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SVM and their use in IR

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May x, 2023

Outline

1. SVM over linearly separable data
2. SVM over non linearly separable data
3. SVM use in IR

Vector Space

Suppose you have to classify whether a document is *relevant* or not. We can think to use *terms* as features to divide properly the data. We will use a *t-dimensional* vector space to represent our documents.

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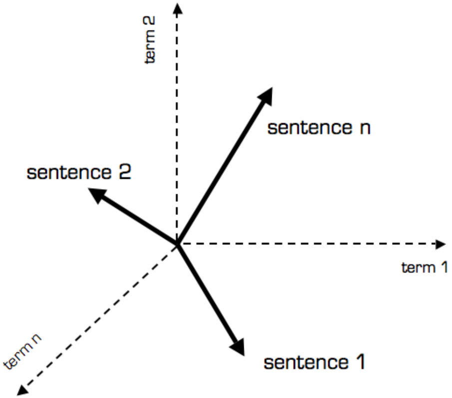

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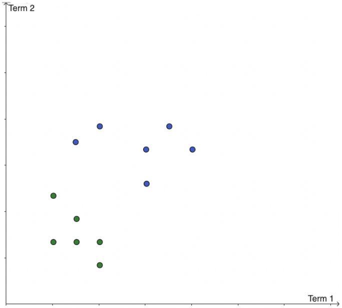
Data can be separated using a *decision boundary*, which is an *hyperplane*.



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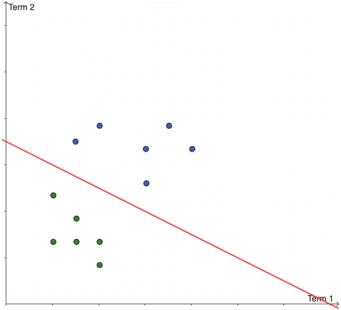

In the previous example the *decision boundary* is a line, represented by the equation $a + bt_1 + ct_2 = 0$.
We can introduce two parallels hyperplanes (lines) to the decision boundary, called *support vectors* whose equations are

$$\begin{cases} a + bt_1 + ct_2 = 1 \\ a + bt_1 + ct_2 = -1 \end{cases} \tag{1}$$

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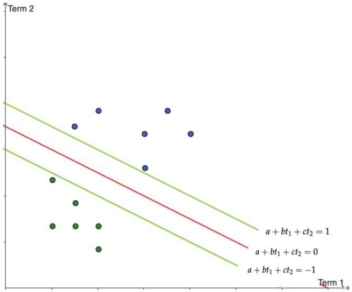

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We can consider two points x_1, x_2 that lie respectively on the two support vectors, their distance is $\lambda \|b\| = \frac{2}{\sqrt{b^T b}}$.

SVM are used to find the *maximum margin linear classifier*, thus we want to maximize the margin.

Cost Function

Remembering we want to classify documents, our goal is to find specific b s.t. given a document x belonging to class y the decision boundary behave the following:

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We can define the *cost function* as a system of equations.

$$\begin{cases} \min_{b,a} \frac{\sqrt{b^T b}}{2} \\ \textit{subject to} \quad y_i(b^T x_i + a) \geq 1 \quad \forall x_i \end{cases} \tag{4}$$

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The new cost function will be

$$\begin{cases} \min_{b,a} \frac{\sqrt{b^T b}}{2} + C \sum_i \xi_i \\ \textit{subject to} \quad y_i(b^T x_i + a) \geq 1 - \xi_i \quad \textit{and } \xi_i > 0 \quad \forall x_i \end{cases} \tag{5}$$

The larger is C the stricter the classification is, since a larger C will give more evidence to slack variables.

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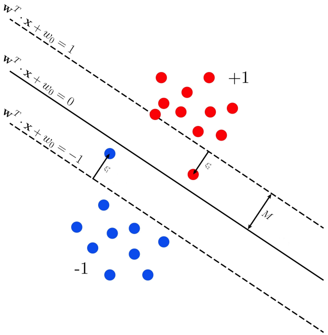

Non linearly separable data

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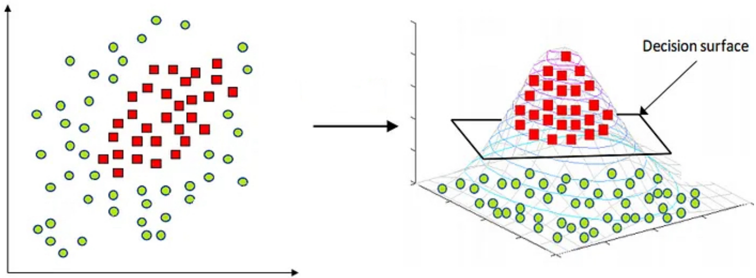

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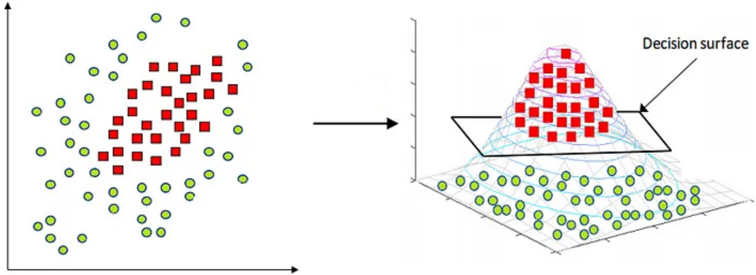

But augmenting dimensions costs a lot…

Kernel Trick

Consider the function $\phi : \mathbb{R}^3 \mapsto \mathbb{R}^{10}$ used to map points in a new vector space. Calculating the *similarity* $\phi(x_i)^T \phi(x_j)$ between each point may be intractable.

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Instead of doing the complex computations in the 10-dimensional space, we reach the same result within the 3-dimensional space by calculating the dot product.

Gaussian Kernel

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The result will be a mapping for each point in a n-dimensional space. Finally, we can identify an hyperplane which can divide correctly the two classes of points.

Hybrid use

An hybrid technique used in the IR field will be presented; it uses two components:

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K-means algorithm is one of the most used clustering algorithms. It consists of partitioning unlabeled objects into k classes, where k is no predefined.
A brief explanation will follow.

K-means

The aim is to create clusters that contain *similar documents*.
Given a training set $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$, the algorithm used is the following:

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$K \leftarrow randomValue$ ▷ multiple K will be tried

procedure K-MEANS(K)

  `initialize cluster centroids` $\mu_1, \dots, \mu_k$ `ramdomly.`

  repeat until convergence{

    for $i \leftarrow 1$ to m **do**

      $c^{(i)} = arg\ min_j \|x^{(i)} - \mu_j\|$ ▷ Closest centroid to $x^{(i)}$

    end for

    for $k \leftarrow 1$ to K **do**

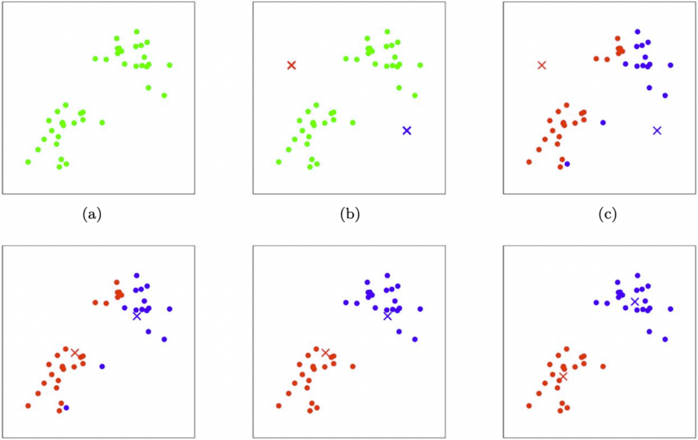
      $\mu_j = \frac{\sum_{i=1} m1\{c^{(i)}=j\}x^{(i)}}{\sum_{i=1} m1\{c^{(i)}=j\}}$ ▷ New centroid of cluster k

    end for

  }

end procedure

Visual explanation



(a) (b) (c) (d) (e) (f)