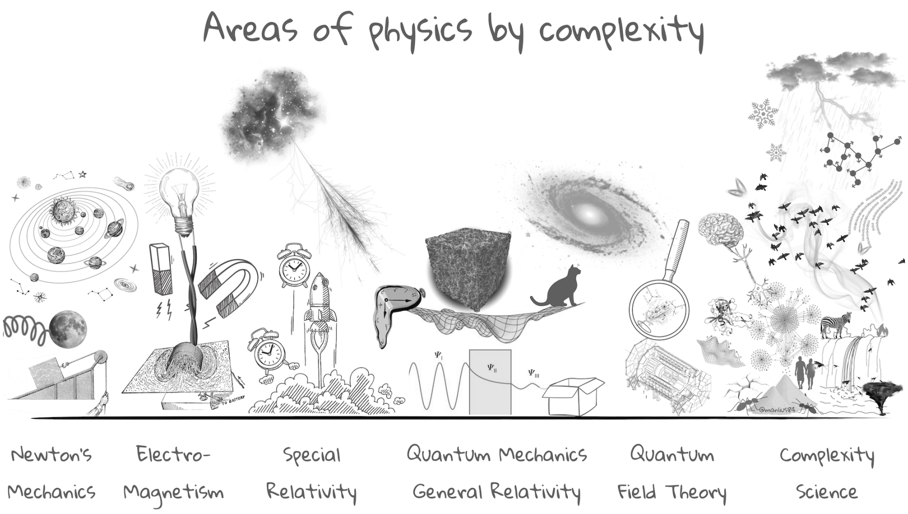
Final Report

Physics of Complex Networks: Structure and Dynamics

Università
degli Studi
di Padova



Project # X: ....

Surname, name

Last update: July 10, 2025

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1 | Explosive percolation

Task leader: *Lorenzo Rizzi*

1.1 | Percolation theory and phase transitions

Percolative processes were initially introduced on specific and regular networks (e.g. D-dimensional lattices) to model physical phenomena such as percolation of water in porous stones. However, nothing prevents us from applying the percolation's paradigm to arbitrary network topologies. Defining p as the probability that a randomly chosen edge (or node) is removed from the network, then a classical result from percolation theory is that, for certain networks topologies, a phase transition (PT) occurs as p is varied. If we define S as the probability that a randomly chosen node belongs to the GCC (our *order parameter*), then, for $N \to \infty$ (*thermodynamic limit*), we have $S = 0$ for $p < p_c$ and $S = S(p) \neq 0$ for $p > p_c$.

It is a standard result [cita!] that classical random percolation on networks displays a *continuous* phase transitions, meaning that, at criticality, $S(p_c) = 0$ and the order parameter has no discontinuous behaviour around the critical point. In addition, there are various other indicators of an incoming phase transition. We'll define n_s as the number of (finite) clusters of size s per node and χ the average cluster size:

$$\chi = \frac{\sum_s n_s s^2}{\sum_s n_s s} \tag{1.1}$$

In second-degree PT, at criticality χ is expected to diverge $\chi \sim |p - p_c|^{-\nu}$ and the cluster distribution n_s reduces to a power law $n_s \sim s^{-\tau}$.

However, in 2009, an inspiring article from Achlioptas et Al. [2] proposed a new type of percolation process that, allegedly, lead to a discontinuous type of transition (called *explosive percolation*). However, various posterior papers [cita!] have proven that PT to be a continuous one. In the present task, we are going to explore and simulate Achlioptas process(es) on various network topologies to recreate the explosive percolation behaviour.

1.2 | Achlioptas processes on random graphs

Classical percolation prescribe the removal (or addition) of randomly selected edges (or nodes). As mentioned in Par 1.1, this random procedure leads to a continuous phase transitions. However, one can modify the rule according to which edges are removed (or added) to obtain new type of percolation processes.

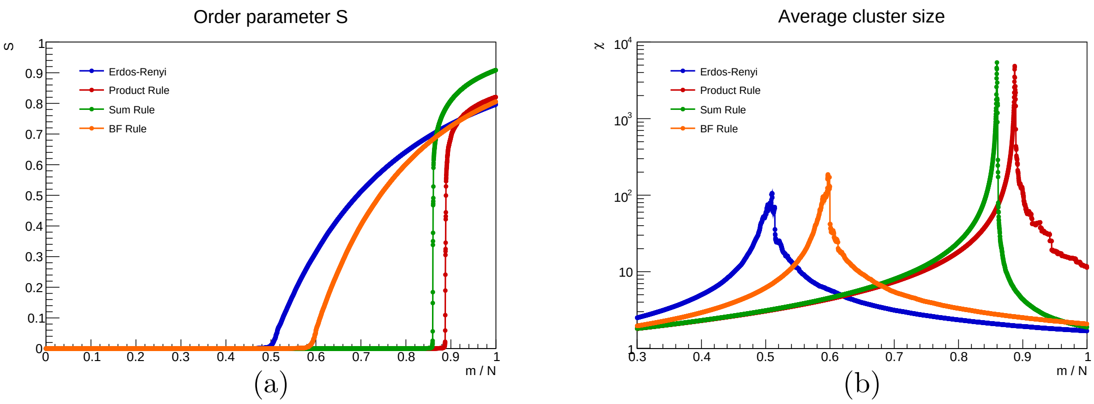
We will proceed as follows. We start with N nodes and no edge connectivity. At each step of our simulation, we are going to add a new link connecting two nodes in the graph. If those nodes are chosen at random, then we are basically performing classical (bond) percolation and building an ER network which is expected to generate a GCC when $\langle k \rangle = 1$, i.e. when the number of added edges m is $\frac{1}{2}N$. However, we can bias the choice of nodes to connect imposing new *selection rules*. In [2], two of such rules are presented: *Product Rule* (PR) and *Sum Rule* (SR). Their mechanism is quite easy: at each iteration, select two pairs of candidate nodes $(u_1, v_1), (u_2, v_2)$. Let U_i, V_i be the dimension of the cluster to which u_i, v_i belongs. Then, PR (SR) rule prescribe to select and connect the pair of nodes that minimizes the quantity $U_i \cdot V_i$ $(U_i + V_i)$, while discarding the other. The idea behind this selection rule is to postpone the formation of a GCC biasing the choice towards small clusters. PR and SR are examples of *unbounded rules*; at variance, *bounded rules* treats all cluster whose size is larger than K equally. When $K = 1$, we talk about the Bohman and Frieze (BF) rule: when possible, always select the edge that connects two isolated nodes.

As of now, we have 4 different protocols with which we can repeatedly add edges and grow a network (ER-like, PR, SR, BF). Simulations over 10 independent realizations when $N = 10^6$ are shown in Fig. 1.1a. As expected, ER growth (i.e. random rule selection) displays a continuous phase transition when $m/N = \frac{1}{2}\langle k \rangle = 0.5$. On the contrary, both PR and SR rules induces a particularly sharp transition called *explosive* because of its abrupt nature. In fact, it is so sharp that it may resembles a first-degree discontinuous PT. This behaviour is due to the fact that the competition mechanism (either SR or PR) tends to suppress the formation of large components, generating the necessary "powder keg" (i.e., abundant small-sized clusters, [**?**]) which then triggers the abrupt transition. In App. 1.4.1, we propose the approach performed in [2] to investigate the nature of this PT.

However, not all selection rule lead to an allegedly discontinuous PT: using BF procedure, a continuous transition is resumed. Along with the LCC, one can also study the behaviour of the average (finite) cluster size χ , defined as in Par.1.1 (Fig.1.1b). Simulations are run for a large but finite value of N, so we can't expect to see a divergence, but results are anyway convincing and show a marked peak in both ER and BF rules around their critical points. Surprisingly enough, PR and SR too display a similar divergent behaviour, which is not to be expected for first-degree PT. To characterize even further what is happening at criticality, we can study the cluster distribution n_s around m_c (discussed in greater details in App. 1.4.2). Again, one can easily show that when $m \approx m_c$, all 4 distributions converge to a power law, suggesting again a continuous-like behaviour for PR and SR rules too. As a matter of fact, explosive percolation was proven to be a continuous PT (cita!). Very steep, but still continuous.

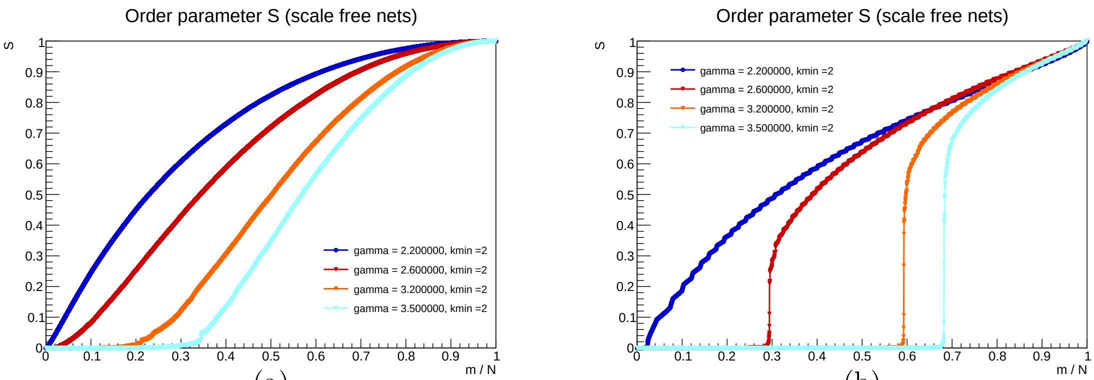
1.3 | Explosive percolation in SF networks

We now turn our attention to a different network topology, i.e. scale-free networks whose degree distribution $P(k) \sim k^{-\gamma}$, using a configuration model. We proceed as illustrated in [4]. Let N be the initial number of initially disconnected nodes. Sampling from a power law, we associate to each node a fixed amount of stubs and we gradually proceed to connect them following either a PR rule either a random choice (of course, when connecting stubs at random we fall back on the classical percolation problem on



(a) (b)

Figure 1.1: *(a) LCC size vs. added edges* m/N*,* $N = 10^6$*. Both ER and BF protocol give rise to a continuous phase transition, while competitive rule like PR and SR generates a seemingly looking discontinuous PT (b) Average cluster size vs.* m/N*,* $N = 10^6$*. As expected,* χ *peaks when* m/N *gets closer and closer to the critical point. Strangely enough, this is true for all generative mechanisms*



(a) (b)

Figure 1.2: *(a) LCC size vs. added edges* m/N*,* $N = 10^6$*, randomly connecting stubs until a power law is reached (classical percolation on SF networks)* (b) *LCC size vs. added edges* m/N*,* $N = 10^6$*, using PR rule to connect stubs. Averages on* 10 *independent realizations*

scale-free networks). Results for S are shown in Fig 1.2a, 1.2b. When performing a random connection between stubs (classical percolation on SF networks), one should expect to see a critical behaviour only when $\gamma > 3$, since $\kappa \to \infty$ for $\gamma < 3$, regardless of k_{min}. In fact, when $\gamma < 3$, the order parameter smoothly increases and is 0 only when $m = 0$. When using PR rule, we observe again the abrupt change when $m \approx m_c(\gamma)$

The most interesting result is, however, that for $\gamma = 2.6 < 3$ a phase transition is now observed. In fact, one can show [4], [3] that, when using PR, the threshold below which the GCC is always observed (thus no PT) is $\gamma = \gamma_c \approx 2.4$. In older papers, researcher pointed out that for $\gamma > 3$, the PT is of the first order (a proper explosive percolation) whereas when $\gamma_c < \gamma < 3$ it is continuous. And in fact, by visually inspecting Fig. 1.2b we can see why this was claimed (when $\gamma = 2.6$, the). However, as already mentioned, looks can be deceiving: even if highly abrupt, those PT are still of the second order.

1.4 | Supplementary material

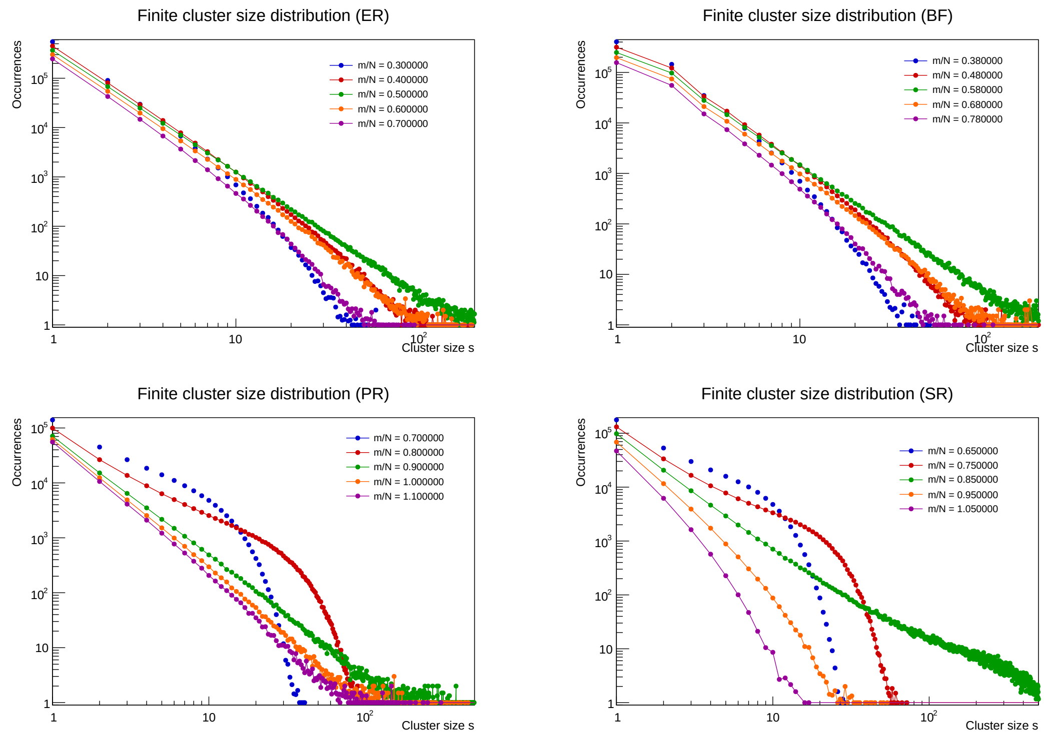


Figure 1.3: *Finite cluster size distribution at and around criticality. As expected, when* $m \approx m_c$*, the distribution appears approximately as a power law. Simulation run on networks with* $N = 10^6$*, over* 10 *independent realizations*

1.4.1 Scaling law for Δm

1.4.2 Cluster size distribution

Let us investigate further the behaviour of the cluster size distribution of continuous PTs. Defining n_s as the number of clusters of size s per node, then out of criticality n_s has exponentially distributed tails. When approaching the critical point, the distribution becomes a power law

$$n_s \sim s^{-\tau} \text{ for large } s$$

where τ is usually called the Fisher exponent. This power law is another strong indicator of the onset of a second order PT.

Thus, we can collect the cluster distribution n_s at different value of m/N (centered around the critical point). Results carried out for each percolative process as those presented in Par. 1.1 (Erdos-Renyi, Product Rule, Sum Rule, BF rule) are shown in Fig. 1.3. And indeed, when $m \sim m_c$, the distribution gets closer and closer to a power law (a straight line in a log-log plot) in all cases, providing more evidence that explosive percolation is a continuous PT.

2 | Task title...

Task leader: *Author name(s)...*

Structure as[1]*:*

- *A short (max 1 page) explanation of the task, including references. Include mathematical concepts.*
- *Max 2 pages for the whole task (including figures)*
- *It is possible to use appendices for supplementary material, at the end of the report. Max 5 pages per task*

A total of 3 pages + 5 supplementary pages per task

2.1 | A section...

Reference examples: book [**?**], article [**?**], website [1]

2.2 | Another section...

[1] Remove this part from the report

3 | Bibliography

[1] Protecting critical infrastructure from a cyber pandemic — weforum.org. https://www.weforum.org/agenda/2021/10/protecting-critical-infrastructure-from-cyber-pandemic/. [Accessed 29-Mar-2023].

[2] Dimitris Achlioptas, Raissa M. D'Souza, and Joel Spencer. Explosive percolation in random networks. *Science*, 323(5920):1453–1455, 2009. doi: 10.1126/science.1167782. URL https://www.science.org/doi/abs/10.1126/science.1167782.

[3] Y. S. Cho, J. S. Kim, J. Park, B. Kahng, and D. Kim. Percolation transitions in scale-free networks under the achlioptas process. *Physical Review Letters*, 103 (13), September 2009. ISSN 1079-7114. doi: 10.1103/physrevlett.103.135702. URL http://dx.doi.org/10.1103/PhysRevLett.103.135702.

[4] Filippo Radicchi and Santo Fortunato. Explosive percolation: A numerical analysis. *Phys. Rev. E*, 81:036110, Mar 2010. doi: 10.1103/PhysRevE.81.036110. URL https://link.aps.org/doi/10.1103/PhysRevE.81.036110.