$AdS_4 \times S^7$ metric and 4-form in compactified spherical coordinates $\left(r = \frac{\rho}{1-\rho},\ \text{i.e.}\ \rho = \frac{r}{1+r} \right)$

$AdS_4 \times S^7$ metric in UNCOMPACTIFIED coords

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right) dt^2 + \left(1+\frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

$$+ 4L^2 \left(d\chi^2 + \sin^2\chi \, d\Omega_6^2 \right)$$

4-form in UNCOMPACTIFIED coords

$$F_{(4)} = \frac{3}{L} \left(r^2 \, dt \wedge dr \wedge d\Omega_2 \right),$$

where $d\Omega_2 = \sin\theta \, d\theta \wedge d\varphi$.

We define a new compactified radial coordinate ρ given by $r = \frac{\rho}{1-\frac{\rho}{\ell}}$, $\ell = 1$.

So $$dr = \left(-\frac{\rho}{(1-\rho)^2}(-1) + \frac{1}{1-\rho} \right) d\rho =$$

$$= \frac{\rho + 1 - \rho}{(1-\rho)^2} d\rho = \frac{1}{(1-\rho)^2} d\rho .$$

So

$$ds^2 = -\left(1 + \frac{\rho^2}{L^2(1-\rho)^2}\right) dt^2 + \frac{(1-\rho)^2}{(1-\rho)^2 + \frac{\rho^2}{L^2}} \frac{d\rho^2}{(1-\rho)^4} + \frac{\rho^2}{(1-\rho)^2} d\Omega_2^2$$

$$+ 4L^2 d\Omega_7^2$$

$$= \frac{1}{(1-\rho)^2}\left(-f(\rho)\, dt^2 + \frac{1}{f(\rho)} d\rho^2 + \rho^2 d\Omega_2^2 \right) + 4L^2 d\Omega_7^2$$

where $f(\rho) = (1-\rho)^2 + \frac{\rho^2}{L^2}$.

If we set $L=1$, $f(\rho) = 1 - 2\rho + 2\rho^2$.

Now the 4-form:

$$F_{(4)} = \frac{3}{L}\frac{\rho^2}{(1-\rho)^4}\, dt\wedge d\rho \wedge d\Omega_2 .$$

$AdS_4\times S^7$ background values for asymptotically $AdS_4\times S^7$ solutions preserving an S^2 in AdS_4 and an S^6 in S^7

We write

$$ds^2 = g^{(3)}_{\alpha\beta}dx^\alpha dx^\beta + g_A\, d\Omega_2^2 + g_B\, d\Omega_6^2 ,$$

where $g^{(3)}_{\alpha\beta}$, g_A, g_B depend on $x^\alpha = \{t,\rho,\chi\}$.

and

$$F_{(4)} = f_2 \wedge d\Omega_2$$

where $f_2 = f_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$.

$f_{\alpha\beta}$ also depend on x^α.

The pure $AdS_4 \times S^7$ values of the functions are

$$g^{(3)} = \frac{1}{(1-\rho)^2} \begin{pmatrix} -f(\rho) & 0 & 0 \\ 0 & \frac{1}{f(\rho)} & 0 \\ 0 & 0 & 4L^2(1-\rho)^2 \end{pmatrix}$$

$$g_A = \frac{\rho^2}{(1-\rho)^2}, \quad g_B = 4L^2 \sin^2\chi,$$

$$f_{t\rho} = \frac{3}{L}\frac{\rho^2}{(1-\rho)^4}, \quad f_{t\chi} = 0, \quad f_{\rho\chi} = 0.$$

Now we define

$$f_1 = *_3 f_2 \implies f_{1\alpha} = \frac{1}{2}\varepsilon^{(3)}{}_{\alpha\beta\gamma} f^{\beta\gamma},$$

where $\varepsilon^{(3)}$ is the volume form associated with $g^{(3)}{}_{\alpha\beta}$.

For pure $AdS_4 \times S^7$, we get

$$F_{1t} = \frac{1}{2}\left(\sqrt{-g^{(3)}}\, F^{\rho x} - \sqrt{-g^{(3)}}\, F^{x\rho}\right) = 0$$

$$F_{1\rho} = \frac{1}{2}\left(-\sqrt{-g^{(3)}}\, F^{tx} + \sqrt{-g^{(3)}}\, F^{xt}\right) = 0$$

$$F_{1x} = \frac{1}{2}\left(\sqrt{-g^{(3)}}\, F^{t\rho} - \sqrt{-g^{(3)}}\, F^{\rho t}\right) = \sqrt{-g^{(3)}}\, F^{t\rho}$$

Now $g^{(3)} = -\frac{4L^2}{(1-\rho)^4} \Rightarrow \sqrt{-g^{(3)}} = \frac{2L}{(1-\rho)^2}$

and $F^{t\rho} = g^{(3)\,tt} g^{(3)\,\rho\rho} F_{t\rho} = -\frac{(1-\rho)^2}{f(\rho)} f(\rho)(1-\rho)^2 \frac{3}{L} \frac{\rho^2}{(1-\rho)^4} =$

$$= -\frac{3}{L}\rho^2 .$$

So $F_{1x} = -\frac{6\rho^2}{(1-\rho)^2}$.

We then define ϕ such that

$$F_{1\alpha} = g_A\, g_B^{-3}\, \partial_\alpha \phi .$$

So

$$\partial_t \phi = 0$$

$$\partial_\rho \phi = 0$$

$$\partial_x \phi = f_{1x}\, g_A^{-1}\, g_B^{3} = -\frac{6\rho^2}{(1-\rho)^2}\,\frac{(1-\rho)^2}{\rho^2}\, 4^3 L^6 \sin^6 x$$

$\Rightarrow$ ϕ can be picked to be

$$\phi = 2L^6\left(-60x + 45\sin 2x - 9\sin 4x + \sin 6x\right).$$

Schwarzschild-AdS$_4 \times S^7$ metric and 4-form in compactified spherical coordinates $\left(r = \frac{\rho}{1-\rho}, \text{ i.e. } \rho = \frac{r}{1+r} \right)$

Schw-AdS$_4 \times S^7$ metric in UNCOMPACTIFIED coords

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\Omega_2^2$$

$$+ 4L^2 \left(d\chi^2 + \sin^2\chi\, d\Omega_6^2 \right),$$

where $f(r) = 1 + \frac{r^2}{L^2} - \frac{r_+}{r}\left(1 + \frac{r_+^2}{L^2}\right)$.

4-form in UNCOMPACTIFIED coords

$$F_{(4)} = \frac{3}{L}\left(r^2\, dt \wedge dr \wedge d\Omega_2 \right),$$

where $d\Omega_2 = \sin\theta\, d\theta \wedge d\varphi$.

We define a new compactified radial coordinate ρ given by $r = \frac{\rho}{1-\frac{\rho}{\ell}}$, $\ell = 1$.

We have

$$f\left(r=\frac{\rho}{1-\rho}\right)=\frac{1}{(1-\rho)^2}\left((1-\rho)^2+\frac{\rho^2}{L^2}\right)-\frac{r_+(1-\rho)}{\rho}\left(1+\frac{r_+^2}{L^2}\right)=$$

$$=\frac{1}{(1-\rho)^2}\left[(1-\rho)^2+\frac{\rho^2}{L^2}-\frac{(1-\rho)^3}{\rho}r_+\left(1+\frac{r_+^2}{L^2}\right)\right]$$

$$=\frac{1}{(1-\rho)^2}\hat{f}_{r_+}(\rho)\text{, where}$$

$$\hat{f}_{r_+}(\rho)=(1-\rho)^2+\frac{\rho^2}{L^2}-\frac{(1-\rho)^3}{\rho}r_+\left(1+\frac{r_+^2}{L^2}\right).$$

Since $dr=\frac{1}{(1-\rho)^2}d\rho$, we have

$$\frac{1}{f}dr^2=\frac{(1-\rho)^2}{\hat{f}_{r_+}}\frac{d\rho^2}{(1-\rho)^4}=\frac{1}{(1-\rho)^2}\frac{1}{\hat{f}_{r_+}}d\rho^2.$$

So, the Schw-$AdS_4\times S^7$ metric in compactified spherical coords is

$$ds^2=\frac{1}{(1-\rho)^2}\left(-\hat{f}_{r_+}dt^2+\frac{1}{\hat{f}_{r_+}}d\rho^2+\rho^2d\Omega_2^2\right)+$$
$$+4L^2\left(d\chi^2+\sin^2\chi\, d\Omega_6^2\right).$$