Exercise 1: Smallest area problem

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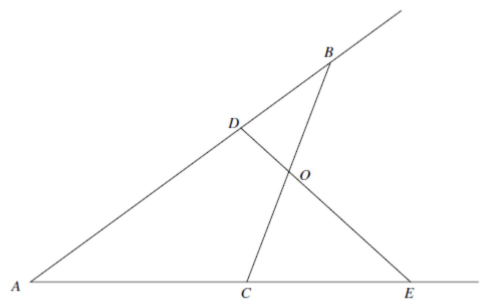
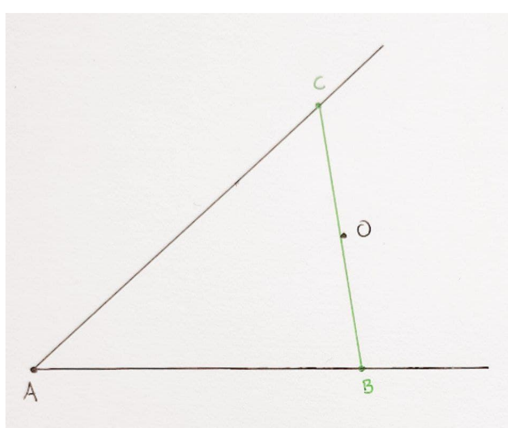
Given an angle with vertex A and a point O in its interior. Pass a line BC through the point O that cuts off from the angle a triangle of minimal area.

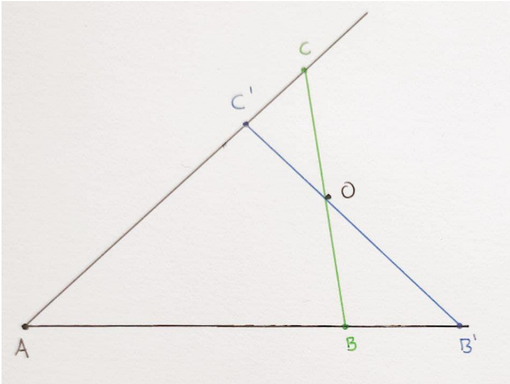


Hint: proof that for a triangle of minimal area the segments OB and OC should be equal.

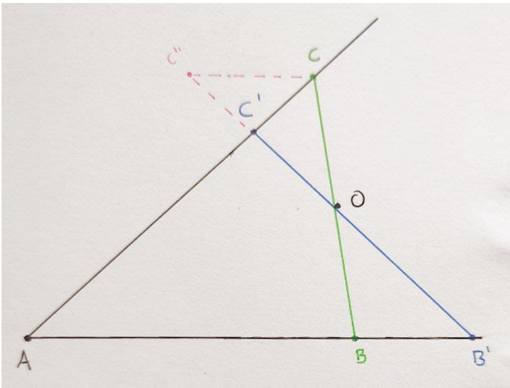
We will start by drawing an arbitrary triangle that fulfills the conditions given in the exercise's hint. From there, we will prove (with drawings) that no other triangle has smaller area.



Now that we have our triangle ABC such that BC contains the point O and such that the segments OC and OB are equal in length, we will define another arbitrary triangle $AB'C'$ such that $B'C'$ contains O and show that its area is larger than the original triangle's area.



Both triangles share the area defined by $ABOC'$ in this case. Therefore, for us to show that ABC has a smaller area than $AB'C'$, we only need to prove that the area of OCC' (only contained in the allegedly minimal area triangle) is smaller than the area of OBB' (only contained in the new cantidate triangle).



Since both triangles share an angle (the one on vertex O) and one of that angle's sides (OB and OC are equally long), the only way for both triangles to have the same area is to be exactly equal.[1] We define our new C'' point such that the length of OC'' and OB is the same.

It is clear now that $Area(OCC') < Area(OBB')$, and therefore: $Area(ABC) < Area(AB'C')$. It could be proved analytically that this happens for any arbitrary triangle $AB'C'$, or just be shown by drawings repeating the same steps we followed during this *proof*.

[1]Due to the following formula of a triangle's area: $Area = \frac{absin(C)}{2}$ where C is a given angle, and a and b are the triangle's sides on that angle.