Exercise 6

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Optimization: 1^{st} November 2021

Exercise 5. To be delivered before 2-XI-2021 as: `Ex05-YourSurname.pdf`

Solve the linear system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

using the conjugate-gradient method.

The Conjugate-Gradient method is useful to minimize functions rather than to solve linear systems. We are going to use in our advantage that the given linear system has the following form: $Ax = b \equiv Ax - b = 0$. $Ax - b$ is the shape of the gradient (∇f) of a quadratic function f when A is definite positive and symmetric, which is the kind of function the mentioned method is able to minimize.

Also, see how $Ax - b = 0$ is equivalent to finding the critical points of the function that has $Ax - b$ as gradient. Therefore, we will define our new function f and minimize it in order to solve the system using the Conjugate-Gradient method.

In order to compute the function to minimize, for any $C \in \mathbb{R}$:

$$\begin{aligned} \nabla f(x, y, z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \nabla f(x, y, z) &= \begin{pmatrix} x - 1 \\ 2y - 1 \\ 3z - 1 \end{pmatrix} \\ f(x, y, z) &= \frac{1}{2}x^2 - x + y^2 - y + \frac{3}{2}z^2 - z + C \end{aligned}$$

As we expected, our function f is quadratic. $f(x, y, z) = \frac{1}{2}x^T Ax + b^T x + c$

where:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$
$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$c = C$$

We will now follow the conjugate gradient method:

$$\begin{aligned}
x^0 &= (0,0,0) \\
\nabla f(x^0) &= (-1,-1,-1)^T \\
z^1 &= -\nabla f(x^0) = (1,1,1)^T \\
\alpha_1^* &= -\frac{(z^1)^T \nabla f(x^0)}{(z^1)^T A z^1} = -\frac{-3}{6} = \frac{1}{2} \\
x^1 &= x^0 + \alpha_1^* * z^1 = (0,0,0)^T + \frac{1}{2} * (1,1,1)^T = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T \\
\nabla f(x^1) &= (-\frac{1}{2}, 0, \frac{1}{2})^T \\
z^2 &= -\nabla f(x^1) + \frac{(z^1)^T A \nabla f(x^1)}{(z^1)^T A (z^1)} z^1 = (\frac{2}{3}, \frac{1}{6}, -\frac{1}{3})^T \\
\alpha_2^* &= -\frac{(z^2)^T \nabla f(x^1)}{(z^2)^T A z^2} = \frac{3}{5} \\
x^2 &= x^1 + \alpha_2^* * z^2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T + \frac{3}{5} * (\frac{2}{3}, \frac{1}{6}, -\frac{1}{3})^T = (\frac{9}{10}, \frac{3}{5}, \frac{3}{10})^T \\
\nabla f(x^2) &= (-\frac{1}{10}, \frac{1}{5}, \frac{1}{10})^T \\
z^3 &= -\nabla f(x^2) + \frac{(z^2)^T A \nabla f(x^2)}{(z^2)^T A (z^2)} z^2 = (\frac{9}{50}, -\frac{9}{50}, \frac{3}{50})^T \\
\alpha_3^* &= -\frac{(z^3)^T \nabla f(x^2)}{(z^3)^T A z^3} = \frac{5}{9} \\
x^3 &= x^2 + \alpha_3^* * z^3 = (\frac{9}{10}, \frac{3}{5}, \frac{3}{10})^T + \frac{5}{9} * (\frac{9}{50}, -\frac{9}{50}, \frac{3}{50})^T = (1, \frac{1}{2}, \frac{1}{3})^T
\end{aligned}$$

x^3 is the solution to our linear system. If we keep iterating, it will show that we have to move in the direction $(0,0,0)$, which means we are already at the critical point and shouldn't move any further.