

# Notes on Photonics IC

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## Preface

This is a non complete collection of notes from the course of Photonics IC held at Politecnico di Milano during a.a. 2022/2023 by Prof. Mauro Nisoli and Prof. Gianluca Valentini. It is intended as a resume of the main topics of the course. Please refer to the **Semiconductor Photonics, Principles and applications** and **Photonics, Part II** [1] textbooks for a in-deep understanding of the concepts.

## 1 Photonics and Radiometry

In paraxial approximation we can derive the Lagrange invariant:

$$nh\theta = n' h' \theta'$$

and in the case of equal refractive index:

$$h\theta = h' \theta'$$

$$M_L = \frac{h}{h'} = \frac{\theta}{\theta'} \quad \text{linear magnification}$$

$$M_\theta = \frac{\theta'}{\theta} \quad \text{angular magnification}$$

The most important radiometric quantity are:

- Radiant flux  $\Phi$  is the power carried by em field emitted from a source and received by a surface.
- Radiant exitance  $M(x, y)$  is the power density emitted by a source.
- Irradiance  $E(x, y)$  is the power density received by a surface.
- Radiant intensity  $I(\theta, \gamma)$  is the power density per solid angle in a given direction.
- Radiance is the most important quantity, defined as the power density per unit solid angle:

$$L = \frac{d\Phi}{dA \cdot d\Omega \cdot \cos(\theta)}$$

We can demonstrate the invariance of the radiance for em field emitted in  $dA_1$  and received in  $dA_2$ :

$$d\Phi = L d\epsilon, \quad \epsilon \quad \text{etendue}$$

The given areas are subtended by the solid angles:

$$d\Omega_2 = \frac{dA_1 \cos(\theta_1)}{r^2}$$

$$d\Omega_1 = \frac{dA_2 \cos(\theta_2)}{r^2}$$

The given radiant flux expressed by the etendue and radiance is:

$$d\Phi = L_1 dA_1 \cos(\theta_1) d\Omega_1 = L_1 dA_1 \cos(\theta_1) \frac{dA_2 \cos(\theta_2)}{r^2}$$

No9w using the definition of radiance  $L_2$ :

$$L_2 = \frac{d\Phi}{dA_2 \cos(\theta_2) d\Omega_2}$$

$$L_2 = \frac{d\Phi}{dA_2 \cos(\theta_2) \frac{dA_1 \cos(\theta_1)}{r^2}}$$

we place the expression for  $d\Phi$ :

$$L_2 = \frac{L_1 dA_1 \cos(\theta_1) \frac{dA_2 \cos(\theta_2)}{r^2}}{dA_2 \cos(\theta_2) \frac{dA_1 \cos(\theta_1)}{r^2}} = L_1$$

Now in the case of refraction at the interface between two media, we considered constant radiant flux:

$$d\Phi_1 = L_1 dA \cos(\theta_1) d\Omega_1 = L_2 dA \cos(\theta_2) d\Omega_2$$

$$\begin{cases} \frac{L_2}{L_1} = \frac{\cos(\theta_1) d\Omega_1}{\cos(\theta_2) d\Omega_2} \\ \frac{n_2}{n_1} = \frac{\sin(\theta_1)}{\sin(\theta_2)}, \end{cases} \quad \text{Snell's law}$$

Now you consider the solid angles in polar coordinates and we get:

$$\frac{d\Omega_1}{d\Omega_2} = \frac{\sin(\theta_1) d\phi_1 d\theta_1}{\sin(\theta_2) d\phi_2 d\theta_2} = \frac{\sin(\theta_1) d\theta_1}{\sin(\theta_2) d\theta_2}$$

we can find that:

$$\frac{L_2}{L_1} = \frac{n_2^2}{n_1^2}$$

i.e.  $L_n/n_n^2$  is conserved.

We now demonstrate the cosine Lambert's law: We start with an emitter place behind a window with a particular inclination  $\theta$ . What is the flux that arrives in  $dA_1$ , i.e. the detector?

$$d\Phi = L \cdot A_j d\Omega = L(\theta, \phi) dA_o \cdot \frac{dA_1}{r^2}$$

The radiant intensity is defined as  $d\Phi/d\Omega$ , for an emitter at  $\theta$ :

$$I = \int_{A_o} L \cos(\theta) dA_o = I_o \cdot \cos(\theta)$$

and the flux is given by:

$$\int d\Phi = \int_{d\Omega} I d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi/2} L A_o \cos(\theta) \sin(\theta) d\theta = \pi I_o$$

If now we consider an extended cylindrical source:

$$dA_o = 2\pi r dr$$

$$r = z \cdot \tan(\theta) \quad dr = \frac{z}{\cos^2(\theta)} d\theta$$

The solid angle on the single element on the source is:

$$d\Omega_o = \frac{dA_1 \cdot \cos(\theta)}{h^2}$$

Thus, the flux is:

$$d\Phi = L dA_o \cos(\theta) d\Omega_o = 2\pi L \sin(\theta) \cos(\theta) d\theta dA_1$$

and the irradiance is:

$$E = \frac{d\Phi}{dA_1} = \pi L \int_0^{\theta_{1/2}} \sin(2\theta) d\theta = \pi L \sin^2(\theta_{1/2})$$

By geometrical argument we can find that:

$$\sin^2(\theta_{1/2}) = \frac{R^2}{R^2 + z^2}$$

We identify two regimes:

- $z \ll R$ :  $E = \pi L = M$
- $z \gg R$ :  $E = \Phi/\pi z^2$  point like source

Let consider a sphere:

$$\Phi \rightarrow M = \frac{d\Phi}{dA_s} \rightarrow \Phi = 4\pi R^2 M$$

And considering it as lambertian:

$$M = \pi L \rightarrow \Phi = 4\pi R^2 \pi L$$

and it is easy to demonstrate:

$$E = \frac{d\Phi}{dA_d} = \frac{R^2 \pi L}{z^2}$$

## Thermal Detectors

All the thermal detectors shares the same scheme. There is an absorbing media which transfer energy to the heat sink. The process is described in the following terms:

$$P_{IN}(t)dt - G_{th}\Delta T(t)dt = C_{th}d\Delta T(t)$$

and therefore we obtain the following differential equation:

$$\frac{d}{dt}\Delta T(t) = \frac{P_{IN}}{C_{th}} - \frac{G_{th}}{C_{th}}\Delta T(t) \quad (1)$$

If we consider  $P_{IN} = 0$ , we obtain an exponential decay:

$$\Delta T(t) = \Delta T_o \exp\left(-\frac{t}{\tau_{th}}\right) \quad \tau_{th} = \frac{C_{th}}{G_{th}}$$

In the case of step function, the solution is:

$$\Delta T(t) = \frac{P_{IN}}{G_{th}}[1 - \exp\left(-\frac{t}{\tau_{th}}\right)] \quad \tau_{th} = \frac{C_{th}}{G_{th}}$$

Consider a general sinusoidal signal,  $P_{IN}(t) = W_o + W_m \sin(2\pi ft)$ . The solution is:

$$\Delta T(t) = \frac{W_o}{G_{th}} + \frac{W_m \sin(2\pi ft)}{G_{th}\sqrt{1 + (2\pi f\tau_{th})^2}}$$

The first type of thermal detector consists in a resistor in which the resistance changes with temperature. Note that for metals the resistivity increase almost linearly with the temperature. In general  $R = R_o(1 + \alpha\Delta T)$ . For superconductor the carrier density increase with temperature but scattering events increase:

$$np = N_c N_v \exp(-E_g/k_B T)$$

This type of device are characterized by Noise Equivalent Temperature Difference of  $NETD \approx 50mK$ . Another possible type of thermal detector is the so called Golay Cell. The energy is absorbed and the gas inside the cell expands. A reflective membrane in the back of the cell is illuminated by a laser. An optical leverage detects the deformation induced by the expansion. The responsivity at 12.5Hz is  $10^4 V/W$ .

One of the most important detectors are the **pyroelectric detectors**. As we can image, we'll use the pyroelectricity of some material, i.e. the dependence of electric polarization with the temperature. All ferroelectric materials are pyroelectric. In the detector we use two slab of conductive separated by a ferroelectric material (as a capacitor). At equilibrium the natural polarization leads to the presence of  $\sigma_P$  densities:

$$Q = A|\sigma_P|$$

Here the current is given by a variation of the temperature and therefore a variation of the polarization of the ferroelectric material:

$$i(t) = \frac{dQ}{dt} = A \frac{d|\sigma|}{dt} = Ap \frac{dT}{dt}$$

where  $p = d|\mathbf{P}|/dt$  is the pyroelectric coefficient. We know the general expression for the  $\Delta T$  on time:

$$\Delta T(t) = \frac{W_o}{G_{th}} + \frac{W_m}{G_{th}} \frac{\sin(2\pi ft)}{\sqrt{1 + (2\pi f\tau_{th})^2}}$$

and the derivative is:

$$\frac{d\Delta T}{dt} = \frac{\omega W_m \cos(\omega t)}{G_{th} \sqrt{1 + \omega^2 \tau_{th}^2}}$$

The **responsivity**  $R_I$  is defined as:

$$R_I = \frac{i(t)}{W_m \cos(\omega t)} = \frac{pA\omega}{G_{th} \sqrt{1 + (\omega\tau_{th})^2}}$$

At high frequency, we obtain:

$$R_I = \frac{pA}{G_{th}\tau_{th}} = \frac{pA}{C_{th}}$$

The most generic transfer function considering the detector capacitance and the shut resistance of the ferroelectric slab is:

$$R_I = \frac{\omega pA}{C_e C_{th}} \frac{\tau_{th}}{\sqrt{1 + (\omega\tau_{th})^2}} \frac{\tau_{RC}}{\sqrt{1 + (\omega\tau_{RC})^2}}$$

two poles at  $1/2\pi\tau_{th}$  and  $1/2\pi\tau_{RC}$ . For frequency well above  $\tau_{th}$  and well below  $\tau_{RC}$ , we can simply derive:

$$R_V = \frac{\tau_{th} pA}{C_e C_{th}} = \frac{pAR_L}{C_{th}}$$

Usually the detectors are used in this region. We can increase the gain using high  $R_L$  but we reduce our band.

## 1.1 Quantum detectors

In quantum detectors, photon in the range between IR and UV interacts with the valence electrons of the active area producing quantum transitions. The most simple type of quantum detectors exploit the photoelectric effect. For a photon absorbed, an electron is emitted from a metal if the transferred energy overcomes the work function of the metal. The threshold condition is therefore:

$$h\nu_{th} = \Phi$$

The remaining energy is converted in kinetic energy:

$$E_k = h\nu - \Phi [eV]$$

In the case of semiconductor active area we have not to consider the work function, i.e. distance between Fermi level and vacuum level, but the sum of energy gap and electron affinity. The threshold frequency becomes:

$$h\nu_{th} = E_g + \chi$$

### 1.1.1 Vacuum photodiode

Now let us consider the vacuum photodiode. Upon illumination of the photocathode region we have the emission of electrons which are accelerated to the anode through a bias of  $\Delta V \approx 100 - 200V$ . The output current is given by:

$$I_\lambda = \Phi_\lambda \eta e = P_\lambda \frac{\eta e \lambda}{hc}$$

since the incoming photon flux is given by:

$$\Phi_\lambda = P_\lambda / h\nu$$

The current and voltage responsivity is simply:

$$R_I = \frac{I_\lambda}{P_\lambda} = \frac{\eta e \lambda}{hc}$$

$$R_V = \frac{\eta e \lambda}{hc} \cdot R_L$$

The dark current in photocathode which define an intrinsic noise source is related to thermoionic emission. We need to consider the Richardson-Dushman equation:

$$J = \alpha T^2 \exp\left(-\frac{\Phi_{WF}}{k_B T}\right)$$

At 300K we have typical values in the range of  $10^{-15} - 10^{-11} A/cm^2$ . From the definition of responsivity, the average current is proportional to the input power  $P_{in}$ :

$$I_s = R_I P_{in}$$

From the theory of shot noise we know that:

$$i_{Shot}^{RMS} = \sqrt{\langle(\delta i)^2\rangle} = \sqrt{\langle[i(t) - I_s]^2\rangle} = \sqrt{2eI_s \Delta f}$$

where  $\Delta f$  is the bandwidth of the detector. It is now simple to calculate the signal to noise ratio in terms of current amplitude:

$$\left(\frac{S}{N}\right)_A = \frac{I_s}{i_{Shot}^{RMS}} = \sqrt{\frac{I_s}{2e\Delta f}} = \sqrt{\frac{P_{in}\eta\lambda}{2hc\Delta f}}$$

Suppose we have a bandwidth of 1Hz, the minimum detectable signal at 400nm is given by  $S/N = 1$  which leads to a minimum power of  $P_{in} = 10^{-17} W$ .

The most dramatic contribution is obviously the thermal noise  $i_{th}$  of the resistor  $R_L$ :

$$i_{th}^{RMS} = \sqrt{\frac{4k_B T}{R_L} \Delta f}$$

Without any input radiation the output current is given by the noise current with a RMS of:

$$i_n = \sqrt{\left(2eI_d + \frac{4k_B T}{R_L}\right) \Delta f}$$

To find the Noise Equivalent Power (NEP), consider a sinusoidal input power in a noise free detector:

$$P_{in}(t) = P_o + P_o \sin(\omega t)$$

with a RMS of:

$$i_s = P^{RMS} R = \frac{P_o}{\sqrt{2}} R$$

The NEP is the calculated for  $S/N = 1$ , i.e.  $i_s^{RMS}/i_n^{RMS} = 1$ :

$$NEP = P^{RMS} = i_n^{RMS}/R = \frac{hc}{\eta e \lambda} i_n^{RMS} = \frac{hc}{\eta e \lambda} \sqrt{\left(2eI_d + \frac{4k_B T}{R_L}\right) \Delta f}$$

### 1.1.2 Photo-multiplier tubes

Now to increase the sensibility of the devise we exploit secondary electrons emitted by dynodes. The photon flux excites a photo-electron in the cathode which scatters with multiple dynodes in series characterized by  $\delta$  secondary yield. The final electron flux arrives to the anode. An high secondary emission ratio is important in the first stages to avoid multiplication noise. It is possible to demonstrate a linear dependence of secondary emission rate with acceleration voltage  $V_B$ :

$$\delta = kV_B$$

Therefore the total gain is:

$$G = \delta^n = k^n V_B^n$$

The responsivity is we we had before times the gain of photomultiplier:  $R_\lambda = \frac{\eta e \lambda}{hc} \cdot G$  A small variation of the supply voltage can seriously affects the overall gain:

$$\frac{\Delta G}{G} = \frac{nkV_B^{n-1}\Delta V_B}{kV_B^n} = n \frac{\Delta V_B}{V_B}$$

Note that also dark current is multiplied. For this reason thermoionic emission control is extremely important. In the case of PMTs an additional noise source regards the uncertainty of emission rate. In fact,  $G$  is not deterministic but has a mean value and variance. A good figure of merit is the **relative variance**:

$$RV_G = \frac{\sigma_G^2}{\langle G \rangle^2}$$

Each multiplication is independent process and we can calculate the relative variance considering  $\delta_1$  for the first dynode and  $\delta$  the other ones:

$$RV_G = RV_1 + \frac{RV_2}{\sigma_1} + \frac{RV_3}{\delta\delta_1} + \frac{RV_4}{\delta_1\delta^2} + \dots + \frac{RV_n}{\delta_1\delta^{n-2}}$$

Suppose Poisson process, i.e.  $\langle \sigma_n \rangle = \sqrt{\langle n \rangle}$ , we obtain:

$$RV_G = \frac{1}{\delta_1} + \frac{1}{\delta_1\delta} + \frac{1}{\delta_1\delta^2} + \frac{1}{\delta_1\delta^{n-1}} = \frac{1}{\delta_1} \left( \sum_{k=0}^{n-1} \frac{1}{\delta^k} \right)$$

This last geometrical series becomes:

$$RV_G = \frac{1}{\delta_1} \frac{\delta^n - 1}{(\delta - 1)\delta^{n-1}}$$

Now considering high multiplication ratio  $\delta^n$  we can neglect the  $-1$  at the numerator:

$$RV_G = \frac{1}{\delta_1} \frac{\delta}{\delta - 1}$$

For homodynodes PMT we have:

$$RV_G = \frac{1}{\delta - 1}$$

Let us now focus on the anode. Suppose to have a mean emitted electron per unit time from the cathode  $\langle N_e \rangle$ . It is simple to demonstrate that the relative variance is simply:

$$RV_K = \frac{1}{\langle N_e \rangle}$$

The variance of the number of electron collected at the anode is linked to variance of emission of photoelectron and the process of amplification:

$$\sigma_a = \langle N_e \rangle \langle G \rangle^2 + \sigma_G \langle N_e \rangle$$

The expected electrons collected at the anode is simply the product of the expected gain and the expected emitted photoelectrons:

$$\langle N_A \rangle = \langle G \rangle \langle N_e \rangle$$

and the relative variance is:

$$RV_A = \frac{\sigma_A^2}{\langle N_A \rangle^2} = \frac{1}{\langle N_e \rangle} + \frac{1}{\langle N_e \rangle} \frac{\sigma_G^2}{\langle G \rangle^2}$$

and finally:

$$RV_A = \frac{1}{\langle N_e \rangle} (1 + RV_G) = RV_K (1 + RV_G)$$

Let find how much the  $(S/N)$  is degraded along the multiplication process finding the noise factor  $F$ :

$$F = \frac{(S/N)_{in}^2}{(S/N)_{out}^2} = \frac{\frac{1}{RV_K}}{\frac{1}{RV_A}} = \frac{RV_A}{RV_K} = 1 + RV_G = \frac{\delta}{\delta - 1}$$

The shot noise defines a RMS current:

$$i_{shot}^{RMS} = \sqrt{2eI_s\Delta f} \quad \rightarrow \quad \frac{S}{N}|_K = \frac{I_s}{i_{shot}^{RMS}} = \sqrt{\frac{I_s}{2e\Delta f}}$$

At the anode:

$$i_{shot}^{RMS} = \sqrt{2eI_s\Delta f G^2 F} \rightarrow \frac{S}{N}|_A = \frac{GI_s}{i_{shot}^{RMS}} = \sqrt{\frac{I_s}{2eF\Delta f}}$$

Considering also the resistance  $R_L$ , the total noise current is:

$$i_n^{RMS} = \sqrt{(2eG^2I_sF + \frac{4k_B T}{R_L})\Delta f}$$

$$\left(\frac{S}{N}\right)_D = \frac{GI_s}{i_n^{RMS}} = \sqrt{\frac{G^2 I_s^2}{\sqrt{(2eG^2I_sF + \frac{4k_B T}{R_L})\Delta f}}}$$

We can distinguish two regime, high light in which the resistive noise is neglected and the low light where the shot noise term is neglected.

### 1.1.3 Junction detectors

We now want to describe photoconductive detectors. Imagine to have a pn junction. The charges generated in the depletion region are separated by the electric field. An electron is extracted from the n-region while holes from the p-region. The current generated by the absorption of light is:

$$i_\lambda = \frac{P_{in}}{h\nu} \eta e = \frac{E_o A \lambda}{hc} \eta e$$

The responsivity is then:

$$R_\lambda^I = \frac{i_\lambda}{P_{in}} = \frac{\eta e \lambda}{hc}$$

We can distinguish two regimes, photovoltaic mode (in which diode is attached to an high impedance  $R_L \rightarrow \infty$  amplifier i.e. the current generated by photons must pass through the diode) and the photoconductive mode (in which a reversed bias is applied across the diode).

We know the expression of the current in the diode:

$$i_d = i_o \left[ \exp\left(\frac{eV_d}{k_B T}\right) - 1 \right] \rightarrow V_d = \frac{k_B T}{e} \log\left(\frac{i_d}{i_o} + 1\right)$$

and in the case of photovoltaic mode  $i_d = i_\lambda$ , and therefore:

$$V_d = \frac{k_B T}{e} \log\left(\frac{E_o A \lambda}{h c i_o} \eta e + 1\right)$$

where  $i_o$  is the reverse current due minority carriers. We can simply say that, at room temperature, the measured bias across the diode  $V_d$  is:

$$V_d = 25\text{mV} \cdot \log(P_{in} \alpha \lambda \eta)$$

In the photoconductive mode, a **reverse bias** is set across the diode and the current passes through the load resistance  $R_L$  producing a signal  $V_\lambda$ :

$$V_\lambda = (i_o + i_\lambda) R_L = (i_o + \frac{P_{in} \eta e \lambda}{hc}) R_L$$

The application of reversed bias increases the depletion region:

$$w_n = \sqrt{\frac{N_a}{N_a + N_d} \cdot \frac{2\varepsilon_o \varepsilon_r V_B}{e N_d}}$$

$$w_p = \sqrt{\frac{N_d}{N_d + N_a} \cdot \frac{2\varepsilon_o \varepsilon_r V_B}{e N_a}}$$

and for high doping level  $N_a \gg N_d$ :

$$w_n = \sqrt{\frac{2\varepsilon_o \varepsilon_r V_B}{e N_d}}$$

$$w_p = \sqrt{\frac{2N_d \epsilon_o \epsilon_r V_B}{e N_a^2}}$$

Note that the depletion layer largely extends on the n-type region. The temporal response of pn junction is severely affected by diffusion tail. It occurs that e-h pair creates in the ohmic region near the junction where no electric field is present. Only few couples can diffuse to the junction and being accelerated by the electric field. The process of diffusion is really slow compared to other processes and creates a tail in the signal.

A particular type of photodiodes are the PIN photodiode. An intrinsic layer I is sandwiched between to doped layers P and N. Since the charge density in the intrinsic layer is reduced, the electric field variation is also reduced:

$$\frac{dE_x}{dx} = \frac{\rho(x)}{\epsilon_o \epsilon_r}$$

The intrinsic layer becomes completely depleted and the active region is extended up to  $20\mu m$ . The bandwidth of PIN photodiode depends mainly to  $\tau_{drift}$  time required for a photon generated electrons to travel across the junction and the RC time constant  $\tau_{RC}$ :

$$\tau_{drift} = \frac{w_i}{v_{drift}} \quad \tau_{RC} = \frac{\epsilon_o \epsilon_r}{w_i} A$$

The total time constant is:

$$\tau = \sqrt{\tau_{RC}^2 + \tau_{drift}^2}$$

Note that there exists an optimal thickness of the intrinsic layer  $w_i$ . Assuming a single pole transfer function, the output voltage for modulated signal is:

$$V_{out}(f) = \frac{i_\lambda(f) R_L}{\sqrt{1 + (2\pi f \tau)^2}}$$

#### 1.1.4 Avalanche photodiode

Low NEP of junction detectors is set by noise resistance and thermal noise. We now want to exploit multiplication mechanism similar to PMTs to obtain an amplification stage. In condition of high reverse bias, the primary electrons can acquire enough energy to create e-h by impact ionization. Sequence of these events can lead to an avalanche of generated electrons. Ionization process is quantitatively described by the impact ionization coefficients  $\alpha_e$  and  $\alpha_h$ . Considering holes and electrons:

$$\begin{aligned} \frac{di_e}{dx} &= \alpha_e i_e + \alpha_h i_h \\ \frac{di_h}{dx} &= -(\alpha_e i_e + \alpha_h i_h) \end{aligned}$$

where the total current is the sum of two and must be constant along x:

$$I = i_e(x) + i_h(x) = \text{const}$$

Using this last equation we can derive a single differential equation in  $i_e(x)$ :

$$\frac{di_e(x)}{dx} = (\alpha_e - \alpha_h) i_e(x) + \alpha_h I$$

Using boundary condition we can solve this equation. With the reference of figure (2), at  $x = d$  the current is only by electrons:

$$i_h(d) = 0 \quad i_e(d) = I$$

We solve the differential equation and we can calculate the gain of electron current  $M$ :

$$M = \frac{i_e(d)}{i_e(0)} = \frac{k_A - 1}{k_A - \exp[(k_A - 1)\alpha_e d]} \quad k_A = \frac{\alpha_h}{\alpha_e}$$

We now consider the limiting case in which the multiplication hole factor is negligible  $\alpha_h = 0$ , i.e.  $k_A \rightarrow 0$ :

$$M = \exp(\alpha_e d)$$



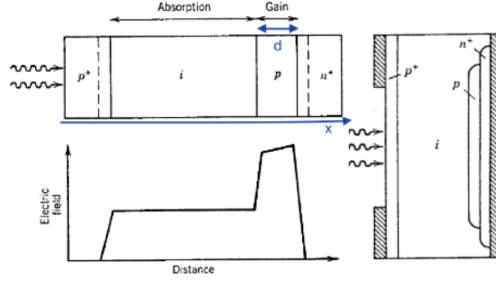


Figure 1: Avalanche PD, image from [1].

The other limiting case is that for equal multiplication factor  $\alpha_e = \alpha_h$ , i.e.  $k_A \rightarrow 1$ :

$$\lim_{k_A \rightarrow 1} M = \frac{1}{1 - \alpha_e d} \quad \text{solved with De L'Hopital}$$

In this case the gain diverges for  $d = 1/\alpha_e$  and is unstable.

The responsivity for APD is the one for simple junction detector times the gain:

$$R_{APD}(f) = M(f) \frac{\eta e \lambda}{hc}$$

where the M factor depends is single pole transfer:

$$M(f) = \frac{M_o}{\sqrt{1 + (2\pi f \tau_{eff} M_o)^2}}$$

For the case in which the multiplication is mainly on electrons ( $k_A \ll 1$ ), the effective transit time is:

$$\tau_{eff} = C_A k_A \tau_{drift} \approx k_A \tau_{drift}$$

### 1.1.5 Noise in Photodiode

Let's now calculate the noise in photodiode. We don't have the thermoionic related current but the thermally e-h couple generation:

$$\sqrt{i_n^2} = \sqrt{\left[ 2e(i_s + i_{dark}) + \frac{4k_B T}{R_L} \right] \Delta f}$$

To give an estimation of the dark current we define the charge generation rate per unit volume:

$$r_g = \frac{n_i}{2\tau_c}$$

and the overall generation rate is therefore:

$$R_g = r_g V$$

A typical value for the dark current is  $I_{dark} = 10^{-14}$ . The one related to thermal noise is much more higher so in the case of low light condition ( $i_s = 0$ ) the dark current term can be neglected:

$$\left( \frac{S}{N} \right) = \frac{i_s^{RMS}}{\sqrt{\langle i_n^2 \rangle}} \approx \frac{i_s^{RMS}}{\sqrt{\frac{4k_B T}{R_L} \Delta f}}$$

$$\left( \frac{S}{N} \right) = \frac{\eta e \lambda}{hc} P_{in} \sqrt{\frac{R_L}{4k_B T \Delta f}}$$

The noise equivalent power divided by the bandwidth (figure of merit independent to bandwidth) is:

$$NEP^* = \frac{NEP}{\sqrt{A \cdot \Delta f}} = \frac{P_{in}}{\sqrt{\Delta f}} = \frac{1}{R_\lambda} \sqrt{\frac{4k_B T}{R_L} \Delta f} \frac{1}{\sqrt{\Delta f}} = \frac{hc}{\eta e \lambda} \sqrt{\frac{4k_B T}{R_L}}$$

For the APD, as in the case of PMTs we must take in account the noise from multiplication. The M factor defines the noise factor  $F_A(M)$  such that:

$$F_A(M) = k_A M + (1 - k_A) \left( 2 - \frac{1}{M} \right)$$

and in the case  $k_A \approx 0$ , i.e.  $\alpha_e \gg \alpha_h$  we have:

$$F_A(M) = 2 - \frac{1}{M} \approx 2 \text{ in the case of high gain } M$$

Finally:

$$\sqrt{\langle i_n^2 \rangle} = \sqrt{[2eM(i_s + i_{dark})F_A + 4k_B T/R_L] \Delta f}$$

and

$$\left( \frac{S}{N} \right) = M R_\lambda P_{in} \sqrt{\frac{R_L}{4k_B T \Delta f}}$$

## 1.2 CMOS detectors

Let us now consider the MOS capacitor in the depletion regime ( $V_G > 0$ ) with a p-type substrate. This means that the charge at the interface metal-insulator is positive charged and a depletion layer is formed in the semiconductor:

$$Q_G = -Q_d = q \cdot N_a \cdot w$$

This region is limited by a threshold value for  $V_G$  for which the concentration of inverse charges (in this case electrons) is analogous to the concentration of majority carriers of the bulk  $N_a$ . This means that, at the interface, the conduction band is really closed to the Fermi level and the charges can be thermally generated (consider  $f_{FD}(\varepsilon)$  for finite temperature). The extension of depleted region reaches a maximum value  $w_T$  and charge  $Q_n$  is "created" at the interface:

$$Q_G = -Q_d - Q_n$$

We now want to work below threshold voltage  $V_T$ . Remember that the positive band bending corresponds to negative potential behaviour. In our case the energy band increases and the potential decreases! We want to derive an expression for the potential in the semiconductor. Let's start considering the Gauss law in one dimension:

$$\nabla \cdot \mathbf{E} = \frac{dE_x}{dx} = \frac{\rho}{\epsilon_s} = \frac{-qN_a}{\epsilon_s}$$

Since  $N_a$  is uniform the electric field is linear in space. We set  $x = 0$  at the interface and if the electric field is zero at  $w$ :

$$E(x) - E(0) = -\frac{qN_a}{\epsilon_s} x$$

$$E(x) = \frac{qN_a w}{\epsilon_s} \left( 1 - \frac{x}{w} \right)$$

$$E_s(0) = \frac{qN_a w}{\epsilon_s}$$

At the interface we know that  $\epsilon_i E_i = \epsilon_s E_s$ : and therefore, the electric field in the oxide present a discontinuity:

$$E_i = E_s(0^+) \frac{\epsilon_s}{\epsilon_i}$$

Considering that  $E(x) = -d\phi(x)/dx$  we can compute the potential:

$$\phi(x) = \frac{qN_a w}{\epsilon_s} \left( \frac{x^2}{2w} - x \right) + \frac{qN_a w^2}{2\epsilon_s}$$

The potential at the interface (which is the potential across the depletion layer) is:

$$\phi_s = \phi(0) = \frac{qN_a w^2}{2\epsilon_s}$$

In the insulator, the electric field is constant  $E_i$  for  $t_{ox}$  thickness, i.e. the potential is linear and the voltage drop across  $t_{ox}$  is:

$$V_i = \frac{qN_a w}{\epsilon_i} t_{ox}$$

The gate voltage is the sum of  $\phi_s$  and  $V_i$ :

$$V_G = \frac{qN_a w}{\epsilon_i} t_{ox} + \frac{qN_a w^2}{2\epsilon_s}$$

$$\epsilon_i q N_a w^2 + 2\epsilon_s q N_a w t_{ox} - 2\epsilon_i \epsilon_s V_G = 0$$

which solution is:

$$w = \frac{-\epsilon_s q N_a t_{ox} \pm \sqrt{(\epsilon_s q N_a t_{ox})^2 - \epsilon_i q N_a \cdot 2\epsilon_i \epsilon_s V_G}}{\epsilon_i q N_a}$$

which can be simplifies:

$$w = \frac{\epsilon_s}{C_{ox}} \left( \sqrt{1 + \frac{2V_G C_i^2}{q\epsilon_s N_a}} - 1 \right)$$

where  $C_{ox} = \epsilon_i/t_{ox}$  and  $C_d = \epsilon_s/w$ . The series of two capacitor determines the total capacity of the MOS structure:

$$C = \left( \frac{1}{C_{ox}} + \frac{1}{C_d} \right)^{-1} = \frac{q\epsilon_s C_{ox} N_a}{\sqrt{q\epsilon_s N_a (2C_{ox}^2 V_G + q\epsilon_s N_a)}}$$

Rearrange everything and defining  $V_o = qN_a \epsilon_s / C_{ox}^2$ :

$$C = \frac{C_{ox}}{\sqrt{1 + 2V_G/V_o}}$$

In the case of non-equilibrium situation, we have to consider quasi-Fermi levels and the mass action law becomes:

$$np = n_i^2 \exp\left(\frac{\varepsilon_{Fc} - \varepsilon_{Fv}}{k_B T}\right)$$

The variation between quasi-Fermi level is given through the external voltage  $\phi_s$  in the semiconductor and the electron concentration becomes:

$$n = \frac{n_i^2}{N_a} \exp\left(\frac{e\phi_s}{k_B T}\right)$$

Near the interface, for inversion condition, we have  $n = N_a$  and the potential is:

$$\left(\frac{N_a}{n_i}\right)^2 = \exp(q\phi_s/k_B T) \longrightarrow \phi_s = 2\frac{k_B T}{q} \ln\left(\frac{N_a}{n_i}\right)$$

The intrinsic concentration is given by the effective density of state  $N_C$  and  $N_V$ :

$$\phi_s = 2\frac{k_B T}{q} \ln\left(\frac{N_a}{\sqrt{N_C N_V}} \exp\left(\frac{\varepsilon_g}{2k_B T}\right)\right) = 2\frac{k_B T}{q} \left(\frac{\varepsilon_g}{2k_B T} + \ln\left(\frac{N_a}{\sqrt{N_C N_V}}\right)\right)$$

and finally:

$$\phi_s^{inv}(T, N_a) = 2\frac{k_B T}{q} \ln\left(\frac{N_a}{\sqrt{N_C N_V}}\right) + \frac{\varepsilon_g}{q}$$

We also know that, in the bulk, the difference between Fermi energy and intrinsic level is:

$$N_a = n_i \exp\left(\frac{\varepsilon_i - \varepsilon_F}{k_B T}\right) \longrightarrow \varepsilon_i - \varepsilon_F = k_B T \ln\left(\frac{N_a}{n_i}\right)$$

and using the previous expression for inversion potential we can conclude that:

$$\phi_s^{inv} = 2\frac{k_B T}{q} \ln\left(\frac{N_a}{n_i}\right) = 2\frac{\varepsilon_i - \varepsilon_F}{q}$$

From definition of  $\phi_s$  we can derive the length  $w = w(\phi_s)$ :

$$w = \sqrt{\frac{2\phi_s\epsilon_s}{qN_a}}$$

and therefore the maximum depletion region (which is obtain at threshold, before inversion) is:

$$w_T = \sqrt{\frac{2\phi_s^{inv}\epsilon_s}{qN_a}}$$

The bias voltage at threshold is then:

$$\begin{aligned} V_T &= \frac{qN_a w_T}{\epsilon_i} t_{ox} + \phi_s^{inv} \\ V_T &= \sqrt{\frac{2\phi_s^{inv}\epsilon_s}{qN_a}} \frac{qN_a}{\epsilon_i} t_{ox} + \phi_s^{inv} \\ V_T &= \sqrt{2qN_a\epsilon_s\phi_s^{inv}} \frac{1}{C_{ox}} + \phi_s^{inv} \end{aligned}$$

Now remember that  $V_o = qN_a\epsilon_s/C_{ox}^2$ :

$$V_T = \sqrt{2V_o\phi_s^{inv}} + \phi_s^{inv}$$

Let's now consider to suddenly increase the gate voltage well above threshold. The MOS is in deep depletion region which is an unstable situation where  $w > w_T$  and the mass action law is not valid  $np \neq n_i^2$ . While time goes by, the system can respond by thermal generation of e-h couple providing inversion charges  $Q_n$ . The system can reach a new equilibrium situation in strong inversion regime. The electric field in the insulator is:

$$E_i = \frac{qN_a w - Q_n}{\epsilon_i} \quad Q_n < 0$$

As usual the depleted region is given by:

$$w = \sqrt{\frac{2\epsilon_s\phi_s}{qN_a}}$$

The gate voltage is the sum of oxide voltage and semiconductor voltage:

$$\begin{aligned} V_G = E_i t_{ox} + \phi_s &= (qN_a w - Q_n) \frac{t_{ox}}{\epsilon_i} + \phi_s = \phi_s + \frac{1}{C_{ox}} (\sqrt{2qN_a\epsilon_s\phi_s} - Q_n) \\ V_G &= \phi_s + \sqrt{2V_o\phi_s} - \frac{Q_n}{C_{ox}} \end{aligned}$$

By solving for  $\phi_s$  we can predict that it decreases with  $Q_n$

The depletion layer can be regarded as a **potential well** capable to store electrons. How many electrons can this region store?

$$Q_n = -(V_G - \phi_s - \sqrt{2V_o\phi_s}) \cdot C_{ox}$$

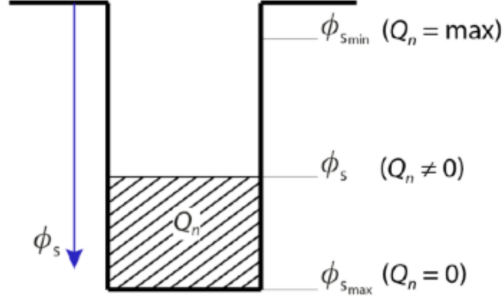


Figure 2: Potential well, image from [1].

As the  $Q_n$  increases, the potential on the depleted region decreases. The minimum  $\phi_s$  to prevent recombination can be  $\phi_s^{inv}$  which set the maximum  $Q_n$ :

$$Q_n^{max} = -C_{ox}(V_G - \phi_s^{inv} - \sqrt{2V_o\phi_s^{inv}}) \quad \text{remember that } \phi_s^{inv} = 2\frac{k_B T}{q} \ln\left(\frac{N_a}{n_i}\right)$$

## 2 Semiconductor Photonics

### 2.1 Introduction to semiconductor properties

The group velocity is defined as follow:

$$\mathbf{v}_g = \frac{\partial \omega_k}{\partial \mathbf{k}} = \frac{1}{\hbar} \frac{\partial \varepsilon_n}{\partial \mathbf{k}}$$

We can demonstrate that the external force is expressed by:

$$\mathbf{F}_{ext} = \frac{d\hbar\mathbf{k}}{dt}$$

The effective mass is inversely proportional to the second derivative of energy:

$$\frac{1}{m^*} \propto \frac{\partial^2 \varepsilon}{\partial k^2}$$

in particular starting from the definition of group velocity and make an analogy with the second equation of the dynamics:

$$\frac{d\mathbf{v}_g}{dt} = \mathbf{a} = \frac{d}{dt} \left( \frac{1}{\hbar} \frac{\partial \varepsilon_n}{\partial \mathbf{k}} \right) = \frac{1}{\hbar} \frac{\partial}{\partial \mathbf{k}} \frac{d\varepsilon}{dt} = \frac{1}{\hbar} \nabla_{\mathbf{k}} \frac{d\varepsilon}{dt}$$

Now let's consider the effect of external force and its related work:

$$d\varepsilon = \mathbf{F} \cdot \mathbf{v}_g dt = \frac{d}{dt} (\hbar\mathbf{k}) \cdot \mathbf{v}_g dt$$

dividing by  $dt$ :

$$\frac{d\varepsilon}{dt} = \frac{d(\hbar\mathbf{k})}{dt} \cdot \mathbf{v}_g = \frac{1}{\hbar} \frac{d(\hbar\mathbf{k})}{dt} \cdot \nabla_{\mathbf{k}} \varepsilon$$

Therefore the definition of acceleration becomes:

$$\frac{d\mathbf{v}_g}{dt} = \frac{1}{\hbar} \nabla_{\mathbf{k}} \frac{d\varepsilon}{dt} = \frac{1}{\hbar} \nabla_{\mathbf{k}} \left( \frac{1}{\hbar} \frac{d(\hbar\mathbf{k})}{dt} \cdot \nabla_{\mathbf{k}} \varepsilon \right)$$

Finally:

$$\mathbf{a} = \frac{1}{\hbar^2} \nabla_{\mathbf{k}} (\mathbf{F} \cdot \nabla_{\mathbf{k}} \varepsilon) = \frac{1}{\hbar^2} \nabla_{\mathbf{k}} \left( F_x \frac{\partial \varepsilon}{\partial k_x} + F_y \frac{\partial \varepsilon}{\partial k_y} + F_z \frac{\partial \varepsilon}{\partial k_z} \right)$$

We can conclude that, since  $\mathbf{a} = \mathbf{F}/m^*$ :

$$\frac{1}{m_{ij}^*} = \frac{1}{\hbar^2} \frac{\partial^2 \varepsilon}{\partial k_i \partial k_j}$$

### 2.2 Carrier density and quasi equilibrium levels

We will now consider semiconductor divided in multidimensional boxes  $\mathbf{L}_i = \sum_i N_i \mathbf{a}_i$ . Since the periodicity of the crystal structure, from the Born-von Karman is possible to state that:

$$\psi_{n\mathbf{k}}(\mathbf{r}) = \psi_{n\mathbf{k}}(\mathbf{r} + N_i \mathbf{a}_i)$$

Take for granted the Bloch's theorem, we derive the quantization of reciprocal lattice:

$$\psi_{n\mathbf{k}}(\mathbf{r}) = \psi_{n\mathbf{k}}(\mathbf{r} + N_i \mathbf{a}_i) = e^{i(\mathbf{k} \cdot N_i \mathbf{a}_i)} \psi_{n\mathbf{k}}(\mathbf{r}) \longrightarrow e^{i(\mathbf{k} \cdot N_i \mathbf{a}_i)} = 1$$

We express the reciprocal vector as  $\mathbf{k} = \kappa_1 \mathbf{b}_1 + \kappa_2 \mathbf{b}_2 + \kappa_3 \mathbf{b}_3$ . Since the scalar product of direct and reciprocal vector is delta-Kronecker ( $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij}$ ), we can write the components of a vector in the reciprocal space as:

$$\mathbf{k} = \frac{m_1}{N_1} \mathbf{b}_1 + \frac{m_2}{N_2} \mathbf{b}_2 + \frac{m_3}{N_3} \mathbf{b}_3, \quad m_i \in \mathbb{Z}$$

The volume occupied by single state ( $m_i = 1$ ) in k-space is then:

$$\Delta k = \frac{\mathbf{b}_1}{N_1} \cdot \left( \frac{\mathbf{b}_2}{N_2} \times \frac{\mathbf{b}_3}{N_3} \right) = \frac{1}{N_1 N_2 N_3} \mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3)$$

Given the metric of the reciprocal space is quite easy to demonstrate that:

$$[\mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3)] \cdot [\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)] = (2\pi)^3$$

$$N \Delta k \cdot V / N = (2\pi)^3 \longrightarrow \Delta k = \frac{(2\pi)^3}{V}$$

where  $V$  is the total volume and  $N = N_1 N_2 N_3$ .

Let us now try to represent the density of state in the energy space. The starting relation regards the number of state which must be the same:

$$\rho(\varepsilon) d\varepsilon = \rho(\mathbf{k}) d\mathbf{k}$$

The number of states contained in the spherical shell with radius  $k - k + dk$  is  $4\pi k^2 dk / \Delta k$  and adding the spin degeneracy we can say that:

$$\rho(k) dk = \frac{4\pi k^2 dk}{8\pi^3} \cdot 2V$$

Now consider  $V = 1$  (number of state per unit volume) and parabolic dispersion:

$$\rho(k) dk = \frac{k^2 dk}{\pi^2}$$

Knowing that

$$\varepsilon = \frac{\hbar^2 k^2}{2m} + \varepsilon_o \longrightarrow d\varepsilon = \frac{\hbar^2 k}{m} dk \longrightarrow \frac{dk}{d\varepsilon} = \frac{m}{\hbar^2 k}$$

Finally we can derive the energy density of state as:

$$\rho(\varepsilon) = \rho(k) \frac{dk}{d\varepsilon} = \frac{m}{\hbar^2 k} \frac{k^2}{\pi^2} = \frac{1}{2\pi^2} \frac{2m}{\hbar^2} \left( \frac{2m}{\hbar^2} \right) \sqrt{\varepsilon - \varepsilon_o}$$

$$\rho(\varepsilon) = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon - \varepsilon_o}$$

The same approach can be used for 2D and 1D case:

$$\rho_{2D}(\varepsilon) = \frac{m}{\pi \hbar^2}$$

$$\rho_{1D}^c(\varepsilon) = \frac{\sqrt{2m_c}}{\pi \hbar} \sqrt{\varepsilon - \varepsilon_c}$$

We can use the density of state to calculate the electron/hole density. For temperature higher than zero we know that the occupation probability is linked to the Fermi-Dirac distribution:

$$f(\varepsilon) = \frac{1}{1 + e^{\varepsilon - \varepsilon_F / k_B T}}$$

The electron density is then:

$$n = \int_{-\infty}^{\varepsilon_F} \rho^c(\varepsilon) f(\varepsilon) d\varepsilon$$

and the hole density:

$$p = \int_{\varepsilon}^{\infty} \rho^v(\varepsilon) (1 - f(\varepsilon)) d\varepsilon$$

$$n = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_{cb} \sqrt{\varepsilon - \varepsilon_c} \frac{1}{1 + e^{\varepsilon - \varepsilon_F / k_B T}} d\varepsilon$$

We can define the new variable:

$$x = \frac{\varepsilon - \varepsilon_c}{k_B T} \quad \text{and} \quad y = \frac{\varepsilon_F - \varepsilon_c}{k_B T}$$

Let rewrite the integral as:

$$n = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \cdot \sqrt{k_B T} \int_{cb} \sqrt{\frac{\varepsilon - \varepsilon_c}{k_B T}} \frac{1}{1 + e^{\varepsilon - \varepsilon_c + \varepsilon_c - \varepsilon_F / k_B T}} d\varepsilon$$

$$\begin{aligned}
n &= \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \cdot \sqrt{k_B T} \int_{cb} \sqrt{\frac{\varepsilon - \varepsilon_c}{k_B T}} \frac{1}{1 + e^{\varepsilon - \varepsilon_c / k_B T} e^{\varepsilon_c - \varepsilon_F / k_B T}} d\varepsilon \\
n &= \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \cdot \sqrt{k_B T} \int_{cb} \sqrt{\frac{\varepsilon - \varepsilon_c}{k_B T}} \frac{1}{1 + e^{\varepsilon - \varepsilon_c / k_B T} e^{\varepsilon_c - \varepsilon_F / k_B T}} d\varepsilon \\
n &= \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \cdot \sqrt{k_B T} \int_{cb} \sqrt{x} \frac{1}{1 + e^x e^{-y}} dx
\end{aligned}$$

where  $d\varepsilon = k_B T dx$

$$n = \frac{1}{2\pi^2} \left( \frac{2mk_B T}{\hbar^2} \right)^{3/2} \cdot \int_0^\infty \frac{\sqrt{x}}{1 + e^{x-y}} dx$$

This integral does not have any primitive by it very similar to the Fermi function of order 1/2:

$$\begin{aligned}
F_{1/2}(y) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{x}}{1 + e^{x-y}} dx \\
n &= \frac{1}{4\pi^{3/2}} \left( \frac{2mk_B T}{\hbar^2} \right)^{3/2} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{x}}{1 + e^{x-y}} dx \\
n &= \frac{1}{4} \left( \frac{2mk_B T}{\pi \hbar^2} \right)^{3/2} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{x}}{1 + e^{x-y}} dx \\
n &= \frac{1}{4} \left( \frac{2mk_B T}{\pi \hbar^2} \right)^{3/2} \cdot F_{1/2}(y)
\end{aligned}$$

By defining the first term as the  $N_c$  effective density of state:

$$n = N_c F_{1/2}(y), \quad y = \frac{\varepsilon_F - \varepsilon_c}{k_B T} \propto \frac{1}{T}$$

The approximation of Fermi function are the following:

$$F_{1/2}(y) = e^y, \quad y < 0, \quad |y| \gg 0$$

$$F_{1/2}(y) = \frac{2}{3} \frac{2}{\sqrt{\pi}} y^{3/2}, \quad y > 0, \quad |y| \gg 0$$

## 2.3 Non-degenerate case

Consider the case in which the Fermi energy is well below of conduction band. The  $y$  value is then negative with a modulus much larger than one. Then, the Fermi function can be approximated as an exponential:

$$F_{1/2}(y) \approx e^y = e^{\frac{\varepsilon_F - \varepsilon_c}{k_B T}}$$

and we can simply calculate the electron density as:

$$n = N_c e^y$$

The same approach can be used to derive the hole density, by considering the  $y$  term related to the distance between Fermi energy and the top of valence band:

$$p = N_v e^y, \quad y = \frac{\varepsilon_v - \varepsilon_F}{k_B T}$$



In this regime, called Maxwell-Boltzmann, we can then calculate the Fermi energy ones we have the electron density and the position of band-edges:

$$\varepsilon_F = \varepsilon_c - k_B T \ln \frac{N_c}{n}$$

$$\varepsilon_F = \varepsilon_v + k_B T \ln \frac{N_v}{p}$$

Moreover, we can derive the well known mass-action law. Take the product of electron and hole densities:

$$np = N_c N_v e^{\varepsilon_F - \varepsilon_c / k_B T} e^{\varepsilon_v - \varepsilon_F / k_B T} = N_c N_v e^{\varepsilon_v - \varepsilon_c / k_B T}$$

$$np = N_c N_v e^{-\varepsilon_g / k_B T}$$

## 2.4 Intrinsic and extrinsic case

In an intrinsic semiconductor the carrier density comes from the thermal generated electrons from the valence band. This process leaves an hole in the conduction band. The two concentrations are then equal and denoted as  $n_i$ :

$$n = p = n_i \longrightarrow n_i^2 = N_c N_v e^{-\varepsilon_g / k_B T}$$

The intrinsic concentration depends then from the effective densities of state, the gap energy and the temperature:

$$n_i = \sqrt{N_c N_v} e^{-\varepsilon_g / 2 k_B T}$$

Using the intrinsic Fermi level as reference, we can derive an expression for the extrinsic case:

$$n = n_i e^{\varepsilon_F - \varepsilon_{Fi} / k_B T}$$

## 2.5 Doped semiconductors

In this section we review a simplify approach to derive the binding energy for donor/acceptor substitutional atom. We need to start from the Bohr hydrogen-model. Let consider the case of acceptor atom (as P atom in Si). The net central attractive potential:

$$U(r) = -\frac{e^2}{4\pi\epsilon_o\epsilon_r r}$$

For the quantization of angular momentum we can derive the energy level ( $n = 1$ ):

$$\varepsilon_H = \frac{e^4 m_o}{2(4\pi\epsilon_o\hbar)^2} = 13.6 \text{ eV}$$

We modify this expression with  $m^*$  and  $\epsilon_r$ :

$$\varepsilon_b = \frac{e^4 m^*}{2(4\pi\epsilon_r\epsilon_o\hbar)^2} = 13.6 \text{ eV} \cdot \left( \frac{m^*}{m_o} \cdot \frac{1}{\epsilon_r^2} \right)$$

For the radius we can derive the following expression:

$$R_H = \frac{4\pi\epsilon_o\hbar^2}{m_o e^2} \longrightarrow R_d = R_H \cdot \epsilon_r \frac{m^*}{m_o}$$

Which effective mass we must use? The conduction effective mass. For silicon, given the six valley minima, the conduction effective mass can be written as:

$$m_\sigma^* = 3 \left( \frac{2}{m_l} + \frac{1}{m_t} \right)^{-1}$$

Note that, in order to effectively control the carrier with external field we need stability of the carrier density. It is for this reason that every semiconductor device works in saturation regime, i.e.

the carrier density is given "only" by the acceptor/donor atoms (and not from thermal excitation). The reason of this is quite obvious, carrier density exponentially increase with temperature.

Let us be more quantitative. From charge neutrality, the mobile and fixed positive charge is equal to mobile and fixed negative charge:

$$p + N_d = n + N_a$$

We consider the case of thermal equilibrium (mass-action law is valid):  $np = n_i^2 \longrightarrow p = \frac{n_i^2}{n}$

$$n - \frac{n_i^2}{n} = N_d - N_a$$

Solving the second degree equation, considering the saturation regime ( $N_d - N_a \gg n_i^2$ ) in n-doped semiconductor ( $N_d > N_a$ ):

$$n = N_d - N_a$$

$$p = \frac{n_i^2}{N_d - N_a}$$

The Fermi level is then shifted:

$$\varepsilon_F = \varepsilon_c - k_B T \ln \frac{N_c}{N_d - N_a}$$

The same approach can be used for p-doped semiconductor. In the usual saturation regime (not consider thermal electrons), we can derive:

$$\varepsilon_F = \varepsilon_v + k_B T \ln \frac{N_v}{N_a - N_d}$$

The equation above make clear that if the effective doping is larger that the effective density of state, the Fermi energy lays above the conduction level. For Si,  $N_c = 3 \cdot 10^{19} \text{cm}^{-3}$  and then if  $N_d - N_a > N_c$  the substrate becomes degenerate. Multiple effect may arise. First the high density of dopants generate new band called impurity bands. Moreover, the energy gap reduces due electron-electron interactions.

## 2.6 Out of equilibrium

All semiconductor devices work in non thermal equilibrium situation. Think on the application of external electric field or exposition of semiconductor to light. Once an electron is excited in conduction band, typically intra-band relaxation time is much faster than radiative recombination (inter-band). The semiconductor reaches an independent equilibrium for the conduction and valence band. We can write two different Fermi-Dirac distribution for the conduction and valence band:

$$f_c(\varepsilon) = \frac{1}{1 + e^{\varepsilon - \varepsilon_{Fc}/k_B T}}$$

$$f_v(\varepsilon) = \frac{1}{1 + e^{\varepsilon - \varepsilon_{Fv}/k_B T}}$$

If Maxwell-Boltzmann statistics can be used, the electron and hole density can be written using the usual approach:

$$n = N_c \exp(\varepsilon_{Fc} - \varepsilon_c/k_B T) = n_i \exp(\varepsilon_{Fc} - \varepsilon_{Fi}/k_B T)$$

$$p = N_v \exp(\varepsilon_v - \varepsilon_{Fv}/k_B T) = n_i \exp(\varepsilon_{Fi} - \varepsilon_{Fv}/k_B T)$$

The quasi-Fermi levels can be written in terms of intrinsic Fermi level:

$$\varepsilon_{Fc} = \varepsilon_{Fi} + k_B T \ln \frac{n}{n_i}$$

$$\varepsilon_{Fv} = \varepsilon_{Fi} - k_B T \ln \frac{p}{n_i}$$

The correction to the mass-action law is linked to the difference between the quasi-Fermi levels:

$$np = n_i^2 \exp(\varepsilon_{Fc} - \varepsilon_{Fv}/k_B T)$$

## 2.7 Electron-Photon Interaction

- Density Matrix definition:

$$\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|$$

- von Neumann equation:

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} \cdot [\hat{H}, \rho]$$

- On diagonal and off diagonal matrix element (longitudinal and transverse time constants):

$$\frac{\partial \rho_{ij}}{\partial t} = \frac{1}{i\hbar} [\hat{H}_o, \rho]_{ij} + \frac{1}{i\hbar} [\hat{H}_{int}, \rho]_{ij} - \frac{\rho_{ij}}{T_2}$$

$$\frac{\partial \rho_{jj}}{\partial t} = \frac{1}{i\hbar} [\hat{H}_{int}, \rho]_{jj} - \frac{\rho_{jj} - \rho_{jj,eq}}{T_1}$$

- Definition of interaction Hamiltonian:

$$\hat{H}_{int} = \frac{e}{m} \mathbf{A} \cdot \mathbf{p}$$

- Approximations (weak field, Coulomb gauge and Electric dipole):

$$\mathbf{A}(\mathbf{r}, t) \approx \mathbf{A}(\mathbf{R}, t)$$

$$\frac{d\mathbf{A}(\mathbf{R}, t)}{dt} = -\mathbf{E}(\mathbf{R}, t)$$

$$\hat{H}_{int} = -\boldsymbol{\mu} \cdot \mathbf{E}$$

- Calculation of  $\rho_{21}$  and  $(\rho_{11} - \rho_{22})$  using the von Neumann equation:

$$\frac{\partial \rho_{21}}{\partial t} = -i\omega_o \rho_{21} + i\frac{\mu}{\hbar} E(t)(\rho_{11} - \rho_{22}) - \frac{\rho_{21}}{T_2}$$

$$\frac{\partial(\rho_{11} - \rho_{22})}{\partial t} = i\frac{\mu}{\hbar} E(t)(\rho_{21}^* - \rho_{21}) - \frac{(\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22})_{eq}}{T_1}$$

- Calculation of the polarization P in terms of  $\rho_{21}$  and  $(\rho_{11} - \rho_{22})$  (no complete calculation):

$$P = \frac{\pi\mu^2}{\hbar} \Delta N E_o g(\omega - \omega_o) [\sin(\omega t) + (\omega_o - \omega) \cdot T_2 \cdot g(\omega - \omega_o) \cos(\omega t)]$$

- Real and imaginary part of atomic susceptibilities from polarization:

$$P = \text{Re}\{\epsilon_o \chi E_o e^{-i\omega t}\}$$

$$\chi'(\omega) = \frac{\pi\mu^2}{\epsilon_o \hbar} \Delta N g(\omega - \omega_o) [(\omega_o - \omega) T_2]$$

$$\chi''(\omega) = \frac{\pi\mu^2}{\epsilon_o \hbar} \Delta N g(\omega - \omega_o)$$

- Resonance and non-resonance polarization contribution to calculate the relative total dielectric constant (total refractive index):

$$P_T = P_{nr} + P_r = \epsilon_o \chi_{nr} \mathbf{E}(t) + \epsilon_o \chi(\omega) \mathbf{E}(t)$$

- First order approximation of refractive index to calculate the real and imaginary parts of wave vector  $k_{tot}$ :

$$n_{tot}^2 = \epsilon_{r,tot} = (1 + \chi_{nr}) + \chi(\omega) = n^2 + \chi(\omega)$$

$$n_{tot}^2 = n^2 \left( 1 + \frac{\chi(\omega)}{n^2} \right)$$

$$n_{tot} = n \sqrt{1 + \frac{\chi(\omega)}{n^2}} \approx n \cdot \left( 1 + \frac{\chi(\omega)}{2n^2} \right)$$

- Plug the  $k_{tot}$  in the electric field expression to derive the absorption coefficient  $\alpha(\omega)$ :

$$k_{tot} = \frac{\omega}{c} n_{tot} = \frac{\omega}{c} n \left( 1 + \frac{\chi(\omega)}{2n^2} \right)$$

$$\alpha(\omega) = 2 \cdot \frac{\omega n}{c} \frac{\chi''(\omega)}{2n^2} = \frac{\omega}{c} \frac{\pi \mu^2}{n \epsilon_o \hbar} \Delta N g(\omega - \omega_o)$$

## 2.8 Optical Properties of Semiconductor

- Transition rate is given by the Fermi golden rule:

$$W_{1 \rightarrow 2} = \frac{2\pi}{\hbar^2} |H'_{21}|^2 \delta(\omega - \omega_o)$$

- Interaction Hamiltonian is compute on Bloch's states and it is not null if and only if:

$$\mathbf{k}_{opt} + \mathbf{k}_1 = \mathbf{k}_2$$

- Transition rate different from zero if:

$$\omega = \omega_o = \omega_2 - \omega_1$$

$$E_{opt} = E_2 - E_1$$

- Electromagnetic field doesn't interact with electron spin:

$$\Delta S = 0$$

- Definition of  $\epsilon_v$  and  $\epsilon_c$  in terms of  $(\epsilon_o - \epsilon_g)$ .
- Definition of joint density of states for the 3D bulk:

$$\rho_j(\epsilon_o) = \frac{1}{2\pi^2} \left( \frac{2m_r}{\hbar^2} \right)^{\frac{3}{2}} \sqrt{\epsilon_o - \epsilon_g}$$

- The net absorption coefficient considering quasi-Fermi levels:

$$dN = \rho_j(\omega_o) [f_v(\epsilon_1) - f_c(\epsilon_2)] d\omega_o$$

- Calculation of absorption coefficient in the case of semiconductor, using the expression of  $dN$ :

$$\alpha = \frac{\omega}{nc} \chi''(\omega)$$

$$d\alpha = \frac{\omega \pi}{\epsilon_o \hbar n c} \mu_{21}^2 g(\omega - \omega_o) dN$$

$$d\alpha = \frac{\omega \pi}{\epsilon_o \hbar n c} \mu_{21}^2 g(\omega - \omega_o) \rho_j(\omega_o) [f_v(\hbar\omega_o) - f_c(\hbar\omega_o)] d\omega_o$$

- Integrate  $d\alpha$  to obtain the absorption coefficient. The Lorentzian is approximated like Delta function:

$$\alpha(\omega) = \int_{\epsilon_g/\hbar}^{\infty} d\alpha$$

$$\alpha(\omega) = \frac{\omega}{nc} \frac{\pi \mu_{21}^2}{\epsilon_o \hbar} \rho_j(\omega) [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

- Calculation of  $\mu_{21}^2$  based on the operators  $\mathbf{r}_{cv}$  and  $\mathbf{p}_{cv}$ , using the definition of Kane energy and considering only the contribution of LH and HH bands:

$$\alpha(\omega) = \frac{\omega}{nc} \frac{\pi}{\epsilon_o \hbar} e^2 x_{cv}^2 \rho_j(\omega) [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

$$\alpha(\hbar\omega) = \frac{\omega}{nc} \frac{\pi}{\epsilon_o} e^2 x_{cv}^2 \rho_j(\omega) [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

$$x_{cv}^2 = \frac{2}{3} |\mathbf{r}_{cv}|^2$$

$$e^2 \cdot |\mathbf{p}_{cv}|^2 = m_o^2 \omega^2 x_{cv}^2$$

- Calculation of absorption coefficient from the power absorbed:

$$dP_{abs} = W_{12} \hbar \omega_o dN$$

If now we divide the power by the electromagnetic intensity we can obtain the absorption coefficient:

$$d\alpha = 2 \cdot \frac{dP_{abs}}{\frac{1}{2} \epsilon_o n c E_o^2}$$

where obviously  $W_{12}$  is the Fermi golden rule transition rate:

$$W_{12} = \frac{2\pi}{\hbar^2} |H'_{21}|^2 g(\omega - \omega_o)$$

We obtain the same result.

- Gain coefficient and Bernard-Durauffourg condition:

$$g(\omega) = -\alpha(\omega)$$

The gain is positive if:

$$f_c(\hbar\omega) > f_v(\hbar\omega)$$

Therefore:

$$\epsilon_2 - \epsilon_1 < \epsilon_{Fc} - \epsilon_{Fv}$$

The threshold condition is that in which we have:

$$\epsilon_g = \epsilon_{Fc} - \epsilon_{Fv}$$

In this case we have  $g = 0$  and the corresponding density is called *transparency density*  $N_{tr}$ . The gain shows a maximum for injected carrier density larger than the transparency density, in function of photon energy. The peak increases almost linearly with the injected carrier density  $N$ :

$$g = \sigma \cdot (N - N_{tr})$$

where  $\sigma$  is the differential gain.

- Model of spontaneous emission based on Einstein coefficient  $A_{cv}$ . Definition of spontaneous emission rate:

$$r_{sp}(\epsilon) = A_{cv} [f_c(\epsilon_2)(1 - f_v(\epsilon_1))]$$

Definition of total emission rate:

$$R_{sp}(\hbar\omega) = \int r_{sp}(\epsilon) \rho_j(\epsilon) g(\epsilon - \hbar\omega) d\epsilon$$

This defines the total emission rate at  $\hbar\omega$ . In the approximation of delta-Dirac, the integral becomes:

$$R_{sp} = r_{sp}(\hbar\omega) \rho_j(\hbar\omega)$$

In the Maxwell approximation:

$$f_c(\epsilon_2)(1 - f_v(\epsilon_1)) \approx \exp\left(-\frac{\hbar\omega}{kT}\right) \cdot \exp\left(\frac{\Delta\epsilon}{kT}\right)$$

In the case of 3D semiconductor, we can derive:

$$R_{sp}(\hbar\omega) = V_{sp} \cdot (\hbar\omega - \epsilon_g)^{\frac{1}{2}} \cdot \exp\left(-\frac{\hbar\omega - \epsilon_g}{kT}\right)$$

$$V_{sp} = \frac{1}{\tau_r} \frac{1}{2\pi^2} \cdot \left(\frac{2m_r}{\hbar^2}\right)^{\frac{1}{2}} \cdot \exp\left(-\frac{\epsilon_g - \Delta\epsilon}{kT}\right)$$

- Calculation of total radiative recombination:

$$R_{sp,T} = \int_{\epsilon_g}^{\infty} R_{sp}(\hbar\omega) d(\hbar\omega)$$

$$R_{sp,T} = \int_{\epsilon_g}^{\infty} r_{sp}(\hbar\omega) \rho_j(\hbar\omega) d(\hbar\omega) = \int_{\epsilon_g}^{\infty} \frac{1}{\tau_r} \exp\left(\frac{-\hbar\omega}{kT}\right) \cdot \exp\left(\frac{\Delta\epsilon}{kT}\right) \rho_j(\hbar\omega) d(\hbar\omega)$$

$$R_{sp,T} = \frac{1}{\tau_r} \frac{1}{4} \left(\frac{2m_r kT}{\pi \hbar^2}\right)^{\frac{3}{2}} \exp\left(\frac{\Delta\epsilon - \epsilon_g}{kT}\right)$$

- Calculation of bimolecular recombination coefficient:

$$R_{sp,T} = \frac{1}{\tau_r} \frac{N_r}{N_c N_v} n \cdot p$$

$$B = \frac{1}{\tau_r} \frac{N_r}{N_c N_v}$$

- Net recombination rate:

$$R_r = B(np - n_o p_o)$$

where  $n = n_o + \Delta n$  and  $p = p_o + \Delta p$

- The recombination dynamics is given by:

$$\frac{d\Delta n}{dt} = -R_r = -B(np - n_o p_o)$$

- Low injection:  $\Delta n = \Delta p \ll n_o, p_o$ :

$$R_r = B(n_o + p_o)\Delta n$$

$$\tau_{rad} = \frac{1}{B(n_o + p_o)}$$

$$\Delta n(t) = \Delta n(0) \exp\left(-\frac{t}{\tau_{rad}}\right)$$

- High injection  $\Delta n, \Delta p \gg n_o, p_o$ :

$$\frac{d\Delta n}{dt} = -B\Delta n^2$$

$$\Delta n(t) = \Delta n(0) \frac{1}{1 + Bt\Delta n(0)}$$

- Total recombination rate:

$$\frac{1}{\tau_T} = \frac{1}{\tau_{rad}} + \frac{1}{\tau_{nr}}$$

and radiative efficiency:

$$\eta_r = \frac{\tau_T}{\tau_{rad}}$$

## 2.9 Quantum Wells

- Electronic wavefunction spatially localized, i.e. superposition of many Bloch's state.
- Envelope function approximation and normalization of periodic and envelope functions.
- QW in z direction, i.e. wave function expressed as:

$$\psi(\mathbf{r}) = F_c(z) \exp(i\mathbf{K} \cdot \mathbf{r}) u_o(x, y)$$

where  $\mathbf{K} = k_x \mathbf{u}_x + k_y \mathbf{u}_y$

- $F_c(z)$  satisfies the one dimensional Schrodinger equation:

$$\left[ -\frac{\hbar^2}{2m_{c,j}} \frac{\partial^2}{\partial z^2} + V_{conf,j}(z) + V_{K,j}(z) \right] F_c(z) = \epsilon_z F_c(z)$$

where  $j$  is the zone considered.

- Solution of envelope function, considering infinite barriers and no momentum on x-y plane:

$$\frac{d^2}{dz^2} F_c = -\frac{2m_{c,w}}{\hbar^2} \epsilon_z F_c = -k_z F_c(z)$$

for the boundary conditions  $F_c(0) = F_c(L_z) = 0$ .

- General solution and unperturbed eigenvalue:

$$F_c(z) = A \sin(k_z z) + B \cos(k_z z)$$

$$\epsilon_z = \epsilon_{nc,0} = n^2 \frac{\hbar^2 \pi^2}{2m_{c,w} L_z^2}$$

- Envelope function:

$$F_c(z) = \sqrt{\frac{2}{L_z}} \sin\left(n \frac{\pi}{L_z} z\right)$$

- General case. The  $V_{K,j}$  potential can be considered as small perturbation of the potential well. Using time-independent perturbation theory:

$$\epsilon_{nc} = \epsilon_{nc,0} + \frac{\hbar^2}{2m_{eff}} K^2$$

where  $m_{eff}$  is given by the confinement factor in well and bulk regions.

- Valence band case. More complicated description. Interaction between HH and LH can not be treated as small perturbation. After some calculation:

$$\epsilon_{nv,lh} = \epsilon_v - \epsilon_{n,lh} - \frac{\hbar^2(k_x^2 + k_y^2)}{2m_{lh}}$$

$$\epsilon_{nv,hh} = \epsilon_v - \epsilon_{n,hh} - \frac{\hbar^2(k_x^2 + k_y^2)}{2m_{hh}}$$

where  $\epsilon_{n,lh/hh} = n^2 \frac{\hbar^2 \pi^2}{2m_{lh/hh} \cdot L_z^2}$  Note that, the degeneracy is removed since the two reduced masses are different: They can be written as:

$$\frac{1}{m_{hh}^z} = \frac{\gamma_1 - 2\gamma_2}{m_o}$$

$$\frac{1}{m_{lh}^z} = \frac{\gamma_1 + 2\gamma_2}{m_o}$$

where  $\gamma_1$  and  $\gamma_2$  are the so called Luttinger parameters. Note that, light hole states have higher confinement, i.e.  $|E_{1lh}| > |E_{1hh}|$ .

- x-y plane effective masses:

$$\frac{1}{m_{hh}^{xy}} = \frac{\gamma_1 + \gamma_2}{m_o}$$

$$\frac{1}{m_{lh}^{xy}} = \frac{\gamma_1 - \gamma_2}{m_o}$$

Therefore, in x-y plane the  $m_{hh} < m_{lh}$  and the energy dispersion cross each others. This degeneracy is then removed also considering higher order perturbation terms.

- Summary. HH and LH energy levels are shifted by  $\epsilon_{nv/nc,hh}$ . The dispersion is then parabolic with  $m_{hh}^{xy} < m_{lh}^{xy}$  and the energy levels should cross. Adding additional perturbation terms we can remove degeneracy.
- Density of state. Consider 2D DOS:

$$\rho(\epsilon)d\epsilon = \rho(k)dk$$

$$\rho(k) = \frac{k}{\pi}$$

and we have previously calculated that:

$$\epsilon_{n,c}(K) = n^2 \frac{\hbar^2 \pi^2}{2m_{c,w} L_z^2} + \frac{\hbar^2 K^2}{2m_{c,w}}$$

Therefore:

$$\rho_{c,1}(k) = \frac{2m_{1c,w}}{\hbar^2 \pi}$$

this is the density of states of the first band.

- Total density of state:

$$\rho_c(\epsilon) = \sum_n \frac{2m_{nc,w}}{\hbar^2 \pi} H(\epsilon - \epsilon_{nc})$$

- The total density of states follows the bulk DOS:

$$\rho^{bulk}(\epsilon_{nc}) = \frac{\rho^{QW}}{L_z}$$

- Density of electrons:

$$n = \int \frac{\rho^{QW}}{L_z} f_c d\epsilon$$

and for a particular subband n, the electron density is:

$$n_n = \int_{\epsilon_{nc}}^{\infty} \frac{2m_{nc,w}}{\hbar^2 \pi} \frac{1}{L_z} \frac{1}{1 + \exp(\frac{\epsilon - \epsilon_{Fc}}{kT})} d\epsilon$$

$$n_n = \frac{kT m_{c,w}}{\hbar^2 \pi L_z} \ln(1 + \exp(\frac{\epsilon_{nc} - \epsilon_{Fc}}{kT}))$$

- Transition from valence band to conduction band in a quantum well.

$$W_{12} = W_{vc} = \frac{2\pi}{\hbar^2} |H'_{cv}|^2 \delta(\epsilon_2 - \epsilon_1 - \hbar\omega)$$

Transition matrix defined as:

$$H' = \frac{e}{m_o} \mathbf{A} \cdot \mathbf{p}$$

where:

$$\mathbf{A}(\mathbf{r}, t) = \hat{a} \frac{1}{2} [A(r) e^{-i\omega t} + A^*(r) e^{i\omega t}]$$

$$A(r) = A e^{i\mathbf{k}_{opt} \cdot \mathbf{r}}$$



- Transition matrix element  $H'_{cv}$

$$H'_{cv} = \langle \psi_c | \frac{e}{m_o} \mathbf{A} \cdot \mathbf{p} | \psi_v \rangle$$

where:

$$\begin{aligned}\psi_c &= F_c(\mathbf{r})u_{co}(\mathbf{r}) \\ F_c(\mathbf{r}) &= F_c(z)e^{i\mathbf{K}\cdot\mathbf{r}}u_{co}(x, y) \\ F_v(\mathbf{r}) &= F_v(z)e^{i\mathbf{K}\cdot\mathbf{r}}u_{vo}(x, y)\end{aligned}$$

We can demonstrate that:

$$H'_{cv} = \frac{e}{2m_o} \int F_c^* u_{co}^* A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p} F_v u_{vo} d\tau$$

- Expansion using the momentum operator:

$$H'_{cv} = \frac{e}{2m_o} \int u_{co}^* u_{vo} F_c^* [A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p}] F_v d\tau + \frac{e}{2m_o} \int F_c^* F_v u_{co}^* [A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p}] u_{vo} d\tau$$

- Lattice is made up of unit cell. We consider the integral on the single unit cell and sum up over all of them. Let us consider the first term:

$$\frac{e}{2m_o} \int u_{co}^* u_{vo} F_c^* [A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p}] F_v d\tau = \frac{e}{2m_o} \sum_j F_c^* [A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p}] F_v \int_{uc} u_{co}^* u_{vo} d\tau$$

where the term

$$F_c^* [A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p}] F_v$$

is considered almost constant along the unit cell. Since the set of cell function is orthonormal,

$$\langle u_{co} | u_{vo} \rangle = 0$$

the first term is null.

- Analysis of second term in which the momenta operator is applied on the cell function:

$$\frac{e}{2m_o} \int F_c^* F_v u_{co}^* [A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p}] u_{vo} d\tau$$

As usual, split the integral fro each unit cell:

$$\frac{e}{2m_o} \int F_c^* F_v u_{co}^* [A(\mathbf{r}) \hat{\mathbf{a}} \cdot \mathbf{p}] u_{vo} d\tau = \frac{e}{2m_o} \sum_j F_c^* A(\mathbf{r}_j) F_v \int_{uc} u_{co}^* [\hat{\mathbf{a}} \cdot \mathbf{p}] u_{vo} d\tau$$

Since the momenta operator can be considered non local:

$$H'_{cv} = \frac{e}{2m_o} \frac{1}{V_c} \int u_{co}^* [\hat{\mathbf{a}} \cdot \mathbf{p}] u_{vo} d\tau \sum_j F_c^* A(\mathbf{r}_j) F_v V_c$$

It becomes:

$$H'_{cv} = \frac{e}{2m_o} \langle u_{co} | [\hat{\mathbf{a}} \cdot \mathbf{p}] | u_{vo} \rangle \cdot \int F_c^*(\mathbf{r}) A(\mathbf{r}) F_v(\mathbf{r}) dV$$

- Momenta conservation and envelope function orthonormality:

$$H'_{cv} \neq 0 \iff \mathbf{K}_c - \mathbf{K}_v = \mathbf{k}_{opt}$$

$$H'_{cv} = \frac{e}{2m_o} A_o \langle u_{co} | [\hat{\mathbf{a}} \cdot \mathbf{p}] | u_{vo} \rangle \langle F_c | F_v \rangle$$

Note that, in the case of infinite well, envelope functions braket introduces another transition selection rule. The transition is admitted if and only if  $n_c = n_v$  (transition between the second conduction band and valence band). When consider exponential decay:

$$\langle F_c | F_v \rangle \approx 0.95 \quad n_c = n_v$$

$$\langle F_c | F_v \rangle \approx 0.05 \quad n_c \neq n_v$$

- Intersubband transition (conduction to conduction). The wavefunctions change only for the envelope function. In this case:

$$\langle u_{co} | u_{co} \rangle = 1$$

Therefore,

$$H'_{cv} = \frac{e}{2m_o} A_o \langle F_2 | \hat{a} \cdot \mathbf{p} | F_1 \rangle$$

We can demonstrate that:

$$\langle F_2 | \hat{a} \cdot \mathbf{p} | F_1 \rangle \neq 0 \iff \hat{a} \cdot \mathbf{u}_z \neq 0$$

- Dipole momenta matrix element. Based on Fourier theory, we can state that:

$$E = -\frac{\partial A}{\partial t} \rightarrow E_o = \omega A_o$$

We have also stated that:

$$|H'_{cv}|^2 = |-\mu \cdot \mathbf{E}|^2 \approx |\mu|^2 \left| \frac{E_o}{2} \right|^2$$

and

$$H'_{cv} = \frac{e}{2m_o} A_o \langle u_{co} | \hat{a} \cdot \mathbf{p} | u_{vo} \rangle \langle F_c | F_v \rangle = \frac{e A_o}{2m_o} M_T$$

It follows that:

$$\left( \frac{e A_o}{2m_o} \right)^2 |M_T|^2 = |\mu_{21}|^2 \left| \frac{\omega A_o}{2} \right|^2$$

hence:

$$|\mu_{21}|^2 = \frac{e^2}{\omega^2 m_o^2} |M_T|^2$$

- Absorption coefficient.

$$\alpha(\omega) = \frac{\omega}{cn} \frac{\pi}{\hbar \epsilon_o} |\mu_{21}|^2 \rho_j(\omega) [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

- Joint Density of States for QW.

$$\epsilon_v = -\epsilon_{1v} - \frac{\hbar^2}{2m_v} K^2$$

and

$$\epsilon_c = \epsilon_g + \epsilon_{1c} + \frac{\hbar^2}{2m_c} K^2$$

where:  $\epsilon_{1v} = \frac{\hbar^2 \pi^2}{2m_v L_z^2}$  and  $\epsilon_{1c} = \frac{\hbar^2 \pi^2}{2m_c L_z^2}$

The energy difference is  $\epsilon_o$ :

$$\epsilon_o = \epsilon_c - \epsilon_v = \epsilon_g + \epsilon_c + \epsilon_v + \frac{\hbar^2}{2m_r} K^2$$

In 2D case, the density of state in k-space is given by:

$$\rho(K) dK = \frac{K}{\pi} dK$$

The joint density of state becomes:

$$\rho_j^{QW} = \rho(K) \frac{dK}{d\epsilon_o} = \frac{k}{\pi} \frac{m_r}{\hbar^2 K}$$

Hence:

$$\rho_j^{QW}(\epsilon_o) = \frac{m_r}{\pi \hbar^2}$$

$$\rho_j^{QW}(\omega_o) = \frac{m_r}{\pi \hbar}$$

- Absorption Coefficient in QW.

$$\alpha(\omega) = \frac{\omega}{cn} \frac{\pi}{\hbar\epsilon_o} \frac{e^2}{\omega^2 m_o^2} |M_T|^2 \rho_j(\omega) [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

$$\alpha(\omega) = \frac{1}{cn} \frac{\pi}{\hbar\epsilon_o} \frac{e^2}{\omega m_o^2} |M_T|^2 \frac{\rho_j^{QW}}{L_z} [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

Therefore:

$$\alpha(\omega) = \frac{1}{cn} \frac{\pi}{\hbar\epsilon_o} \frac{e^2}{\omega m_o^2} |M_T|^2 \frac{m_r}{\hbar\pi L_z} [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

- Gain and transparency condition.

$$\alpha(\omega) \frac{1}{cn} \frac{\pi}{\hbar\epsilon_o} \frac{e^2}{\omega m_o^2} |M_T|^2 \frac{\rho_j^{QW}}{L_z} [f_v(\hbar\omega) - f_c(\hbar\omega)] < 0$$

$$\Longleftrightarrow$$

$$f_v(\hbar\omega) - f_c(\hbar\omega) < 0$$

This means that, in order to have positive gain:

$$\epsilon_2 - \epsilon_1 < \epsilon_{Fc} - \epsilon_{Fv}$$

Transparency condition is obtained when:

$$\epsilon_{Fc} - \epsilon_{Fv} = \epsilon_g + \epsilon_{1c} + \epsilon_{1v}$$

- Intersubband transition. An example is the case of transition between two conduction bands:

$$d\alpha = \frac{\omega}{cn} \frac{\pi}{\hbar\epsilon_o} |\mu_{mn}|^2 \rho_j(\omega_o) (f_n(\hbar\omega_o) - f_m(\hbar\omega_o)) g(\omega - \omega_{mn}) d\omega_o$$

where the bracket term represents the difference between the population in the n-th and m-th band:

$$\alpha(\omega) = \frac{\omega}{cn} \frac{\pi}{\hbar\epsilon_o} \rho_j(\omega) (N_n - N_m) g(\omega - \omega_{mn})$$

We can write that:

$$\alpha(\omega) = \frac{\omega}{cn} \frac{\pi}{\hbar\epsilon_o} \rho_j(\omega) \frac{m_c}{\pi\hbar^2 L_z} (\epsilon_n - \epsilon_m) g(\omega - \omega_{mn})$$

## 2.10 Excitons

So far we have discussed semiconductor optical properties based on single particle approach, i.e. without considering electron-electron correlations. When an electron is excited to the conduction band a hole is created in the valence band. Attraction between hole and electron creates a correlated motion of this pair. This electron-hole quasiparticle is called exciton. We can define two types of excitons **Frenkel excitons** and **Wannier-Mott excitons**. The first one is localized in few unit cells and therefore is delocalized in k-space. It must be analyzed with a full band approach and becomes important for material with low dielectric constant where the screening effect is weak. In material with large dielectric constant, the interaction between electron and hole can be screened by the valence electrons. The exciton binding energy is much more smaller (0.01eV for Wannier-Mott with respect 1-0.1eV for Frenkel) and this quasiparticle is delocalized on hundreds of Angstrom, i.e. well defined in k-space.

We can use the Hydrogen-like approach already discussed. We assume that the creation of excitons leads to the formation of discrete energy levels in the forbidden gap:

$$\epsilon = \epsilon_g - \epsilon_{Bn} \quad \epsilon_{Bn} \text{ exciton binding energy}$$

$$\epsilon_{Bn} = \frac{m_r e^4}{2(4\pi\epsilon\hbar)^2} \frac{1}{n^2}$$

with a radius.

$$R_B = \frac{4\pi\epsilon\hbar^2}{m_r e^2}$$

For the first energy level  $n = 1$ , the binding energy is  $\epsilon_{B1} = 4.7meV$  and  $R_B = 1170nm$ . The exciton motion in terms of center of mass is described by two body Schroedinger equation for the electron and hole:

$$\left( -\frac{\hbar^2}{2m_e}\nabla_e^2 - \frac{\hbar^2}{2m_h}\nabla_h^2 - \frac{e^2}{4\pi\epsilon|\mathbf{r}_e - \mathbf{r}_h|} \right) \psi(\mathbf{r}_e, \mathbf{r}_h) = \epsilon_{eh} \cdot \psi(\mathbf{r}_e, \mathbf{r}_h)$$

In terms of center of mass:

$$\mathbf{R} = \frac{m_e\mathbf{r}_e + m_h\mathbf{r}_h}{M}$$

$$\mathbf{r} = \mathbf{r}_e - \mathbf{r}_h$$

we can derive the following Schroedinger equation, equivalent to the previous one:

$$\left( -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}}^2 - \frac{\hbar^2}{2m_r}\nabla_{\mathbf{r}}^2 - \frac{e^2}{4\pi\epsilon r} \right) \psi(\mathbf{R}, \mathbf{r}) = \epsilon_{eh} \psi(\mathbf{R}, \mathbf{r})$$

The eigenstates are described by the electron-hole distance  $\mathbf{r}$  and the center of mass  $\mathbf{R}$ , such that:

$$\psi(\mathbf{R}, \mathbf{r}) = F(\mathbf{R})g(\mathbf{r})$$

are decoupled:

$$\left( -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}}^2 \right) F(\mathbf{R}) = \epsilon_R F(\mathbf{R}) \quad \text{Free electron model}$$

$$\left( \frac{\hbar^2}{2m_r}\nabla_{\mathbf{r}}^2 - \frac{e^2}{4\pi\epsilon r} \right) g(\mathbf{r}) = \epsilon_r g(\mathbf{r}) \quad \text{Hydrogen like equation}$$

Hence, the solutions are in the following terms, where  $\mathbf{K} = \mathbf{k}_e + \mathbf{k}_h$ :

$$F(\mathbf{R}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k} \cdot \mathbf{R}}$$

$$\epsilon_R = \frac{\hbar^2 K^2}{2M}$$

The g-function is expressed using Laguerre polynomials and spherical harmonics:

$$g_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\theta, \phi)$$

$$\epsilon_r = \epsilon_g - \frac{Ry^*}{n^2} \quad Ry^* = \frac{m_r e^4}{2(4\pi\epsilon\hbar)^2}$$

Let plug everything together:

$$\psi(\mathbf{R}, \mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k} \cdot \mathbf{R}} R_{nl}(r)Y_{lm}(\theta, \phi)$$

$$\epsilon_{eh} = \epsilon_g - \frac{Ry^*}{n^2} - \frac{\hbar^2 K^2}{2M}$$

Presence of exciton implies discrete states (which can absorb) with a density growing up to continuous level near  $\epsilon_g$ .

Light Emitting Diode

- Homojunction: p-n junction where the minority carriers are injected into the p and n regions. Recombination takes place and photons are emitted.
- Several recombination processes take place. SRH and Auger recombination are non radiative processes:

$$\frac{1}{\tau_{nr}} = A_{nr} + Cn^3$$

- The total radiative recombination efficiency is thus:

$$\eta_r = \frac{\frac{1}{\tau_{rad}}}{\frac{1}{\tau_{rad}} + \frac{1}{\tau_{nr}}}$$

- We must take in account also the injection efficiency:

$$\phi_{carrier} = \eta_{inj} \frac{J}{ed}$$

- The total photon flux emitted will be:

$$\phi = \eta_{inj} \eta_r \frac{J}{e}$$

- Heterojunction: reduce the active region to increase carrier density: few consideration about Burstein-Moss shift, carrier overflow and carrier leakage.
- An active region between two clapping region at different potentials, will be characterized by different Fermi energies. Near the edges of Fermi energies the concentration of electrons and holes will be high. Therefore the energy levels at the bottom of conduction band are occupied by electrons and at top of valence band by holes. This lead to an increase of absorbed photon energies and therefore shift of absorption spectrum. This is called **Burstein-Moss shift**.
- If the difference between the quasi-Fermi energy in the bottom of conduction band is equal to the barrier potential an increase of injected electrons doesn't lead to an increase of emitted light! This is called carrier overflow.
- If the difference between quasi-Fermi energy in bottom of conduction band is lower than the barrier potential but the injection is sufficiently high we can have carrier leakage. The electron with energies higher than barrier potential are:

$$n_B = \int_{\epsilon_B}^{\infty} \rho_c(\epsilon) f_c(\epsilon) d\epsilon$$

which can be approximated as:

$$n_B = N_c \exp\left(-\frac{\epsilon_B - \epsilon_{Fc}}{kT}\right)$$

Since the movement of electrons are random only a portion of these move to the cladding regions:

$$n_c(x) = n_B \cdot \exp\left(-\frac{x}{L_n}\right)$$

The diffusion current will be:

$$J_n = -eD_n \frac{dn}{dx} = e \frac{L_n}{\tau_n} n_B$$

## External efficiency

The external efficiency considering internal reflection is given by:

$$\eta_{ext} = \eta_t \cdot \eta_{int}$$

and the external photon flux is:

$$\Phi_{ext} = \eta_{ext} \frac{J}{e} \left[ \frac{photon}{cm^2} \right]$$

from which we can derive the output power and the transfer function:

$$P_{ext} = \Phi_{ext} \cdot \eta_{ext} \cdot S \cdot h\nu$$

$$R = \frac{P_{ext}}{I}$$

Another large fraction of photons cannot escape from the LED due internal reflection. We can define a critical angle  $\theta_c$  such that:

$$n_1 \sin(\theta_c) = n_2 \sin(\pi/2), \quad \theta_c = \arcsin \frac{n_2}{n_1}$$

So the light can escape for an angle between 0 and  $\theta_c$ . For the *GaAs* case,  $n_1 = 3.4$  while  $n_2 = 1$ . We can obtain a critical angle of  $\theta_c = 17^\circ$ .

The solid angle of escape:

$$\Omega_c = \int_0^{\theta_c} \frac{2\pi r \cdot r \sin \theta d\theta}{r^2} = 2\pi(1 - \cos(\theta_c))$$

For small angle approximation,  $\cos x \approx 1 - x^2/2$ :

$$\Omega_c = 2\pi(1 - 1 + \theta_c^2/2)$$

and finally:

$$\Omega_c = \pi\theta_c^2 = \pi \left( \frac{n_2}{n_1} \right)^2$$

Of the entire solid angle  $4\pi$ , the emitted region is  $\Omega_c/4\pi$  and the power that can escape is:

$$P_{ext} = P_{source} T(0) \Omega_c / 4\pi$$

where  $T(0) = T(\theta)$  is the transmission and can be defined as:

$$T(0) = 1 - \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

We can calculate the extraction efficiency:

$$F = \frac{P_{ext}}{P_{source}} = \frac{1}{4} \left[ 1 - \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 \right] \left( \frac{n_2}{n_1} \right)^2$$

## 2.11 Lasers and DFL

We have seen that a semiconductor far from equilibrium ( $\varepsilon_{Fc} \neq \varepsilon_{Fv}$ ) can present a net gain if the Bernard-Duraffourg condition is archived:

$$\varepsilon_2 - \varepsilon_1 < \varepsilon_{Fc} - \varepsilon_{Fv}$$

For example, in our discussion:

$$\alpha(\omega) = \frac{\omega}{nc} \frac{\pi}{\epsilon_0 \hbar} e^2 x_{cv}^2 \rho_j(\omega) [f_v(\hbar\omega) - f_c(\hbar\omega)]$$

and the gain has been defined as:

$$g(\omega) = -\alpha(\omega) > 0 \iff f_c(\hbar\omega) - f_v(\hbar\omega) > 0$$

but since:

$$f(\hbar\omega) = \frac{1}{1 + e^{(\hbar\omega - \varepsilon_F)/k_B T}}$$

we have:

$$e^{(\hbar\omega_2 - \varepsilon_{Fc})/k_B T} < e^{(\hbar\omega_1 - \varepsilon_{Fv})/k_B T} \longrightarrow \varepsilon_2 - \varepsilon_1 < \varepsilon_{Fc} - \varepsilon_{Fv} \quad \square$$

The most simple model of laser can be described using the following equations:

$$\begin{aligned} \frac{dN}{dt} &= \eta_i \frac{I}{eV} - \frac{N}{\tau} - R_{st} \\ \frac{d\phi}{dt} &= \Gamma R_{st} + \Gamma \beta_{sp} R_{sp} - \frac{\phi}{\tau_c} \end{aligned}$$

where  $N$  and  $\phi$  are the carrier density and photon flux,  $R_{st}$  is the net stimulated recombination rate and  $\eta_i$  is the internal injection efficiency.  $\Gamma$  is the confinement factor defined on the field intensity distribution:

$$\Gamma = \frac{\int_{-d/2}^{d/2} |u(x)|^2 dx}{\int_{-\infty}^{\infty} |u(x)|^2 dx}$$

A DFB (distributed feedback laser) consists of an active medium in which a periodic thickness variation of the cladding layers forms a grating. The modulation has periodicity  $\Lambda$ . The effective refractive index can be calculated assuming sinusoidal modulation:

$$n(z) = \langle n(x, z) \rangle_x = n_o + \delta n \cos \left( \frac{2\pi}{\Lambda} z + \varphi \right)$$

and the total effective refractive index:

$$\bar{n} = \langle n(z) \rangle_z$$

Electric field in forward and backward direction are coupled when Bragg condition is satisfied:

$$\frac{\lambda_o}{4\bar{n}} = \frac{\Lambda}{2}$$

where  $\lambda_o$  is free-space wavelength radiation. Note that  $\Lambda$  determines the wavelength of the mode which can oscillates in the cavity. The **propagation constant**  $\beta_o$  at the Bragg condition, is:

$$\beta_o = k_o \bar{n} = \frac{2\pi}{\lambda_o} \bar{n} = \frac{\pi}{\Lambda}$$

Calculation can demonstrate that the transmittance in function of detuning from Bragg condition shows maxima whose value increase with the gain. The minima is obtained for  $\beta = \beta_o$  and the first two maxima correspond to the first two modes (the lower order).

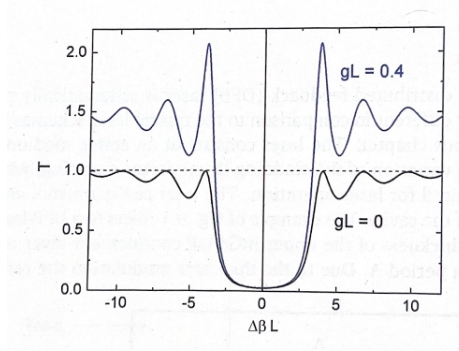


Figure 3: Transmittance in function of detuning  $\Delta\beta L$  for different gains from [2].

In general double mode operation is not advantageous for application. To solve this problem the so called  $\lambda/4$ -shifted DFB laser is introduced where a periodic modulation of the thickness of the cladding layer is created. The refractive index undergoes a phase shift of  $\Lambda/2$  at the center of the structure. The maximum of intensity transmittance is obtained for  $\beta = \beta_o$ .

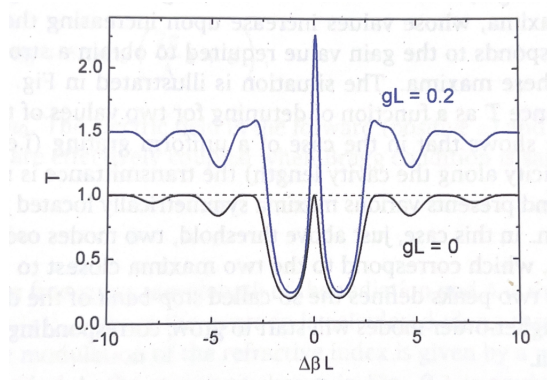


Figure 4: Transmittance in function of detuning  $\Delta\beta L$  for different gains in the case of non-uniform grating from [2].

First consider a passive structure ( $g = 0$ ). The refractive index is given by:

$$n(z) = n_o + \delta n \cos \left( \frac{2\pi}{\Lambda} z + \phi(z) \right) \cdot h(z)$$

where  $h(z)$  is a "variational parameter". The general wave equation is:

$$\nabla^2 \tilde{\mathbf{E}} = \frac{n^2}{c^2} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2}$$

We now assume that the field distribution in the xy plane is given by the fundamental mode of the wave-guide:

$$\tilde{\mathbf{E}}(x, y, z, t) = F(x, y) \cdot \tilde{E}(z, t) \mathbf{e}$$

$$\frac{\partial^2}{\partial z^2} \tilde{E}(z, t) = \frac{n^2}{c^2} \frac{\partial^2 \tilde{E}(z, t)}{\partial t^2}$$

The wave function has wave solution, therefore we can assume:

$$\tilde{E}(z, t) = A(z, t)e^{i(\beta_o z - \omega t)} + B(z, t)e^{-i(\beta_o z - \omega t)}$$

sum of forward and backward waves. The propagation constant at Bragg condition is:

$$\beta_o = k_o \bar{n} = \frac{2\pi}{\lambda_o} \bar{n} = \frac{\pi}{\lambda} = \frac{\omega_o \bar{n}}{c} \approx \frac{\omega_o n_o}{c}$$

Let assume slowly varying field in space and time:

$$\left| \frac{\partial A}{\partial z} \right| \Lambda \ll |A| \quad \text{and} \quad \left| \frac{\partial^2 A}{\partial z^2} \right| \ll \frac{1}{\Lambda} \left| \frac{\partial A}{\partial z} \right|$$

$$\left| \frac{\partial A}{\partial t} \right| \frac{1}{\nu_o} \ll |A| \quad \text{and} \quad \left| \frac{\partial^2 A}{\partial t^2} \right| \ll \omega_o \left| \frac{\partial A}{\partial t} \right|$$

We can now calculate  $\partial^2 E / \partial z^2$  and  $\partial^2 E / \partial t^2$  and plug them into the wave equation considering:

$$n^2(z) \approx n_o^2 + 2n_o \delta n \cos \left( \frac{2\pi}{\Lambda} z + \phi(z) \right) \cdot h(z)$$

where:

$$\beta_o = \pi / \Lambda \rightarrow \frac{2\pi}{\Lambda} = 2\beta_o$$

$$n^2(z) \approx n_o^2 + 2n_o \delta n \cos (2\beta_o z + \phi) \cdot h(z)$$

In exponential terms:

$$\cos (2\beta_o z + \phi) = \frac{1}{2} \left( e^{i(2\beta_o z + \phi)} + e^{-i(2\beta_o z + \phi)} \right)$$

and therefore:

$$n^2(z) \approx n_o^2 + n_o \delta n \left( e^{i(2\beta_o z + \phi)} + e^{-i(2\beta_o z + \phi)} \right) \cdot h(z)$$

Finally we can obtain:

$$\frac{\partial A}{\partial z} + \frac{n_o}{c} \frac{\partial A}{\partial t} = i \frac{\delta n}{2n_o} \beta_o h(z) e^{i\phi(z)} B$$

$$\frac{\partial B}{\partial z} - \frac{n_o}{c} \frac{\partial B}{\partial t} = i \frac{\delta n}{2n_o} \beta_o h(z) e^{-i\phi(z)} A$$

For just matter of simplicity we define:

$$q(z) = i \frac{\delta n}{2n_o} \beta_o h(z) e^{i\phi(z)}$$

and the **coupled equation** becomes:

$$\frac{\partial A}{\partial z} + \frac{n_o}{c} \frac{\partial A}{\partial t} = q(z) B$$

$$\frac{\partial B}{\partial z} - \frac{n_o}{c} \frac{\partial B}{\partial t} = q^*(z) A$$



and adding the gain term:

$$\frac{\partial A}{\partial z} + \frac{n_o}{c} \frac{\partial A}{\partial t} = q(z)B + \frac{g}{2}A$$

$$\frac{\partial B}{\partial z} - \frac{n_o}{c} \frac{\partial B}{\partial t} = q^*(z)A - \frac{g}{2}B$$

We now derive the threshold conditions (below which spontaneous emission, no stimulated) by performing a perturbative analysis of the coupled-mode equation. Let assume that the structure is not excited (no external signal):

$$A(0, t) = B(L, t) = 0$$

The solution are in the form:

$$A(z, t) = \tilde{A}(z)e^{at}$$

$$B(z, t) = \tilde{B}(z)e^{at}$$

and it is easy to derive:

$$\frac{\partial \tilde{A}}{\partial z} = iq(z)\tilde{B} + \frac{g}{2}\tilde{A} - \frac{n_o}{c}a\tilde{A}$$

$$\frac{\partial \tilde{B}}{\partial z} = \frac{n_o}{c}a\tilde{B} - iq^*(z)\tilde{A} - \frac{g}{2}\tilde{B}$$

The general solution is:

$$\tilde{A}(z) = M_{11}(z)\tilde{A}(0) + M_{12}(z)\tilde{B}(0)$$

$$\tilde{B}(z) = M_{21}(z)\tilde{A}(0) + M_{22}(z)\tilde{B}(0)$$

From our boundary conditions:

$$\tilde{A}(L) = M_{12}(L)\tilde{B}(0)$$

$$\tilde{B}(L) = M_{22}(L)\tilde{B}(0) = 0 \iff M_{22}(L) = 0$$

The parameter  $a$  is fixed for fixed gain  $g$  and the solution become:

$$A(z, t) = \tilde{A}e^{Re(a)t}e^{iIm(a)t}$$

Threshold is reached when  $Re(a) = 0$ . The corresponding gain is the threshold gain  $g_{th}$ :

$$a = Im(a) = -i\Omega = -i(\omega - \omega_o)$$

If we define  $\Delta\kappa$  as:

$$\Delta\kappa = \frac{n_o}{c}\Omega - i\frac{g}{2} = \frac{n_o}{c}(\omega - \omega_o) - i\frac{g}{2} = -i(\frac{g}{2} - i\Delta\beta)$$

We can rewrite the coupled equation as:

$$\frac{\partial \tilde{A}}{\partial z} = iq(z)\tilde{B} + i\Delta\kappa\tilde{A}$$

$$\frac{\partial \tilde{B}}{\partial z} = -i\Delta\kappa\tilde{B} - iq^*(z)\tilde{A}$$

We now make a modification in order to take into account also the confinement:

$$\Delta\kappa = \frac{n_o}{c}(\omega - \omega_o) - i\Gamma\frac{g}{2}$$

Consider the case in which a signal  $A$  is injected in  $z = 0$  but no signal  $B$  at  $z = L$  has been injected.

$$A(L) = M_{11}A(0) + M_{12}B(0)$$

$$B(L) = M_{21}A(0) + M_{22}B(0) = 0$$

We define the field transmission and reflection coefficient as:

$$r = \frac{B(0)}{A(0)} \quad t = \frac{A(L)}{B(0)}$$

and it is easy to demonstrate that:

$$r = -\frac{M_{21}(L)}{M_{22}(L)}$$

$$t = \frac{\det(M)}{M_{22}(L)}$$

Since the structure is reciprocal  $\det(M) = 1$ :

$$T = \frac{1}{|M_{22}(L)|^2}$$

At threshold we have seen that  $M_{22}(L) \rightarrow 0$  and the transmittance becomes infinity.

A particular case of laser is DBR (Distributed Bragg Reflector) lasers. Here the modulation of the refractive index is added using two Bragg reflectors at the extremities of the active region. Therefore the fabrication process is much cheaper but the frequency selectivity is decreased due to the presence of multiple longitudinal modes. We can investigate the performance of a DBRL using the coupled mode equations. The kappa parameter becomes:

$$\Delta\kappa = \Delta\beta + i\frac{\alpha}{2}$$

The field reflective coefficient becomes:

$$r_g = \frac{A(0)}{B(0)} = -\frac{M_{21}(L)}{M_{22}(L)}$$

The maximum of reflectivity is obtained for  $\beta = \beta_o$ .

## Quantum Dots

In order to success in describing the quantum dots electronic states, we first consider confinement along x-y-z direction, i.e. the potential is zero outside:

$$-\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\psi(x, y, z) = \varepsilon \cdot \psi(x, y, z)$$

with the boundary condition:

$$\psi(0, y, z) = \psi(L, y, z)$$

$$\psi(x, 0, z) = \psi(x, L, z)$$

$$\psi(x, y, 0) = \psi(x, y, L)$$

Since the Hamiltonian is the sum of separated variables, the solution is the product:

$$\psi(x, y, z) = X(x) \cdot Y(y) \cdot Z(z)$$

Let take this solution and take its second derivative. By dividing for  $\psi(x, y, z) = X(x) \cdot Y(y) \cdot Z(z)$  we obtain the following equation:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\varepsilon \cdot \frac{2m}{\hbar^2}$$

Sum of variational terms are constant if and only if each term is constant:

$$\frac{X''(x)}{X(x)} = -k_x^2 = \text{const} \quad \frac{Y''(y)}{Y(y)} = -k_y^2 = \text{const} \quad \frac{Z''(z)}{Z(z)} = -k_z^2 = \text{const}$$

It is the simple harmonic equation:

$$f''(x) = -k^2 f(x) \rightarrow f(x) = A \sin(kx)$$

and the separated solution are:

$$X(x) = \sqrt{\frac{2}{L}} \sin(k_x x)$$

$$Y(y) = \sqrt{\frac{2}{L}} \sin(k_y y)$$

$$Z(z) = \sqrt{\frac{2}{L}} \sin(k_z z)$$

The k-values are calculated using  $X(0) = X(L)$ :

$$X(0) = \sqrt{\frac{2}{L}} \sin(k_x 0) = \sqrt{\frac{2}{L}} \sin(k_x L) = 0 \iff k_x L = n_x \pi$$

We can conclude that:

$$k_i = n_i \frac{\pi}{L}$$

Since the eigenvalues depend on k-values:

$$k_x^2 + k_y^2 + k_z^2 = \frac{2m}{\hbar^2} \varepsilon$$

the epsilon becomes:

$$\varepsilon = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2} (n_x^2 + n_y^2 + n_z^2)$$

The lowest level is  $n_x^2 + n_y^2 + n_z^2 = 3$

Let consider now the case of spherical potential well. The solution can be written as:

$$\psi(\mathbf{r}) = R_{nl}(r) Y_{lm}(\theta)$$

The solution is expressed in term of spherical Bessel functions  $j_l(z)$ :

$$\psi(r, \theta, \phi) = \sqrt{\frac{2}{a^3}} \frac{1}{j_{l+1}(z_{nl})} j_l\left(\frac{r}{a} z_{nl}\right) Y_{ml}(\theta, \phi)$$

$$\varepsilon = \varepsilon_{nl} = z_{nl}^2 \frac{\hbar^2}{2ma^2} \quad (2l+1)\text{-fold degenerate}$$

where  $z_{nl}$  is the n-th zero of the  $j_{nl}$  function.

## References

- [1] Valentini Gianluca. *PHOTONICS, Part II, Notes for the MS Students in Engineering Physics at Politecnico di Milano*.
- [2] M. Nisoli. *Semiconductor photonics. Principles and Applications*. Società Editrice Esculapio, 2016.