# NLA 2021-2022
# Special linear systems

Martin Sombra

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# Band matrices

$A$ is a *band matrix* with *lower bandwith* $b_L$ and *upper bandwith* $b_U$ if

$$a_{i,j} = 0$$

whenever $i > j - b_L$ or $i < j + b_U$:

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,b_U+1} & & \\ \vdots & & & \ddots & \\ a_{b_L+1,1} & & & & a_{n-b_U,n} \\ & \ddots & & & \vdots \\ & & a_{n,n-b_L} & \cdots & a_{n,n} \end{bmatrix}.$$

# GE on band matrices

**Key observation:** the Schur complement of a band matrix is also banded, with the same upper and lower bandwiths

$\rightsquigarrow$ GEWP preserves the band structure:

$$A = L\,U$$

where $L$ is unit lower triangular with lower bandwith $b_L$ and $U$ is upper triangular with upper bandwith $b_U$

This factorization can be computed with

$$2\,n\,b_L\,b_U + O(n\,(b_L + b_U)) \text{ flops}$$

# Example

Let

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \qquad (b_L = b_U = 1)$$

Then

$$L(1:4,1) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & -1 & 0 & 0 \end{bmatrix}$$

The *Schur complement* is given by $a_{j,k} \leftarrow a_{j,k} - l_{j,1}\, u_{1,k}$ for $j, k = 2, 3, 4$ and so

$$S_1 = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -2 & 1 \\ 0 & 3 & 4 \end{bmatrix}$$

# Example (cont.)

Next

$$L(2:4,2) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad U(2,2:4) = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

and then

$$L(3:4,3) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad U(3,3:4) = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad S_3 = \begin{bmatrix} 1 \end{bmatrix}$$

Finally

$$L(4:4,4) = \begin{bmatrix} 1 \end{bmatrix} \quad \text{and} \quad U(4,4:4) = \begin{bmatrix} 1 \end{bmatrix}$$

# Example (cont.)

Thus

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} = L\,U = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ & 1 & 3 & 0 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

# GEPP on band matrices

GEPP can exploit band structure, but in a more involved way:

$$A = P\,L\,U$$

where $U$ is banded with upper bandwith $b_L + b_U$ and $L$ has at most $b_L + 1$ nonzero entries per column

**Why?**

At each step pivoting can only be done within the first $b_L$ rows, and later permutations can reorder the earlier columns of $L$

# Example

Let again

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

and swap rows 1 and 2:

$$A = \begin{bmatrix} 4 & -1 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Then

$$L(1:4,1) = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, \quad U(1,1:4) = \begin{bmatrix} 4 & -1 & 3 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} \frac{-1}{2} & \frac{-3}{2} & 0 \\ -1 & -2 & 1 \\ 0 & 3 & 4 \end{bmatrix}$$

# Example (cont.)

We then swap the rows 2 and 3 of $A$ to obtain

$$L(2:4,2) = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad U(2,2:4) = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 3 & 4 \end{bmatrix}$$

Swap the rows 3 and 4 of $A$:

$$L(3:4,3) = \begin{bmatrix} 1 \\ \frac{-1}{6} \end{bmatrix}, \quad U(3,3:4) = \begin{bmatrix} 3 & 4 \end{bmatrix}, \quad S_3 = \begin{bmatrix} \frac{1}{6} \end{bmatrix}$$

Finally

$$U(4,4:4) = \begin{bmatrix} \frac{1}{6} \end{bmatrix}$$

# Example (cont.)

Hence

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{6} & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 & 0 \\ & -1 & -2 & 1 \\ & & 3 & 4 \\ & & & \frac{1}{6} \end{bmatrix}$$

$L$ has at most 2 nonzero elements per column, and $U$ has upper bandwith 2

# Sparse matrices

GEPP does not preserve the sparse structure:

$$
\begin{bmatrix}
1 & 0.1 & 0.1 & \dots & 0.1 \\
0.1 & 1 & 0 & \dots & 0 \\
0.1 & 0 & 1 & & 0 \\
\vdots & \vdots & & \ddots & \vdots \\
0.1 & 0 & 0 & \dots & 1
\end{bmatrix}
$$
$$
= \begin{bmatrix}
1 & & & & \\
0.1 & 1 & & & \\
0.1 & -0.01 & 1 & & \\
\vdots & \vdots & & \ddots & \\
0.1 & -0.01 & \dots & \dots & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0.1 & 0.1 & \dots & 0.1 \\
 & 0.99 & -0.01 & \dots & -0.01 \\
 & & 0.99 & & -0.01 \\
 & & & \ddots & \vdots \\
 & & & & 0.99
\end{bmatrix}
$$