**Analisi del tempo medio**

Sia $\alpha$ il rango del pivot

$$T_M(n) = \begin{cases} k_1 & \text{se } n \le 1 \\ \frac{1}{n} \sum_{\alpha=1}^{n} [T_M(q_\alpha) + T_M(n - q_\alpha) + k_2\, n] & \text{se } n > 1 \end{cases}$$

| rango | $q$ | $\mid [p,\, q] \mid$ |
|---|---|---|
| 1 | $p$ | 1 |
| 2 | $p$ | 1 |
| 3 | $p+1$ | 2 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | $p+n-2$ | |

$\frac{1}{n}$ perché sto calcolando la media $\frac{\sum_{\alpha=1}^{n}}{n}$

$k_2\, n$ viene da $\Theta(n)$ (tempo di *Partition*)

$q_\alpha$ è l'indice del pivot quando il rango è $\alpha$

$$T_M(n) = \frac{1}{n} \sum_{\alpha=1}^{n} [T_M(q_\alpha) + T_M(n - q_\alpha) + k_2\, n]$$

$$T_M(n) = \frac{1}{n} \left[ T_M(q_1) + T_M(n - q_1) + k_2\, n + \sum_{\alpha=2}^{n} [T_M(q_\alpha) + T_M(n - q_\alpha) + k_2\, n] \right]$$

Estraggo il primo elemento della sommatoria.

$\alpha$ partirà da 2

$q_1 = 1$ dalla tabella sopra

$$\underbrace{T_M(1)}_{k_1} + \underbrace{T_M(n-1)}_{k_3\, n^2} + k_2\, n$$

approssimo ad un andamento quadratico

$$k_1 + k_3\, n^2 + k_2\, n = n\left(\frac{k_1}{n} + k_3\, n + k_2\right)$$

$$T_M(n) = \underbrace{\frac{1}{n}\left[n\left(\frac{k_1}{n} + k_3\, n + k_2\right)\right]}_{\text{semplifico le due } n \text{ a sinistra}} + \frac{1}{n} \sum_{\alpha=2}^{n} [T_M(q_\alpha) + T_M(n - q_\alpha) + k_2\, n] =$$

$$= \underbrace{\left(\frac{k_1}{n} + k_3\, n + k_2\right)}_{k_4} + \frac{1}{n} \sum_{\alpha=2}^{n} [T_M(q_\alpha) + T_M(n - q_\alpha) + k_2\, n]$$

$$= k_4 + \frac{1}{n} \sum_{\alpha=2}^{n} [T_M(q_\alpha) + T_M(n - q_\alpha) + k_2\, n] \quad \le$$

$$\le\ k_4 + \frac{1}{n} \sum_{\alpha=2}^{n} [T_M(\alpha - 1) + T_M(n - (\alpha - 1)) + k_2\, n] \ =$$

Imposto $\alpha - 1 = i$

$$k_4 + \frac{1}{n} \sum_{i=1}^{n-1} [T_M(i) + T_M(n - i) + k_2\, n] =$$

$$k_4 + \frac{1}{n}\, k_2\, n(n-1) + \frac{1}{n} \sum_{i=1}^{n-1} [T_M(i) + T_M(n - i)] =$$

Ho portato fuori dalla sommatoria $k_2\, n$ per $(n-1)$ volte

$$(k_4\, k_2)n - k_2 + \frac{1}{n} \sum_{i=1}^{n-1} [T_M(i) + T_M(n - i)] =$$

$$(k_4\, k_2)n - k_2 + \frac{2}{n} \sum_{i=1}^{n-1} T_M(i)$$

$\frac{1}{n} \sum_{i=1}^{n-1} [T_M(i) + T_M(n-i)] = \frac{2}{n} \sum_{i=1}^{n-1} T_M(i)$ perche $T_M(i)$ compare due volte nella sommatoria

$i = 1,\quad T(i) = T(1) \qquad i = n-1,\quad T(n-i) = T(n-(n-1)) = T(1)$

$i = 2,\quad T(i) = T(2) \qquad i = n-2,\quad T(n-i) = T(n-(n-2)) = T(2)$

e così via

$$T_M(n) = \begin{cases} k_1 & \text{se } n \le 1 \\ (k_4\, k_2)n - k_2 + \frac{2}{n} \sum_{i=1}^{n-1} T_M(i) & \text{altrimenti} \end{cases}$$

Ora si vuole dimostrare (per **induzione**) che $T(n) \leq c\, n\, log\, n$ oppure che $T(n) = O(n\, log\, n)$

**Passo base**: $n = 2$

$$T_M(2) = (k_4\, k_2)2 - k_2 + \frac{2}{2} \underbrace{\sum_{i=1}^{2-1} T_M(i)}_{T_M(1)=k_1} = (k_4\, k_2)2 - k_2 + k_1$$

Per una costante $c \geq (k_4\, k_2)2 - \frac{k_2}{2} + \frac{k_1}{2}$

$T_M(2) \leq c\, 2\, log\, 2$

**Ipotesi induttiva**: $\forall\, p \leq n, \quad T_M(p) \leq c\, p\, log\, p \Rightarrow T_M(n+1) \leq c\, n\, log\, n$

Sia $(k_4\, k_2) = k_5$

$$T_M(n) = k_5\, n - k_2 + \frac{2}{n} \sum_{p=1}^{n-1} T_M(p) \leq k_5\, n - k_2 + \frac{2}{n} c \underbrace{\sum_{p=1}^{n-1} [p\, log\, p]}_{\text{sviluppiamo questa sommatoria a parte}}$$

$$\sum_{p=1}^{n-1} [p\, log\, p] = \underbrace{\sum_{p=1}^{\frac{n-1}{2}-1} [p\, log\, p]}_{log(p) \mapsto log\, \frac{n}{2}} + \underbrace{\sum_{p=\frac{n-1}{2}}^{n-1} [p\, log\, p]}_{log(p) \mapsto log\, (n)} \leq$$

divido la sommatoria in due per cercare una buona approssimazione **superiore**

sia $h = \dfrac{n-1}{2}$

$$\leq \sum_{p=1}^{h-1} \left[p\, log\, \frac{n}{2}\right] + \sum_{p=h}^{n-1} [p\, log\, n] = (log\, n - \underbrace{log\, 2}_{=1}) \sum_{p=1}^{h-1} p + log\, n \sum_{p=h}^{n-1} p =$$

$$log\, n \sum_{p=1}^{h-1} p - \sum_{p=1}^{h-1} p + log\, n \sum_{p=h}^{n-1} p = log\, n \underbrace{\left( \sum_{p=1}^{h-1} p + \sum_{p=h}^{n-1} p \right)}_{\underbrace{\sum_{p=1}^{n-1} p}_{\frac{n(n-1)}{2}}} - \underbrace{\sum_{p=1}^{h-1} p}_{\frac{h(h-1)}{2}} =$$

$$(log\, n) \left( \frac{n(n-1)}{2} \right) - \frac{h(h-1)}{2} \leq \frac{1}{2} (log\, n)\, n^2 - \frac{n^2}{8}$$

Svolta quella sommatoria, abbiamo infine:

$$T_M(n) = k_5\, n - k_2 + \frac{2}{n} \sum_{p=1}^{n-1} T_M(p) \leq k_5\, n - k_2 + \frac{2c}{n} \left[ \frac{n^2}{2} (log\, n) - \frac{n^2}{8} \right] = c\, n\, log\, n + k_5\, n - \frac{c\, n}{4} - k_2$$

Si ricorda che si vuole dimostrare che $T_M(n) \leq c\, n\, log\, n$, ovvero che $c\, n\, log\, n + k_5\, n - \frac{c\, n}{4} - k_2 \leq c\, n\, log\, n$

Questo è vero quando $k_5\, n - \frac{c\, n}{4} - k_2 \leq 0 \Rightarrow c \geq 4k_5 - \frac{4k_2}{n}$