Gullstrand - Painlevé (GP) - like coords in Schwarzschild - AdS (SAdS) in 4 dimensions

Set the AdS radius to $L=1$.

Consider the Schwarzschild - AdS (SAdS) metric in usual Schwarzschild (S) coords (t, r, θ, φ):

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2$$

where $f = 1 + r^2 - \frac{r_0}{r}$ and $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$.

The event horizon (EH) is at $r = r_H$ where r_H is the real solution of $f(r_H) = 0$ i.e. $r_H(1 + r_H^2) = r_0$.

In asymptotically flat Schwarzschild, it is possible to define horizon - penetrating coordinates whose slices at constant time are flat. These are the so - called Gullstrand - Painlevé (GP) coords and are defined by following ingoing radial timelike geodesics starting from rest at large r.

These coords (v, r, θ, φ) are defined by

$$dv = dt + \frac{\sqrt{1-f}}{f_s} dr = dt + \frac{1}{f_s}\sqrt{\frac{2M}{r}}\, dr \quad \text{, where } f_s = 1 - \frac{2M}{r} .$$

The Schwarzschild metric in these coords reads

$$ds_s^2 = -f_s dv^2 + 2\sqrt{\frac{2M}{r}}\, dv\, dr + dr^2 + r^2 d\Omega^2 .$$

In the presence of a cosmological constant Λ, GP coordinates can only be defined by looking at outgoing radial timelike geodesics. However, when $\Lambda < 0$, such observers never reach the boundary so they do not define global coordinates. In other words, GP coords cannot be defined in asymptotically globally AdS spacetimes. This discussion can be found in arXiv:2006.10827v2.

However, it is possible to define coords (v, r, θ, φ) in SAdS that resemble GP coords.

These are given by

$$dv = dt + \frac{1}{f}\sqrt{\frac{r_0}{r(1+r^2)}}\, dr$$

and the SAdS metric in these coords reads

$$ds^2 = -f dv^2 + 2\sqrt{\frac{r_0}{r(1+r^2)}}\, dv dr + \frac{1}{1+r^2} dr^2 + r^2 d\Omega^2 ,$$

which is manifestly non-singular at $r = r_H$.

Clearly $v = \text{const.}$ slices are not flat (due to the fact that $g_{rr} = \frac{1}{1+r^2} \neq 1$).

Let us now define the compactified radial coordinate ρ as $r = \frac{2\rho}{1-\rho^2}$

$$\Rightarrow dr = \left(\frac{4\rho^2}{(1-\rho^2)^2} + \frac{2}{1-\rho^2}\right) d\rho = \frac{2(1+\rho^2)}{(1-\rho^2)^2} d\rho .$$

We have

$$f = 1 + \frac{4\rho^2}{(1-\rho^2)^2} - \frac{r_0}{2\rho}(1-\rho^2) =$$

$$= \frac{1}{(1-\rho^2)^2}\left(\underbrace{(1-\rho^2)^2 + 4\rho^2}_{\overset{\shortparallel}{(1+\rho^2)^2}} - \frac{r_0}{2\rho}(1-\rho^2)^3\right) =$$

$$= \frac{(1+\rho^2)^2}{(1-\rho^2)^2}\left(1 - \chi(\rho)\right) , \text{ where } \chi(\rho) := \frac{r_0}{2\rho}\frac{(1-\rho^2)^3}{(1+\rho^2)^2} .$$

So, in ~~coords~~ $(v, \rho, \theta, \varphi)$,

$$g_{vv} = -\frac{1}{(1-\rho^2)^2}(1+\rho^2)^2\left(1-\chi(\rho)\right).$$

then

$$g_{v\rho} = \sqrt{\frac{r_0}{\frac{2\rho}{1-\rho^2}\left(1+\frac{4\rho^2}{(1-\rho^2)^2}\right)}}\,\frac{2(1+\rho^2)}{(1-\rho^2)^2} = \sqrt{\frac{r_0(1-\rho^2)^3}{2\rho(1+\rho^2)^2}}\,\frac{2(1+\rho^2)}{(1-\rho^2)^2}$$

$$= \frac{1}{(1-\rho^2)^2}\,2(1+\rho^2)\sqrt{\chi(\rho)}$$

and

$$g_{\rho\rho} = \frac{1}{1+\frac{4\rho^2}{(1-\rho^2)^2}}\left(\frac{2(1+\rho^2)}{(1-\rho^2)^2}\right)^2 = \frac{4}{(1-\rho^2)^2}.$$

So the SAdS metric in $(v, \rho, \theta, \varphi)$ ~~coords~~ reads

$$ds^2 = \frac{1}{(1-\rho^2)^2}\Big\{-(1+\rho^2)^2\left(1-\chi(\rho)\right)dv^2 + 4(1+\rho^2)\sqrt{\chi(\rho)}\,dv\,d\rho + 4\left[d\rho^2+\rho^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right)\right]\Big\}.$$

We see that $v = $ const. slices in these coords are conformally flat.

Finally we define compactified Cartesian coordinates (v, x, y, z) where

$$x = \rho \cos\theta$$
$$y = \rho \sin\theta \cos\varphi$$
$$z = \rho \sin\theta \sin\varphi .$$

We have $\rho^2 = x^2 + y^2 + z^2 \Rightarrow d\rho = \frac{1}{\rho}\left(x\,dx + y\,dy + z\,dz\right)$.

The SAdS metric in these coords reads

$$ds^2 = \frac{1}{(1-\rho^2)^2}\left\{ -(1+\rho^2)^2\left(1-\chi(\rho)\right)dv^2 + 4\frac{(1+\rho^2)}{\rho}\sqrt{\chi}\,dv\left(x\,dx + y\,dy + z\,dz\right) + 4\left(dx^2 + dy^2 + dz^2\right)\right\}$$

with $\chi(\rho) = \frac{r_0}{2\rho}\frac{(1-\rho^2)^3}{(1+\rho^2)^2}$.