nodes belonging to the sink component and 0 for the others, so the invariant distribution of GG is $\pi = [0, 0, 0, 0, 1]$
Since $\alpha = \pi' x(0)$,

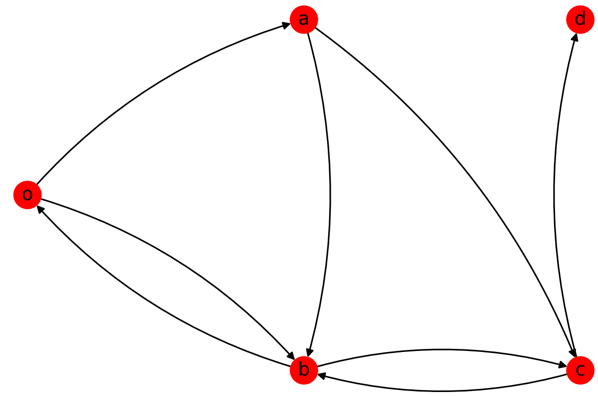

Fig. 2. Initial Network without (d,a) and (d,c)

$var(\alpha) = var(x_i(0)) \sum_i \pi_i^2 = var(x_i(0)) = 8.3333$

As expected the simulated value is close to the theoretical one, in this case the consensus has the same variance of the initial state, because it depends only on the initial state of one node.

H. Question

In this task is asked to remove edges (c, b) and (d, a) form G and to describe the asymptotic behaviour of the dynamic on the obtained graph.

The obtained graph is represented by Fig. 3.

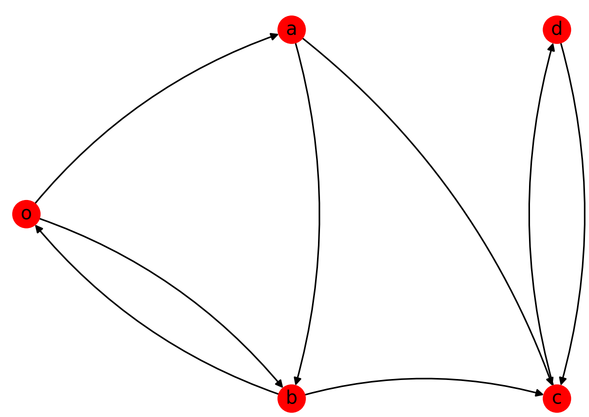

Fig. 3. Initial network without (c,b) and (d,a)

The condensation graph has 1 sink component, but differently from the previous point its subgraph is periodic.

Because of this, in general, consensus is not reached because the conditions of the nodes fluctuate between those of d and those of c.

This behaviour is represented by Fig. 4.

Consensus can be reached only if c and d have the same initial condition.

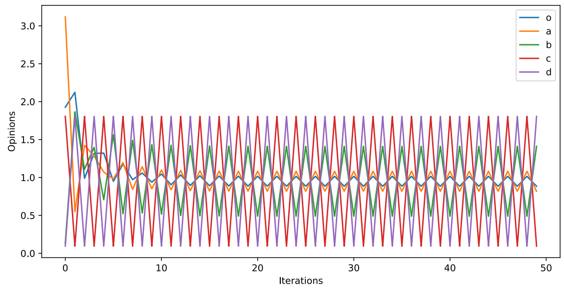

Fig. 4. Opinions with respect to the time over Network in Fig. 3.

II. EXERCISE

This problem aims to simulate the system described in the previous one in 2 different ways: from node perspective and from particle perspective.

A. Particle Perspective

Specifically, is asked to simulate from the particle perspective 100 particles departing from node a, to compute the average time for a particle to return back in a and to compare the results with the ones obtained in Problem 1.

The simulation from the particle perspective has been performed attaching to each particle its own Poisson clock. The simulation can be performed separately for each particle as done in Exercise 1, given the fact that each particle moves independently from the others.
According to that, the average return time is 7.1543, which is close to the results obtained in 1.A.

B. Node Perspective

Specifically, is asked to simulate from the node perspective 100 particles starting from the node o for 60 time units, to plot the number of particle in each node during the simulation, and to compare the results with the stationary distribution of the continuous-time random walk of a single particle.

The simulation from the particle perspective has been performed attaching to each particle its own Poisson clock with a rate proportional to the number of particle in that node at the given time. When the first clock ticks, the transition of a particle is performed and the rates are updated according to the new distribution of particles along the network.

The distribution of particles along the network is represented in Fig. 4.

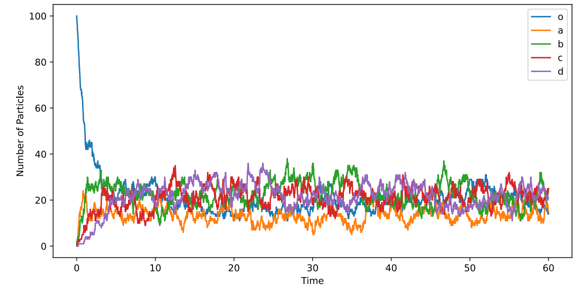

Fig. 5. Distribution of particles over the network in node perspective.

The average number of particles per node over the simulation is:
$N_{avg} = [21.1147, 13.4872, 22.6500, 21.1797, 21.5684]$.
The stationary distribution of the continuous time random walk is $\bar{\pi} = [0.1852, 0.1481, 0.2222, 0.2222, 0.2222]$, which represents the probability of the particles to be in one node. As expected particles are distributed along the network with this probability, in fact for each i it happens that $N_{avg,i} \simeq \bar{\pi}_i$.

III. EXERCISE

This problem aims to study how different particles affect each other moving around in the open network in Fig. 6 with transition rate matrix Λ_{open} in continuous time.

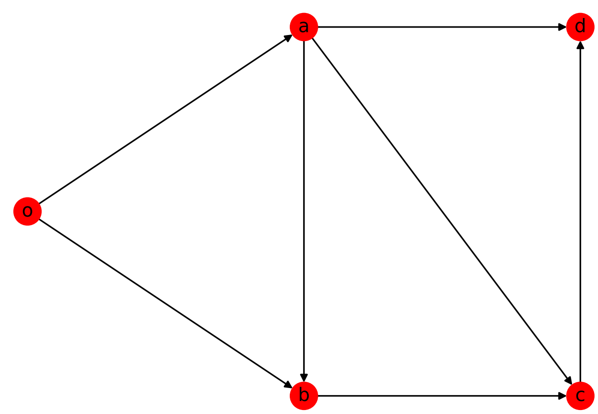

Fig. 6. Open Network

The input rate in node o is 1 and the rate of the d Poisson clock is assumed to be 2.
The system has been simulated form the node perspective for 60 time units in 2 different ways: with proportional rate and fixed rate.
Is also asked to find for both cases the largest input rate that the network can handle without blowing up.

A. Proportional Rate

In this case each node i will pass along particle with rate $n_{it}\omega_i$ where n_{it} is the number of particles in node i at time

t.

With input rate 1 the behaviour of the network is represented by Fig. 7

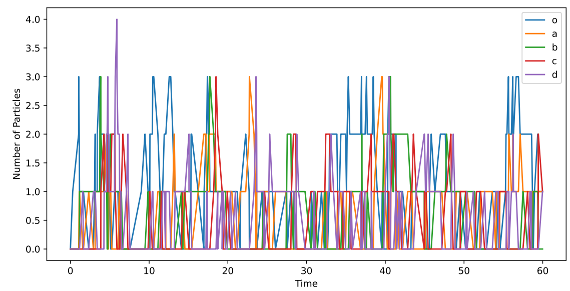

Fig. 7. Distribution of particles over the proportional rate network with input rate 1.

With input rate 1000 the behaviour of the network is represented by Fig. 8

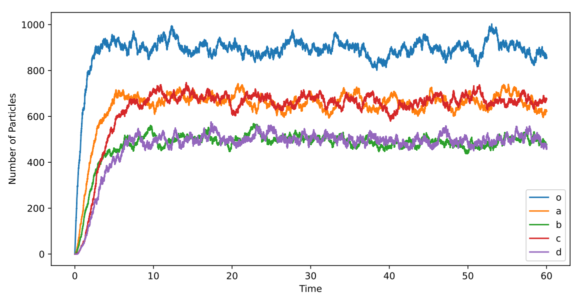

Fig. 8. Distribution of particles over the proportional rate network with input rate 1000.

It can be deduced from the plots that, as the number of particles in the input node increases, the rate of its clock increases together with the probability to move a particle to one of the adjacent node, and so on for other nodes.
For this reason the system can handle any input rate without blowing up.

B. Fixed Rate

In this case each node i will pass along particles with a fixed rate ω_i.

It happens that as the input rate increases, the number of particles in node o grows quickly but ω_0 remains fixed differently from the previous case.
By this fact it can be deduced that the network tends to blow up when input rate is greater than ω_0.

This fact is confirmed by the following graphs.
With input rate $\omega_0 - 0.2$ the behaviour of the network is represented by Fig. 9.

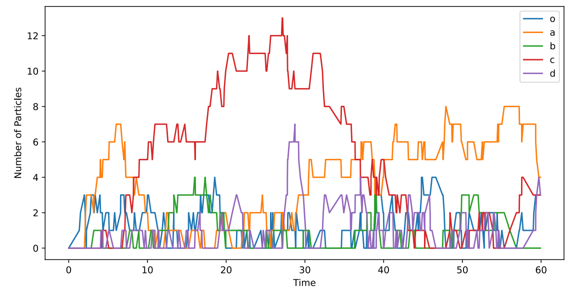

Fig. 9. Distribution of particles over the fixed rate network with input rate $\omega_0 - 0.2$.

With input rate $\omega_0 + 0.2$ the behaviour of the network is represented by Fig. 10

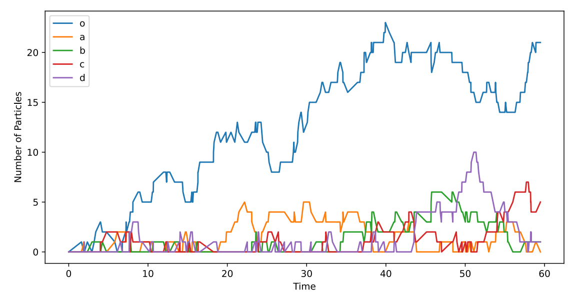

Fig. 10. Distribution of particles over the fixed rate network with input rate $\omega_0 + 0.2$.