This is the seventh part of a series of notes in which we compute expressions relevant for the modified Cartoon reduction of the equations of motion and other quantities, when preserving an $SO(n+1)$ symmetry.

In this note we calculate the expression of the quasi-local energy-momentum tensor $T^{(\rho)}_{AB}$ at a fixed ρ hypersurface Σ_ρ in an asymptotically locally $AdS_{D=5}$ spacetime, where ρ is a radial coordinate that determines the distance from the AdS boundary and on the hypersurface S' at $z \geq 0$, $w^p = 0$ $\forall p = 1,2$, i.e. at $\Sigma_\rho \cap S'$ at fixed ρ, $z \geq 0$, $w^p = 0$ $\forall p = 1,2$, when preserving an $SO(3)$ symmetry.

Other expressions for scalar, vector and $\binom{0}{2}$ tensor densities at S' can be obtained from the expressions of APPENDIX A of 1603.00362 or eq. (27)-(31) of 1004.4970, by treating the t coord just like one of the x^i coords.

These expressions tell us that all tensorial objects and their derivatives with an odd number of indices associated with w^p coords vanish. In particular, this is given in APPENDIX A for scalars, vectors and $\binom{0}{2}$ tensors. In the following, we see this in particular for $T^{(\rho)}_{AB}$.

Following Balasubramanian and Kraus,
given a timelike hypersurface Σ_ρ at fixed radial coordinate ρ, the quasi-local energy-momentum tensor on asymptotically locally AdS_5 spacetimes V of AdS radius L is a tensor at Σ_ρ given by

$$T^{(\rho)(D)}_{AB} = \frac{1}{8\pi}\left(\Theta^{(D)}_{AB} - \Theta^{(D)}\Omega_{AB} - \frac{3}{L}\Omega_{AB} + \frac{L}{2}G^{(D)}_{AB}\right),$$

where $\Theta^{(D)}_{AB} = -\Omega^{C}_{A}\Omega^{D}_{B}\nabla^{(D)}_{C}S^{(D)}_{D}$ is the extrinsic curvature of Σ_ρ,

$\Omega^{(D)}_{AB} = g_{AB} - S^{(D)}_{A}S^{(D)}_{B}$ is the induced metric on Σ_ρ,

$S^{(D)A}$ is the spacelike, outward pointing, unit vector normal to Σ_ρ, $\Theta^{(D)} = \Omega^{AB}\Theta^{(D)}_{AB} = g^{AB}\Theta^{(D)}_{AB}$ is the trace of $\Theta^{(D)}_{AB}$ and $G^{(D)}_{AB}$ is the Einstein tensor of Σ_ρ.

We want to find the modified Cartoon reduction of $T^{(\rho)(D)}_{AB}$ on V_D^1 the d-dimensional hypersurface at $z=0$, $w^1 = w^2 = 0$.

Let's consider each piece separately.

1. $\xi^{(D)A} = \frac{1}{\sqrt{g^{\rho\rho}}} (d\rho)^A = \frac{1}{\sqrt{g^{\rho\rho}}} \delta^A_\rho . \Rightarrow \xi^{(D)a} = \frac{1}{\sqrt{g^{\rho\rho}}} \delta^a_\rho = \xi^{(d)a} , \xi^{(D)p} = 0.$

$\xi^{(D)}_A = g_{AB}\,\xi^{(D)B} = \frac{1}{\sqrt{g^{\rho\rho}}} g_{\rho A} . \Rightarrow \xi^{(D)}_a = \frac{1}{\sqrt{g^{\rho\rho}}} g_{\rho a} = \xi^{(d)}_a , \quad \xi^{(D)}_p = 0.$

2. $\Omega_{AB} = g_{AB} - \xi^{(D)}_A \xi^{(D)}_B$. In matrix form at Σ

$$\Omega_{AB} = \begin{array}{c} \\ d=3 \\ n=2 \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{d=3} & \overbrace{\qquad}^{n=2} \end{array} \\ \left(\begin{array}{c|c} \Omega_{ab} & 0 \\ \hline 0 & \delta_{pq}\,\Omega_{ww} \end{array}\right) \end{array} ,$$

where $\Omega_{ab} = g_{ab} - \xi^{(d)}_a \xi^{(d)}_b$ and $\Omega_{ww} = g_{ww}$.

$\Omega^b{}_a = g^{bc}\,\Omega_{ac} = g^{bc}\,\Omega_{ac} = \delta^b_a - \xi^{(d)}_a \xi^{b(d)}$, $\Omega^p{}_a = 0$, $\Omega^a{}_p = 0$,

$\Omega^q{}_p = g^{qc}\,\Omega_{pc} = g^{qr}\,\Omega_{pr} = \delta^{qr} g^{ww} \delta_{pr}\,\Omega_{ww} = \delta^q_p$, $\Omega^{pq} = \delta^{pq}\,\Omega^{ww}$, where $\Omega^{ww} = \frac{1}{\Omega_{ww}} = g^{ww}$.

3. $\nabla^{(D)}_A \xi^{(D)}_B = \partial_A \xi^{(D)}_B - \Gamma^{C\,(D)}_{AB}\, \xi^{(D)}_C$

CASE $(A,B) = (a,b)$

$\nabla^{(D)}_a \xi^{(D)}_b = \partial_a \xi^{(D)}_b - \Gamma^{c\,(D)}_{ab}\, \xi^{(D)}_c = \partial_a \xi^{(d)}_b - \Gamma^{c\,(d)}_{ab}\, \xi^{(d)}_c =$

∂_a does not affect
the values at $z=0, w^p=0$
and we previously found $\Gamma^{c\,(D)}_{ab} = \Gamma^{c\,(d)}_{ab}$

$= \nabla^{(d)}_a \xi^{(d)}_b$

CASE $(A,B) = (a,p)$

$$\nabla^{(D)}_a S^{(D)}_p = \underbrace{\partial_a S^{(D)}_p}_{=0} - \Gamma^{C\,(D)}_{ap} \underbrace{S^{(D)}_C}_{S^{(D)}_p = 0 \text{ as found in previous note}} = - \underbrace{\Gamma^{c\,(D)}_{ap}}_{=0} S^{(d)}_c = 0$$

CASE $(A,B) = (p,a)$

$$\nabla^{(D)}_p S^{(D)}_a = \partial_p S^{(D)}_a - \Gamma^{C\,(D)}_{pa} S^{(D)}_C = - \underbrace{\Gamma^{c\,(D)}_{pa}}_{=0} S^{(d)}_c = 0$$

$$\partial_p S^{(D)}_a = (\partial_p g_{aB}) S^{B(D)} + g_{aB}\, \partial_p S^{B(D)} = \underbrace{\partial_p g_{ab}}_{\substack{=0 \\ \text{from A.7} \\ \text{of 1603.06362}}} S^{b(d)} + g_{ab} \underbrace{\partial_p S^{b(D)}}_{\substack{=0 \\ \text{from A.3} \\ \text{of 1603.06362}}}$$

CASE $(A,B) = (p,q)$

$$\nabla^{(D)}_p S^{(D)}_q = \partial_p S^{(D)}_q - \Gamma^{A\,(D)}_{pq} S^{(D)}_A = \delta_{pq} \frac{S^{(D)}_z}{z} - \Gamma^{a\,(D)}_{pq} S^{(d)}_a =$$

$\partial_p S^{(D)}_q = \delta_{pq} \dfrac{S^{(D)}_z}{z}$ → from A.4 of 1603.06362 ($= 0$) but we still include this term

$$= \delta_{pq} \frac{S^{(D)}_z}{z} - \delta_{pq} \Gamma^a_{ww} S^{(d)}_a$$

$$= \delta_{pq} \nabla_w S_w ,$$

where we defined $\nabla_w S_w = \dfrac{S^{(D)}_z}{z} - \Gamma^a_{ww} S^{(d)}_a .$

4. $\overset{(D)}{\Theta}_{AB} = -\Omega_A{}^C \Omega_B{}^D \overset{(D)}{\nabla}_C \overset{(D)}{S}_D$

CASE $(A,B)=(a,b)$

$$\overset{(D)}{\Theta}_{ab} = -\Omega_a{}^C \Omega_b{}^D \overset{(D)}{\nabla}_C \overset{(D)}{S}_D =$$

$$= -\Omega_a{}^c \Omega_b{}^d \overset{(d)}{\nabla}_c \overset{(d)}{S}_d = \overset{(d)}{\Theta}_{ab}, \text{ where } \overset{(d)}{\Theta}_{ab} \text{ is}$$

the extrinsic curvature of $(\Sigma_\rho \cap S, \Omega_{ab} dx^a dx^b)$.

CASE $(A,B)=(a,p)$

$\overset{(D)}{\Theta}_{ap} = 0$ from A.7 of 1603.00362.

CASE $(A,B)=(p,q)$

$$\overset{(D)}{\Theta}_{pq} = -\Omega_p{}^C \Omega_q{}^D \overset{(D)}{\nabla}_C \overset{(D)}{S}_D = -\delta_p{}^r \delta_q{}^s \overset{(D)}{\nabla}_r \overset{(D)}{S}_s =$$

$$= -\delta_p{}^r \delta_q{}^s \delta_{rs} \nabla_w S_w = \delta_{pq} \Theta_{ww},$$

where we defined $\Theta_{ww} = -\nabla_w S_w$.

5. $\overset{(D)}{\Theta} = \Omega^{AB} \overset{(D)}{\Theta}_{AB} = \Omega^{aB} \overset{(D)}{\Theta}_{aB} + \Omega^{pB} \overset{(D)}{\Theta}_{pB} =$

$$= \Omega^{ab} \overset{(D)}{\Theta}_{ab} + \Omega^{pq} \overset{(D)}{\Theta}_{pq} = \overset{(d)}{\Theta} + n\,\Omega^{ww} \Theta_{ww},$$

where $\Theta^{d} = \Omega^{ab}\,\Theta^{(d)}_{ab} = g^{ab}\,\Theta^{(d)}_{ab}$ is the trace of $\Theta^{(d)}_{ab}$.

Finally, we determine the modified Cartan reduction of the Einstein tensor of $\left(\Sigma_\rho \cap S^d, \Omega_{ab}\,dx^a dx^b\right)$.

$$G^{(D)}_{AB} = R^{(D)}_{AB} - \frac{1}{2} R^{(D)}\,\Omega_{AB},$$

where $R^{(D)} = \Omega^{AB} R^{(D)}_{AB}$, $R^{(D)}_{AB} = R^{C\,(D)}{}_{ACB}$,

$$R^{A\,(D)}{}_{BCD} = \partial_C \Gamma^{(D)A}_{BD} - \partial_D \Gamma^{(D)A}_{BC} + \Gamma^{(D)E}_{BD}\,\Gamma^{(D)A}_{EC} - \Gamma^{(D)E}_{BC}\,\Gamma^{(D)A}_{ED},$$

where $\Gamma^{(D)A}_{BC} = \frac{1}{2}\Omega^{AD}\left(\Omega_{BD,C} - \Omega_{BC,D} + \Omega_{CD,B}\right)$ are the Christoffel symbols of the Levi-Civita connection associated with $\Omega_{AB}\,dx^A dx^B$.

Let's study

$$R^{(D)}_{AB} = R^{(D)C}{}_{ACB} = \partial_C \Gamma^{(D)C}_{AB} - \partial_B \Gamma^{(D)C}_{AC} + \Gamma^{(D)D}_{AB}\,\Gamma^{(D)C}_{DC} - \Gamma^{(D)D}_{AC}\,\Gamma^{(D)C}_{DB}.$$

CASE $(A,B) = (a,b)$

$$R^{(D)}_{ab} = \partial_C \Gamma^{(D)C}_{ab} - \partial_b \Gamma^{(D)C}_{aC} + \Gamma^{(D)D}_{ab}\Gamma^{(D)C}_{DC} - \Gamma^{(D)D}_{aC}\Gamma^{(D)C}_{Db}$$

$$= \partial_c \Gamma^{(d)c}_{ab} + \partial_p \Gamma^{(D)p}_{ab} - \partial_b \Gamma^{(d)c}_{ac} - \partial_b \Gamma^{(D)p}_{ap}$$
$$+ \Gamma^{(d)d}_{ab}\Gamma^{(d)c}_{dc} + \Gamma^{(d)d}_{ab}\Gamma^{(D)p}_{dp}$$
$$- \Gamma^{(d)d}_{ac}\Gamma^{(d)c}_{db} - \Gamma^{(D)q}_{ap}\Gamma^{(D)p}_{qb} =$$

$$= \partial_c \Gamma^{(d)c}_{ab} + h\,\partial_w \Gamma^{w}_{ab} - \partial_b \Gamma^{(d)c}_{ac} - h\,\partial_b \Gamma^{w}_{aw}$$
$$+ \Gamma^{(d)d}_{ab}\Gamma^{(d)c}_{dc} + h\,\Gamma^{(d)c}_{ab}\Gamma^{w}_{cw}$$
$$- \Gamma^{(d)d}_{ac}\Gamma^{(d)c}_{db} - h\,\Gamma^{w}_{aw}\Gamma^{w}_{wb}$$
$$= R^{(d)}_{ab} + h\left(\partial_w \Gamma^{w}_{ab} - \partial_b \Gamma^{w}_{aw} + \Gamma^{(d)c}_{ab}\Gamma^{w}_{cw} - \Gamma^{w}_{aw}\Gamma^{w}_{wb}\right)$$

where

$$R^{(d)}_{ab} = \partial_c \Gamma^{(d)c}{}_{ab} - \partial_b \Gamma^{(d)c}{}_{ac} + \Gamma^{(d)d}{}_{ab}\Gamma^{(d)c}{}_{dc} - \Gamma^{(d)d}{}_{ac}\Gamma^{(d)c}{}_{db}$$

is the Ricci tensor of $\left(\Sigma_p \cap S, \Omega_{ab} dx^a dx^b\right)$. See expression for $\partial_w \Gamma^w{}_{ab}$, $\partial_b \Gamma^w{}_{aw}$, $\Gamma^w{}_{aw}$ in previous notes.

CASE $(A,B) = (a,p)$

$$R^{(D)}_{ap} = 0$$

from A.7 of 1603.00362.

CASE $(A,B) = (p,q)$

$$R^{(D)}_{pq} = \partial_C \Gamma^{(D)C}{}_{pq} - \partial_q \Gamma^{(D)C}{}_{pC} + \Gamma^{(D)D}{}_{pq}\Gamma^{(D)C}{}_{DC} - \Gamma^{(D)D}{}_{pC}\Gamma^{(D)C}{}_{qD}$$

$$= \delta_{pq}\partial_c \Gamma^c{}_{ww} - \delta_{pq}\partial_w \Gamma^c{}_{wc}$$

$$+ \Gamma^{(D)d}{}_{pq}\Gamma^{(D)c}{}_{dc} + \Gamma^{(D)d}{}_{pq}\Gamma^{(D)r}{}_{dr}$$

$$- \Gamma^{(D)r}{}_{pc}\Gamma^{(D)c}{}_{qr} - \Gamma^{(D)a}{}_{pr}\Gamma^{(D)r}{}_{qa}$$

$$= \delta_{pq}\partial_c \Gamma^c{}_{ww} - \delta_{pq}\partial_w \Gamma^c{}_{wc}$$

$$+ \delta_{pq}\Gamma^b{}_{ww}\Gamma^{(d)a}{}_{ba} + n\,\delta_{pq}\Gamma^a{}_{ww}\Gamma^w{}_{aw}$$

$$- \delta_{pq}\Gamma^w{}_{wa}\Gamma^a{}_{ww} - \delta_{pq}\Gamma^a{}_{ww}\Gamma^w{}_{wa} =$$

$$= \delta_{pq} R_{ww},$$

where we defined

$$R_{ww} = \partial_c \Gamma^c{}_{ww} - \partial_w \Gamma^c{}_{wc}$$
$$+ \Gamma^a{}_{ww}\Gamma^{(d)b}{}_{ab} + (n-2)\Gamma^a{}_{ww}\Gamma^w{}_{aw}.$$

Recall, from previous notes that,

$$\Gamma^{(d)a}{}_{bc} = \frac{1}{2}\Omega^{ad}\left(\Omega_{bd,c} - \Omega_{bc,d} + \Omega_{cd,b}\right)$$

$$\Rightarrow \Gamma^{(d)b}{}_{ab} = \frac{1}{2}\Omega^{bd}\left(\Omega_{ad,b} - \Omega_{ab,d} + \Omega_{bd,a}\right) = \frac{1}{2}\Omega^{bc}\Omega_{bc,a}.$$

Then

$$\Gamma^a{}_{ww} = -\frac{1}{2}\Omega^{ab}\Omega_{ww,b} + \Omega^{ab}\Omega_{bw,w}$$

$$\Gamma^w{}_{aw} := \frac{1}{2}\Omega^{ww}\Omega_{ww,a}.$$

Then

$$\partial_a \Gamma^b{}_{ww} = \partial_a \Omega^{bc}\left(-\frac{1}{2}\partial_c \Omega_{ww} + \partial_w \Omega_{cw}\right)$$

$$+ g^{bc}\left(-\frac{1}{2}\partial_a \partial_c \Omega_{ww} + \partial_a \partial_w \Omega_{cw}\right)$$

$$\partial_w \Gamma^a{}_{wb} = \frac{1}{2}\partial_w \Omega^{aw} \partial_b \Omega_{ww}$$

$$+\frac{1}{2}\Omega^{ac}\left(\partial_w \partial_w \Omega_{bc} - \partial_c \partial_w \Omega_{bw} + \partial_b \partial_w \Omega_{cw}\right).$$

See previous notes for the expression of the ∂_w derivatives of the metric Ω_{AB} on $\Sigma_\rho \cap S'$.

We then consider

$$R^{(D)} = \Omega^{AB} R^{(D)}{}_{AB} =$$

$$= \Omega^{ab} R^{(D)}{}_{ab} + \Omega^{pq} R^{(D)}{}_{pq} =$$

$$= \Omega^{ab}\left(R^{(d)}{}_{ab} + h\left(\partial_w \Gamma^w{}_{ab} - \partial_b \Gamma^w{}_{aw} + \Gamma^{(d)c}{}_{ab}\Gamma^w{}_{cw} - \Gamma^w{}_{aw}\Gamma^w{}_{wb}\right)\right)$$

$$+ h\,\Omega^{ww} R_{ww}$$

$$= R^{(d)} + h\Big(\Omega^{ab}\partial_w \Gamma^{w}{}_{ab} - \Omega^{ab}\partial_b \Gamma^{w}{}_{aw}$$

$$+ \Omega^{ab}\Gamma^{(d)\,c}{}_{ab}\Gamma^{w}{}_{cw} - \Omega^{ab}\Gamma^{w}{}_{aw}\Gamma^{w}{}_{wb}\Big)$$

$$+ h\,\Omega^{ww} R_{ww}\,.$$

These results can be combined to compute

$$G^{(D)}_{ab} = R^{(D)}_{ab} - \frac{1}{2}\Omega_{ab} R^{(D)}$$

$$G^{(D)}_{ap} = 0$$

$$G^{(D)}_{pq} = \delta_{pq}\, G_{ww}\,,$$

where we defined $G_{ww} = R_{ww} - \frac{1}{2}\Omega_{ww} R^{(D)}$.

Therefore,

$$T^{(p)(D)}{}_{ab} = \frac{1}{8\pi}\left(\Theta^{(d)}{}_{ab} - \left(\Theta^{(d)} + n\Omega^{ww}\Theta_{ww}\right)\Omega_{ab} - \frac{3}{L}\Omega_{ab} + \frac{L}{2}G^{(D)}{}_{ab}\right),$$

$$T^{(p)(D)}{}_{ap} = 0,$$

$$T^{(p)(D)}{}_{pq} = \delta_{pq}T^{(p)}_{ww},$$

where

$$T^{(p)}_{ww} = \frac{1}{8\pi}\left(\Theta_{ww} - \left(\Theta^{(d)} + n\Omega^{ww}\Theta_{ww}\right)\Omega_{ww} - \frac{3}{L}\Omega_{ww} + \frac{L}{2}G_{ww}\right).$$