This is the fifth part of a series of notes in which we compute expressions relevant for the modified Cartoon reduction of the equations of motion.

In this note we calculate the expression of the energy-momentum tensor $T^{(D)}_{AB}$ of a real scalar field and its trace-reversed form $\bar{T}^{(D)}_{AB}$ on the hypersurface S at $z \geq 0$, $w^p = 0\ \forall p$.

Other expressions for scalar, vector and $\binom{0}{2}$ tensor densities at S can be obtained from the expressions of APPENDIX A of 1603.00362 or eq. (27)-(31) of 1004.4970, by treating the t coord just like one of the x^i coords.

These expressions tell us that all tensorial objects and their derivatives with an odd number of indices associated with w^p coords vanish. In particular, this is given in APPENDIX A for scalars, vectors and $\binom{0}{2}$ tensors. In the following, we see this in particular for $\bar{T}^{(D)}_{AB}$.

For a real scalar field ϕ of mass m, the energy-momentum tensor is

$$T^{(D)}_{AB} = \partial_A \phi \, \partial_B \phi + g_{AB}\left(-\frac{1}{2} g^{CD} \partial_C \phi \, \partial_D \phi - \frac{1}{2} m^2 \phi^2\right),$$

and the energy-momentum ~~in~~ trace-reversed form with cosmological constant Λ is

$$\bar{T}^{(D)}_{AB} = \frac{2}{D-2} \Lambda g_{AB} + 8\pi \left(T^{(D)}_{AB} - \frac{1}{D-2} T^{(D)} g_{AB}\right),$$

where $T^{(D)} = g^{AB} T^{(D)}_{AB}$ is the trace of T_{AB}.

Using the expressions of APPENDIX A of 1603.00362, we find the expression of these quantities at S.

Let's start from $T^{(D)}_{AB}$.

CASE $(A,B) = (a,b)$

$$T^{(D)}_{ab} = \partial_a\phi\,\partial_b\phi + g_{ab}\left(-\frac{1}{2}g^{cd}\partial_c\phi\,\partial_d\phi - \frac{1}{2}g^{pq}\underbrace{\partial_p\phi}_{=0}\underbrace{\partial_q\phi}_{=0} - \frac{1}{2}m^2\phi^2\right)$$

$$= \partial_a\phi\,\partial_b\phi + g_{ab}\left(-\frac{1}{2}g^{cd}\partial_c\phi\,\partial_d\phi - \frac{1}{2}m^2\phi^2\right)$$

$$= T^{(d)}_{ab},$$

where $T^{(d)}_{ab}$ is a tensor of the same form as $T^{(D)}_{AB}$ but applied to the metric $g_{ab}dx^a dx^b$ on S^d.

CASE $(A,B)=(\alpha,\beta)$,

from A.7 of 1603.00362,

$$T^{(D)}{}_{\alpha\beta}=0$$

CASE $(A,B)=(p,q)$

$$T^{(D)}{}_{pq}=\underset{\underset{0}{\shortparallel}}{\partial_p}\phi\,\partial_q\phi+g_{pq}\left(-\frac{1}{2}g^{AB}\partial_A\phi\,\partial_B\phi-\frac{1}{2}m^2\phi^2\right)$$

$$=\delta_{pq}\,g_{ww}\left(-\frac{1}{2}g^{ab}\partial_a\phi\,\partial_b\phi-\frac{1}{2}m^2\phi^2\right)$$

$$=\delta_{pq}T_{ww},$$

where we defined

$$T_{ww} = g_{ww}\left(-\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - \frac{1}{2} m^2 \phi^2\right).$$

Let's now consider

$$T^{(D)} = g^{AB} T^{(D)}_{AB} = g^{ab} T^{(D)}_{ab} + g^{pq} T^{(D)}_{pq}$$

$$= g^{ab} T^{(d)}_{ab} + \delta^{pq} g^{ww} \delta_{pq} T_{ww}$$

$$= T^{(d)} + n g^{ww} T_{ww} =$$

$$= T^{(d)} - \frac{n}{2}\left(g^{ab} \partial_a \phi \partial_b \phi + m^2 \phi^2\right)$$

Finally, we study

$$\overline{T}^{(D)}_{AB}.$$

CASE $(A,B)=(a,b)$

$$\overline{T}^{(D)}_{ab}=\frac{2}{D-2}\Lambda g_{ab}+8\pi\left(T^{(D)}_{ab}-\frac{1}{D-2}T^{(D)}g_{ab}\right)$$

$$=\frac{2}{D-2}\Lambda g_{ab}+8\pi\left[T^{(d)}_{ab}-\frac{1}{D-2}g_{ab}\left(T^{(d)}+h g^{mn}T_{mn}\right)\right]$$

CASE $(A,B) = (a,p)$

$$\bar{T}^{(D)}_{ap} = 0$$

CASE $(A,B) = (p,q)$

$$\bar{T}^{(D)}_{pq} = \frac{2}{D-2}\Lambda g_{pq} + 8\pi\left(T^{(D)}_{pq} - \frac{1}{D-2}T^{(D)} g_{pq}\right)$$

$$= \delta_{pq}\left\{\frac{2}{D-2}\Lambda g_{ww} + 8\pi\left(T_{ww} - \frac{1}{D-2} g_{ww}\left(T^{(d)} + \bar{h} g^{ww} T_{ww}\right)\right)\right\}$$

$$= \delta_{pq}\overline{T}_{ww},$$

where we defined

$$\tilde{T}_{\mu\nu} = \frac{2}{D-2} \Lambda g_{\mu\nu} + 8\pi \left(T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} \left(T^{(d)} + \bar{h} g^{\mu\nu} T_{\mu\nu} \right) \right)$$