FYS 3150 - Project 3

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In this project the evolution of many-body configurations in the context of our Solar system is studied numerically. Starting by considering a simple two-body system (Earth-Sun), it is used to check the efficiency and stability of the two ODE "*solvers*": Euler and Velocity Verlet (VV). This last one proves itself to be more suitable to execute simulations. The escape velocity for Earth is analysed and is found to be in agreement with theoretical predictions. The radial dependence, r^{-b}, of Newtonian gravity is investigated in conjunction with the stability of orbits by modifying b, for different integers . Later we switch to a three-body system by adding Jupiter and see how it influences the previous system, additionally its mass is varied by a factor of 10 and then 1000. Eventually, we build a model of the Solar system with all the planets and Pluto, and simulate its time evolution. Furthermore, the historically relevant problem of the perihelion precession of Mercury's orbit is discussed and analysed by introducing a relativistic correction and calculating the perihelion angle $\theta_p = (43.206 \pm 0.27)''$, which is in agreement with the value predicted by General Relativity.

- URL to GitHub folder of the code: `https://github.com/DavideSaccardo/Project3`

I. INTRODUCTION

We want to analyse the solar system from a numerical point of view. But first let us consider the simple case of a two body problem, where Earth is attracted by the Sun according to the Newtonian force:

$$\vec{F} = -G\,\frac{M_\odot\, M_{Earth}}{r^3}\,\vec{r} \tag{1}$$

where $M_\odot$ is the solar mass, $\vec{r}$ has the direction from Sun to Earth and its module is the distance between the two bodies, $G = 6.67 \times 10^{-11}\,\mathrm{Nm}^2/\mathrm{kg}^2$ is the gravitational constant.

We implement an object oriented code able to solve coupled ODE$_s$ relative to a system with two solver: one based on the Euler algorithm and the other based on the Velocity Verlet (VV). In our case, the code is specialized to the solar system, but it can be easily applied to other systems (molecular dynamics). After testing the two solvers with the Sun-Earth system with respect to the stepsize dt and performances, we identify in Velocity Verlet a better solver for our simulations since it is more stable even for bigger step-sizes and guarantees conservation of mechanical energy and angular momentum even for a longer period of time. Hence we analyze the two-body system: we find numerically an interval for escape velocity and compare it to the analytical result and then we modify the Newtonian potential to get to a short-range one by changing its dependence on the radial distance r.

Later we add Jupiter and we switch to a three-body problem. At first, we study the problem with Sun, Earth, Jupiter, with the Sun fixed as CM to the origin of reference system. The presence of Jupiter (the most massive planet of solar system) allows us to understand how the other planets affect the motion of Earth. Then, we fix the center of mass of the system as origin of our reference frame and we give the Sun a velocity to make the linear momentum of the system to be 0. In this way, we are able to consider a real three-body problem, which is then enlarged to include all the other planets (plus Pluto). Each time the stability of the system is analysed through the conservation of energy and angular momentum.

II. METHODS AND ALGORITHMS

In our approximation all celestial objects are governed by Newtonian force eq.(1). Following this reasoning, firstly we express all the planets' mass in terms of the planetary mass using as unit of measure the solar mass $\mathrm{M}_\odot \simeq 2 \times 10^{30}\,\mathrm{kg}$ and then we scale the equations. As for the masses, we decide to use the astronomical units also for lengths ($1\,\mathrm{AU} = 149\,597\,870\,700\,\mathrm{m} \simeq 150 \times 10^6\,\mathrm{km}$), which is approximately the distance between Earth and Sun. As regards time, considering that our simulation are going to last for centuries, we decide to use as unit the year (yr).

At this point, let us consider Earth-Sun system and assume the Earth's orbit to be circular with a radius $r = 1\,\mathrm{AU}$, its acceleration is given by:

$$a = \frac{v^2}{r} = \frac{|\vec{F}|}{M_{Earth}} = \frac{G\,M_\odot}{r^2},$$

while its velocity can be written as

$$v = \frac{2\,\pi\,r}{T} = \frac{2\,\pi\,1\,\mathrm{AU}}{1\,\mathrm{yr}}, \tag{2}$$

combining the two expressions, we can express the constant $G\,\mathrm{M}_\odot$ as

$$G\,\mathrm{M}_\odot = \frac{4\,\pi^2\,(1\,\mathrm{AU})^3}{(1\,\mathrm{yr})^2}, \tag{3}$$

According to this we rewrite eq.(1):

$$\vec{F} = -\frac{M_{Earth}}{r^3}\,\vec{r}$$

and we will use this for all the numerical analysis.

Let us consider an inertial reference frame (x, y, z), and lets us fix Sun's position in $(0, 0, 0)$. In general, using second law of mechanics the differential equations for the motion of the Earth along $x-$direction are:

$$\begin{cases} a_x = \dfrac{d}{dt}v_x = -\dfrac{4\,\pi^2}{r^3}\,(x - x_\odot) \\ v_x = \dfrac{d}{dt}x \\ x(t=0) = x_0 \\ v_x(t=0) = v_{x_0}. \end{cases} \tag{4}$$

Along y and z the differential equations assume the same structure. Now, we can add one more planet, for example Jupiter, which is the biggest in the solar system, with a mass $M_J = 9.5 \times 10^{-4}\,\mathrm{M}_\odot$. To abbreviate notation we keep only the first letter of the planets. Hence, the force acting on Earth from Jupiter is

$$F_x^{EJ} = -G\,\frac{M_J\,M_E}{r_{EJ}^3}(x_E - x_J),$$
$$r_{EJ} = \sqrt{(x_E - x_J)^2 + (y_E - y_J)^2 + (z_E - z_J)^2}.$$

The total acceleration of Earth in the $x-$direction (analogous structure for y, z) is

$$\begin{aligned} a_x^E = \frac{dv_x^E}{dt} &= -G\frac{M_\odot}{r^3}x_E - G\frac{M_J}{r_{EJ}^3}(x_E - x_J) \\ &= -\frac{4\,\pi^2\,x_E}{r_E^2} - \frac{4\,\pi^2\,M_J/M_\odot}{r_{EJ}^2}(x_E - x_J) \end{aligned}$$

where we assume the Sun to be fixed in the origin of our reference system. The structure of Jupiter's equations is the same. Per each planet there are 3 differential equations referred to the position and other 3 referred to velocity. We can add other planets following the above structure.

To solve the equations, let us start to discretize eq.(4):

$$\begin{aligned} x(t) &\to x(t_i) = x_i \\ t &\to t_i = t_0 + i \cdot dt \\ dt &= \frac{t_{final} - t_0}{n}, \end{aligned}$$

where dt is the stepsize, and n = numberOfSteps in our code. Taking into account this discretization we introduce two methods to solve the ODE$_s$:

a. Euler algorithm It uses Taylor approximation in order to find the subsequent position x_{i+1} and velocity v_{i+1} after a step size dt:

$$\begin{aligned} x_{i+1} &= x_i + dt\,x_i^{(1)} + dt^2\,\frac{x_i^{(2)}}{2} + O(dt^3) \\ v_{i+1} &= v_i + dt\,v_i^{(1)} + dt^2\,\frac{v_i^{(2)}}{2} + O(dt^3) \end{aligned} \tag{5}$$

where $^{(1)}$ and $^{(2)}$ are respectively the first and second order of derivative. Approximating

$$\begin{aligned} x_{i+1} \simeq x_i + dt\,v_i, \quad v_i &= \frac{x_{i+1} - x_i}{dt} \simeq \frac{dx}{dt} \\ v_{i+1} \simeq v_i + dt\,a_i, \quad a_i &= \frac{v_{i+1} - v_i}{dt} \simeq \frac{dv}{dt} \end{aligned}$$

and using as starting points the values given by the initial conditions x_0 and v_0, we obtain the *forward Euler formula*

$$\begin{aligned} x_{i+1} &= x_i + dt\,v_i + O(dt^2) \\ v_{i+1} &= v_i + dt\,a_i + O(dt^2). \end{aligned} \tag{6}$$

Per each step i the acceleration $a_i = a(x_i, t_i)$ is calculated using Newton's law and $t_i = dt \cdot i$.

b. Velocity Verlet (VV) algorithm It is also built using Taylor expansion, but the velocity's approximation is improved thanks to Euler formula:

$$v_{i+1}^{(1)} = v_i^{(1)} + dt\,v_i^{(2)} + O(dt^2)$$

We can rewrite eq.(5) using $v_i^{(2)}$ obtained from the above formula:

$$v_{i+1} = v_i + \frac{dt}{2}\left(v_{i+1}^{(1)} + v_i^{(1)}\right) + O(dt^3).$$

In this case the velocity has an error that goes like $O(dt^3)$ and we need the value of the acceleration at the next step. Given two initial conditions x_0 and v_0 and using Newton's law to compute $a_0 = a(x_0, t_0)$, the Velocity Verlet algorithm is given by

$$\begin{aligned} x_{i+1} &= x_i + dt\,v_i + dt^2\,\frac{a_i}{2} + O(dt^3) \\ a_{i+1} &= a(x_{i+1}, t_{i+1}) \\ v_{i+1} &= v_i + \frac{dt}{2}\left(a_{i+1} + a_i\right) + O(dt^3). \end{aligned} \tag{7}$$

Using Euler's and Verlet's algorithm we develop a code to solve the coupled ODE. To structure our code in an object oriented way (and to learn how to use properly the classes in c++), we used as an example Morten's program [1]. This program is structured in several classes and subclasses to make the code as general as possible:

- potential.cpp contains how the force and the potential are defined;
- integrator.cpp contains Euler and Velocity Verlet (VV) algorithms and it allows us to solve our problem choosing between the two solvers (polymorphism?);
- planet.cpp is used to organize the planets and their properties (velocity, position, force acting on the planet, angular momentum);
- solarsystem.cpp coordinates all the previous classes in order to solve all the differential equations and write to a file the planets' positions.

The main advantage of object orientation is that it is possible to modify the system's configurations without changing the core of the program. In addition several classes can be reused for analogous computational problems. For example, the potential.cpp class allows us to choose what kind of force and potential affects our system. We used the newtonian force, but this one can be replaced by another one (i.e. molecular dynamics).

Firstly, we implement the easiest two-body system: Earth and Sun with Sun fixed at the origin of our frame. This configuration is used to check the stability of our solvers and their performances. Moreover we find a value for the escape velocity of Earth and we compare this result with the theoretical prevision. Furthermore we change the functional form of the forces from $1/r^2$ to $1/r^3$ and we see what happens.

After that, we also include Jupiter in our system. We solve this three-body problem using the VV solver in two cases: firstly, approximating the sun as center of mass of the system and fixing it as origin of our coordinates system, secondly, fixing the position of the center of mass as origin and giving the Sun a velocity to make the momentum of the whole system null. This allows us to discuss how the presence of Jupiter affects the motion of Earth, in particular in the first case through the increment of the mass of Jupiter by 10 and by 1000. Then we extend the program to include all the planets of the solar system plus Pluto in both cases. In each case the stability of the solver is studied. To initialize the planets we use the positions (in AU) and velocity (converted in AU/yr) provided by NASA [4].

Lastly, we analyze the precession of Mercury's perihelion after 100yr in order to confirm the discrepancy between Newtonian force and a relativistic correction to that one.

A. Newton's force analysis

Now, our aim is to analyse from a theoretical point of view the Gravitational Newton's force and why its functional form leads to the conservation of energy E and total angula momentum $|\vec{L}|$ [3].
Let us consider a body of mass m influenced by a gravitational field generated by a mass M fixed in th center of our inertial reference frame, so the Newton's force is given by:

$$\vec{F} = -G\,\frac{M\,m}{r^2}\,\hat{r}$$

where $\hat{r}$ is the versor given by $\vec{r}/|\vec{r}|$. The above functional form depends only on the position $\vec{r}$ of the mass m, so $\vec{F}$ is a conservative force because the work done between two points A and B is independent of the taken path

$$\begin{aligned} W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{s} = -G\,M\,m \int_A^B \frac{1}{r^2}\,dr = \\ = -\left[-\frac{G\,M\,m}{r_B} + \frac{G\,M\,m}{r_A}\right] \end{aligned}$$

where we used the fact the projection of the versor $hatr$ on the infinitesiaml path $d\vec{s}$ is given by the infinitesimal variation of the distance:$\hat{r}\cdot d\vec{s} = dr$. In addition we need Work-Energy theorem:

$$W_{A\to B} = E_k^B - E_k^A = \frac{m\,v_B^2}{2} - \frac{m\,v_A^2}{2}$$

the work W done on an object by a force between two points A and B equals the change in kinetic energy E_k of the object.
Take into account these properties and defining $V(r) = -G\,M\,m/r$ we write:

$$\begin{aligned} -\left[-\frac{G\,M\,m}{r_B} + \frac{G\,M\,m}{r_A}\right] &= E_k^B - E_k^A \\ E_k^A - \frac{G\,M\,m}{r_A} &= E_k^B - \frac{G\,M\,m}{r_B} \\ E_k^A + V(r_A) &= E_k^B + V(r_B) \end{aligned}$$

Because of the arbitrary of the two points the total energy $E = E_k + V(r)$ is conserved.
Let us now prove the conservation of the total angu-

lar momentum:

$$\frac{d}{dt}\vec{L} = m\,\frac{d}{dt}\,[\vec{r}\times\vec{v}]$$
$$= m\left(\frac{d}{dt}\vec{r}\times\vec{v}+\vec{r}\times\frac{d}{dt}\vec{v}\right)$$
$$= m\left(\vec{0}+\vec{r}\times\frac{\vec{F}}{m}\right)=\vec{0}$$

where we used the second law of mechanics and the cross product between vectors along the same direction is the zero vector. The conservation of Angular momentum implies that the orbit of the object m remains always in the plane orthogonal to $\vec{L}$. We want to stress that the conservation of energy and angular momentum is due to the fact that newtonian force is a central force $= f(r)\hat{r}$.
The energy E does not change in time and this is the key point in the following discussion, so we can study it in the limit $r \to \infty$:

$$E = -\frac{G\,M\,m}{r} + \frac{m\,v^2}{2}$$
$$if\, r \to \infty \qquad E \to \frac{m\,v^2}{2} \geqslant 0$$

So if we request that the object should not be more influenced by the gravitational field we need $E \geqslant 0$. The velocity v_e needed by an object of mass m at a distance R to escape the gravitational influence of M can be found setting $E = 0$:

$$-\frac{G\,M\,m}{R} + \frac{m\,v_e^2}{2} = 0$$
$$v_e = \sqrt{\frac{2\,G\,M}{R}}$$

III. OUR RESULTS

A. Earth-Sun system

We first run a simulation of the two body system Sun-Earth with both Euler and VV solver (Fig. (1)). According to eq.2, we set the velocity of Earth to $v_0 = 2\pi\,AU\,yr^{-1}$ to get a circular orbit of radius $r = 1\,AU$ around the Sun.

We see clearly that the orbit we get from Euler solver can be considered stable only for small values of dt ($dt \le 10^{-5}$), and if we check the value of total energy we see that it is not conserved (as it should be since we are now considering Sun-Earth as an isolated system).

Velocity Verlet solver however enhances the stability of the system and gives a stable circular orbit even for bigger values of dt ($dt \le 10^{-3}$); furthermore

	ΔE [$\mathrm{M_\odot\,AU^2/yr^2}$]	ΔL [$\mathrm{M_\odot\,AU^2/yr}$]
Euler	-3.8890×10^{-5}	0
Verlet	1.8000×10^{-9}	0.000

Table I: The table displays the difference between the first and the last value of energy and total angular momentum

energy balance shows that now mechanical energy is conserved under tolerance of about 0.5 % as well as angular momentum (Tab. I). This difference is due to the fact that Euler's method estimates velocity with an error that scales as $O(dt^2)$, while for VV the error scales as $O(dt^3)$.

If we focus on the algorithms themselves we see that each cycle of Euler solver requires 4 flops, while for VV 7 flops are needed. This means that with N steps our Euler solver will perform $4(N-1)d$ flops and VV $7(N-1)d$ flops -d is the dimensionality of our system (depending on whether it is 2D or 3D) [2]. VV needs in fact more time to perform than Euler (table ref -for $dt = 10^-3$ and $N = 10^-5$), but provides a more stable and reliable result without dramatically increasing the number of flops that remains $O(N)$. As shown in Fig. (**??**) the N-dependence of the running time is in fact linear in both cases and, as expected, the plot for VV solver has a higher slope. Therefore, for further simulations we will use the VV solver.

To test the Earth-Sun system we try to increase the kinetic energy of Earth (keeping the Sun fixed) by a factor k and look numerically for a value for its escape velocity through a 1000yr simulation.

As shown in Fig. (2), the orbit becomes open for a value of $v \in [1.39v_0, 1.42v_0]$ where v_0 is the value set previously to 2π. We therefore try to compute that value using the formula already derived:

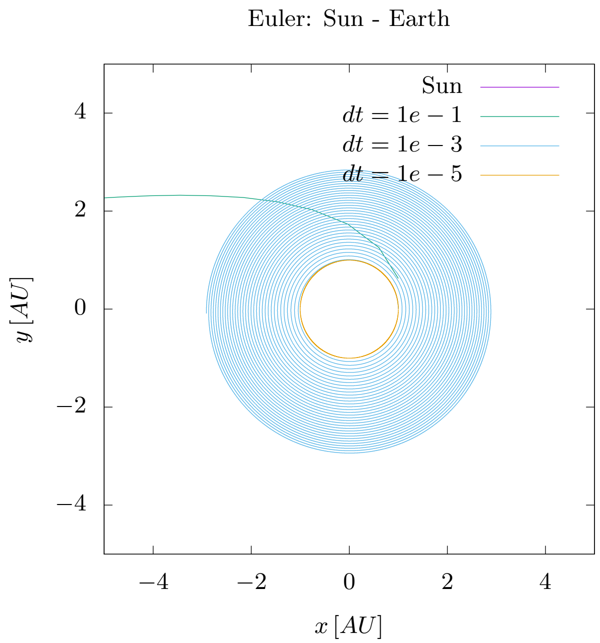
$$v_e = \sqrt{2\frac{4\pi^2}{1}}\,AU/yr = \sqrt{2}(2\pi)\,AU/yr = \sqrt{2}v_0\,AU/yr$$

thus, $k = \sqrt{(}2) \sim 1.414$. The analytical value is clearly compatible with the interval we identified numerically.

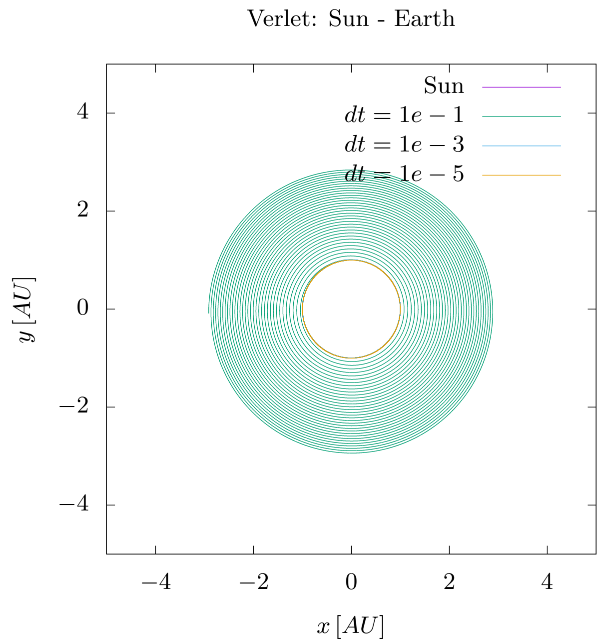
Furthermore, we modify the functional form of of the forces changing their dependence from $\frac{1}{r^2}$ to $\frac{1}{r^\beta}$, with $\beta \in [2,3]$ and we see how the system evolves for different values of β.

What we see (Fig. (**??**)) is that for $\beta \to 3$ the orbit becomes larger and less stable and for $\beta = 3$ it is not stable anymore. By increasing β in fact we make gravitational potential becomes a *short-range* potential,

$$V(r) \propto \frac{1}{r^n} \quad n > 1$$



(a) Euler solver



(b) VV solver

Figure 1: Comparison between Euler (a) and Verlet's (b) solver: it's required a step size of $1e-3$ with Verlet to get conservation of energy, while with Euler a lower stepsize ($1e-5$) is needed.

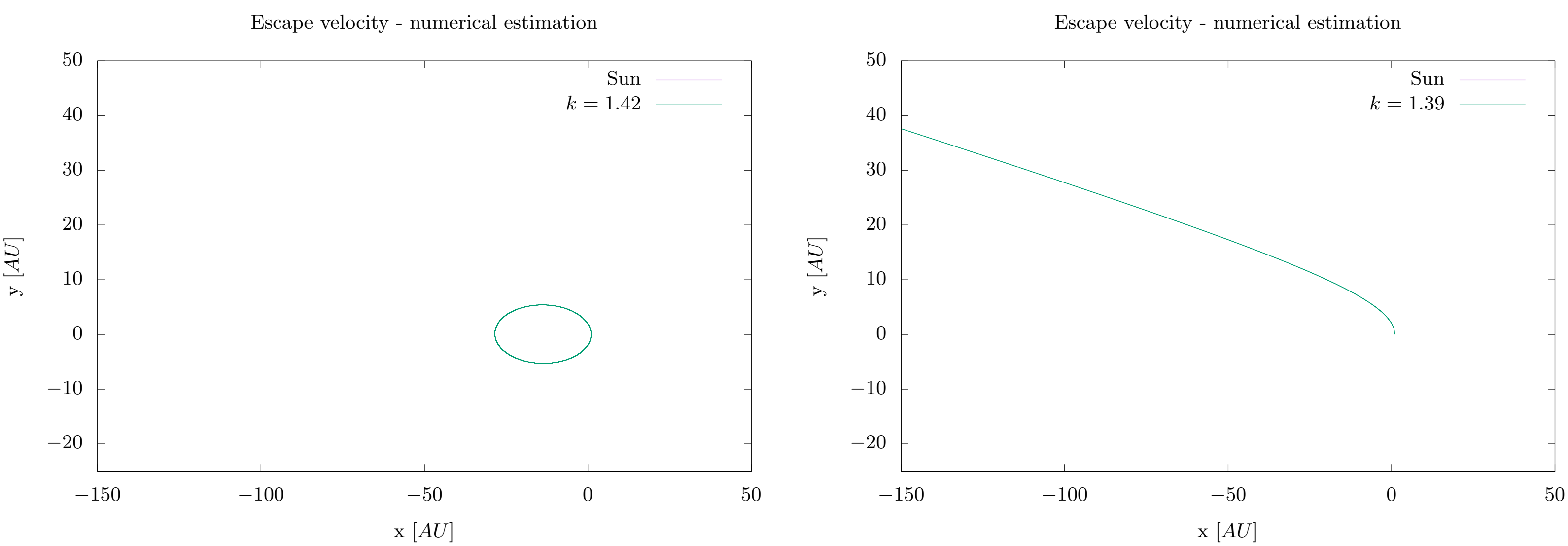


Figure 2: In these figures the limit of the interval of escapes velocity are represented.

An important aspect is that the if the initial conditions are set up with a negative energy E the Earth will never escape from the Sun, as discussed in the theoretical part. The energy will be conserved and the new escape velocity for a body at a distance r is given by:

$$v_e(n) \propto \sqrt{\left(\frac{4\pi^2}{r^n}\right)}$$

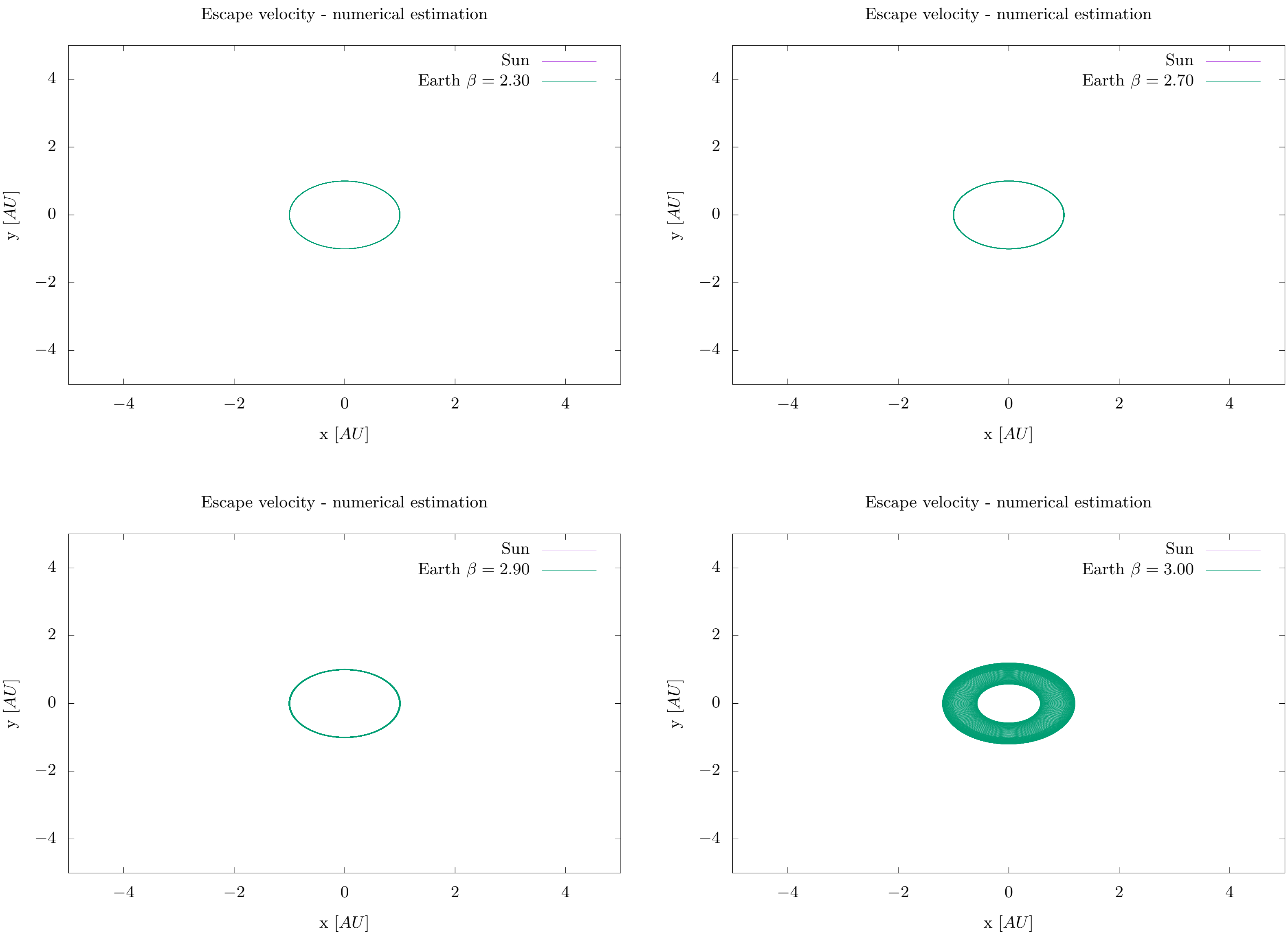


Figure 3: In these figures are shown the positions of Earth with a different functional dependence of the force r^{β}. As we can see the terrestrial orbit is stable but it becomes larger as β increases.

B. Jupiter: three-body system

Now, we include Jupiter, which is the most massive planet in the solar system, to evaluate a three-body problem. This allows us to understand how Jupiter affects Earth's motion. We compute all the calculations using Velocity Verlet algorithm.
At first we fix the Sun as center of the mass and as origin of our system. With this setup, if we plot the $x - y$ position of Sun, Earth and Jupiter, we obtain Fig. (4). To evaluate properly the stability of the algo, we consider the conservation of angular momentum and mechanical energy. In Tab. (II) are shown the initial and final values of angular momentum (L) and energy (E) after a 100 yr. As we can see the quantities are conserved as we expect.

At this point we increase the mass of Jupiter by a factor of 10 (Fig. (5a)) and, then, by a factor of 1000 (Fig. (5b)). In Fig. (5a), the motion of Earth seems quite stable yet, but during the years it doesn't stay properly in the same orbit: Earth's eccentricity increases from $e_E = 0.0018$ to $e_E = 0.0090$ after 100yr. We note that these value are still not comparable with the data from NASA. In Fig. (5b)

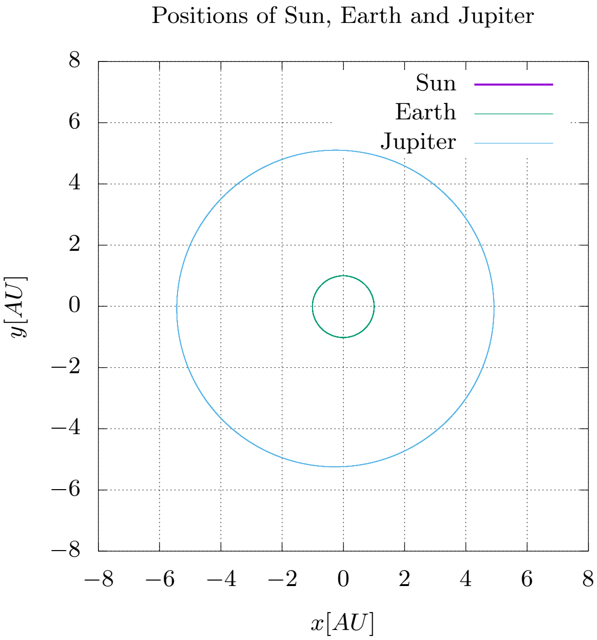


Figure 4: In this plot is shown the positions of Sun, Earth and Jupiter in the XY plane, during 100yr using VV algorithm with numberOfSteps $= 1 \times 10^5$ and $dt = 1 \times 10^{-3}$, when Sun is fixed. As we can see, the orbits are stable.

	L [$M_\odot$ AU^2/yr]	E [$M_\odot$ AU^2/yr^2]
Initial	0.0135869884	-0.00367816594
After 100 yr	0.0135869884	-0.0036782002
Difference	0	-3.4260$\times10^{-8}$

Table II: In this table are shown the initial and after 100yr values of the angular momentum and mechanical energy in the case of three-body (Sun, Earth, Jupiter) with Sun fixed. The difference of their value demonstrates that these values are conserved.

it is shown that the Earth orbits two times around the Sun before being influenced by Jupiter's attraction. After 42 yr of orbiting around Jupiter, Earth leaves the system. Instead, Jupiter revolutions are undisturbed. To understand the stability of the algo, we consider the conservation of angular momentum and energy. A change of mass doesn't affect the conservation of energy and angular momentum. Hence our algorithm must provide results compatible with this prevision: as we can see from Tab.(III), our algorithm works well.

Hence, Jupiter doesn't affect much Earth's motion if we increase its mass by 10, this could be a consequence of the facts that it is very far from Earth, so the $1/r^2$ in the force kills the increase contribution of the mass. On the other hand, increasing Jupiter's mass by 1000 makes mass of Jupiter compatible with Sun's mass. Indeed the gravitational force that Jupiter exerts on Earth increases with the same factor, this changes completely Earth's motion.

$M = 10 \times M_J$	L [$M_\odot$ AU^2/yr]	E [$M_\odot$ AU^2/yr^2]
Initial	0.135699595	-0.0362530921
After 100 yr	0.135699595	-0.0362533793
Difference	0	-2.8720$\times10^{-7}$

$M = 10^3 \times M_J$	L [$M_\odot$ AU^2/yr]	E [$M_\odot$ AU^2/yr^2]
Initial	13.5680863	-3.61949497
After 100 yr	13.5680863	-3.6191442173
Difference	0	3.5075$\times10^{-4}$

Table III: In the tables are shown the initial and after 100yr values of the angular momentum and mechanical energy in the case of three-body (Sun, Earth, Jupiter) with Sun fixed, above for $M = 10 \times M_J$ and below for $M = 10^3 \times M_J$. Energy and angular momentum are conserved as we expect.

At this point, we proceed to consider a real three body problem. To do this, we fix the center of mass of the system as the origin and give to the Sun an initial velocity to make the linear momentum of all the system to be zero. In Fig. (6) are shown the position of Sun, Earth and Jupiter and a zoom on the motion of the Sun. From Fig. (6a), we can see that the motion of the planets seems still stable. In particular, during 100yr, Earth increases its eccentricity from $e_E = 0.0009$ to $e_E = 0.0066$, still not comparable with the data from NASA. From Fig. (6b), we can see that revolution Sun around the center of mass has a radius of ~ 0.0043AU. However, Sun has a radius of ~ 0.0046AU, hence, fixing the Sun as center of mass of the system is an excellent approximation. To understand the stability of our VV solver, again we consider the conservation of angular momentum and energy (see Tab.IV). The table shows that the quantities are conserved as it has to be. In this case, we consider also the position of the center of mass and the momentum of all system. We expect the first to stay still for all the period considered and the second to be zero as there aren't external forces acting on the system, therefore momentum is conserved. In Tab.(V) is shown that our expectation are respected with really good approximation mostly with momentum (at least 6 decimal digits for the coordinates of CM, and 16 for momentum).

	L [$M_\odot$ AU^2/yr]	E [$M_\odot$ AU^2/yr^2]
Initial	0.0135869884	-0.00367509849
After 100 yr	0.0135869884	-0.0036751328
Difference	0	-3.4310$\times 10^{-8}$

Table IV: In the table are shown the initial and after 100yr values of the angular momentum and mechanical energy in the case of three-body (Sun, Earth, Jupiter) with CM fixed. Angular momentum and energy are conserved with good precision.

C. Solar system model

Then, we expand our program to count all the planets (including Pluto) and we fix the center of mass as the origin. In Fig.(7) are shown a $3D$ representation of motion of all the planets and a zoom on the motion of the Sun during 300yr (this arc of time is chosen to let Pluto perform a complete revolution around Sun). From Fig. (7a), we can see how the orbits of the planets are quiet planar except for Pluto as we expect. However Earth's eccentricity increases from $e_E = 0.0015$ to $e_E = 0.0087$, still far from NASA data. On the other hand, from Fig. (7b), we can see that the motion of the Sun around the center of mass is not trivial. This could be caused on the relative position between Jupiter, Saturn and the Sun itself since it has to balance the displacement of CM. However, the diameter of the motion is close to the diameter of Sun (~ 0.01AU), hence the center of mass of the system is nearly on the surface of Sun. This shows again how the approximation of Sun as center of mass is indeed good. In Tab.(VII) and Tab. (VIII), are shown the results about conservation of angular momentum, energy, position of center of mass, and momentum of the system. As from the previous case, the quantities are conserved with good precisions (the conservation is "perfect" for angular momentum, up to 8 decimal digits for energy, and at least 6 for the coordinates of CM and 15 for momentum).

D. Precession of Mercury's perihelion

During the study of Mercury's orbits and the precession of its perihelion, it has been observed a difference of $\approx 43'' = 43$ arcseconds per century between the experimental data and the theoretical ones (obtained via Newton's force, eq. (1)). This phenomenon was after explained by the General Theory of Relativity and represented a text for the theory itself.

Our aim is to simulate Mercury's orbit for more than one century and compare the results obtained through Newton's force and the following relativistic correction:

$$F_R = -G\,\frac{M_{Mercury}\,M_\odot}{r^2}\,\left(1+\frac{3\,l^2}{c^2\,r^2}\right) \tag{8}$$

where $l = |\vec{r}\times\vec{v}|$ is Mercury's total angular momentum over its mass, and c is the speed of light in the vacuum. Using the above formula for Mercury's interaction we analyse the precession of its perihelion with respect to the Sun and for simplicity we neglect the interactions due to the other planets of the solar system. In order to have enough precision and evaluate the desired discrepancy of $\approx 43''$ we run the program with a step-size of 0.5×10^{-7}yr. The initial conditions of the two celestial objects are reported in Tab. (IX). We let the system evolve for more than one century and we study the distance r between the Sun and Mercury as a function of time (Fig. (**??**)).

We use the x_P and y_P positions at perihelion for the Newtonian and the relativistic interactions to calculate the desired discrepancy. Taking into account that the Sun is $\approx 1.2\times10^{37}$ heavier than Mercury, we calculate the angle of precession per each theory as $\phi_P = \tan^{-1}\left(\frac{y_P}{x_P}\right)$. After subtracting them we get an angle discrepancy of $\phi = |137.813'' - 181.019''| = 43.206''$. It is possible to analyse if our result ϕ is compatible with the theoretical ones. The orbital period of the planet is roughly 0.241 yr , our step size is 0.5×10^{-7} yr, hence, within the simulation, we can reach the minimum sensitive variation of angle of

$$d\phi = \frac{0.5\times10^{-7}\,\text{yr}}{0.241\,\text{yr}}\,360\cdot3600 \approx 0^\circ\,0'\,0.27''$$

We conclude that our discrepancy ϕ is compatible with the expected one.
We highlight that we didn't call F_R (eq.(8) a force because according to the General Theory of Relativity the gravitational interaction is a result of space-time's deformation due to the presence of masses.

IV. CONCLUSION

The analysis of the Newtonian gravitational force has shown that its form leads to the conservation of energy and angular momentum. The study of solver's stability is based on this theoretical aspect. It is evident from the analytical expression of Euler's method that the Velocity Verlet has a better approximation, although it is slower than Euler's algorithm. Moreover our analysis confirms the stability of VV is dependent of step-size. We tested VV stability with all the different cases, checking the conservation of energy, angular momentum and also the position of CM and momentum of the system (the last two, when it made sense). The solver passed all the tests.

As regards the two body problem, the Earth affected by a force that scales as $1/r^{\beta}$ does not escapes from the gravitational attraction of the Sun but its orbits cover a bigger range of values than the "real" case. Furthermore we confirm the theoretical derivation of the escape velocity by numerically determining for it the interval $[1.39v_0,\ 1.42v_0]$ performing a simulation for 1000 yr.

Jupiter is especially important for the stability of Earth's orbit since it is the most massive planet. We were able to implement a three body problem with the Sun fixed at the centre and also with the Sun free to move. The motion of Sun was restricted in a region smaller than the Sun itself: the approximation of the Sun as CM is indeed good. Furthermore, we increase Jupiter's mass by ten and by a thousand to check the action of Jupiter on Earth. In the second case, we saw that the terrestrial orbit is not stable as the force acting on Earth from Jupiter increases of the same factor as the mass.

Then, we were able to develop a program with all the planets (plus Pluto) and the Sun free to move. As before the motion of Sun was restricted to a region smaller than the its dimensions (Sun as CM is a good approximation), but its motion is not trivial. We tried to give an explanation of this, saying that it's probably due to the relative position of Jupiter and Saturn, which has to compensate the CM displacement.

Finally, we showed how the General Relativity is able to predict the perihelion procession of Mercury. The discrepancy of the perihelion angles between the Newtonian force and its relativistic correction is $(43.206 \pm 0.27)''$ compatible with the discrepancy obtained between the observed value and the value predicted from Newton's theory $(43'')$.

[1] `https://github.com/mortele/solar-system-fys3150`.

[2] M. H. Jensen, *Computational physics - Lecture notes Fall 2015*, University of Oslo - Department of Physics, 2015.

[3] J. M. Knudsen, P. G. Hjorth, *Elements of Newtonian Mechanics*, Springer-Verlag 1995. XVI, 413 pp.

[4] `https://ssd.jpl.nasa.gov/horizons.cgi#top`

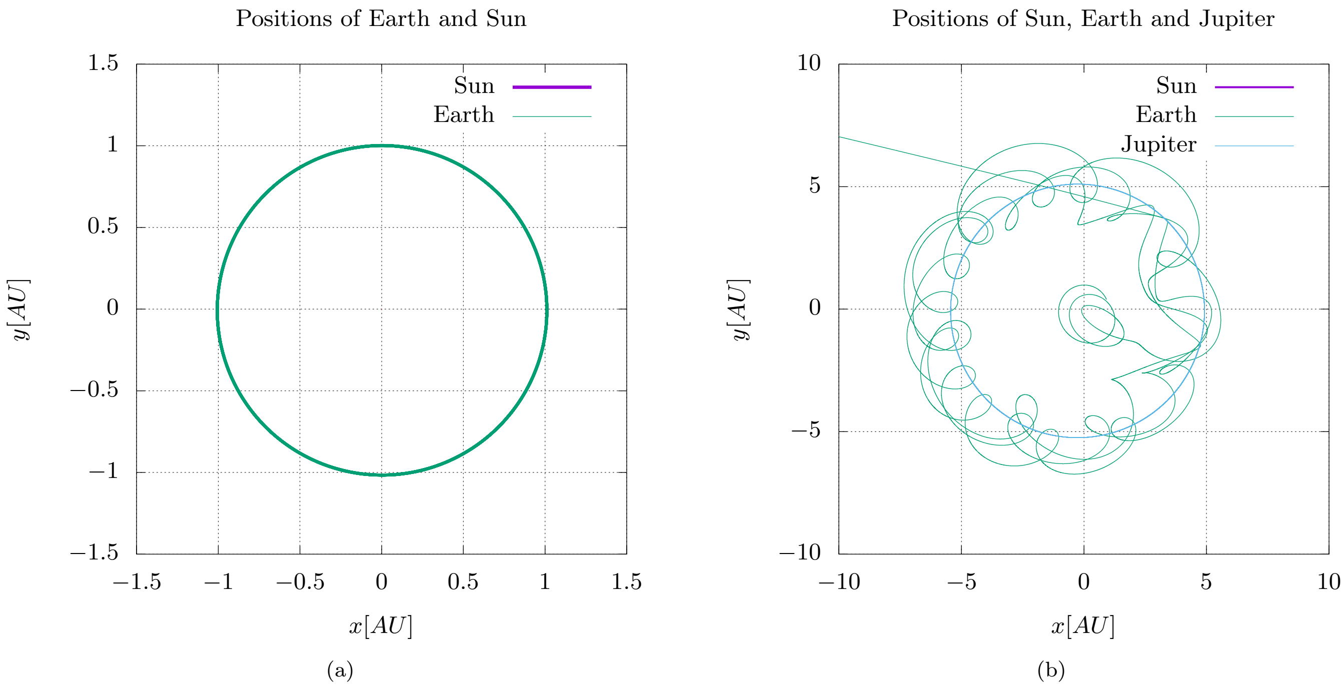


Figure 5: In these plots are shown the position of Sun and Earth when the mass of Jupiter is increased by 10 (a), and the position of Sun, Earth and Jupiter when the mass of Jupiter is increased by 1000 (b) during 100yr using VV algorithm. In (a) the orbit of Earth is still stable even if it slightly varies during the arc of time. While in (b) Earth's orbit is unstable and after rotating around Jupiter, it escapes the gravitational attraction of the other two planets.

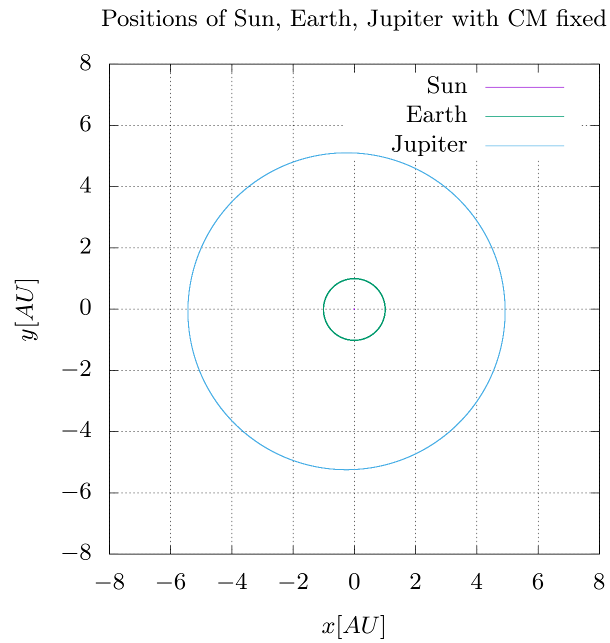
Coordinates of CM	x [AU]	y [AU]	z [AU]
Initial	-0.00354024×10^{-6}	0.0972923×10^{-8}	-0.197645×10^{-9}
After 100 yr	-2.64291744×10^{-7}	-3.89889898×10^{-8}	-1.30758661×10^{-9}
Difference	0.26075×10^{-6}	-3.9962×10^{-8}	-1.1099×10^{-9}

Momentum of the system	p_x [$M_\odot$ AU/yr]	p_y [$M_\odot$ AU/yr]	p_z [$M_\odot$ AU/yr]
Initial	-2.61×10^{-9}	-4.0×10^{-10}	-1.111×10^{-11}
After 100 yr	-2.61000011×10^{-9}	-3.99999975×10^{-10}	-1.11099951×10^{-11}
Difference	-1.1×10^{-16}	2.500$10^{-17}$	4.900$10^{-18}$

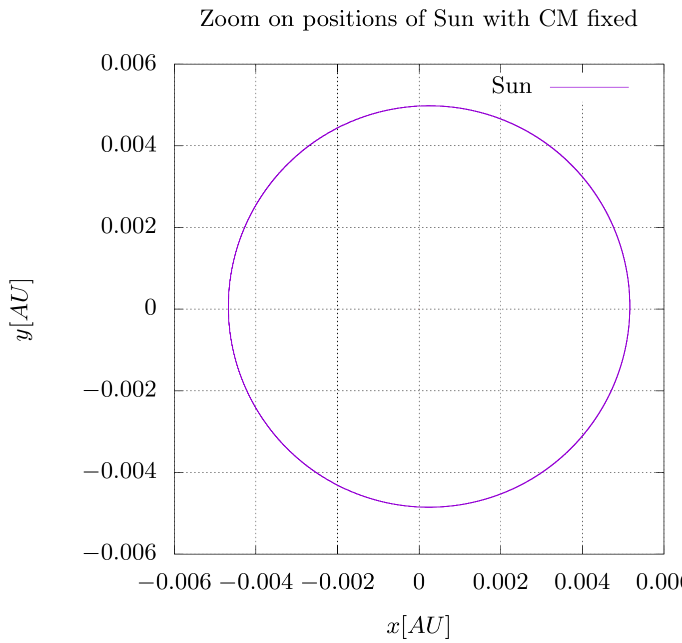
Table V: In the tables are reported the initial and after 100yr values of the position of the center of mass and the momentum in all the directions of all the system for the three-body case. Considering the difference of the values, we can state that the CM is fixed, and momentum is conserved as we expect.

	L [$M_\odot$ AU^2/yr]	E [$M_\odot$ AU^2/yr^2]
Initial	156.900371	-0.00440779305
After 300 yr	156.8994454127	-0.0044078175
Difference	-9.2559×10^{-4}	-2.4450×10^{-8}

Table VI: stepsize= 300000, $dt = 1e-3$



(a)



(b)

Figure 6: In these figure are shown the position of Sun, Earth, Jupiter (a) and a zoom on the position of Sun (b), when the center of mass is fixed as origin, during 100yr using VV algorithm. From (a), we note that the motion of the planets are stable and from (b), we see that the motion of the sun is less than its diameter ~ 0.009AU: our approximation of Sun as CM is good.

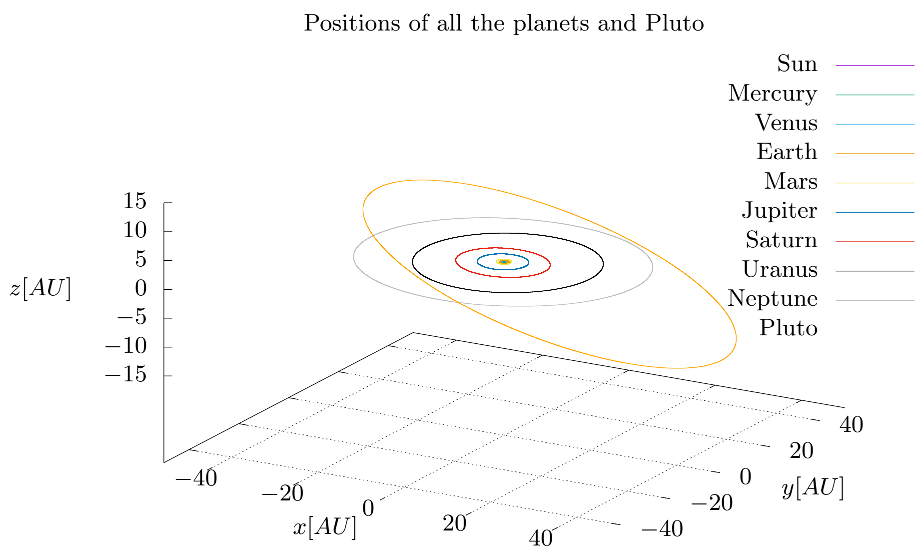
	L [$M_\odot$ AU^2/yr]	E [$M_\odot$ AU^2/yr^2]
Initial	0.0219130463	-0.00440404285
After 300 yr	0.0219130463	-0.0044040677
Difference	0	-2.485$\times10^{-8}$

Table VII: In the tables are shown the initial and after 300yr values of the angular momentum and mechanical energy for all planets system. The energy and angular momentum are conserved as we expect.

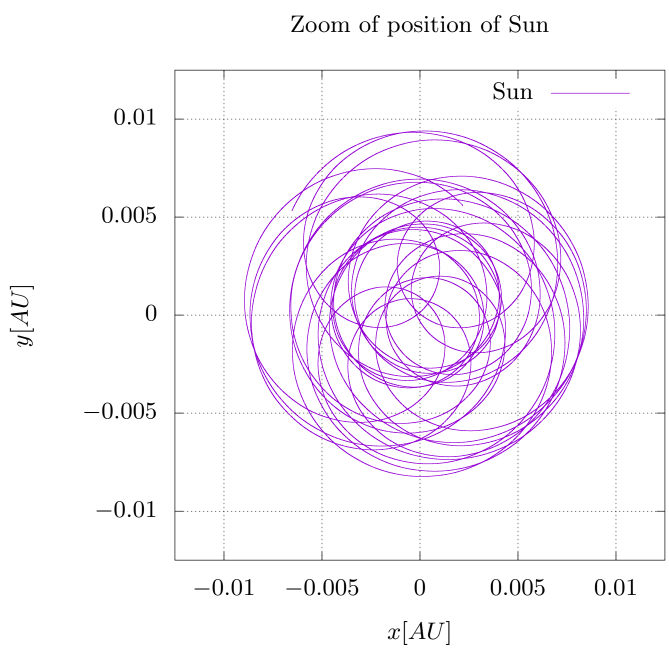
Coordinates of CM	x [AU]	y [AU]	z [AU]
Initial	-0.005$\times10^{-6}$	-0.003$\times10^{-6}$	-0.103$\times10^{-9}$
After 300 yr	1.135$\times10^{-6}$	1.482$\times10^{-6}$	9.284$\times10^{-9}$
Difference	1.140$\times10^{-6}$	1.485$\times10^{-6}$	9.387$\times10^{-9}$

Momentum of the system	p_x [$M_\odot$ AU/yr]	p_y [$M_\odot$ AU/yr]	p_z [$M_\odot$ AU/yr]
Initial	3.80$\times10^{-9}$	4.958$\times10^{-9}$	3.13$\times10^{-11}$
After 300 yr	3.80$\times10^{-9}$	4.958$\times10^{-9}$	3.13$\times10^{-11}$
Difference	-4.00$\times10^{-17}$	-3.2310^{-15}	+5.4010^{-17}

Table VIII: In these tables are reported the initial and after 300yr values of the position of the center of mass and the momentum in all the directions of all the system in the all planets case. Considering the difference of the values, we can state that the CM is fixed, and momentum is conserved as we expect.

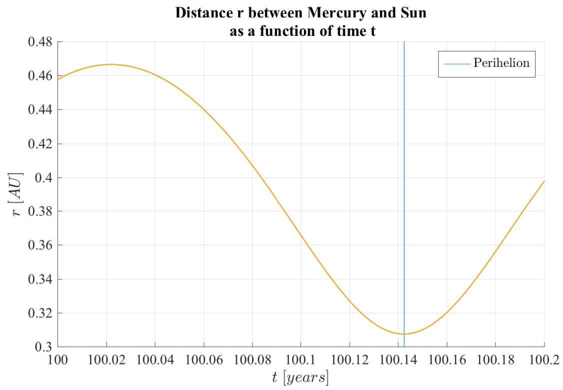


(a)



(b)

Figure 7: In these figures are shown the position of all the planets (plus Pluto) (a) and a zoom on the orbit of Sun (b) during 300yr. From (a), we can see how all the planets have quiet planar orbits except for Pluto as we expect. As regard (b), we note that Sun's orbit is complicated, but it remains in a region of diameter less than 1AU. Hence the center of mass is close to the surface of Sun. Once again this proves that our approximation (Sun as CM) is good.



Object	(x, y, z) $[AU]$	(v_x, v_y, v_z) $[AU/y]$
Mercury	$(0.3075, 0, 0)$	$(0, 12.44, 0)$
Sun	$(0, 0, 0)$	$(0, 0, 0)$

Table IX: Initial conditions of the simulation of Mercury-Sun system.