EFE for asymptotically $AdS_5 \times S^5$ spacetimes without background terms (which must cancel)

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$g^{(3)}_{\mu\nu}$ - EFE:

$$\text{term}1_{\mu\nu} = -\frac{1}{2} g^{(3)\rho\sigma} \partial_\rho \partial_\sigma g^{(3)}{}_{\mu\nu} \xrightarrow{\text{only non-background terms}}$$

$$\to -\frac{1}{2} \hat{g}^{(3)\rho\sigma} \partial_\rho \partial_\sigma h_{\mu\nu} - \frac{1}{2} h^{\rho\sigma} \partial_\rho \partial_\sigma \hat{g}^{(3)}{}_{\mu\nu} - \frac{1}{2} h^{\rho\sigma} \partial_\rho \partial_\sigma h_{\mu\nu},$$

where $h^{\mu\nu} \equiv g^{(3)\mu\nu} - \hat{g}^{(3)\mu\nu}$.

$$\text{term}2_{\mu\nu} = -\frac{1}{2} \partial_\rho g^{(3)}{}_{\sigma\mu} \partial_\nu g^{(3)\rho\sigma} \longrightarrow$$

$$\to -\frac{1}{2} \partial_\rho \hat{g}^{(3)}{}_{\sigma\mu} \partial_\nu h^{\rho\sigma} - \frac{1}{2} \partial_\rho h_{\sigma\mu} \partial_\nu \hat{g}^{(3)\rho\sigma} - \frac{1}{2} \left( \partial_\rho h_{\sigma\mu} \right) \left( \partial_\nu h^{\rho\sigma} \right)$$

$$\text{term}3_{\mu\nu} = \text{term}2_{\nu\mu}$$

$$\text{term}4_{\mu\nu} = -\frac{1}{2} \partial_\mu H_\nu \longrightarrow -\frac{1}{2} \partial_\mu A_\nu$$

$$\text{term}5_{\mu\nu} = \text{term}4_{\nu\mu}$$

$$\mathrm{term6}_{\mu\nu} = H_\rho \Gamma^\rho{}_{\mu\nu} \longrightarrow A_\rho \Gamma_{\hat{g}\hat{g}}{}^\rho{}_{\mu\nu} + \hat{H}_\rho \Gamma_{h\hat{g}}{}^\rho{}_{\mu\nu} +$$

$$+ \hat{H}_\rho \Gamma_{\hat{g}h}{}^\rho{}_{\mu\nu} + \hat{H}_\rho \Gamma_{hh}{}^\rho{}_{\mu\nu}$$

$$\mathrm{term7}_{\mu\nu} = -\Gamma^\rho{}_{\mu b} \Gamma^b{}_{\nu\rho} \longrightarrow$$

$$\longrightarrow -\Gamma_{\hat{g}\hat{g}}{}^\rho{}_{\mu\nu} \Gamma_{h\hat{g}}{}^b{}_{\nu\rho} - \Gamma_{\hat{g}\hat{g}}{}^\rho{}_{\mu\nu} \Gamma_{\hat{g}h}{}^b{}_{\nu\rho} - \Gamma_{\hat{g}\hat{g}}{}^\rho{}_{\mu\nu} \Gamma_{hh}{}^b{}_{\nu\rho}$$

$$-\Gamma_{h\hat{g}}{}^\rho{}_{\mu\nu} \Gamma_{\hat{g}\hat{g}}{}^b{}_{\nu\rho} - \Gamma_{h\hat{g}}{}^\rho{}_{\mu\nu} \Gamma_{h\hat{g}}{}^b{}_{\nu\rho} - \Gamma_{h\hat{g}}{}^\rho{}_{\mu\nu} \Gamma_{\hat{g}h}{}^b{}_{\nu\rho} - \Gamma_{h\hat{g}}{}^\rho{}_{\mu\nu} \Gamma_{hh}{}^b{}_{\nu\rho}$$

$$-\Gamma_{\hat{g}h}{}^\rho{}_{\mu\nu} \Gamma_{\hat{g}\hat{g}}{}^b{}_{\nu\rho} - \Gamma_{\hat{g}h}{}^\rho{}_{\mu\nu} \Gamma_{h\hat{g}}{}^b{}_{\nu\rho} - \Gamma_{\hat{g}h}{}^\rho{}_{\mu\nu} \Gamma_{\hat{g}h}{}^b{}_{\nu\rho} - \Gamma_{\hat{g}h}{}^\rho{}_{\mu\nu} \Gamma_{hh}{}^b{}_{\nu\rho}$$

$$-\Gamma_{hh}{}^\rho{}_{\mu\nu} \Gamma_{\hat{g}\hat{g}}{}^b{}_{\nu\rho} - \Gamma_{hh}{}^\rho{}_{\mu\nu} \Gamma_{h\hat{g}}{}^b{}_{\nu\rho} - \Gamma_{hh}{}^\rho{}_{\mu\nu} \Gamma_{\hat{g}h}{}^b{}_{\nu\rho} - \Gamma_{hh}{}^\rho{}_{\mu\nu} \Gamma_{hh}{}^b{}_{\nu\rho}$$

$$\mathrm{term8}_{\mu\nu} = -\frac{n_A}{4g_A^2} \left(\nabla_\mu g_A\right)\left(\nabla_\nu g_A\right) \longrightarrow$$

$$\longrightarrow -\frac{n_A}{4g_A^2}$$

$\phi$ equation of motion :

$$g^{(3)\mu\nu}\partial_\mu\partial_\nu\phi + H_\mu g^{(3)\mu\nu}\nabla_\nu\phi - \frac{(n_A-3)}{2g_A}g^{(3)\mu\nu}(\nabla_\mu g_A)(\nabla_\nu\phi)$$

$$-\frac{(n_B+4)}{2g_B}g^{(3)\mu\nu}(\nabla_\mu g_B)(\nabla_\nu\phi)=0$$