Exercise 2. (Concrete mixing)

Find the best possible approximation $\boldsymbol{x} = (x_1, ..., x_m)$ *of the ideal mixture,* $\boldsymbol{c} = (c_1, ..., c_n)$*, by using the material from the* m *mines.*

Show that the optimal mixture will be the point $\boldsymbol{x}$ *such that:*

$$\min \ (C\boldsymbol{x} - \boldsymbol{c})^T(C\boldsymbol{x} - \boldsymbol{c}),$$
$$s.t. \quad \sum_{j=1}^{m} x_j = 1, \quad \text{and} \quad x_j \geq 0,$$

where the matrix $C = (C_1, ..., C_m)$ *has* C_j *as columns, and* $\boldsymbol{c} = (c_1 \cdots c_n)^T$.

Solution

We assume that $n > m$ (more gravel sizes than mines), and write C as

$$C = \begin{pmatrix} c_1^1 & c_1^2 & \dots & c_1^m \\ c_2^1 & c_2^2 & \dots & c_2^m \\ \vdots & \vdots & \ddots & \vdots \\ c_n^1 & c_n^2 & \dots & c_n^m \end{pmatrix},$$

where c_i^j is the fraction of gravel of size i provided by the mine M_j to the mixture.

Let x_j be the fraction of material rovided by the mine M_j to the mixture and $\boldsymbol{x} = (x_1, x_2, ..., x_m)^T$, so

$$x_j \geq 0, \quad \text{and} \quad \sum_{j=1}^{m} x_j = 1.$$

Then, the optimal concrete mixture must satisfy

$$\begin{pmatrix} c_1^1 & c_1^2 & \dots & c_1^m \\ c_2^1 & c_2^2 & \dots & c_2^m \\ \vdots & \vdots & \ddots & \vdots \\ c_n^1 & c_n^2 & \dots & c_n^m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \Leftrightarrow \quad C\boldsymbol{x} = \boldsymbol{c}.$$

Since $n > m$, the above linear system has more equations than unknowns, so we must look for the least squares solution $\boldsymbol{c}^*$

$$\boldsymbol{c}^* = \min_x \|C\boldsymbol{x} - \boldsymbol{c}\|,$$

or, equivalently by

$$\boldsymbol{c}^* = \min_x \|C\boldsymbol{x} - \boldsymbol{c}\|^2 = \min_x (C\boldsymbol{x} - \boldsymbol{c})^T(C\boldsymbol{x} - \boldsymbol{c}),$$

with $\boldsymbol{x}$ such that

$$x_j \geq 0, \quad \text{and} \quad \sum_{j=1}^{m} x_j = 1.$$