This is the third part of a series of notes in which we compute expressions relevant for the modified Cartoon reduction of the equations of motion.

In this note we calculate the expression of the generalized harmonic source functions $H^{(D)A}$, $H^{(D)}{}_A$ and their first derivatives $H^{(D)}{}_{A,B}$ on the hypersurface S' at $z \geq \sigma$, $w^p = 0\ \forall p$.

Other expressions for scalar, vector and $\binom{0}{2}$ tensor densities at S' can be obtained from the expressions of APPENDIX A of 1603.00362 or eq. (27)-(31) of 1004.4970, by treating the t coord just like one of the x^i coords.

These expressions tell us that all tensorial objects and their derivatives with an odd number of indices associated with w^p coords vanish. In particular, this is given in APPENDIX A for scalars, vectors and $\binom{0}{2}$ tensors. In the following, we show that this also holds for $H^{(D)A}$, $H^{(D)}{}_A$ and $H^{(D)}{}_{A,B}$.

By definition

$$H^{(D)A} = -g^{BC}\,\Gamma^{(D)A}{}_{BC},$$ where the Christoffel symbols $\Gamma^{(D)A}{}_{BC} = \frac{1}{2} g^{AD}\left(g_{BD,C} - g_{BC,D} + g_{CD,B}\right)$ at S were computed in a previous note.

Notice that $H^{(D)} = \Box^{(D)} X^A$, where $\Box^{(D)} := g^{AB}\nabla^{(D)}_A \nabla^{(D)}_B$ and $\nabla^{(D)}$ is the Levi-Civita covariant derivative associated with g_{AB}.

So

$$H^{(D)}{}_A = -g_{AB}\, g^{CD}\,\Gamma^{(D)B}{}_{CD}$$

and

$$H^{(D)}{}_{A,B} = -g_{AC,B}\, g^{DE}\,\Gamma^{(D)C}{}_{DE} - g_{AC}\, g^{DE}{}_{,B}\,\Gamma^{(D)C}{}_{DE}$$
$$- g_{AC}\, g^{DE}\,\Gamma^{(D)C}{}_{DE,B}.$$

Let's compute the expression of these quantities at S, one by one.

1. $H^{(D)A}$

CASE $A = a$

$$H^{(D)a} := -g^{CD}\Gamma^{(D)a}{}_{CD} =$$

$$= -g^{cd}\Gamma^{(D)a}{}_{cd} - 2\underset{\overset{\shortparallel}{0}}{g^{cp}}\Gamma^{(D)a}{}_{cp} - g^{pq}\Gamma^{(D)a}{}_{pq}.$$

Now, from a previous note,

$$\Gamma^{(D)a}{}_{bc} = \Gamma^{(d)a}{}_{bc}$$

$$\Gamma^{(D)a}{}_{pq} = \delta_{pq}\Gamma^{a}{}_{ww}, \text{ where } \Gamma^{a}{}_{ww} = -\frac{1}{2}g^{ab}g_{ww,b} + g^{ab}g_{bw,w}.$$

Hence,

$$H^{(D)a} = -g^{cd}\Gamma^{(d)a}{}_{cd} - g^{pq}\delta_{pq}\Gamma^{a}{}_{ww} = H^{(d)a} - n g^{ww}\Gamma^{a}{}_{ww}$$

$$= H^{(d)a} + \frac{n}{2}g^{ww}g^{ab}g_{ww,b} - n g^{ww}g^{ab}g_{bw,w};$$

where $H^{(d)a} = -g^{bc}\Gamma^{(d)a}{}_{bc}$ are the source functions associated with the d-dimensional metric $g_{ab}dx^a dx^b$ on S.

CASE $A = p$

$$H^{(D)p} = -g^{AB}\Gamma^{(D)p}{}_{AB} =$$

$$= -g^{ab}\Gamma^{(D)p}{}_{ab} - 2\underset{\overset{\shortparallel}{0}}{g^{aq}}\Gamma^{(D)p}{}_{aq} - 2g^{qr}\Gamma^{(D)p}{}_{qr}$$

Now $\Gamma^{(D)p}{}_{ab} = 0$, $\Gamma^{(D)p}{}_{qr} = 0$.

So $H^{(D)p} = 0$.

2. $H^{(D)}{}_A$

CASE $A = a$

$$H^{(D)}{}_a = g_{aB} H^{(D)B} = g_{ab} H^{(D)b} + g_{ap} \underset{\overset{\shortparallel}{0}}{H^{(D)p}} =$$

$$= g_{ab} H^{(d)b} - n g^{ww} g_{ab} \Gamma^b{}_{ww} = H^{(d)}{}_a - n g^{ww} g_{ab} \Gamma^b{}_{ww} =$$

$$= H^{(d)}{}_a + \frac{n}{2} g^{ww} g_{ab} g^{bc} g_{ww,c} - n g^{ww} g_{ab} g^{bc} g_{cw,w} =$$

$$= H^{(d)}{}_a + \frac{n}{2} g^{ww} g_{ww,a} - n g^{ww} g_{aw,w}.$$

CASE $A = p$

$$H^{(D)}{}_p = g_{pA} H^{(D)A} = g_{pq} \underset{\overset{\shortparallel}{0}}{H^{(D)q}} = 0.$$

3. $H^{(D)}_{A,B}$

CASE $(A,B) = (a,b)$

$H^{(D)}_{a,b}$ can be simply computed as the $\frac{\partial}{\partial x^b}$ derivative of $H^{(D)}_a$ at $z=0, w^p=0\ \forall p$. That's because in computing $\frac{\partial}{\partial x^b} H^{(D)}_a$ we are keeping all coords fixed except x^b, in particular all w^p stay 0.

$$H^{(D)}_{a,b} = H^{(d)}_{a,b} + \frac{n}{2} g^{ww}{}_{,b}\, g_{ww,a} + \frac{n}{2} g^{ww} g_{ww,ab} - n g^{ww}{}_{,b}\, g_{aw,w} - n g^{ww} g_{aw,bw}$$

CASE

$(A, B) = (a, p)$

$$H^{(D)}{}_{a,p} = -g_{aC,p}\, g^{DE}\, \Gamma^{(D)C}{}_{DE} - g_{aC}\, g^{DE}{}_{,p}\, \Gamma^{(D)C}{}_{DE}$$
$$- g_{aC}\, g^{DE}\, \Gamma^{(D)C}{}_{DE,p}$$

$$t1^{(D)}{}_{ap} = -g_{aC,p}\, g^{DE}\, \Gamma^{(D)C}{}_{DE}$$

$$= -\underbrace{g_{ac,p}}_{\substack{= \\ 0}}\, g^{DE}\, \Gamma^{(D)c}{}_{DE} - g_{aq,p}\, g^{DE}\, \Gamma^{(D)q}{}_{DE}$$

from A.7
of 1603.00362

$$= -g_{aq,p}\, g^{de}\, \underbrace{\Gamma^{(D)q}{}_{de}}_{\substack{= \\ 0}} - g_{aq,p}\, g^{rs}\, \underbrace{\Gamma^{(D)q}{}_{rs}}_{\substack{= \\ 0}}$$

$$= 0$$

$$t\, 2^{(D)}{}_{ap} = -g_{ac}\, g^{DE}{}_{,p}\, \Gamma^{(D)c}{}_{DE} =$$

$$= -g_{ac}\, g^{DE}{}_{,p}\, \Gamma^{(D)c}{}_{DE}$$

$$= -g_{ac}\, g^{de}{}_{,p}\, \Gamma^{(D)c}{}_{de} - 2 g_{ac}\, g^{dq}{}_{,p}\, \Gamma^{(D)c}{}_{dq}$$

$$- g_{ac}\, g^{qr}{}_{,p}\, \Gamma^{(D)c}{}_{qr} .$$

Now, from previous notes,

$$g^{ab}{}_{,p} \tfrac{1}{2} \delta_{pq} - g_{ww}\, g^{aw}{}_{,w}\, \Gamma^{w}{}_{aw} ,$$

$$g^{pq}{}_{,r} = 0 ,$$

and

$$\Gamma^{(D)q}{}_{bp} = 0 .$$

So

$$t2^{(D)}{}_{ap} = 0.$$

$$t3^{(D)}{}_{ap} = -g_{aC} g^{DE} \Gamma^{(D)C}{}_{DE,p}$$

$$= -g_{ac} g^{DE} \Gamma^{(D)c}{}_{DE,p}$$

$$= -g_{ac} g^{de} \Gamma^{(D)c}{}_{de,p} - g_{ac} g^{qr} \Gamma^{(D)c}{}_{qr,p}$$

From a previous note

$$\Gamma^{(D)a}{}_{bc,p} = 0$$

and

$$\Gamma^{(D)a}{}_{pq,r} = 0.$$

So

$$t3^{(D)}{}_{ap} = 0.$$

Hence

$$H^{(D)}{}_{a,p} = 0.$$

CASE $(A, B) = (p, a)$

$H^{(D)}_{p,a}$ can be simply computed as the $\frac{\partial}{\partial x^a}$ derivative of $H^{(D)}_p$ at $z \geq 0$, $w^p = 0$ $\forall p$. That's because in computing $\frac{\partial}{\partial x^a} H^{(D)}_p$ we are keeping all ~~coords~~ fixed except x^a, in particular all w^p stay 0.

Hence $H^{(D)}_{p,a} = 0$

CASE $(A, B) = (p, q)$

$$H^{(D)}_{p,q} = -g_{pC,q} g^{DE} \Gamma^{(D)C}_{DE} - g_{pC} g^{DE}_{,q} \Gamma^{(D)C}_{DE} - g_{pC} g^{DE} \Gamma^{(D)C}_{DE,q} .$$

$$t1^{(D)}_{pq} = -g_{pC,q} g^{DE} \Gamma^{(D)C}_{DE}$$

$$= -g_{pc,q} g^{DE} \Gamma^{(D)c}_{DE}$$

$$= -g_{pc,q} g^{de} \Gamma^{(D)c}_{de} - g_{pc,q} g^{rs} \Gamma^{(D)c}_{rs} =$$

$$= -\delta_{pq} g_{cw,w} g^{de} \Gamma^{(d)c}_{de} - \delta_{pq} g_{cw,w} \delta^{rs} g^{ww} \delta_{rs} \Gamma^{c}_{ww}$$

$$= -\delta_{pq}\, g_{aw,w} \left(g^{bc}\, \Gamma^{(d)\,a}{}_{bc} + n\, g^{ww}\, \Gamma^{a}{}_{ww} \right).$$

$$t2^{(D)}{}_{pq} = - g_{pC}\, g^{DE}{}_{,q}\, \Gamma^{(D)\,C}{}_{DE} =$$

$$= - g_{pr}\, g^{DE}{}_{,q}\, \Gamma^{(D)\,r}{}_{DE} =$$

$$= - g_{pr}\, g^{de}{}_{,q}\, \underbrace{\Gamma^{(D)\,r}{}_{de}}_{=0} - g_{pr}\, g^{ds}{}_{,q}\, \Gamma^{(D)\,r}{}_{ds}$$

$$- g_{pr}\, g^{se}{}_{,q}\, \Gamma^{(D)\,r}{}_{se} - g_{pr}\, g^{st}{}_{,q}\, \underbrace{\Gamma^{(D)\,r}{}_{st}}_{=0}$$

$$= - 2\, g_{pr}\, g^{as}{}_{,q}\, \Gamma^{(D)\,r}{}_{as}.$$

From a previous note, $g^{ap}{}_{,q} = \delta^{p}_{q}\, g^{aw}{}_{,w}$ and $\Gamma^{(D)\,p}{}_{aq} = \delta^{p}_{q}\, \Gamma^{w}{}_{aw}$, where $g^{aw}{}_{,w} = \dfrac{g^{az} - \delta^{az} g^{ww}}{z}$ and $\Gamma^{w}{}_{aw} = \dfrac{1}{2} g^{ww} g_{ww,a}$.

So, $t2^{(D)}{}_{pq} = - 2\, \delta_{pr}\, g_{ww}\, \delta^{s}_{q}\, g^{aw}{}_{,w}\, \delta^{r}_{s}\, \Gamma^{w}{}_{aw} =$

$$= - 2\, \delta_{pq}\, g_{ww}\, g^{aw}{}_{,w}\, \Gamma^{w}{}_{aw}.$$

$$t3^{(D)}{}_{pq} = - g_{pC}\, g^{DE}\, \Gamma^{(D)\,C}{}_{DE,q} =$$

$$= - g_{pr}\, g^{DE}\, \Gamma^{(D)\,r}{}_{DE,q}$$

$$= - g_{pr}\, g^{de}\, \Gamma^{(D)\,r}{}_{de,q} - g_{pr}\, g^{st}\, \Gamma^{(D)\,r}{}_{st,q}.$$

From a previous note

$$\Gamma^{(D)p}{}_{ab,q} = \delta^p_q \, \Gamma^w{}_{ab,w} \, ,$$

where

$$\Gamma^w{}_{ab,w} = \frac{1}{2} g^{cw}{}_{,w} \left(g_{ac,b} - g_{ab,c} + g_{bc,a} \right) + \frac{1}{2} g^{ww} \left(g_{aw,bw} - g_{ab,ww} + g_{bw,aw} \right).$$

Furthermore

$$\Gamma^{(D)p}{}_{qr,s} = \left(\delta^p_q \delta_{rs} + \delta^p_r \delta_{qs} \right) \Gamma^{w(1)}{}_{ww,w} + \delta^p_s \delta_{qr} \Gamma^{w(2)}{}_{ww,w} \, ,$$

where

$$\Gamma^{w(2)}{}_{ww,w} = -\frac{1}{2} g^{aw}{}_{,w} g_{ww,a} + g^{aw}{}_{,w} g_{aw,w} + g^{ww} g^{(1)}_{ww,ww} - \frac{1}{2} g^{ww} g^{(2)}_{ww,ww}$$

and $\Gamma^{w(1)}{}_{ww,w} = \frac{1}{2} g^{ww} g^{(2)}_{ww,ww}$.

Therefore

$$t3^{(D)}{}_{pq} = -\delta_{pr} g_{ww} g^{de} \delta^r_q \Gamma^w{}_{de,w}$$

$$-\delta_{pr}\delta^{st}\left(\left(\delta^r_s\delta_{tq}+\delta^r_t\delta_{sq}\right)\Gamma^{w(1)}_{ww,w}+\delta^r_q\delta_{st}\,\Gamma^{w(2)}_{ww,w}\right)$$

$$=-\delta_{pq}\,g_{ww}\,g^{ab}\,\Gamma^w_{ab,w}$$

$$-\left(\delta_{pq}2\Gamma^{w\,(1)}_{ww,w}+n\,\delta_{pq}\Gamma^{w\,(2)}_{ww,w}\right)=$$

$$=-\delta_{pq}\left(g_{ww}\,g^{ab}\,\Gamma^w_{ab,w}+2\Gamma^{w(1)}_{ww,w}+n\Gamma^{w(2)}_{ww,w}\right).$$

To summarise

$$H^{(D)}_{p,q}=-\delta_{pq}\Big[g_{aw,w}\left(g^{bc}\Gamma^{(d)\,a}_{\quad bc}+n\,g^{ww}\,\Gamma^a_{ww}\right)$$

$$+2\,g_{ww}\,g^{aw}{}_{,w}\,\Gamma^w_{aw}$$

$$+g_{ww}\,g^{ab}\,\Gamma^w_{ab,w}+2\Gamma^{w(1)}_{ww,w}+n\Gamma^{w(2)}_{ww,w}\Big].$$

We rewrite this as

$$H^{(D)}_{p,q} = \delta_{pq} \; H_{w,w} \,,$$

where

$$H_{w,w} = -\Big[g_{aw,w} \left(g^{bc} \Gamma^{(d)a}{}_{bc} + n g^{ww} \Gamma^{a}{}_{ww} \right)$$

$$+ 2 g_{ww} g^{aw}{}_{,w} \Gamma^{w}{}_{aw}$$

$$+ g_{ww} g^{ab} \Gamma^{w}{}_{ab,w} + 2 \Gamma^{w\,(1)}{}_{ww,w} + n \Gamma^{w\,(2)}{}_{ww,w} \Big] .$$