Kerr-Schild coordinates for Schwarzschild-AdS in 4 dimensions

Set the AdS radius to $L=1$.

Consider the Schwarzschild-AdS (SAdS) metric in usual Schwarzschild (S) coords (t, r, θ, φ):

$$ds^2 = -f\,dt^2 + \frac{1}{f}\,dr^2 + r^2 d\Omega^2$$

where $f = 1 + r^2 - \frac{r_0}{r}$ and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$.

The event horizon (EH) is at $r = r_H$ where r_H is the real solution of $f(r_H) = 0$ i.e. $r_H(1+r_H^2) = r_0$.

Kerr-Schild (KS) coords are $(\tau, r, \theta, \varphi)$ where

$$d\tau = dt + \frac{r_0}{r}\,\frac{1}{(1+r^2)f}\,dr\,.$$

Notice that these are NOT ingoing Eddington-Finkelstein (in EF) coords (v, r, θ, φ).

In fact, inEF coords are defined by

$dv = dt + dr^*$ where the Tortoise coordinate r^* is defined by requiring that

$$-f\,dt^2+\frac{1}{f}dr^2 = f\left(-dt^2+dr^{*2}\right)$$

$$\Rightarrow \quad \frac{1}{f}dr^{*} = f\,dr^{*2} \quad \Rightarrow \quad dr^{*} = \frac{dr}{f}.$$

So $dv = dt + \frac{dr}{f}$, which is clearly different from $d\tau$.

SAdS in KS coords is given by

$$ds^2 = -f\,d\tau^2 + 2\frac{r_0}{r}\frac{1}{(1+r^2)f}\,d\tau\,dr$$

$$-\left(\frac{r_0}{r}\right)^2\frac{1}{(1+r^2)^2 f}dr^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2$$

$$= -f\,d\tau^2 + \frac{2r_0}{r}\frac{1}{(1+r^2)f}\,d\tau\,dr + \frac{1}{f}\left(1-\left(\frac{r_0}{r(1+r^2)}\right)^2\right)dr^2$$

$$+r^2 d\Omega^2$$

Notice that the g_{rr} component is no longer singular at $r=r_H$, hence KS coords are horizon-penetrating (also inEF are). In fact, $f = A(r-r_H)$ for some finite $A \neq 0$.

So, for $r = r_H + \varepsilon$, we get $f = A\varepsilon$. Then in the small ε limit

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$$\frac{r_0}{r(1+r^2)} \simeq \frac{r_0}{(r_H+\varepsilon)(1+r_H^2+2\varepsilon r_H+O(\varepsilon^2))} \simeq \frac{r_0}{r_H(1+r_H^2)+B\varepsilon+O(\varepsilon^2)}$$

ignoring $O(\varepsilon^2)$ terms

for some finite $B \neq 0$

$$\simeq \frac{r_0}{r_H(1+r_H^2)}\left(1-\frac{B}{r_H(1+r_H^2)}\varepsilon+O(\varepsilon^2)\right)$$

$$= \frac{r_0}{r_H(1+r_H^2)} + C\varepsilon + O(\varepsilon^2) \quad \text{for some finite } C \neq 0.$$

Since $r_H(1+r_H^2) = r_0$, then we get $1 + C\varepsilon + O(\varepsilon^2)$.

So

$$1-\left(\frac{r_0}{r(1+r^2)}\right)^2 \simeq 1-\left(1+C\varepsilon+O(\varepsilon^2)\right)^2$$

$$\simeq 1-\left(1+2C\varepsilon+O(\varepsilon^2)\right) = -2C\varepsilon+O(\varepsilon^2)$$

Hence, in the small ε limit,

$$g_{rr} \simeq \frac{-2C\varepsilon}{A\varepsilon} \longrightarrow -\frac{2C}{A} \quad \text{which is non } 0.$$

To see this even more explicitly, we can rewrite

$$g_{rr} = \frac{1}{f}\left(1 - \left(\frac{r_0}{r(1+r^2)}\right)^2\right) = \frac{1}{1+r^2-\frac{r_0}{r}}\left(\frac{r^2(1+r^2)^2 - r_0^2}{r^2(1+r^2)^2}\right) =$$

$$= \frac{r}{r(1+r^2)-r_0}\left(\frac{r^2(1+r^2)^2-r_0^2}{r^2(1+r^2)^2}\right) = \frac{r(1+r^2)+r_0}{r(1+r^2)^2} =$$

$$= \frac{1+r^2+\frac{r_0}{r}}{(1+r^2)^2},$$ which is non-vanishing at r_H.

In summary, we found that the SAdS metric in KS coords can be written as

$$ds^2 = -f d\tau^2 + 2\frac{r_0}{r(1+r^2)} d\tau dr + \frac{1}{(1+r^2)^2}\left(1+r^2+\frac{r_0}{r}\right)dr^2 + r^2 d\Omega^2$$

Now we define the compactified coordinate $\rho \in (0,1)$ via $r = \frac{2\rho}{1-\rho^2}$.

$$dr = \left(\frac{4\rho^2}{(1-\rho^2)^2} + \frac{2}{1-\rho^2} \right) d\rho .$$

In $(\tau, \rho, \theta, \varphi)$ ~~coords~~,

$$f = 1 + \frac{4\rho^2}{(1-\rho^2)^2} - \frac{r_0 (1-\rho^2)}{2\rho} = \frac{1}{(1-\rho^2)^2} \left((1-\rho^2)^2 + 4\rho^2 - \frac{r_0}{2\rho} (1-\rho^2)^3 \right)$$

Since $\hat{f}(\rho) := (1-\rho^2)^2 + 4\rho^2 = (1+\rho^2)^2$, we get

$$f = \frac{(1+\rho^2)^2}{(1-\rho^2)^2} \left(1 - \chi(\rho) \right), \text{ where } \chi(\rho) := \frac{r_0}{2\rho} \frac{(1-\rho^2)^3}{(1+\rho^2)^2} .$$

So $$g_{\tau\tau} = \frac{1}{(1-\rho^2)^2} \left[-(1+\rho^2)^2 (1 - \chi(\rho)) \right] .$$

Then

$$g_{\tau\rho} = \frac{r_0}{2\rho} (1-\rho^2) \frac{1}{1 + \frac{4\rho^2}{(1-\rho^2)^2}} \left(\frac{4\rho^2}{(1-\rho^2)^2} + \frac{2}{1-\rho^2} \right) =$$

$$= \frac{r_0}{2\rho} \frac{(1-\rho^2)^3}{(1+\rho^2)^2} \frac{1}{(1-\rho^2)^2} \left(2(1+\rho^2) \right) = \frac{2(1+\rho^2)}{(1-\rho^2)^2} \chi(\rho)$$

and

$$g_{\rho\rho} = \frac{1 + \frac{4\rho^2}{(1-\rho^2)^2} + \frac{r_0}{2\rho}(1-\rho^2)}{\left(1 + \frac{4\rho^2}{(1-\rho^2)^2}\right)^2} \cdot \left(\frac{4\rho^2}{(1-\rho^2)^2} + \frac{2}{1-\rho^2}\right)^2 =$$

$$= \frac{(1+\rho^2)^2 + \frac{r_0}{2\rho}(1-\rho^2)^3}{(1+\rho^2)^4} (1-\rho^2)^2 \left(\frac{2(1+\rho^2)}{(1-\rho^2)^2}\right)^2 =$$

$$= \frac{1}{(1-\rho^2)^2} \frac{4}{(1+\rho^2)^2} \left((1+\rho^2)^2 + \frac{r_0}{2\rho}(1-\rho^2)^3\right) = \frac{1}{(1-\rho^2)^2} 4\left(1 + \chi(\rho)\right)$$

Hence, the SAdS metric in compactified KS coords $(\tau, \rho, \theta, \varphi)$ as

$$ds^2 = \frac{1}{(1-\rho^2)^2}\left\{-(1+\rho^2)^2(1-\chi(\rho))\,d\tau^2 + 4(1+\rho^2)\chi(\rho)\,d\tau\,d\rho + 4(1+\chi(\rho))\,d\rho^2 + 4\rho^2\,d\Omega^2\right\}$$

with $\chi(\rho) = \frac{r_0}{2\rho}\frac{(1-\rho^2)^3}{(1+\rho^2)^2}$.

Notice that when $r_0 = 0 \Rightarrow \chi(\rho) = 0$ and we recover the pure AdS metric in the form of eq. II.5 of arXiv: 2011.12970v2, as expected.

Finally we define KS compactified Cartesian coordinates (τ, x, y, z) where

$$x = \rho \cos\theta$$
$$y = \rho \sin\theta \cos\varphi$$
$$z = \rho \sin\theta \sin\varphi .$$

We have $\rho^2 = x^2 + y^2 + z^2 \Rightarrow d\rho = \frac{1}{\rho}(x\,dx + y\,dy + z\,dz)$.

The SAdS metric in these coords reads (with Mathematica)

$$ds^2 = \frac{1}{(1-\rho^2)^2}\Big\{ -(1+\rho^2)^2(1-\chi(\rho))\,d\tau^2 + 4\frac{(1+\rho^2)}{\rho}\chi(\rho)\,d\tau(x\,dx + y\,dy + z\,dz)$$

$$+ 4\Big[\Big(1+\chi(\rho)\frac{x^2}{\rho^2}\Big)dx^2 + \Big(1+\chi(\rho)\frac{y^2}{\rho^2}\Big)dy^2 + \Big(1+\chi(\rho)\frac{z^2}{\rho^2}\Big)dz^2 +$$

$$+ \frac{2}{\rho^2}\chi(\rho)\,(xy\,dx\,dy + xz\,dx\,dz + yz\,dy\,dz)\Big]\Big\}$$