The metric of the black string in AdS_5 with the horizon of Schwarzschild-AdS_4 is given in arxiv:0104213

$$ds^2 = \frac{1}{H(z)^2}\left(dz^2 + \gamma_{\mu\nu}\, dx^\mu dx^\nu\right)$$

where $\gamma = \gamma_{\mu\nu}\, dx^\mu dx^\nu$ is the Schwarzschild-AdS_4 metric, $\gamma = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$, $f(r) = 1 + \frac{r^2}{\ell_4^2} - \frac{r_0}{r}$, and the warping factor $H(z)$ is

$$H(z) = \frac{\ell_4}{L}\sin\left(\frac{z}{\ell_4}\right). \qquad z \in (0, \ell_4 \pi).$$

The value of ℓ_4 is arbitrary (the dependence on ℓ_4 can be made disappear by a rescaling of coordinates $z \to \ell_4 z$, $t \to \ell_4 t$, $r \to \ell_4 t$ (so $r_0 \to \ell_4 r_0$), as shown explicitly in a Mathematica notebook). So we pick $\ell_4 = 1$.

L is the AdS_5 radius.

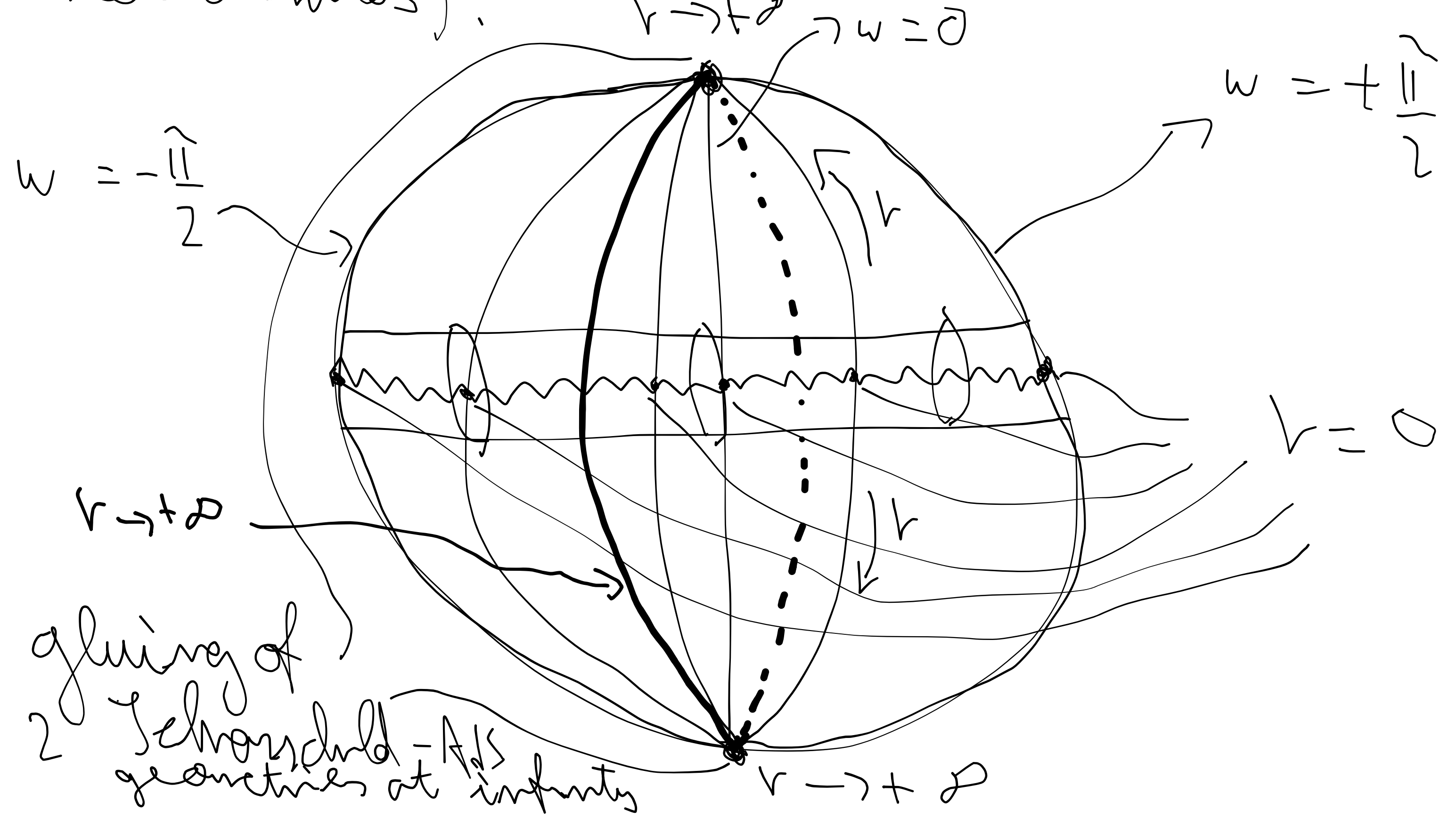
If one now define a new w coordinate via

$$w = z - \frac{\hat{\pi}}{2} \in \left(-\frac{\hat{\pi}}{2}, +\frac{\hat{\pi}}{2}\right),$$

we get

$$ds^2 = \frac{L^2}{\cos w^2}\left(dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu\right).$$

This is the metric of the black string in the form given by eq. (1) of arxiv: 2112.07967 (to obtain the very same expression, one also needs to write γ in ingoing Eddington-Finkelstein coordinates).



This Schwarzschild-AdS black string in AdS (also called uniform AdS black string in AdS) is an asymptotically locally AdS spacetime with conformal boundary, $\rho = 1$, given by gluing two Schwarzschild-AdS$_4$ at their conformal boundaries, $r \to +\infty$.

If we now define

$\tilde{z} = e^{2\,\mathrm{arctanh}\left(\tan\left(\frac{w}{2}\right)\right)}$, which goes from $\tilde{z} \to 0$ (for $w \to -\frac{\pi}{2}$) to $\tilde{z} \to +\infty$ (for $w \to +\frac{\pi}{2}$) and can be inverted as

$w = 2\arctan\left(\tanh\left(\frac{\ln \tilde{z}}{2}\right)\right)$, we get the metric in Fefferman-Graham gauge,

$$ds^2 = \frac{L^2}{\tilde{z}^2}\left[d\tilde{z}^2 + \left(\frac{1}{4} + \frac{\tilde{z}^2}{2} + \frac{\tilde{z}^4}{4}\right)\gamma_{\mu\nu}\,dx^\mu dx^\nu\right].$$

We can also define a different coordinate that takes the metric to Fefferman-Graham (FG) form,

$$\bar{z} = 2\left(\frac{1}{\cos w} + \tan w\right) = 2\left(\frac{1+\sin w}{\cos w}\right)$$

(this also satisfies $\cos w = \dfrac{4\bar{z}}{4+\bar{z}^2}$ and is inverted as

$$w = \begin{cases} -\arccos\left(\frac{4\bar{z}}{4+\bar{z}^2}\right) & \text{for } \bar{z} \leq 2 \\ \arccos\left(\frac{4\bar{z}}{4+\bar{z}^2}\right) & \text{for } \bar{z} > 2 \end{cases} .$$

$\bar{z}$ goes from $\bar{z} \to 0$ (for $w \to -\frac{\pi}{2}$) to $\bar{z} \to +\infty$ (for $w \to +\frac{\pi}{2}$).

The metric using the coordinate $\bar{z}$ reads

$$ds^2 = \frac{L^2}{\bar{z}^2}\left[d\bar{z}^2 + \left(1 + \frac{\bar{z}^2}{2} + \frac{\bar{z}^4}{16}\right)\gamma_{\mu\nu}\, dx^\mu dx^\nu\right].$$

We can now use $\bar{z}$ to define a compactified radial coordinate ρ, which is 1 at the AdS boundary and 0 at the centre, as $\rho = \sqrt{1 - \frac{\bar{z}}{2}}$ $\left(\bar{z} = 2(1-\rho^2)\right)$.

In coords $(t, \rho, r, \theta, \varphi)$, the black string metric reads

$$ds^2 = \frac{L^2}{(1-\rho^2)^2}\left(4\rho^2 d\rho^2 + \frac{1}{4}\left(2 - 2\rho^2 + \rho^4\right)^2 \gamma_{\mu\nu}\, dx^\mu dx^\nu\right).$$

Notice, however, that $\bar{z} = 4(1-\rho) + O((1-\rho)^2)$ near the boundary $\rho=1$. $(1-\rho)$ does not coincide with the FG radial coordinate near the boundary due to a factor of 4. We thus say that ρ is an IMPROPER RADIAL COORDINATE. This is a crucial observation when computing the energy-momentum tensor of the boundary theory following the prescription by Balasubramanian and Kraus. That prescription gives the correct boundary energy-momentum tensor, i.e. the one matching the prescription by de Haro, Skenderis, Solodukhin (arXiv:000 2230), only if the radial coordinate used to write the metric matches $1-\bar{z}$, where $\bar{z}$ is a radial FG coordinate.

In our study, such a PROPER RADIAL

COORDINATE ζ can be defined, for instance, by

$$\zeta = \frac{2-\bar z}{2+\bar z} \quad \left(\text{inverted as } \bar z = \frac{2(1-\zeta)}{(1+\zeta)}\right).$$

Notice that $\zeta = 1$ at $\bar z = 0$, $\zeta = 0$ at $\bar z = 2$ and

$$\bar z = (1-\zeta) + O\left((1-\zeta)^2\right) \quad \text{near } \zeta = 1.$$

The black string metric in coordinates $(t, \zeta, r, \theta, \varphi)$ reads

$$ds^2 = \frac{L^2}{(1-\zeta^2)^2}\left(4 d\zeta^2 + (1+\zeta^2)^2 \gamma_{\mu\nu} dx^\mu dx^\nu\right).$$