Exercise 10.

Prove that the number of faces of dimension p of a n-dimensional simplex is equal to

$$\binom{n+1}{p+1} = \frac{(n+1)!}{(p+1)!(n-p)!}.$$

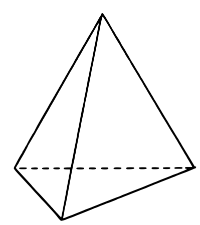
Solution

Consider an n-dimensional simplex S_n determined by $n+1$ points. Then, its elements are simplexes formed by subsets of these $n+1$ points.

In this way, to get a p-dimensional simplex, we need to select $p+1$ of the $n+1$ points, and this can be done in $\binom{n+1}{p+1}$ different forms.

$$\binom{n+1}{p+1}.$$

For example, let S_3 be the tetrahedron shown below.



It has 4 vertex, $\binom{4}{6} = 6$ edges and $\binom{4}{3} = 4$ triangles.