Fundamentals of Theoretical Computer Science

S 2023 Exercise Sheet I

Hand in: 2.5.2023, 10:00 a.m.

Note: Only the tasks marked with ⋆ are to be handed in. A total of 24 points is to be achieved. The tasks will be discussed/solved in the tutorials.

Task 1 — Induction proof (repetition)

Prove the following statements by induction:

(a) For any positive integer n holds:

$$\sum_{k=1}^{n} k^3 = \frac{n^2 (n+1)^2}{4}.$$

(b) The number of edges in a tree with n nodes is $n - 1$.

(c) (optional) Every connected graph G with at least two nodes has at least two nodes x, y such that both $G - x$ and $G - y$ are connected.

Task 2 — Kleene star

Let Σ be an alphabet. A formal language $L \subseteq \Sigma^*$ is a set of words over Σ. The *Kleene closure* L^* of L is the set of all words obtained by writing finitely many words from L in sequence (including the empty word ε). If $L_1, L_2 \subseteq \Sigma^*$ are formal languages, then the concatenation $L_1 \circ L_2$ of L_1 and L_2 is the set of all words that can be obtained by writing a word from L_2 after a word from L_1. Prove or disprove:

(a) $(L_1 \cup L_2)^* = L_1^* \cup L_2^*$;

(b) $(L_1 \cap L_2)^* = L_1^* \cap L_2^*$;

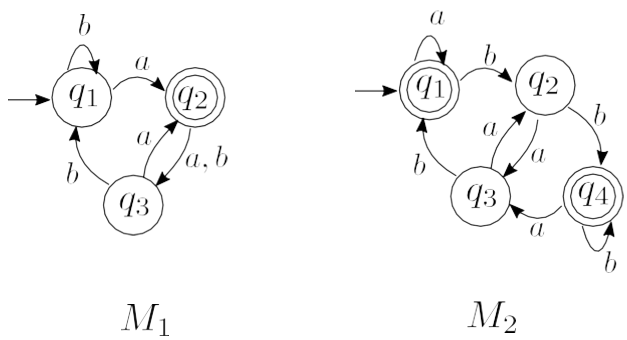
(c) $(L_1 \circ L_2)^* = L_1^* \circ L_2^*$;

(d) $(L)^{**} = L$;

(e) $L_1 \subseteq L_2 \Longrightarrow L_1^* \subseteq L_2^*$.

Task 3 **Deterministic Finite Automata (DEA)**

Consider the following state graphs of deterministic finite automata M_1 and M_2 .



(a) What are the start states of M_1 and M ?$_2$

(b) Give the formal description of M_1 and M_2 .

(c) What state sequence do M_1 and M_2 go through for input aabb?

(d) What sequence of states do M_1 and M_2 go through for input ε?

(e) Does M_1 or M_2 accept the input aabb?

(f) Does M_1 or M_2 accept the input ε?

Task 4 **DEA** 4×2 **points**

Give deterministic finite automata for the following languages over the alphabet $\{0, 1\}$. Briefly explain your automaton.

(a) The set of all words ending with 001.

(b) The set of all words that contain 0110 as a partial word.

(c) The set of all words that do **not** contain the partial word 01.

(d) The set of all words whose first and last letters are the same.

(e) The set of all words whose penultimate letter is 0. (⋆)

(f) The set of all words that contain an odd number of 1s.

(g) The set of all words that contain at least two 1s and exactly three 0s. (⋆)

(h) The set of all words that contain even numbers of 1s and end with 01. (⋆)

(i) The set of all words that contain neither 0001 nor 1110 as a partial word. (⋆)

(j) The set of all words except the empty word.

(k) The set of all words that do **not** contain exactly three 1s.

Task 5 **Regular Languages** + 4 **points** A language L is called a *regular language if there* exists a DEA that accepts L. Prove the following statements:

(a) Every finite language is regular.

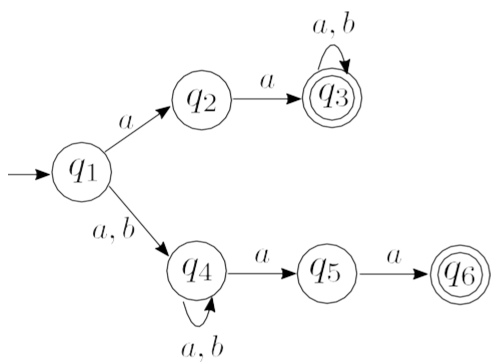
(b) Let L be a regular language. Then the language L' is also regular, which is obtained by reading each word in L from the back.

(c) Let L be a regular language. Then the language L' is also regular, which is obtained by doubling each letter in each word of L. (⋆)

(d) Let L_1 and L_2 be regular languages. Then the language $L_1 \circ L_2$ is also regular. (⋆)

Task 6 **Non-deterministic finite automaton (NEA) (⋆)** 4 **points**

Which language does the following machine recognize? Give reasons for your answer.



Task **7NEAs with multiple start states**

In the lecture we defined that non-deterministic finite automata (NEA) have exactly one initial state. In this task we want to consider the variant that there can be several initial states.

(a) Give a reasonable definition for NEAs with multiple initial states. Explain how the language recognized by such an automaton is defined.

(b) Let $L \subseteq \Sigma^*$ be a language. Show: There exists an NEA with exactly one initial state that accepts L exactly if there exists an NEA with multiple initial states that accepts L.

Task 8 Regular expressions 4 points

(a) Give regular expressions that describe some of the examples in Task 4.

(b) Give regular expressions describing the languages accepted by M_1 or M_2 (see Task 3).

(c) Specify a regular expression that describes the sums of positive decimal fixed-point numbers. For example, the strings 3.14 or even 3 + 4.2 + 7 + 1 should occur in the language. (⋆)

(d) Briefly describe in words sets characterized by the following regular expressions.
(i) $0^* (0^* 10^* 10)^{**}$;
(ii) $(00 \cup 11 \cup (01 \cup 10)(00 \cup 11)^* (01 \cup 10))^*$.

(e) Construct finite automata for the languages described by the regular expressions in point (d).

Task 9 Regular expressions

Two regular expressions x, y are called *equivalent*, written $x \sim y$, exactly if they describe the same language, i.e., $L(x) = L(y)$. Show:

(a) $(x \cup y)z \sim xz \cup yz$;

(b) $(x)^{**} \sim x$;

(c) $(x \cup y)^* \sim (x y)^{***}$;

(d) $x\emptyset \sim \emptyset$;

(e) $x\emptyset^* \sim x$.