Department of Mathematics, University of Udine

SVM and their use in IR

Lorenzo Zanolin.
lorenzo.zanolin@spes.uniud.it

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Outline

1. SVM over linearly separable data
2. SVM over non linearly separable data

Suppose you have to classify whether a document is *relevant* or not. We can think to use *terms* as features to divide properly the data. We will use a *t-dimensional* vector space to represent our documents.

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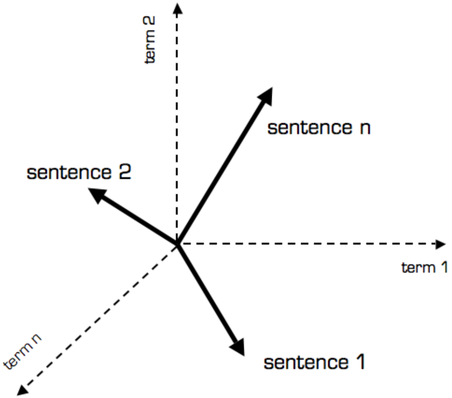
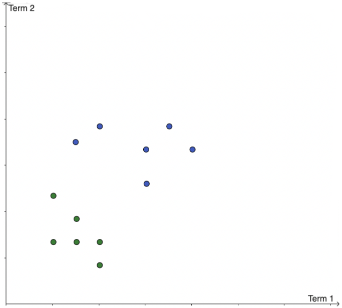


Classification

For simplicity, we can reason using a 2D Space.
Data can be separated using a *decision boundary*, which is an *hyperplane*.



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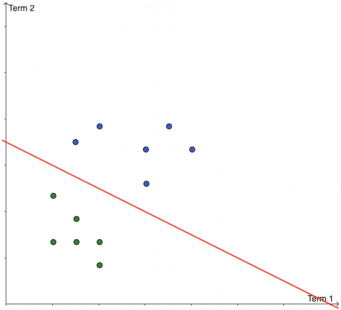

Support Vectors

In the previous example the *decision boundary* is a line, represented by the equation $a + bt_1 + ct_2 = 0$.
We can introduce two parallels hyperplanes (lines) to the decision boundary, called *support vectors* whose equations are

$$\begin{cases} a + bt_1 + ct_2 = 1 \\ a + bt_1 + ct_2 = -1 \end{cases} \tag{1}$$

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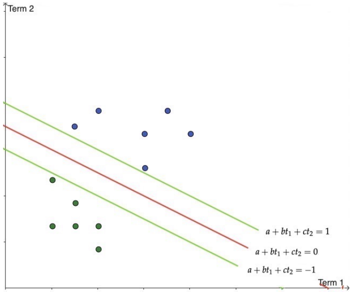

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We can consider two points x_1, x_2 that lie respectively on the two support vectors, their distance is $\lambda \|b\| = \frac{2}{\sqrt{b^T b}}$.

SVM are used to find the *maximum margin linear classifier*, thus we want to maximize the margin.

Cost Function

Remembering we want to classify documents, our goal is to find specific b s.t. given a document x belonging to class y the decision boundary behave the following:

$$\begin{cases} b^T x + a \geq 1 & \text{\textit{if} } y = 1 \\ b^T x + a \leq -1 & \text{\textit{if} } y = -1 \end{cases} \tag{3}$$

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We can define the *cost function* as a system of equations.

$$\begin{cases} \min_{b,a} \frac{\sqrt{b^T b}}{2} \\ \textit{subject to} \quad y_i(b^T x_i + a) \geq 1 \quad \forall x_i \end{cases} \tag{4}$$

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The new cost function will be

$$\begin{cases} \min_{b,a} \frac{\sqrt{b^T b}}{2} + C \sum_i \xi_i \\ \textit{subject to} \quad y_i(b^T x_i + a) \geq 1 - \xi_i \quad \textit{and}\ \xi_i > 0 \quad \forall x_i \end{cases} \tag{5}$$

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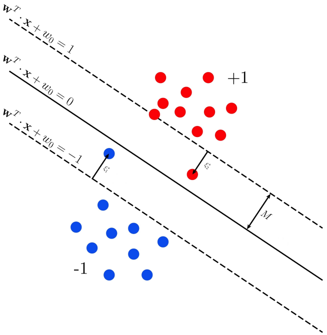

Non linearly separable data

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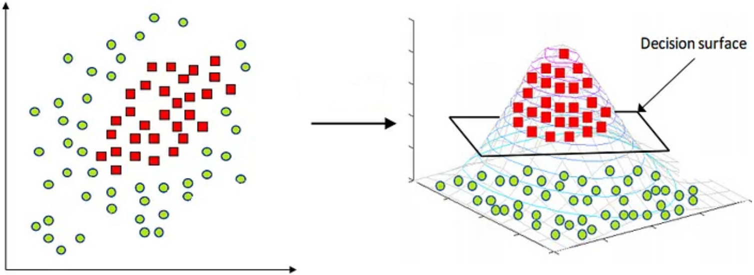

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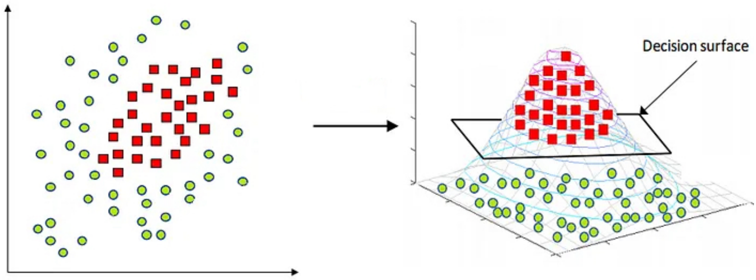

But augmenting dimensions costs a lot...

Consider the function $\phi : \mathbb{R}^3 \mapsto \mathbb{R}^{10}$ used to map points in a new vector space. Calculating the *similarity* $\phi(x_i)^T \phi(x_j)$ between each point may be intractable.

Kernel Trick

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Instead of doing the complex computations in the 10-dimensional space, we reach the same result within the 3-dimensional space by calculating the dot product.

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The result will be a mapping for each point in a n-dimensional space. Finally, we can identify an hyperplane which can divide correctly the two classes of points.