# The five colour theorem

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**Abstract**

The four colour theorem, inspired by a colour minimisation problem in cartography, states that the vertices of a planar graph can take on at least 4 colours, such that no vertices with the same colour are adjacent to each other. Due to the highly exhaustive nature of the proof, a related yet easier theorem will be proved. This paper will focus on the necessary background and the proof of the five colour theorem

## 1 Introduction

Consider a geographic map which is to be coloured. The smallest number of colours needed to separate two regions on the map is known to be 4. This fact was initially conjectured by August Möbious, and was only proved in the last century through an exhaustive computer assisted proof. A simpler related problem involves 5 colours instead of 4, which will be proved in the course of the paper. In order to build a mathematical proposition, we will transform a coloured geographic map into a graph. First consider the following:

**Definition 1.1** (Plane dual). *Let $G = (V, E)$ and $G^* = (V^*, E^*)$ be planar graphs. $G$ is said to be a plane dual of $G^*$ if every face in $G$ contains exactly one vertex in $G^*$, and any two vertices in $G^*$ share an edge when the two faces are separated by an edge in $G$*

In the context of coloured geographic maps, each coloured region is separated by an "edge", where every point of intersection holds the place of a vertex, thus forming the graph $G$. In reality, the notion of an edge and vertex needs a stronger definition in a space like $\mathbb{R}^2$, but for colouring regions this will suffice. It is clear that $G$ then has a plane dual $G^*$, whose vertices take on the colours of the faces in $G$

To properly state the five colour theorem with respect to the vertices of a graph, we need to define the following:

**Definition 1.2** (Vertex colouring)**.** *A graph $G$ has a vertex colouring, defined by the map $c : V \longrightarrow S$, where $S$ is the set of available colours and $\forall v, w \in V(G)$. $c(v) \neq c(w)$.*

**Definition 1.3** (k-colouring)**.** *A vertex colouring of $G$ is said to be a k-colouring when $c : V \longrightarrow \{i \in \mathbb{N} \ : \ 1 \leq i \leq k\}$.*

**Definition 1.4** (Chromatic number)**.** *The chromatic number of a graph $G$, denoted $\chi(G)$, is the smallest k-colouring of $G$.*

**Definition 1.5** (k-colourability)**.** *A graph $G$ is k-colourable if $\chi(G) \leq k$*

We can now introduce the five colour theorem as a vertex colouring problem as follows:

**Theorem 1.6** (Five colour theorem)**.** *Every planar graph $G$ is 5-colourable*

For the proof of Theorem 1.6, we will consider an important theorem (assumed to be true) and its corollary:

**Theorem 1.7** (Euler's formula)**.** *Let a graph $G$ be planar-connected with $v$ vertices, $e$ edges and $f$ faces. It follows that: $v - e + f = 2$*

**Corollary 1.8.** *If a planar-connected graph $G$ has $v \geq 3$ vertices and $e$ edges, it follows that: $e \leq 3v - 6$*

For the proof of the corollary as well as the five colour theorem, we need to introduce (and prove) a few lemmas:

**Lemma 1.9** (Handshaking lemma)**.**

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

*Proof.* The smallest number of vertices required to form an edge is 2. Since the sum of vertex degrees counts every edge exactly twice, the lemma holds $\square$

**Lemma 1.10.** *If a graph $G$ is planar and has $e$ edges and $f$ faces, it follows that: $2e \geq 3f$*

*Proof.* The smallest number of edges required to form a face is 3. In this case, e = 3 and f = 2 (considering the outer face), and since two contiguous faces share an edge, it follows that $2e = 3f$. If we add an edge to one of the vertices, it follows that $2e \geq 3f$ $\square$

We can now prove Corollary 1.8

*Proof (Corollary 1.8).*

$$\begin{aligned}
& 2 = v - e + f && \text{(Theorem 1.7)} \\
\iff\ & f = 2 + e - v \\
\implies\ & 2e \geq 3(2 + e - v) && \text{(Lemma 1.10)} \\
& \quad = 6 + 3e - 3v \\
\iff\ & e \leq 3v - 6
\end{aligned}$$

$\square$

**Lemma 1.11.** *For a planar-connected graph $G$ with $v$ vertices and $e$ edges, there exists a vertex $\tilde{v} \in V(G)$ such that* $\deg(\tilde{v}) < 6$

*Proof.*

$$\begin{aligned}
& e \leq 3v - 6 && \text{(Corollary 1.8)} \\
\implies\ & \sum_{\tilde{v} \in V(G)} \deg(\tilde{v}) = 2e \leq 2(3v - 6) && \text{(Lemma 1.9)} \\
\implies\ & \underbrace{\frac{1}{v} \sum_{\tilde{v} \in V(G)} \deg(\tilde{v})}_{\deg(\tilde{v}')} = \frac{2e}{v} \leq \frac{6v - 12}{v} \\
\implies\ & \deg(\tilde{v}') = \frac{2e}{v} \leq 6 - \frac{12}{v} < 6 && \text{(Since } v \in \mathbb{N}) \\
\implies\ & \deg(\tilde{v}) < 6
\end{aligned}$$

$\square$

The last step is clearly true since the average degree of a vertex can only be $< 6$ when there exists a vertex degree $< 6$. If the every vertex degree is $\geq 6$ (i.e. a 6-regular graph), then it cannot be planar (and will not hold for Corollary 1.8)

# 2 Proof of the five colour theorem

In order to prove Theorem 1.6, we restate it as a Proposition, on which we can perform induction:

**Proposition 2.1** (Five colour theorem)**.** *A graph $G_n = (V, E)$, which is planar-connected with $n$ vertices and $m$ edges, is 5-colourable* $\forall n \in \mathbb{N}^{>0}$

We proceed by induction on $n$:

*Proof.* Let Proposition 2.1 be labeled $\mathrm{P}(n)$

**Base case**: *Show that* $\mathrm{P}(n)$ *holds for* $1 \leq n \leq 5$
test

**Induction hypothesis**: *Assume* $\mathrm{P}(n)$ *holds for* $n \geq 5$*, show that* $\mathrm{P}(n+1)$ *follows*

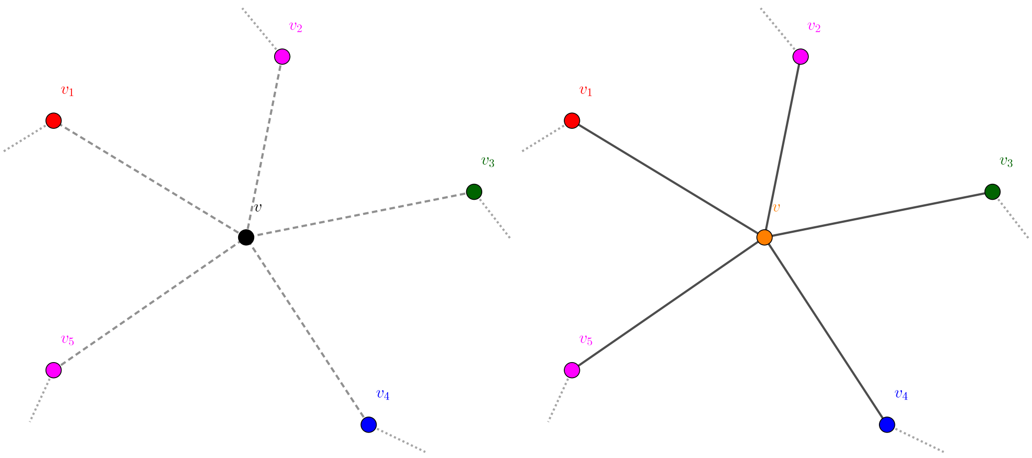
We assume that $G_n$ is 5-colourable, and we need to inductively show that the same is true for $G_{n+1}$

**Induction step**: *Prove* $\mathrm{P}(n+1)$

By Lemma 1.11, $\exists v \in V(G_{n+1})$ such that the $\deg(v) < 6$. For the sake of the proof, let us construct $H_n := G_{n+1} \setminus \{v\}$ where $\deg(v) = 5$. By the hypothesis, $H_n$ is 5-colourable. The goal is to show that based on our construction of $H_n$, that $H_n \cup \{v\} =: G_{n+1}$ is always 5-colourable. Let us consider the neighbourhood of $v$ (which are all the adjacent vertices of $v$) and denote it as $N(v) = \{v_1, v_2, v_3, v_4, v_5\} \subseteq H_n$. We define the 5-colouring of the graph $H_n$ as $c : V(H_n) \longrightarrow \{i \in \mathbb{N} : 1 \leq i \leq 5\}$. By our assumption, $N(v) \subseteq V(H_n)$ admits either 5 or less than 5 colours, therefore:

**case 1**: $|\{c(v_i) : v_i \in N(v)\}| < 5$

Let $G_{n+1} := H_n \cup \{v\}$ and $c(v) := k$, where $k \in \{c(v_i)\}_{v_i \in H_n} \setminus \{c(v_i)\}_{v_i \in N(v)}$ i.e $v$ gets assigned any remaining colour and we are done




(a) Diagram of $N(v) \subseteq H_n$ (b) Diagram of $N(v) \subseteq G_{n+1}$

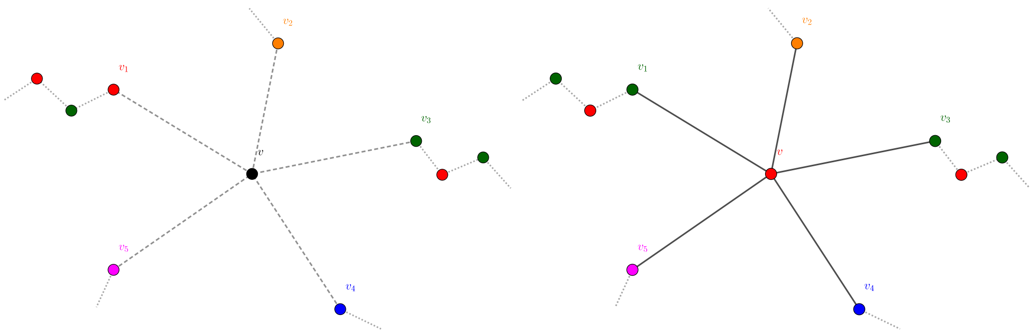
Figure 1: Graph focussed around $N(v)$. The dotted lines represent arbitrary edges, whereas the vertices connected by the dashed edge are disjoint

**case 2**: $|\{c(v_i) : v_i \in N(v)\}| = 5$

Since all vertices of $v$ are uniquely coloured, we cannot construct $G_{n+1}$ such that it has a 5-colouring. We must reduce the colouring of $H_n$ so $v$ can be assigned the missing colour. For this let us consider a subgraph $H_{i,j} \subset H_n$, where $c(v_i)$ and $c(v_j)$ are the only possible colourings. This allows us to generalise the graph in such a way that vertices can be assigned different colours while maintaining a 5-colouring. Let us arbitrarily consider the vertices $v_1$ and $v_3$ from the previous illustration with $H_{1,3} \subset H_n$. We now consider two more cases:

**case 2a**: Suppose $v_1$ and $v_3$ are not connected by a path in $H_n$

All vertices $v_i, v_j \in V(H_{i,j})$ where $c(v_i) \neq c(v_j)$ can be swapped with respect to each other without the need to modify the colouring of any other vertices outside of $H_{i,j}$. Since both $v_1, v_3 \in V(H_{1,3})$, we are allowed to set $c(v_1) := c(v_3)$ (or vice versa), such that $G_{n+1} := H_n \cup \{v\}$ and $v$ gets assigned the missing colour (as explained in case 1).

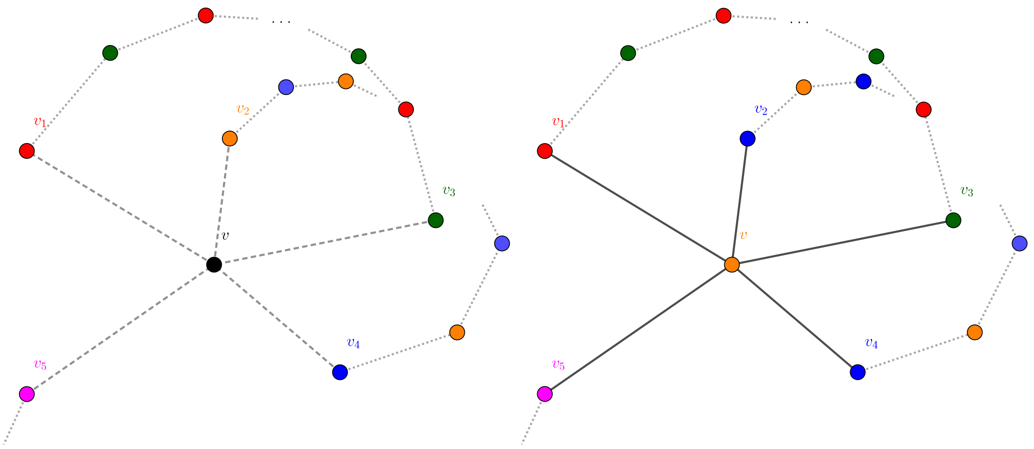


(a) Diagram of $N(v) \subseteq H_n$ (b) Diagram of $N(v) \subseteq G_{n+1}$

Figure 2: Comparison of the planar graph before and after recolouring and union of $v$

**case 2b**: Suppose $v_1$ and $v_3$ are connected by a path in $H_n$

Here the path of vertices connecting $v_1$ and $v_3$ in $H_{1,3} \cup \{v\}$ forms a cycle which contains $v_2$. We can safely assume that by our construction of $H_{1,3}$, the vertices $v_2$ and $v_4$ cannot be connected by a path in $H_{2,4}$, since this would disrupt the planarity of the graph $H_n$ (by intersecting edges). Because of this, $c(v_2) := c(v_4)$ (as in case 2a). Therefore $G_{n+1} := H_n \cup \{v\}$ and $v$ gets assigned the missing colour.




(a) Diagram of $N(v) \subseteq H_n$ (b) Diagram of $N(v) \subseteq G_{n+1}$

Figure 3: Comparison of the planar graph before and after recolouring and the union of $v$

Since we can always find a way to recolour $H_n$ so that $H_n \cup \{v\} =: G_{n+1}$ is 5-colourable, by the principle of mathematical induction, $G_n$ is 5-colourable $\forall n \in \mathbb{N}$ $\square$

# References

[1] Reinhard Diestel. *Graph Theory*. 2nd. New York: Springer-Verlag, 2000, pp. 87, 95–97. ISBN: 0387989765.

[2] Brown Kevin. *The Four Color Theorem*. URL: `https://www.mathpages.com/home/kmath266/kmath266.htm` (visited on 03/14/2019).

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