4-form components for bosonic 11D supergravity in asymptotically $AdS_4 \times S^7$ spacetimes, preserving an S^2 in AdS_4 and an S^6 in S^7, encoded in a scalar field.

ANSATZ:

$$F_{(4)} = f_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \wedge d\Omega_{(A)} =$$

$$= f_{(2)} \wedge \mathrm{vol}(S^2),$$

where $f_{(2)} = f_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$, $f_{\alpha\beta} = -f_{\beta\alpha}$ and depends on $x^\alpha = \{t, r, x\}$, and $d\Omega_{(A)} = \sin\theta\, d\theta \wedge d\varphi$.

So the only non-vanishing components of $F_{(4)}$ are $F_{(4)\,\alpha\beta\theta\varphi} = f_{\alpha\beta} \sin\theta$, and those related to this by anti-symmetry.

$F_{(4)}$ must satisfy the Bianchi identity

$dF = 0$ and the E.O.M. $d{*F} = 0$.

From $dF=0$, we have

$$0 = df_{(2)} \wedge d\Omega_{(A)} \Rightarrow df_{(2)} = 0.$$

$$d(d\Omega_{(A)}) = d(\sin\theta \, d\theta \wedge d\varphi) = 0$$

Let's define f_1 by

$$f_2 = -*_3 f_1 \quad \left(f_{\alpha\beta} = -\varepsilon_{\alpha\beta\gamma} f_1^{\gamma}\right)$$

where $*_3$ is the Hodge dual associated with the volume form of the 3-metric

$$ds^2_{(3)} = g^{(3)}_{\alpha\beta} dx^{\alpha} dx^{\beta}.$$

We have $-*_3 f_2 = *_3 *_3 f_1$.

Since $*_3 *_3 f_1 = -(-1)^2 f_1 = -f_1$,

we get $f_1 = *_3 f_2$ $\left(\text{i.e. } f_{1\alpha} = \frac{1}{2} \varepsilon_{\alpha\beta\gamma} f^{\beta\gamma}\right)$.

Then $df_2 = 0$ gives $d(*_3 f_1) = 0$

$$\Rightarrow \quad *_3 d(*_3 f_1) = 0 \quad \Rightarrow \quad \nabla^{(3)\alpha} f_{1\alpha} = 0, \text{ i.e.}$$

$f_{1\alpha}$ is divergenceless w.r.t. the Levi-Civita connection associated with $g^{(3)}{}_{\alpha\beta}$.

Let's now consider $d * F_{(4)} = 0$

$$\Rightarrow \quad * d * F_{(4)} = 0 \quad \Rightarrow \quad \nabla^M F_{(4)NPQM} = 0$$

$$\Rightarrow \nabla^M F_{(4)MNPQ} = 0 \quad \Rightarrow \nabla_M F_{(4)}{}^{MNPQ} = 0$$

$$\Rightarrow \frac{1}{\sqrt{-g}} \partial_M \left(\sqrt{-g}\, F_{(4)}{}^{MNPQ} \right) = 0 \quad \Rightarrow \quad \partial_M \left(\sqrt{-g}\, F_{(4)}{}^{MNPQ} \right) = 0$$

Let's consider the $(N,P,Q) = (\alpha, \theta, \varphi)$ components:

$$\partial_M \left(\sqrt{-g}\, F_{(4)}{}^{M\alpha\theta\varphi}\right) = 0 \implies$$

$$\implies \partial_M \left(\sqrt{-g}\, g^{MN} g^{\alpha\beta} g^{\theta\theta} g^{\varphi\varphi} F_{N\beta\theta\varphi}\right) = 0$$

$$\implies \partial_M \left(\sqrt{-g}\, g^{MN} g^{(3)\alpha\beta} \frac{1}{g_A} b^{(A)\theta\theta} \frac{1}{g_A} b^{(A)\varphi\varphi} F_{(4)N\beta\theta\varphi}\right) = 0$$

$$\implies \partial_\gamma \left(\sqrt{-g}\, g^{(3)\gamma\delta} g^{(3)\alpha\beta} \frac{1}{g_A{}^2} \frac{1}{\sin^2\theta} F_{(4)\delta\beta\theta\varphi}\right) = 0 .$$

Now $\sqrt{-g} = \sqrt{-g^{(3)}}\, g_A \sqrt{\det b^{(A)}}\, g_B^3 \sqrt{\det b^{(B)}}$,

and $F_{(4)\delta\beta\theta\varphi} = f_{\delta\beta} \sin\theta$

Since $\sqrt{\det b^{(A)}}$ and $\sqrt{\det b^{(B)}}$ do not depend on the coords $x^\gamma = \{t, r, z\}$, we get

$$\partial_\gamma\left(\sqrt{-g^{(3)}}\; g_A\, g_B^3\, \frac{1}{g_A^2}\, g^{(3)\gamma b}\, g^{(3)\alpha\beta}\, f_{b\beta}\right) = 0$$

$$\Rightarrow \partial_\gamma\left(\sqrt{-g^{(3)}}\; g_A^{-1} g_B^3\, f^{\gamma\alpha}\right) = 0,$$

i.e. $*_3 d *_3 \left(g_A^{-1} g_B^3\, f_2\right) = 0$

Since $*_3 f_2 = f_1,$

we get $*_3 d\left(g_A^{-1} g_B^3 f_1\right) = 0 \Rightarrow d\left(g_A^{-1} g_B^3 f_1\right) = 0$

$\Rightarrow g_A^{-1} g_B^3\, f_1 = d\phi$ locally,

i.e. $f_{1\alpha} = g_A\, g_B^{-3}\, \partial_\alpha \phi$.

The expression

$$*_3 d(*_3 F_1) = 0 \quad , \text{i.e.} \ \nabla^{(3)\alpha} F_{1\alpha} = 0,$$

gives

$$*_3 d *_3 g_A g_B^{-3} d\phi = 0, \text{ i.e. } \nabla^{(3)\alpha}\left(g_A g_B^{-3} \nabla^{(3)}_\alpha \phi\right) = 0$$

or $\nabla^{(3)}{}_\alpha \nabla^\alpha \phi + \frac{1}{g_A} \nabla^{(3)\alpha} g_A \nabla^{(3)}_\alpha \phi - \frac{3}{g_B} \nabla^{(3)\alpha} g_B \nabla_\alpha \phi = 0$

Let's rewrite this in terms of partial derivatives:

$$\nabla^{(3)}{}_\alpha \nabla^{(3)\alpha} \phi = \frac{1}{\sqrt{-g^{(3)}}} \partial_\alpha \left(\sqrt{-g^{(3)}} g^{(3)\alpha\beta} \partial_\beta \phi \right) =$$

$$= \frac{1}{\sqrt{-g^{(3)}}} \partial_\alpha \left(\sqrt{-g^{(3)}} g^{(3)\alpha\beta} \right) \partial_\beta \phi + g^{(3)\alpha\beta} \partial_\alpha \partial_\beta \phi .$$

Since $H^{(3)\beta} = \Box^{(3)} x^\beta = \frac{1}{\sqrt{-g^{3}}} \partial_\alpha \left(\sqrt{-g^{3}} g^{(3)\alpha\beta} \right),$

we get

$$\nabla^{(3)}{}_\alpha \nabla^{(3)\alpha} \phi = g^{(3)\alpha\beta} \partial_\alpha \partial_\beta \phi + H^{(3)\beta} \partial_\beta \phi .$$

Therefore, the condition on ϕ reads

$$g^{(3)\alpha\beta}\partial_\alpha\partial_\beta\phi + H^{(3)\beta}\partial_\beta\phi + \frac{1}{g_A}g^{(3)\alpha\beta}\partial_\alpha g_A\partial_\beta\phi - \frac{3}{g_B}g^{(3)\alpha\beta}\partial_\alpha g_B\partial_\beta\phi = 0$$

Using

$$H_\alpha = H^{(3)}_\alpha + \frac{n_A}{2g_A}\partial_\alpha g_A + \frac{n_B}{2g_B}\partial_\alpha g_B,$$

we finally get

$$g^{(3)\alpha\beta}\left(\partial_\alpha\partial_\beta\phi + H_\alpha\partial_\beta\phi - \frac{(n_A-2)}{2g_A}\partial_\alpha g_A\partial_\beta\phi - \frac{(n_B+6)}{2g_B}\partial_\alpha g_B\partial_\beta\phi\right) = 0.$$