Essential OCL - A Study for a Consistent Semantics of UML/OCL 2.2 in HOL.

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2.1 Notations for the option type

First of all, we will use a more compact notation for the library option type which occur all over in our definitions and which will make the presentation more "textbook"-like:

notation *Some* $(\lfloor(\text{-})\rfloor)$
notation *None* $(\bot)$

The following function (corresponding to *the* in the Isabelle/HOL library) is defined as the inverse of the injection *Some*.

fun *drop* :: $'\alpha$ *option* $\Rightarrow$ $'\alpha$ $(\lceil(\text{-})\rceil)$
where *drop-lift*[*simp*]: $\lceil\lfloor v\rfloor\rceil = v$

2.2 Minimal Notions of State and State Transitions

Next we will introduce the foundational concept of an object id (oid), which is just some infinite set.

type-synonym *oid* = *ind*

States are just a partial map from oid's to elements of an object universe $'\mathfrak{A}$, and state transitions pairs of states...

type-synonym $('\mathfrak{A})$*state* = *oid* $\rightharpoonup$ $'\mathfrak{A}$

type-synonym $('\mathfrak{A})$*st* = $'\mathfrak{A}$ *state* $\times$ $'\mathfrak{A}$ *state*

2.3 Prerequisite: An Abstract Interface for OCL Types

In order to have the possibility to nest collection types, such that we can give semantics to expressions like *Set*{*Set*{**2**},*null*}, it is necessary to introduce a uniform interface for types having the *invalid* (= bottom) element. The reason is that we impose a data-invariant on raw-collection `types_code` which assures that the *invalid* element is not allowed inside the collection; all raw-collections of this form were identified with the *invalid* element itself. The construction requires that the new collection type is un-comparable with the raw-types (consisting of nested option type constructions), such that the data-invariant mussed be expressed in terms of the interface. In a second step, our base-types will be shown to be instances of this interface.

This uniform interface consists in a type class requiring the existence of a bot and a null element. The construction proceeds by abstracting the null (which is defined by $\lfloor\ \bot\ \rfloor$ on $'a$ *option option* to a null - element, which may have an abritrary semantic structure, and an undefinedness element $\bot$ to an abstract undefinedness element *bot* (also written $\bot$ whenever no confusion arises). As a consequence, it is necessary to redefine the notions of invalid, defined, valuation etc. on top of this interface.

This interface consists in two abstract type classes *bot* and *null* for the class of all types comprising a bot and a distinct null element.

instance *option* :: (*plus*) *plus* **by** *intro-classes*
instance *fun* :: (*type*, *plus*) *plus* **by** *intro-classes*

class *bot* =
  fixes *bot* :: $'a$
  assumes *nonEmpty* : $\exists\ x.\ x \neq bot$

class *null* = *bot* +
  fixes *null* :: $'a$
  assumes *null-is-valid* : $null \neq bot$

2.4 Accomodation of Basic Types to the Abstract Interface

In the following it is shown that the option-option type type is in fact in the *null* class and that function spaces over these classes again "live" in these classes. This motivates the default construction of the semantic domain for the basic types (Boolean, Integer, Reals, ...).

instantiation *option* :: (*type*)*bot*
begin
  definition *bot-option-def*: (bot::$'a$ *option*) $\equiv$ (*None*::$'a$ *option*)
  instance proof show $\exists\, x$::$'a$ *option*. $x \neq bot$
        by(*rule-tac* x=*Some* x **in** *exI*, *simp add*:*bot-option-def*)
      qed
end

instantiation *option* :: (*bot*)*null*
begin
  definition *null-option-def*: (*null*::$'a$::*bot option*) $\equiv$ $\lfloor$ *bot* $\rfloor$
  instance proof show (*null*::$'a$::*bot option*) $\neq$ *bot*
        by(*simp add*:*null-option-def bot-option-def*)
      qed
end

instantiation *fun* :: (*type*,*bot*) *bot*
begin
  definition *bot-fun-def*: *bot* $\equiv$ (λ x. *bot*)

  instance proof show $\exists (x$::$'a \Rightarrow\ 'b)$. $x \neq bot$
        apply(*rule-tac* x=λ -. (*SOME* y. $y \neq bot$) **in** *exI*, *auto*)
        apply(*drule-tac* x=x **in** *fun-cong*,*auto simp*:*bot-fun-def*)

apply(*erule contrapos-pp*, *simp*)
apply(*rule some-eq-ex*[*THEN iffD2*])
apply(*simp add*: *nonEmpty*)
done
qed
end

instantiation *fun* :: (*type*,*null*) *null*
begin
definition *null-fun-def*: $(null::'a \Rightarrow 'b::null) \equiv (\lambda\ x.\ null)$

instance proof
show $(null::'a \Rightarrow 'b::null) \neq bot$
apply(*auto simp*: *null-fun-def bot-fun-def*)
apply(*drule-tac x*=*x* **in** *fun-cong*)
apply(*erule contrapos-pp*, *simp add*: *null-is-valid*)
done
qed
end

A trivial consequence of this adaption of the interface is that abstract and concrete versions of null are the same on base types (as could be expected).

2.5 The Semantic Space of OCL Types: Valuations.

Valuations are now functions from a state pair (built upon data universe $'\mathfrak{A}$) to an arbitrary null-type (i.e. containing at least a destinguished *null* and *invalid* element.

type-synonym $('\mathfrak{A},'\alpha)\ val = '\mathfrak{A}\ st \Rightarrow '\alpha$

All OCL expressions *denote* functions that map the underlying

type-synonym $('\mathfrak{A},'\alpha)\ val' = '\mathfrak{A}\ st \Rightarrow '\alpha\ option\ option$

As a consequence of semantic domain definition, any OCL type will have the two semantic constants *invalid* (for exceptional, aborted computation) and *null*; the latter, however is either defined

definition *invalid* :: $('\mathfrak{A},'\alpha::bot)\ val$
where $invalid \equiv \lambda\ \tau.\ bot$

The definition :

```
definition null     :: "('\<AA>,'\<alpha>::null) val"
where      "null    \<equiv> \<lambda> \<tau>. null"
```

is not necessary since we defined the entire function space over null types again as null-types; the crucial definition is $null \equiv \lambda x.\ null$.

3 Boolean Type and Logic

The semantic domain of the (basic) boolean type is now defined as standard: the space of valuation to *bool option option*:

type-synonym $('\mathfrak{A})$*Boolean* = $('\mathfrak{A}$,*bool option option*) *val*

3.1 Basic Constants

lemma *bot-Boolean-def* : $(bot::('\mathfrak{A})Boolean) = (\lambda\ \tau.\ \bot)$
by(*simp add*: *bot-fun-def bot-option-def*)

lemma *null-Boolean-def* : $(null::('\mathfrak{A})Boolean) = (\lambda\ \tau.\ \lfloor\bot\rfloor)$
by(*simp add*: *null-fun-def null-option-def bot-option-def*)

definition *true* :: $('\mathfrak{A})$*Boolean*
where $true \equiv \lambda\ \tau.\ \lfloor\lfloor True\rfloor\rfloor$

definition *false* :: $('\mathfrak{A})$*Boolean*
where $false \equiv\ \lambda\ \tau.\ \lfloor\lfloor False\rfloor\rfloor$

lemma *bool-split*: $X\ \tau = invalid\ \tau \lor X\ \tau = null\ \tau \lor$
$X\ \tau = true\ \tau\quad \lor X\ \tau = false\ \tau$
apply(*simp add*: *invalid-def null-def true-def false-def*)
apply(*case-tac X* τ,*simp-all add*: *null-fun-def null-option-def bot-option-def*)
apply(*case-tac a*,*simp*)
apply(*case-tac aa*,*simp*)
apply *auto*
done

lemma [*simp*]: $false\ (a,\ b) = \lfloor\lfloor False\rfloor\rfloor$
by(*simp add*:*false-def*)

lemma [*simp*]: $true\ (a,\ b) = \lfloor\lfloor True\rfloor\rfloor$
by(*simp add*:*true-def*)

The definitions above for the constants *true* and *false* are geared towards a format that Isabelle can check to be a "conservative" (i.e. logically safe) axiomatic definition. By introducing an explicit interpretation function (which happens to be defined just as the identity since we are using a shallow embedding of OCL into HOL), all these definions can be rewritten into the conventional semantic "textbook" format as follows:

definition *Sem* :: $'a \Rightarrow 'a\ (I[\![\text{-}]\!])$
where $I[\![x]\!] \equiv x$

lemma *textbook-true*: $I[\![true]\!]\ \tau = \lfloor\lfloor True\rfloor\rfloor$
by(*simp add*: *Sem-def true-def*)

lemma *textbook-false*: $I[\![false]\!]\ \tau = \lfloor\lfloor False \rfloor\rfloor$
by(*simp add*: *Sem-def false-def*)

3.2 Fundamental Predicates I: Validity and Definedness

However, this has also the consequence that core concepts like definedness, validness and even cp have to be redefined on this type class:

definition *valid* :: $('\mathfrak{A},'a::null)val \Rightarrow ('\mathfrak{A})Boolean$ (υ - [*100*]*100*)
where $\upsilon\ X \equiv\ \lambda\ \tau\ .\ if\ X\ \tau = bot\ \tau\ then\ false\ \tau\ else\ true\ \tau$

lemma *valid1*[*simp*]: υ *invalid* = *false*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def invalid-def true-def false-def*)

lemma *valid2*[*simp*]: υ *null* = *true*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def null-is-valid null-fun-def invalid-def true-def false-def*)

lemma *valid3*[*simp*]: υ *true* = *true*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def null-is-valid null-fun-def invalid-def true-def false-def*)

lemma *valid4*[*simp*]: υ *false* = *true*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def null-is-valid null-fun-def invalid-def true-def false-def*)

lemma *cp-valid*: $(\upsilon\ X)\ \tau = (\upsilon\ (\lambda\ \text{-}.\ X\ \tau))\ \tau$
by(*simp add*: *valid-def*)

definition *defined* :: $('\mathfrak{A},'a::null)val \Rightarrow ('\mathfrak{A})Boolean$ (δ - [*100*]*100*)
where $\delta\ X \equiv\ \lambda\ \tau\ .\ if\ X\ \tau = bot\ \tau\ \ \lor\ X\ \tau = null\ \tau\ then\ false\ \tau\ else\ true\ \tau$

The generalized definitions of invalid and definedness have the same properties as the old ones :

lemma *defined1*[*simp*]: δ *invalid* = *false*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def null-def invalid-def true-def false-def*)

lemma *defined2*[*simp*]: δ *null* = *false*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def null-def null-option-def null-fun-def invalid-def true-def false-def*)

lemma *defined3*[*simp*]: δ *true* = *true*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def null-is-valid null-option-def*

null-fun-def invalid-def true-def false-def)

lemma *defined4*[*simp*]: δ *false* = *true*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def null-is-valid null-option-def null-fun-def invalid-def true-def false-def*)

lemma *defined5*[*simp*]: δ δ X = *true*
by(*rule ext*,*auto simp*: *defined-def true-def false-def bot-fun-def bot-option-def null-option-def null-fun-def*)

lemma *defined6*[*simp*]: δ υ X = *true*
by(*rule ext*,
auto simp: *valid-def defined-def true-def false-def bot-fun-def bot-option-def null-option-def null-fun-def*)

lemma *defined7*[*simp*]: δ δ X = *true*
by(*rule ext*,
auto simp: *valid-def defined-def true-def false-def bot-fun-def bot-option-def null-option-def null-fun-def*)

lemma *valid6*[*simp*]: υ δ X = *true*
by(*rule ext*,
auto simp: *valid-def defined-def true-def false-def bot-fun-def bot-option-def null-option-def null-fun-def*)

lemma *cp-defined*:$(\delta\ X)\tau = (\delta\ (\lambda\ \text{-}.\ X\ \tau))\ \tau$
by(*simp add*: *defined-def*)

The definitions above for the constants *defined* and *valid* can be rewritten into the conventional semantic "textbook" format as follows:

lemma *textbook-defined*: $I[\![\delta(X)]\!]\ \tau = (\textit{if}\ I[\![X]\!]\ \tau = I[\![\textit{bot}]\!]\ \tau\ \lor I[\![X]\!]\ \tau = I[\![\textit{null}]\!]\ \tau$
$\textit{then}\ I[\![\textit{false}]\!]\ \tau$
$\textit{else}\ I[\![\textit{true}]\!]\ \tau)$
by(*simp add*: *Sem-def defined-def*)

lemma *textbook-valid*: $I[\![\upsilon(X)]\!]\ \tau = (\textit{if}\ I[\![X]\!]\ \tau = I[\![\textit{bot}]\!]\ \tau$
$\textit{then}\ I[\![\textit{false}]\!]\ \tau$
$\textit{else}\ I[\![\textit{true}]\!]\ \tau)$
by(*simp add*: *Sem-def valid-def*)

3.3 Fundamental Predicates II: Logical (Strong) Equality

Note that we define strong equality extremely generic, even for types that contain an *null* or $\perp$ element:

definition *StrongEq*::$[{}'\mathfrak{A}\ st \Rightarrow {}'\alpha, {}'\mathfrak{A}\ st \Rightarrow {}'\alpha] \Rightarrow ({}'\mathfrak{A})Boolean$ (**infixl** $\triangleq$ *30*)
where $X \triangleq Y \equiv \lambda\ \tau.\ \lfloor\lfloor X\ \tau = Y\ \tau\ \rfloor\rfloor$

Equality reasoning in OCL is not humpty dumpty. While strong equality is clearly an equivalence:

lemma *StrongEq-refl* [*simp*]: $(X \triangleq X) = true$
by(*rule ext*, *simp add*: *null-def invalid-def true-def false-def StrongEq-def*)

lemma *StrongEq-sym* [*simp*]: $(X \triangleq Y) = (Y \triangleq X)$
by(*rule ext*, *simp add*: *eq-sym-conv invalid-def true-def false-def StrongEq-def*)

lemma *StrongEq-trans-strong* [*simp*]:
assumes *A*: $(X \triangleq Y) = true$
and *B*: $(Y \triangleq Z) = true$
shows $(X \triangleq Z) = true$
apply(*insert A B*) **apply**(*rule ext*)
apply(*simp add*: *null-def invalid-def true-def false-def StrongEq-def*)
apply(*drule-tac x=x* **in** *fun-cong*)+
by *auto*

... it is only in a limited sense a congruence, at least from the point of view of this semantic theory. The point is that it is only a congruence on OCL- expressions, not arbitrary HOL expressions (with which we can mix Essential OCL expressions. A semantic — not syntactic — characterization of OCL-expressions is that they are *context-passing* or *context-invariant*, i.e. the context of an entire OCL expression, i.e. the pre-and poststate it referes to, is passed constantly and unmodified to the sub-expressions, i.e. all sub-expressions inside an OCL expression refer to the same context. Expressed formally, this boils down to:

lemma *StrongEq-subst* :
assumes *cp*: $\bigwedge X.\ P(X)\tau = P(\lambda\ \text{-}.\ X\ \tau)\tau$
and *eq*: $(X \triangleq Y)\tau = true\ \tau$
shows $(P\ X \triangleq P\ Y)\tau = true\ \tau$
apply(*insert cp eq*)
apply(*simp add*: *null-def invalid-def true-def false-def StrongEq-def*)
apply(*subst cp*[*of X*])
apply(*subst cp*[*of Y*])
by *simp*

3.4 Fundamental Predicates III

And, last but not least,

lemma *defined8*[*simp*]: $\delta\ (X \triangleq Y) = true$

by(*rule ext*,
auto simp: *valid-def defined-def true-def false-def StrongEq-def*
bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *valid5*[*simp*]: υ $(X \triangleq Y) = true$
by(*rule ext*,
auto simp: *valid-def true-def false-def StrongEq-def*
bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *cp-StrongEq*: $(X \triangleq Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau) \triangleq (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
by(*simp add*: *StrongEq-def*)

The semantics of strict equality of OCL is constructed by overloading: for each base type, there is an equality.

3.5 Logical Connectives and their Universal Properties

It is a design goal to give OCL a semantics that is as closely as possible to a "logical system" in a known sense; a specification logic where the logical connectives can not be understood other that having the truth-table aside when reading fails its purpose in our view.

Practically, this means that we want to give a definition to the core operations to be as close as possible to the lattice laws; this makes also powerful symbolic normalizations of OCL specifications possible as a pre-requisite for automated theorem provers. For example, it is still possible to compute without any definedness- and validity reasoning the DNF of an OCL specification; be it for test-case generations or for a smooth transition to a two-valued representation of the specification amenable to fast standard SMT-solvers, for example.

Thus, our representation of the OCL is merely a 4-valued Kleene-Logics with *invalid* as least, *null* as middle and *true* resp. *false* as unrelated top-elements.

definition *not* :: $('\mathfrak{A})Boolean \Rightarrow ('\mathfrak{A})Boolean$
where $not\ X \equiv \lambda\ \tau\ .\ case\ X\ \tau\ of$

$$\begin{array}{ll} \bot & \Rightarrow \bot \\ \mid \lfloor \bot \rfloor & \Rightarrow \lfloor \bot \rfloor \\ \mid \lfloor\lfloor x \rfloor\rfloor & \Rightarrow \lfloor\lfloor \neg\ x \rfloor\rfloor \end{array}$$

lemma *cp-not*: $(not\ X)\tau = (not\ (\lambda\ \text{-}.\ X\ \tau))\ \tau$
by(*simp add*: *not-def*)

lemma *not1*[*simp*]: *not invalid* = *invalid*
by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def bot-option-def*)

lemma *not2*[*simp*]: *not null* = *null*
  by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def*
                          bot-option-def null-fun-def null-option-def)

lemma *not3*[*simp*]: *not true* = *false*
  by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def*)

lemma *not4*[*simp*]: *not false* = *true*
  by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def*)

lemma *not-not*[*simp*]: *not* (*not X*) = *X*
  apply(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def*)
  apply(*case-tac X x*, *simp-all*)
  apply(*case-tac a*, *simp-all*)
  done

definition *ocl-and* :: $[('\mathfrak{A})Boolean, ('\mathfrak{A})Boolean] \Rightarrow ('\mathfrak{A})Boolean$ (**infixl** *and 30*)
where $X\ and\ Y \equiv (\lambda\, \tau\, .\ case\ X\ \tau\ of$
$\quad \bot \Rightarrow (case\ Y\ \tau\ of$
$\qquad \bot \Rightarrow \bot$
$\qquad | \lfloor\bot\rfloor \Rightarrow \bot$
$\qquad | \lfloor\lfloor True\rfloor\rfloor \Rightarrow \bot$
$\qquad | \lfloor\lfloor False\rfloor\rfloor \Rightarrow \lfloor\lfloor False\rfloor\rfloor)$
$\quad | \lfloor\bot\rfloor \Rightarrow (case\ Y\ \tau\ of$
$\qquad \bot \Rightarrow \bot$
$\qquad | \lfloor\bot\rfloor \Rightarrow \lfloor\bot\rfloor$
$\qquad | \lfloor\lfloor True\rfloor\rfloor \Rightarrow \lfloor\bot\rfloor$
$\qquad | \lfloor\lfloor False\rfloor\rfloor \Rightarrow \lfloor\lfloor False\rfloor\rfloor)$
$\quad | \lfloor\lfloor True\rfloor\rfloor \Rightarrow (case\ Y\ \tau\ of$
$\qquad \bot \Rightarrow \bot$
$\qquad | \lfloor\bot\rfloor \Rightarrow \lfloor\bot\rfloor$
$\qquad | \lfloor\lfloor y\rfloor\rfloor \Rightarrow \lfloor\lfloor y\rfloor\rfloor)$
$\quad | \lfloor\lfloor False\rfloor\rfloor \Rightarrow \lfloor\lfloor False\rfloor\rfloor)$

Note that *not* is *not* defined as a strict function; proximity to lattice laws implies that we *need* a definition of *not* that satisfies $not(not(x))=x$.

In textbook notation, the logical core constructs *not* and *op and* were represented as follows:

lemma *textbook-not*:
$$I[\![not(X)]\!]\ \tau = (case\ I[\![X]\!]\ \tau\ of\ \ \bot \Rightarrow \bot \\ | \lfloor\bot\rfloor \Rightarrow \lfloor\bot\rfloor \\ | \lfloor\lfloor x\rfloor\rfloor \Rightarrow \lfloor\lfloor \neg\, x\rfloor\rfloor)$$
by(*simp add*: *Sem-def not-def*)

lemma *textbook-and*:
$I[\![X\ and\ Y]\!]\ \tau = (case\ I[\![X]\!]\ \tau\ of$

$$
\begin{array}{l}
\quad\quad \bot \ \Rightarrow (case\ I[\![Y]\!]\ \tau\ of \\
\quad\quad\quad\quad\quad \bot \Rightarrow\ \bot \\
\quad\quad\quad\quad \mid \lfloor\bot\rfloor \Rightarrow \bot \\
\quad\quad\quad\quad \mid \lfloor\lfloor True\rfloor\rfloor \Rightarrow\ \bot \\
\quad\quad\quad\quad \mid \lfloor\lfloor False\rfloor\rfloor \Rightarrow\ \lfloor\lfloor False\rfloor\rfloor) \\
\quad \mid \lfloor\ \bot\ \rfloor \Rightarrow (case\ I[\![Y]\!]\ \tau\ of \\
\quad\quad\quad\quad\quad \bot \Rightarrow\ \bot \\
\quad\quad\quad\quad \mid \lfloor\bot\rfloor \Rightarrow \lfloor\bot\rfloor \\
\quad\quad\quad\quad \mid \lfloor\lfloor True\rfloor\rfloor \Rightarrow \lfloor\bot\rfloor \\
\quad\quad\quad\quad \mid \lfloor\lfloor False\rfloor\rfloor \Rightarrow\ \lfloor\lfloor False\rfloor\rfloor) \\
\quad \mid \lfloor\lfloor True\rfloor\rfloor \Rightarrow (case\ I[\![Y]\!]\ \tau\ of \\
\quad\quad\quad\quad\quad \bot \Rightarrow\ \bot \\
\quad\quad\quad\quad \mid \lfloor\bot\rfloor \Rightarrow \lfloor\bot\rfloor \\
\quad\quad\quad\quad \mid \lfloor\lfloor y\rfloor\rfloor \Rightarrow\ \lfloor\lfloor y\rfloor\rfloor) \\
\quad \mid \lfloor\lfloor False\rfloor\rfloor \Rightarrow\ \lfloor\lfloor\ False\ \rfloor\rfloor)
\end{array}
$$

by(*simp add*: *Sem-def ocl-and-def split*: *option.split*)

definition *ocl-or* :: $[('\mathfrak{A})Boolean,\ ('\mathfrak{A})Boolean] \Rightarrow ('\mathfrak{A})Boolean$
(**infixl** *or 25*)
where $X\ or\ Y \equiv not(not\ X\ and\ not\ Y)$

definition *ocl-implies* :: $[('\mathfrak{A})Boolean,\ ('\mathfrak{A})Boolean] \Rightarrow ('\mathfrak{A})Boolean$
(**infixl** *implies 25*)
where $X\ implies\ Y \equiv not\ X\ or\ Y$

lemma *cp-ocl-and*:$(X\ and\ Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau)\ and\ (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
by(*simp add*: *ocl-and-def*)

lemma *cp-ocl-or*:$((X{::}('\mathfrak{A})Boolean)\ or\ Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau)\ or\ (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
apply(*simp add*: *ocl-or-def*)
apply(*subst cp-not*[*of not* $(\lambda\text{-}.\ X\ \tau)$ *and not* $(\lambda\text{-}.\ Y\ \tau)$])
apply(*subst cp-ocl-and*[*of not* $(\lambda\text{-}.\ X\ \tau)$ *not* $(\lambda\text{-}.\ Y\ \tau)$])
by(*simp add*: *cp-not*[*symmetric*] *cp-ocl-and*[*symmetric*])

lemma *cp-ocl-implies*:$(X\ implies\ Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau)\ implies\ (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
apply(*simp add*: *ocl-implies-def*)
apply(*subst cp-ocl-or*[*of not* $(\lambda\text{-}.\ X\ \tau)$ $(\lambda\text{-}.\ Y\ \tau)$])
by(*simp add*: *cp-not*[*symmetric*] *cp-ocl-or*[*symmetric*])

lemma *ocl-and1*[*simp*]: (*invalid and true*) = *invalid*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def*)
lemma *ocl-and2*[*simp*]: (*invalid and false*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def*)
lemma *ocl-and3*[*simp*]: (*invalid and null*) = *invalid*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def*

```
                        null-fun-def null-option-def)
lemma ocl-and4[simp]: (invalid and invalid) = invalid
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def bot-option-def)

lemma ocl-and5[simp]: (null and true) = null
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def bot-option-def
                        null-fun-def null-option-def)
lemma ocl-and6[simp]: (null and false) = false
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def bot-option-def
                        null-fun-def null-option-def)
lemma ocl-and7[simp]: (null and null) = null
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def bot-option-def
                        null-fun-def null-option-def)
lemma ocl-and8[simp]: (null and invalid) = invalid
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def bot-option-def
                        null-fun-def null-option-def)

lemma ocl-and9[simp]: (false and true) = false
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def)
lemma ocl-and10[simp]: (false and false) = false
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def)
lemma ocl-and11[simp]: (false and null) = false
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def)
lemma ocl-and12[simp]: (false and invalid) = false
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def)

lemma ocl-and13[simp]: (true and true) = true
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def)
lemma ocl-and14[simp]: (true and false) = false
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def)
lemma ocl-and15[simp]: (true and null) = null
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def bot-option-def
                        null-fun-def null-option-def)
lemma ocl-and16[simp]: (true and invalid) = invalid
  by(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def bot-option-def
                        null-fun-def null-option-def)

lemma ocl-and-idem[simp]: (X and X) = X
  apply(rule ext,simp add: ocl-and-def null-def invalid-def true-def false-def)
  apply(case-tac X x, simp-all)
  apply(case-tac a, simp-all)
  apply(case-tac aa, simp-all)
  done

lemma ocl-and-commute: (X and Y) = (Y and X)
  by(rule ext,auto simp:true-def false-def ocl-and-def invalid-def
                  split: option.split option.split-asm
                         bool.split bool.split-asm)
```

lemma *ocl-and-false1*[*simp*]: (*false and X*) = *false*
  apply(*rule ext*, *simp add*: *ocl-and-def*)
  apply(*auto simp*:*true-def false-def invalid-def*
            split: *option.split option.split-asm*)
  done

lemma *ocl-and-false2*[*simp*]: (*X and false*) = *false*
  by(*simp add*: *ocl-and-commute*)

lemma *ocl-and-true1*[*simp*]: (*true and X*) = *X*
  apply(*rule ext*, *simp add*: *ocl-and-def*)
  apply(*auto simp*:*true-def false-def invalid-def*
            split: *option.split option.split-asm*)
  done

lemma *ocl-and-true2*[*simp*]: (*X and true*) = *X*
  by(*simp add*: *ocl-and-commute*)

lemma *ocl-and-assoc*: (*X and* (*Y and Z*)) = (*X and Y and Z*)
  apply(*rule ext*, *simp add*: *ocl-and-def*)
  apply(*auto simp*:*true-def false-def null-def invalid-def*
            split: *option.split option.split-asm*
                  bool.split bool.split-asm)
done

lemma *ocl-or-idem*[*simp*]: (*X or X*) = *X*
  by(*simp add*: *ocl-or-def*)

lemma *ocl-or-commute*: (*X or Y*) = (*Y or X*)
  by(*simp add*: *ocl-or-def ocl-and-commute*)

lemma *ocl-or-false1*[*simp*]: (*false or Y*) = *Y*
  by(*simp add*: *ocl-or-def*)

lemma *ocl-or-false2*[*simp*]: (*Y or false*) = *Y*
  by(*simp add*: *ocl-or-def*)

lemma *ocl-or-true1*[*simp*]: (*true or Y*) = *true*
  by(*simp add*: *ocl-or-def*)

lemma *ocl-or-true2*: (*Y or true*) = *true*
  by(*simp add*: *ocl-or-def*)

lemma *ocl-or-assoc*: (*X or* (*Y or Z*)) = (*X or Y or Z*)
  by(*simp add*: *ocl-or-def ocl-and-assoc*)

lemma *deMorgan1*: $not(X\ and\ Y) = ((not\ X)\ or\ (not\ Y))$
by(*simp add*: *ocl-or-def*)

lemma *deMorgan2*: $not(X\ or\ Y) = ((not\ X)\ and\ (not\ Y))$
by(*simp add*: *ocl-or-def*)

3.6 A Standard Logical Calculus for OCL

Besides the need for algebraic laws for OCL in order to normalize

definition *OclValid* :: $[('\mathfrak{A})st, ('\mathfrak{A})Boolean] \Rightarrow bool\ ((1(\text{-})/ \models (\text{-}))\ 50)$
where $\tau \models P \equiv ((P\ \tau) = true\ \tau)$

4 Global vs. Local Judgements

lemma *transform1*: $P = true \Longrightarrow \tau \models P$
by(*simp add*: *OclValid-def*)

lemma *transform1-rev*: $\forall\ \tau.\ \tau \models P \Longrightarrow P = true$
by(*rule ext*, *auto simp*: *OclValid-def true-def*)

lemma *transform2*: $(P = Q) \Longrightarrow ((\tau \models P) = (\tau \models Q))$
by(*auto simp*: *OclValid-def*)

lemma *transform2-rev*: $\forall\ \tau.\ (\tau \models \delta\ P) \wedge (\tau \models \delta\ Q) \wedge (\tau \models P) = (\tau \models Q) \Longrightarrow P = Q$
apply(*rule ext*,*auto simp*: *OclValid-def true-def defined-def*)
apply(*erule-tac x=a* **in** *allE*)
apply(*erule-tac x=b* **in** *allE*)
apply(*auto simp*: *false-def true-def defined-def bot-Boolean-def null-Boolean-def split*: *option.split-asm HOL.split-if-asm*)
done

However, certain properties (like transitivity) can not be *transformed* from the global level to the local one, they have to be re-proven on the local level.

lemma *transform3*:
assumes $H : P = true \Longrightarrow Q = true$
shows $\tau \models P \Longrightarrow \tau \models Q$
apply(*simp add*: *OclValid-def*)
apply(*rule H*[*THEN fun-cong*])
apply(*rule ext*)
oops

4.0.1 Local Validity and Meta-logic

lemma *foundation1*[*simp*]: $\tau \models true$
by(*auto simp*: *OclValid-def*)

lemma *foundation2*[*simp*]: $\neg(\tau \models \mathit{false})$
by(*auto simp*: *OclValid-def true-def false-def*)

lemma *foundation3*[*simp*]: $\neg(\tau \models \mathit{invalid})$
by(*auto simp*: *OclValid-def true-def false-def invalid-def bot-option-def*)

lemma *foundation4*[*simp*]: $\neg(\tau \models \mathit{null})$
by(*auto simp*: *OclValid-def true-def false-def null-def null-fun-def null-option-def bot-option-def*)

lemma *bool-split-local*[*simp*]:
$(\tau \models (x \triangleq \mathit{invalid})) \vee (\tau \models (x \triangleq \mathit{null})) \vee (\tau \models (x \triangleq \mathit{true})) \vee (\tau \models (x \triangleq \mathit{false}))$
apply(*insert bool-split*[*of x* τ], *auto*)
apply(*simp-all add*: *OclValid-def StrongEq-def true-def null-def invalid-def*)
done

lemma *def-split-local*:
$(\tau \models \delta\ x) = ((\neg(\tau \models (x \triangleq \mathit{invalid}))) \wedge (\neg\ (\tau \models (x \triangleq \mathit{null}))))$
by(*simp add*:*defined-def true-def false-def invalid-def null-def*
StrongEq-def OclValid-def bot-fun-def null-fun-def)

lemma *foundation5*:
$\tau \models (P\ \mathit{and}\ Q) \Longrightarrow (\tau \models P) \wedge (\tau \models Q)$
by(*simp add*: *ocl-and-def OclValid-def true-def false-def defined-def*
split: *option.split option.split-asm bool.split bool.split-asm*)

lemma *foundation6*:
$\tau \models P \Longrightarrow \tau \models \delta\ P$
by(*simp add*: *not-def OclValid-def true-def false-def defined-def*
null-option-def null-fun-def bot-option-def bot-fun-def
split: *option.split option.split-asm*)

lemma *foundation7*[*simp*]:
$(\tau \models \mathit{not}\ (\delta\ x)) = (\neg\ (\tau \models \delta\ x))$
by(*simp add*: *not-def OclValid-def true-def false-def defined-def*
split: *option.split option.split-asm*)

lemma *foundation7'*[*simp*]:
$(\tau \models \mathit{not}\ (\upsilon\ x)) = (\neg\ (\tau \models \upsilon\ x))$
by(*simp add*: *not-def OclValid-def true-def false-def valid-def*
split: *option.split option.split-asm*)

Key theorem for the Delta-closure: either an expression is defined, or it can be replaced (substituted via `StrongEq_L_subst2`; see below) by invalid or null. Strictness-reduction rules will usually reduce these substituted terms drastically.

lemma *foundation8*:
$(\tau \models \delta\ x) \vee (\tau \models (x \triangleq \mathit{invalid})) \vee (\tau \models (x \triangleq \mathit{null}))$

proof –
  have *1* : $(\tau \models \delta\ x) \vee (\neg(\tau \models \delta\ x))$ **by** *auto*
  have *2* : $(\neg(\tau \models \delta\ x)) = ((\tau \models (x \triangleq \mathit{invalid})) \vee (\tau \models (x \triangleq \mathit{null})))$
    by(*simp only*: *def-split-local*, *simp*)
  show *?thesis* **by**(*insert 1*, *simp add*:*2*)
qed

lemma *foundation9*:
$\tau \models \delta\ x \Longrightarrow (\tau \models \mathit{not}\ x) = (\neg\ (\tau \models x))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *not-def null-fun-def null-option-def bot-option-def*
    OclValid-def invalid-def true-def null-def StrongEq-def)

lemma *foundation10*:
$\tau \models \delta\ x \Longrightarrow \tau \models \delta\ y \Longrightarrow (\tau \models (x\ \mathit{and}\ y)) = (\ (\tau \models x) \wedge (\tau \models y))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *ocl-and-def OclValid-def invalid-def*
    true-def null-def StrongEq-def null-fun-def null-option-def bot-option-def
  split:*bool.split-asm*)

lemma *foundation11*:
$\tau \models \delta\ x \Longrightarrow\ \tau \models \delta\ y \Longrightarrow (\tau \models (x\ \mathit{or}\ y)) = (\ (\tau \models x) \vee (\tau \models y))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *not-def ocl-or-def ocl-and-def OclValid-def invalid-def*
    true-def null-def StrongEq-def null-fun-def null-option-def bot-option-def
  split:*bool.split-asm bool.split*)

lemma *foundation12*:
$\tau \models \delta\ x \Longrightarrow\ \tau \models \delta\ y \Longrightarrow (\tau \models (x\ \mathit{implies}\ y)) = (\ (\tau \models x) \longrightarrow (\tau \models y))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *not-def ocl-or-def ocl-and-def ocl-implies-def bot-option-def*
    OclValid-def invalid-def true-def null-def StrongEq-def null-fun-def
null-option-def
  split:*bool.split-asm bool.split*)

lemma *foundation13*:$(\tau \models A \triangleq \mathit{true}) \quad = (\tau \models A)$
by(*auto simp*: *not-def OclValid-def invalid-def true-def null-def StrongEq-def*
    split:*bool.split-asm bool.split*)

lemma *foundation14*:$(\tau \models A \triangleq \mathit{false}) \quad = (\tau \models \mathit{not}\ A)$
by(*auto simp*: *not-def OclValid-def invalid-def false-def true-def null-def StrongEq-def*

  split:*bool.split-asm bool.split option.split*)

lemma *foundation15*:$(\tau \models A \triangleq \mathit{invalid}) = (\tau \models \mathit{not}(\upsilon\ A))$

by(*auto simp*: *not-def OclValid-def valid-def invalid-def false-def true-def null-def*
StrongEq-def bot-option-def null-fun-def null-option-def bot-option-def bot-fun-def
split:*bool.split-asm bool.split option.split*)

lemma *foundation16*: $\tau \models (\delta\ X) = (X\ \tau \neq bot \wedge X\ \tau \neq null)$
by(*auto simp*: *OclValid-def defined-def false-def true-def bot-fun-def null-fun-def*
split:*split-if-asm*)

lemmas *foundation17* = *foundation16*[*THEN iffD1*,*standard*]

lemma *foundation18*: $\tau \models (\upsilon\ X) = (X\ \tau \neq invalid\ \tau)$
by(*auto simp*: *OclValid-def valid-def false-def true-def bot-fun-def invalid-def*
split:*split-if-asm*)

lemma *foundation18'*: $\tau \models (\upsilon\ X) = (X\ \tau \neq bot)$
by(*auto simp*: *OclValid-def valid-def false-def true-def bot-fun-def*
split:*split-if-asm*)

lemmas *foundation19* = *foundation18*[*THEN iffD1*,*standard*]

lemma *foundation20* : $\tau \models (\delta\ X) \Longrightarrow \tau \models \upsilon\ X$
by(*simp add*: *foundation18 foundation16 invalid-def*)

lemma *foundation21*: $(not\ A \triangleq not\ B) = (A \triangleq B)$
by(*rule ext*, *auto simp*: *not-def StrongEq-def*
split: *bool.split-asm HOL.split-if-asm option.split*)

lemma *defined-not-I* : $\tau \models \delta\ (x) \Longrightarrow \tau \models \delta\ (not\ x)$
by(*auto simp*: *not-def null-def invalid-def defined-def valid-def OclValid-def*
true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def
split: *option.split-asm HOL.split-if-asm*)

lemma *valid-not-I* : $\tau \models \upsilon\ (x) \Longrightarrow \tau \models \upsilon\ (not\ x)$
by(*auto simp*: *not-def null-def invalid-def defined-def valid-def OclValid-def*
true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def
split: *option.split-asm option.split HOL.split-if-asm*)

lemma *defined-and-I* : $\tau \models \delta\ (x) \Longrightarrow \tau \models \delta\ (y) \Longrightarrow \tau \models \delta\ (x\ and\ y)$
apply(*simp add*: *ocl-and-def null-def invalid-def defined-def valid-def OclValid-def*
true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def
split: *option.split-asm HOL.split-if-asm*)
apply(*auto simp*: *null-option-def split*: *bool.split*)

by(*case-tac ya*,*simp-all*)

lemma *valid-and-I* : $\tau \models \upsilon\ (x) \Longrightarrow\ \tau \models \upsilon\ (y) \Longrightarrow \tau \models \upsilon\ (x\ and\ y)$
apply(*simp add*: *ocl-and-def null-def invalid-def defined-def valid-def OclValid-def true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def split*: *option.split-asm HOL.split-if-asm*)
by(*auto simp*: *null-option-def split*: *option.split bool.split*)

5 Local Judgements and Strong Equality

lemma *StrongEq-L-refl*: $\tau \models (x \triangleq x)$
by(*simp add*: *OclValid-def StrongEq-def*)

lemma *StrongEq-L-sym*: $\tau \models (x \triangleq y) \Longrightarrow \tau \models (y \triangleq x)$
by(*simp add*: *OclValid-def StrongEq-def*)

lemma *StrongEq-L-trans*: $\tau \models (x \triangleq y) \Longrightarrow \tau \models (y \triangleq z) \Longrightarrow \tau \models (x \triangleq z)$
by(*simp add*: *OclValid-def StrongEq-def true-def*)

In order to establish substitutivity (which does not hold in general HOL-formulas we introduce the following predicate that allows for a calculus of the necessary side-conditions.

definition *cp* $:: (('\mathfrak{A},'\alpha)\ val \Rightarrow ('\mathfrak{A},'\beta)\ val) \Rightarrow bool$
where $cp\ P \equiv (\exists\ f.\ \forall\ X\ \tau.\ P\ X\ \tau = f\ (X\ \tau)\ \tau)$

The rule of substitutivity in HOL-OCL holds only for context-passing expressions - i.e. those, that pass the context τ without changing it. Fortunately, all operators of the OCL language satisfy this property (but not all HOL operators).

lemma *StrongEq-L-subst1*: $\bigwedge \tau.\ cp\ P \Longrightarrow \tau \models (x \triangleq y) \Longrightarrow \tau \models (P\ x \triangleq P\ y)$
by(*auto simp*: *OclValid-def StrongEq-def true-def cp-def*)

lemma *StrongEq-L-subst2*:
$\bigwedge \tau.\ \ cp\ P \Longrightarrow \tau \models (x \triangleq y) \Longrightarrow \tau \models (P\ x) \Longrightarrow \tau \models (P\ y)$
by(*auto simp*: *OclValid-def StrongEq-def true-def cp-def*)

lemma *cpI1*:
$(\forall\ X\ \tau.\ f\ X\ \tau = f(\lambda\text{-}.\ X\ \tau)\ \tau) \Longrightarrow cp\ P \Longrightarrow cp(\lambda X.\ f\ (P\ X))$
apply(*auto simp*: *true-def cp-def*)
apply(*rule exI*, (*rule allI*)+)
by(*erule-tac x=P X* **in** *allE*, *auto*)

lemma *cpI2*:
$(\forall\ X\ Y\ \tau.\ f\ X\ Y\ \tau = f(\lambda\text{-}.\ X\ \tau)(\lambda\text{-}.\ Y\ \tau)\ \tau) \Longrightarrow$
$cp\ P \Longrightarrow cp\ Q \Longrightarrow cp(\lambda X.\ f\ (P\ X)\ (Q\ X))$
apply(*auto simp*: *true-def cp-def*)
apply(*rule exI*, (*rule allI*)+)

by(*erule-tac x=P X* **in** *allE*, *auto*)

lemma *cp-const* : *cp*(λ-. *c*)
  by (*simp add*: *cp-def*, *fast*)

lemma *cp-id* : *cp*(λX. X)
  by (*simp add*: *cp-def*, *fast*)

lemmas *cp-intro*[*simp*,*intro*!] =
  cp-const
  cp-id
  cp-defined[*THEN allI*[*THEN allI*[*THEN cpI1*], *of defined*]]
  cp-valid[*THEN allI*[*THEN allI*[*THEN cpI1*], *of valid*]]
  cp-not[*THEN allI*[*THEN allI*[*THEN cpI1*], *of not*]]
  cp-ocl-and[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of op and*]]
  cp-ocl-or[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of op or*]]
  cp-ocl-implies[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of op implies*]]
  cp-StrongEq[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]],
    of StrongEq]]

6 Laws to Establish Definedness (Delta-Closure)

For the logical connectives, we have — beyond $?\tau \models ?P \Longrightarrow ?\tau \models \delta\ ?P$ — the following facts:

lemma *ocl-not-defargs*:
$\tau \models (not\ P) \Longrightarrow \tau \models \delta\ P$
by(*auto simp*: *not-def OclValid-def true-def invalid-def defined-def false-def*
  bot-fun-def bot-option-def null-fun-def null-option-def
  split: *bool.split-asm HOL.split-if-asm option.split option.split-asm*)

So far, we have only one strict Boolean predicate (-family): The strict equality.

7 Miscellaneous: OCL's if then else endif

definition *if-ocl* :: $[(\mathfrak{A})Boolean\ ,\ (\mathfrak{A},'\alpha{::}null)\ val,\ (\mathfrak{A},'\alpha)\ val] \Rightarrow (\mathfrak{A},'\alpha)\ val$
  (*if* (-) *then* (-) *else* (-) *endif* [*10*,*10*,*10*]*50*)
where (*if C then* B_1 *else* B_2 *endif*) = ($\lambda\ \tau$. *if* (δ *C*) τ = *true* τ
    then (*if* (*C* τ) = *true* τ
      then B_1 τ
      else B_2 τ)
    else invalid τ)

lemma *cp-if-ocl*:((*if C then* B_1 *else* B_2 *endif*) τ =
  (*if* (λ -. *C* τ) *then* (λ -. B_1 τ) *else* (λ -. B_2 τ) *endif*) τ)

by(*simp only*: *if-ocl-def*, *subst cp-defined*, *rule refl*)

lemma *if-ocl-invalid* [*simp*]: (*if invalid then* B_1 *else* B_2 *endif*) = *invalid*
by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-null* [*simp*]: (*if null then* B_1 *else* B_2 *endif*) = *invalid*
by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-true* [*simp*]: (*if true then* B_1 *else* B_2 *endif*) = B_1
by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-false* [*simp*]: (*if false then* B_1 *else* B_2 *endif*) = B_2
by(*rule ext*, *auto simp*: *if-ocl-def*)

end

theory *OCL-lib*
imports *OCL-core*
begin

8 Simple, Basic Types like Void, Boolean and Integer

Since Integer is again a basic type, we define its semantic domain as the valuations over *int option option*

type-synonym $('\mathfrak{A})$*Integer* = $('\mathfrak{A}$,*int option option*) *val*

type-synonym $('\mathfrak{A})$*Void* = $('\mathfrak{A}$,*unit option*) *val*

Note that this *minimal* OCL type contains only two elements: undefined and null. For technical reasons, he does not contain to the null-class yet.

9 Strict equalities.

Note that the strict equality on basic types (actually on all types) must be exceptionally defined on null — otherwise the entire concept of null in the language does not make much sense. This is an important exception from the general rule that null arguments — especially if passed as "self"-argument — lead to invalid results.

consts *StrictRefEq* :: $[('\mathfrak{A},'a)val,('\mathfrak{A},'a)val] \Rightarrow ('\mathfrak{A})Boolean$ (**infixl** $\doteq$ *30*)

syntax

notequal :: $('\mathfrak{A})$*Boolean* $\Rightarrow$ $('\mathfrak{A})$*Boolean* $\Rightarrow$ $('\mathfrak{A})$*Boolean* (**infix** <> *40*)
translations
$a <> b == CONST\ not(\ a \doteq b)$

defs *StrictRefEq-int*[*code-unfold*] :
$(x::('\mathfrak{A})Integer) \doteq y \equiv \lambda\,\tau.\ if\ (\upsilon\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ (x \triangleq y)\ \tau$
$else\ invalid\ \tau$

defs *StrictRefEq-bool*[*code-unfold*] :
$(x::('\mathfrak{A})Boolean) \doteq y \equiv \lambda\,\tau.\ if\ (\upsilon\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ (x \triangleq y)\tau$
$else\ invalid\ \tau$

lemma *RefEq-int-refl*[*simp*,*code-unfold*] :
$((x::('\mathfrak{A})Integer) \doteq x) = (if\ (\upsilon\ x)\ then\ true\ else\ invalid\ endif)$
by(*rule ext*, *simp add*: *StrictRefEq-int if-ocl-def*)

lemma *RefEq-bool-refl*[*simp*,*code-unfold*] :
$((x::('\mathfrak{A})Boolean) \doteq x) = (if\ (\upsilon\ x)\ then\ true\ else\ invalid\ endif)$
by(*rule ext*, *simp add*: *StrictRefEq-bool if-ocl-def*)

lemma *StrictRefEq-int-strict1*[*simp*] : $((x::('\mathfrak{A})Integer) \doteq invalid) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-int true-def false-def*)

lemma *StrictRefEq-int-strict2*[*simp*] : $(invalid \doteq (x::('\mathfrak{A})Integer)) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-int true-def false-def*)

lemma *StrictRefEq-bool-strict1*[*simp*] : $((x::('\mathfrak{A})Boolean) \doteq invalid) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-bool true-def false-def*)

lemma *StrictRefEq-bool-strict2*[*simp*] : $(invalid \doteq (x::('\mathfrak{A})Boolean)) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-bool true-def false-def*)

lemma *strictEqBool-vs-strongEq*:
$\tau \models(\upsilon\ x) \Longrightarrow \tau \models(\upsilon\ y) \Longrightarrow (\tau \models ((x::('\mathfrak{A})Boolean) \doteq y)) = (\tau \models (x \triangleq y))$
by(*simp add*: *StrictRefEq-bool OclValid-def*)

lemma *strictEqInt-vs-strongEq*:
$\tau \models(\upsilon\ x) \Longrightarrow \tau \models(\upsilon\ y) \Longrightarrow (\tau \models ((x::('\mathfrak{A})Integer) \doteq y)) = (\tau \models (x \triangleq y))$
by(*simp add*: *StrictRefEq-int OclValid-def*)

lemma *strictEqBool-defargs*:
$\tau \models ((x::('\mathfrak{A})Boolean) \doteq y) \Longrightarrow (\tau \models (\upsilon\ x)) \wedge (\tau \models(\upsilon\ y))$
by(*simp add*: *StrictRefEq-bool OclValid-def true-def invalid-def*
bot-option-def
split: *bool.split-asm HOL.split-if-asm*)

lemma *strictEqInt-defargs*:
$\tau \models ((x::('\mathfrak{A})\mathit{Integer}) \doteq y) \Longrightarrow (\tau \models (\upsilon\ x)) \wedge (\tau \models (\upsilon\ y))$
by(*simp add*: *StrictRefEq-int OclValid-def true-def invalid-def valid-def bot-option-def*
split: *bool.split-asm HOL.split-if-asm*)

lemma *strictEqBool-valid-args-valid*:
$(\tau \models \upsilon((x::('\mathfrak{A})\mathit{Boolean}) \doteq y)) = ((\tau \models(\upsilon\ x)) \wedge (\tau \models(\upsilon\ y)))$
by(*auto simp*: *StrictRefEq-bool OclValid-def true-def valid-def false-def StrongEq-def*

defined-def invalid-def valid-def bot-option-def bot-fun-def
split: *bool.split-asm HOL.split-if-asm option.split*)

lemma *strictEqInt-valid-args-valid*:
$(\tau \models \upsilon((x::('\mathfrak{A})\mathit{Integer}) \doteq y)) = ((\tau \models(\upsilon\ x)) \wedge (\tau \models(\upsilon\ y)))$
by(*auto simp*: *StrictRefEq-int OclValid-def true-def valid-def false-def StrongEq-def*

defined-def invalid-def bot-fun-def bot-option-def
split: *bool.split-asm HOL.split-if-asm option.split*)

lemma *StrictRefEq-int-strict* :
assumes *A*: $\upsilon\ (x::('\mathfrak{A})\mathit{Integer}) = \mathit{true}$
and *B*: $\upsilon\ y = \mathit{true}$
shows $\upsilon\ (x \doteq y) = \mathit{true}$
apply(*insert A B*)
apply(*rule ext*, *simp add*: *StrongEq-def StrictRefEq-int true-def valid-def defined-def*
bot-fun-def bot-option-def)
done

lemma *StrictRefEq-int-strict′* :
assumes *A*: $\upsilon\ (((x::('\mathfrak{A})\mathit{Integer})) \doteq y) = \mathit{true}$
shows $\upsilon\ x = \mathit{true} \wedge \upsilon\ y = \mathit{true}$
apply(*insert A*, *rule conjI*)
apply(*rule ext*, *drule-tac x=xa* **in** *fun-cong*)
prefer *2*
apply(*rule ext*, *drule-tac x=xa* **in** *fun-cong*)
apply(*simp-all add*: *StrongEq-def StrictRefEq-int*
false-def true-def valid-def defined-def)
apply(*case-tac y xa*, *auto*)
apply(*simp-all add*: *true-def invalid-def bot-fun-def*)
done

lemma *StrictRefEq-int-strict″* : $\upsilon\ ((x::('\mathfrak{A})\mathit{Integer}) \doteq y) = (\upsilon(x)\ \mathit{and}\ \upsilon(y))$
by(*auto intro*!: *transform2-rev defined-and-I simp*:*foundation10 strictEqInt-valid-args-valid*)

lemma *StrictRefEq-bool-strict″* : $\upsilon\ ((x::('\mathfrak{A})\mathit{Boolean}) \doteq y) = (\upsilon(x)\ \mathit{and}\ \upsilon(y))$

by(*auto intro*!: *transform2-rev defined-and-I simp*:*foundation10 strictEqBool-valid-args-valid*)

lemma *cp-StrictRefEq-bool*:
$((X::('\mathfrak{A})Boolean) \doteq Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau) \doteq (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
by(*auto simp*: *StrictRefEq-bool StrongEq-def defined-def valid-def cp-defined*[*symmetric*])

lemma *cp-StrictRefEq-int*:
$((X::('\mathfrak{A})Integer) \doteq Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau) \doteq (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
by(*auto simp*: *StrictRefEq-int StrongEq-def valid-def cp-defined*[*symmetric*])

lemmas *cp-intro*[*simp*,*intro*!] =
cp-intro
cp-StrictRefEq-bool[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of StrictRefEq*]]
cp-StrictRefEq-int[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of StrictRefEq*]]

definition *ocl-zero* ::$('\mathfrak{A})$*Integer* (**0**)
where **0** = $(\lambda$ - . $\lfloor\lfloor 0::int \rfloor\rfloor)$

definition *ocl-one* ::$('\mathfrak{A})$*Integer* (**1**)
where **1** = $(\lambda$ - . $\lfloor\lfloor 1::int \rfloor\rfloor)$

definition *ocl-two* ::$('\mathfrak{A})$*Integer* (**2**)
where **2** = $(\lambda$ - . $\lfloor\lfloor 2::int \rfloor\rfloor)$

definition *ocl-three* ::$('\mathfrak{A})$*Integer* (**3**)
where **3** = $(\lambda$ - . $\lfloor\lfloor 3::int \rfloor\rfloor)$

definition *ocl-four* ::$('\mathfrak{A})$*Integer* (**4**)
where **4** = $(\lambda$ - . $\lfloor\lfloor 4::int \rfloor\rfloor)$

definition *ocl-five* ::$('\mathfrak{A})$*Integer* (**5**)
where **5** = $(\lambda$ - . $\lfloor\lfloor 5::int \rfloor\rfloor)$

definition *ocl-six* ::$('\mathfrak{A})$*Integer* (**6**)
where **6** = $(\lambda$ - . $\lfloor\lfloor 6::int \rfloor\rfloor)$

definition *ocl-seven* ::$('\mathfrak{A})$*Integer* (**7**)
where **7** = $(\lambda$ - . $\lfloor\lfloor 7::int \rfloor\rfloor)$

definition *ocl-eight* ::$('\mathfrak{A})$*Integer* (**8**)
where **8** = $(\lambda$ - . $\lfloor\lfloor 8::int \rfloor\rfloor)$

definition *ocl-nine* ::$('\mathfrak{A})$*Integer* (**9**)

where $\mathbf{9} = (\lambda\ \text{-}\ .\ \lfloor\lfloor \mathit{9::int} \rfloor\rfloor)$

definition *ten-nine* ::('$\mathfrak{A}$)*Integer* (**10**)
where $\mathbf{10} = (\lambda\ \text{-}\ .\ \lfloor\lfloor \mathit{10::int} \rfloor\rfloor)$

Here is a way to cast in standard operators via the type class system of Isabelle.

Here follows a list of code-examples, that explain the meanings of the above definitions by compilation to code and execution to "True".

Elementary computations on Booleans

value $\tau_0 \models \upsilon(\mathit{true})$
value $\tau_0 \models \delta(\mathit{false})$
value $\neg(\tau_0 \models \delta(\mathit{null}))$
value $\neg(\tau_0 \models \delta(\mathit{invalid}))$
value $\tau_0 \models \upsilon((\mathit{null}::(\prime\mathfrak{A})\mathit{Boolean}))$
value $\neg(\tau_0 \models \upsilon(\mathit{invalid}))$
value $\tau_0 \models (\mathit{true\ and\ true})$
value $\tau_0 \models (\mathit{true\ and\ true} \triangleq \mathit{true})$
value $\tau_0 \models ((\mathit{null\ or\ null}) \triangleq \mathit{null})$
value $\tau_0 \models ((\mathit{null\ or\ null}) \doteq \mathit{null})$
value $\tau_0 \models ((\mathit{true} \triangleq \mathit{false}) \triangleq \mathit{false})$
value $\tau_0 \models ((\mathit{invalid} \triangleq \mathit{false}) \triangleq \mathit{false})$
value $\tau_0 \models ((\mathit{invalid} \doteq \mathit{false}) \triangleq \mathit{invalid})$

Elementary computations on Integer

value $\tau_0 \models \upsilon(\mathbf{4})$
value $\tau_0 \models \delta(\mathbf{4})$
value $\tau_0 \models \upsilon((\mathit{null}::(\prime\mathfrak{A})\mathit{Integer}))$
value $\tau_0 \models (\mathit{invalid} \triangleq \mathit{invalid}\ )$
value $\tau_0 \models (\mathit{null} \triangleq \mathit{null}\ )$
value $\tau_0 \models (\mathbf{4} \triangleq \mathbf{4})$
value $\neg(\tau_0 \models (\mathbf{9} \triangleq \mathbf{10}\ ))$
value $\neg(\tau_0 \models (\mathit{invalid} \triangleq \mathbf{10}\ ))$
value $\neg(\tau_0 \models (\mathit{null} \triangleq \mathbf{10}\ ))$
value $\neg(\tau_0 \models (\mathit{invalid} \doteq (\mathit{invalid}::(\prime\mathfrak{A})\mathit{Integer})))$
value $\tau_0 \models (\mathit{null} \doteq (\mathit{null}::(\prime\mathfrak{A})\mathit{Integer})\ )$
value $\tau_0 \models (\mathit{null} \doteq (\mathit{null}::(\prime\mathfrak{A})\mathit{Integer})\ )$
value $\tau_0 \models (\mathbf{4} \doteq \mathbf{4})$
value $\neg(\tau_0 \models (\mathbf{4} \doteq \mathbf{10}\ ))$

lemma $\delta(\mathit{null}::(\prime\mathfrak{A})\mathit{Integer}) = \mathit{false}$ **by** *simp*
lemma $\upsilon(\mathit{null}::(\prime\mathfrak{A})\mathit{Integer}) = \mathit{true}$ **by** *simp*

lemma [*simp*,*code-unfold*]:$\delta\ \mathbf{0} = \mathit{true}$
by(*simp add*:*ocl-zero-def defined-def true-def*
bot-fun-def bot-option-def null-fun-def null-option-def)

lemma [*simp*,*code-unfold*]:υ **0** = *true*
by(*simp add*:*ocl-zero-def valid-def true-def*
bot-fun-def bot-option-def null-fun-def null-option-def)

lemma [*simp*,*code-unfold*]:δ **1** = *true*
by(*simp add*:*ocl-one-def defined-def true-def*
bot-fun-def bot-option-def null-fun-def null-option-def)

lemma [*simp*,*code-unfold*]:υ **1** = *true*
by(*simp add*:*ocl-one-def valid-def true-def*
bot-fun-def bot-option-def null-fun-def null-option-def)

lemma [*simp*,*code-unfold*]:δ **2** = *true*
by(*simp add*:*ocl-two-def defined-def true-def*
bot-fun-def bot-option-def null-fun-def null-option-def)

lemma [*simp*,*code-unfold*]:υ **2** = *true*
by(*simp add*:*ocl-two-def valid-def true-def*
bot-fun-def bot-option-def null-fun-def null-option-def)

lemma *zero-non-null* [*simp*]: (**0** $\doteq$ *null*) = *false*
by(*rule ext*,*auto simp*:*ocl-zero-def null-def StrictRefEq-int valid-def invalid-def*
bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

lemma *null-non-zero* [*simp*]: (*null* $\doteq$ **0**) = *false*
by(*rule ext*,*auto simp*:*ocl-zero-def null-def StrictRefEq-int valid-def invalid-def*
bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

lemma *one-non-null* [*simp*]: (**1** $\doteq$ *null*) = *false*
by(*rule ext*,*auto simp*:*ocl-one-def null-def StrictRefEq-int valid-def invalid-def*
bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

lemma *null-non-one* [*simp*]: (*null* $\doteq$ **1**) = *false*
by(*rule ext*,*auto simp*:*ocl-one-def null-def StrictRefEq-int valid-def invalid-def*
bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

lemma *two-non-null* [*simp*]: (**2** $\doteq$ *null*) = *false*
by(*rule ext*,*auto simp*:*ocl-two-def null-def StrictRefEq-int valid-def invalid-def*
bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

lemma *null-non-two* [*simp*]: (*null* $\doteq$ **2**) = *false*
by(*rule ext*,*auto simp*:*ocl-two-def null-def StrictRefEq-int valid-def invalid-def*
bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

Here is a common case of a built-in operation on built-in types. Note that the arguments must be both defined (non-null, non-bot).

Note that we can not follow the lexis of standard OCL for Isabelle- technical reasons; these operators are heavily overloaded in the library that a further overloading would lead to heavy technical buzz in this document...

definition *ocl-add-int* ::$('\mathfrak{A})Integer \Rightarrow ('\mathfrak{A})Integer \Rightarrow ('\mathfrak{A})Integer$ (**infix** $\oplus$ *40*)
where $x \oplus y \equiv \lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\delta\ y)\ \tau = true\ \tau$
$then\ \lfloor\lfloor\lceil\lceil x\ \tau\rceil\rceil + \lceil\lceil y\ \tau\rceil\rceil\rfloor\rfloor$
$else\ invalid\ \tau$

definition *ocl-less-int* ::$('\mathfrak{A})Integer \Rightarrow ('\mathfrak{A})Integer \Rightarrow ('\mathfrak{A})Boolean$ (**infix** $\prec$ *40*)
where $x \prec y \equiv \lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\delta\ y)\ \tau = true\ \tau$
$then\ \lfloor\lfloor\lceil\lceil x\ \tau\rceil\rceil < \lceil\lceil y\ \tau\rceil\rceil\rfloor\rfloor$
$else\ invalid\ \tau$

definition *ocl-le-int* ::$('\mathfrak{A})Integer \Rightarrow ('\mathfrak{A})Integer \Rightarrow ('\mathfrak{A})Boolean$ (**infix** $\preceq$ *40*)
where $x \preceq y \equiv \lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\delta\ y)\ \tau = true\ \tau$
$then\ \lfloor\lfloor\lceil\lceil x\ \tau\rceil\rceil \leq \lceil\lceil y\ \tau\rceil\rceil\rfloor\rfloor$
$else\ invalid\ \tau$

Here follows a list of code-examples, that explain the meanings of the above definitions by compilation to code and execution to "True".

value $\tau_0 \models (\mathbf{9} \preceq \mathbf{10}\ )$
value $\tau_0 \models ((\ \mathbf{4} \oplus \mathbf{4}\ ) \preceq \mathbf{10}\ )$
value $\neg(\tau_0 \models ((\mathbf{4} \oplus(\ \mathbf{4} \oplus \mathbf{4}\ )) \prec \mathbf{10}\ ))$

9.1 Example: The Set-Collection Type on the Abstract Interface

no-notation *None* $(\bot)$
notation *bot* $(\bot)$

For the semantic construction of the collection types, we have two goals:

1. we want the types to be *fully abstract*, i.e. the type should not contain junk-elements that are not representable by OCL expressions.

2. We want a possibility to nest collection types (so, we want the potential to talking about $Set(Set(Sequences(Pairs(X,Y))))$), and

The former principe rules out the option to define $'\alpha$ *Set* just by $('\mathfrak{A}, ('\alpha$ *option option*) *set*) *val*. This would allow sets to contain junk elements such as $\{\bot\}$ which we need to identify with undefinedness itself. Abandoning fully abstractness of rules would later on produce all sorts of problems when quantifying over the elements of a type. However, if we build an own type, then it must conform to our abstract interface in order to have nested types:

arguments of type-constructors must conform to our abstract interface, and the result type too.

The core of an own type construction is done via a type definition which provides the raw-type *'α Set-0*. it is shown that this type "fits" indeed into the abstract type interface discussed in the previous section.

typedef $'\alpha$ *Set-0* $= \{X::('\alpha::null)$ *set option option*.
$X = bot \vee X = null \vee (\forall x \in \lceil\lceil X \rceil\rceil.\ x \neq bot)\}$
by (*rule-tac x=bot* **in** *exI*, *simp*)

instantiation *Set-0* :: (*null*)*bot*
begin

definition *bot-Set-0-def*: $(bot::('a::null)\ Set\text{-}0) \equiv Abs\text{-}Set\text{-}0\ None$

instance proof show $\exists\, x::'a\ Set\text{-}0.\ x \neq bot$
apply(*rule-tac x=Abs-Set-0* $\lfloor None \rfloor$ **in** *exI*)
apply(*simp add:bot-Set-0-def*)
apply(*subst Abs-Set-0-inject*)
apply(*simp-all add: Set-0-def bot-Set-0-def*
null-option-def bot-option-def)
done
qed
end

instantiation *Set-0* :: (*null*)*null*
begin

definition *null-Set-0-def*: $(null::('a::null)\ Set\text{-}0) \equiv Abs\text{-}Set\text{-}0\ \lfloor\ None\ \rfloor$

instance proof show $(null::('a::null)\ Set\text{-}0) \neq bot$
apply(*simp add:null-Set-0-def bot-Set-0-def*)
apply(*subst Abs-Set-0-inject*)
apply(*simp-all add: Set-0-def bot-Set-0-def*
null-option-def bot-option-def)
done
qed
end

... and lifting this type to the format of a valuation gives us:

type-synonym $('\mathfrak{A},'\alpha)$ *Set* $= ('\mathfrak{A},\ '\alpha\ Set\text{-}0)$ *val*

lemma *Set-inv-lemma*: $\tau \models (\delta\ X) \Longrightarrow (X\ \tau = Abs\text{-}Set\text{-}0\ \lfloor bot \rfloor)$
$\vee\ (\forall x \in \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau) \rceil\rceil.\ x \neq bot)$
apply(*insert OCL-lib.Set-0.Rep-Set-0* [*of X* τ], *simp add:Set-0-def*)
apply(*auto simp: OclValid-def defined-def false-def true-def cp-def*
bot-fun-def bot-Set-0-def null-Set-0-def null-fun-def
split:split-if-asm)

apply(*erule contrapos-pp* [*of Rep-Set-0* (X τ) = *bot*])
apply(*subst Abs-Set-0-inject*[*symmetric*], *simp add*:*Rep-Set-0*)
apply(*simp add*: *Set-0-def*)
apply(*simp add*: *Rep-Set-0-inverse bot-Set-0-def bot-option-def*)
apply(*erule contrapos-pp* [*of Rep-Set-0* (X τ) = *null*])
apply(*subst Abs-Set-0-inject*[*symmetric*], *simp add*:*Rep-Set-0*)
apply(*simp add*: *Set-0-def*)
apply(*simp add*: *Rep-Set-0-inverse null-option-def*)
done

lemma *invalid-set-not-defined* [*simp*,*code-unfold*]:δ(*invalid*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *false*
by *simp*
lemma *null-set-not-defined* [*simp*,*code-unfold*]:δ(*null*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *false*
by(*simp add*: *defined-def null-fun-def*)
lemma *invalid-set-valid* [*simp*,*code-unfold*]:υ(*invalid*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *false*
by *simp*
lemma *null-set-valid* [*simp*,*code-unfold*]:υ(*null*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *true*
apply(*simp add*: *valid-def null-fun-def bot-fun-def bot-Set-0-def null-Set-0-def*)
apply(*subst Abs-Set-0-inject*,*simp-all add*: *Set-0-def null-option-def bot-option-def*)
done

... which means that we can have a type ($'\mathfrak{A}$,($'\mathfrak{A}$,($'\mathfrak{A}$) *Integer*) *Set*) *Set* corresponding exactly to Set(Set(Integer)) in OCL notation. Note that the parameter $\mathfrak{A}$ still refers to the object universe; making the OCL semantics entirely parametric in the object universe makes it possible to study (and prove) its properties independently from a concrete class diagram.

definition *mtSet*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set* (*Set*{})
where *Set*{} $\equiv$ (λ τ. *Abs-Set-0* $\lfloor\lfloor${}::$'\alpha$ *set*$\rfloor\rfloor$)

lemma *mtSet-defined*[*simp*,*code-unfold*]:δ(*Set*{}) = *true*
apply(*rule ext*, *auto simp*: *mtSet-def defined-def null-Set-0-def*
bot-Set-0-def bot-fun-def null-fun-def)
apply(*simp-all add*: *Abs-Set-0-inject Set-0-def bot-option-def null-Set-0-def null-option-def*)
done

lemma *mtSet-valid*[*simp*,*code-unfold*]:υ(*Set*{}) = *true*
apply(*rule ext*,*auto simp*: *mtSet-def valid-def null-Set-0-def*
bot-Set-0-def bot-fun-def null-fun-def)
apply(*simp-all add*: *Abs-Set-0-inject Set-0-def bot-option-def null-Set-0-def null-option-def*)
done

Note that the collection types in OCL allow for null to be included; however, there is the null-collection into which inclusion yields invalid.

This section of foundational operations on sets is closed with a paragraph on equality. Strong Equality is inherited from the OCL core, but we have to consider the case of the strict equality. We decide to overload strict equality in the same way we do for other value's in OCL:

defs *StrictRefEq-set* :

$$(x{::}('\mathfrak{A},'\alpha{::}null)Set) \doteq y \equiv \lambda\ \tau.\ if\ (\upsilon\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau \\ then\ (x \triangleq y)\tau \\ else\ invalid\ \tau$$

One might object here that for the case of objects, this is an empty definition. The answer is no, we will restrain later on states and objects such that any object has its id stored inside the object (so the ref, under which an object can be referenced in the store will represented in the object itself). For such well-formed stores that satisfy this invariant (the WFF - invariant), the referential equality and the strong equality — and therefore the strict equality on sets in the sense above) coincides.

To become operational, we derive:

lemma *StrictRefEq-set-refl* :
$((x{::}('\mathfrak{A},'\alpha{::}null)Set) \doteq x) = (if\ (\upsilon\ x)\ then\ true\ else\ invalid\ endif)$
by(*rule ext*, *simp add*: *StrictRefEq-set if-ocl-def*)

The key for an operational definition if OclForall given below.

The case of the size definition is somewhat special, we admit explicitly in Essential OCL the possibility of infinite sets. For the size definition, this requires an extra condition that assures that the cardinality of the set is actually a defined integer.

definition *OclSize* $:: ('\mathfrak{A},'\alpha{::}null)Set \Rightarrow '\mathfrak{A}\ Integer$
where $OclSize\ x = (\lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge finite(\lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil) \\ then\ \lfloor\lfloor\ int(card\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil)\ \rfloor\rfloor \\ else\ \bot\ )$

definition *OclIncluding* $:: [('\mathfrak{A},'\alpha{::}null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow ('\mathfrak{A},'\alpha)\ Set$
where $OclIncluding\ x\ y = (\lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau \\ then\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil\ \cup \{y\ \tau\}\ \rfloor\rfloor \\ else\ \bot\ )$

definition *OclIncludes* $:: [('\mathfrak{A},'\alpha{::}null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow '\mathfrak{A}\ Boolean$
where $OclIncludes\ x\ y = (\lambda\ \tau.\ \ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau \\ then\ \lfloor\lfloor(y\ \tau) \in \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil\ \rfloor\rfloor \\ else\ \bot\ \ )$

definition *OclExcluding* $:: [('\mathfrak{A},'\alpha{::}null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow ('\mathfrak{A},'\alpha)\ Set$
where $OclExcluding\ x\ y = (\lambda\ \tau.\ \ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau \\ then\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil - \{y\ \tau\}\ \rfloor\rfloor \\ else\ \bot\ )$

definition *OclExcludes* $:: [('\mathfrak{A},'\alpha{::}null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow '\mathfrak{A}\ Boolean$
where $OclExcludes\ x\ y = (not(OclIncludes\ x\ y))$

definition $\mathit{OclIsEmpty}$ :: $({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set} \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$
where $\mathit{OclIsEmpty}\ x = ((\mathit{OclSize}\ x) \doteq \mathbf{0})$

definition $\mathit{OclNotEmpty}$ :: $({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set} \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$
where $\mathit{OclNotEmpty}\ x = \mathit{not}(\mathit{OclIsEmpty}\ x)$

definition $\mathit{OclForall}$ :: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\mathit{Set},({}'\mathfrak{A},{}'\alpha)\mathit{val}{\Rightarrow}({}'\mathfrak{A})\mathit{Boolean}] \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$
where $\mathit{OclForall}\ S\ P = (\lambda\ \tau.\ \mathit{if}\ (\delta\ S)\ \tau = \mathit{true}\ \tau$
$\mathit{then}\ \mathit{if}\ (\forall\, x{\in}\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil.\ P\ (\lambda\ \text{-}.\ x)\ \tau = \mathit{true}\ \tau)$
$\mathit{then}\ \mathit{true}\ \tau$
$\mathit{else}\ \mathit{if}\ (\forall\, x{\in}\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil.\ P(\lambda\ \text{-}.\ x)\ \tau = \mathit{true}$
$\tau \vee$
$P(\lambda\ \text{-}.\ x)\ \tau = \mathit{false}\ \tau)$
$\mathit{then}\ \mathit{false}\ \tau$
$\mathit{else}\ \bot$
$\mathit{else}\ \bot)$

definition $\mathit{OclExists}$ :: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set},({}'\mathfrak{A},{}'\alpha)\mathit{val}{\Rightarrow}({}'\mathfrak{A})\mathit{Boolean}] \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$
where $\mathit{OclExists}\ S\ P = \mathit{not}(\mathit{OclForall}\ S\ (\lambda\ X.\ \mathit{not}\ (P\ X)))$

syntax
$\text{-}\mathit{OclForall}$ :: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set},\mathit{id},({}'\mathfrak{A})\mathit{Boolean}] \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$ $((\text{-}){-}{>}\mathit{forall}'(\text{-}|\text{-}'))$
translations
$X{-}{>}\mathit{forall}(x \mid P) == \mathit{CONST}\ \mathit{OclForall}\ X\ (\%x.\ P)$

syntax
$\text{-}\mathit{OclExist}$ :: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set},\mathit{id},({}'\mathfrak{A})\mathit{Boolean}] \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$ $((\text{-}){-}{>}\mathit{exists}'(\text{-}|\text{-}'))$
translations
$X{-}{>}\mathit{exists}(x \mid P) == \mathit{CONST}\ \mathit{OclExists}\ X\ (\%x.\ P)$

consts

$\mathit{OclUnion}$ :: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set},({}'\mathfrak{A},{}'\alpha)\ \mathit{Set}] \Rightarrow ({}'\mathfrak{A},{}'\alpha)\ \mathit{Set}$
$\mathit{OclIntersection}$:: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set},({}'\mathfrak{A},{}'\alpha)\ \mathit{Set}] \Rightarrow ({}'\mathfrak{A},{}'\alpha)\ \mathit{Set}$
$\mathit{OclIncludesAll}$ :: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set},({}'\mathfrak{A},{}'\alpha)\ \mathit{Set}] \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$
$\mathit{OclExcludesAll}$ :: $[({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set},({}'\mathfrak{A},{}'\alpha)\ \mathit{Set}] \Rightarrow {}'\mathfrak{A}\ \mathit{Boolean}$
$\mathit{OclComplement}$ :: $({}'\mathfrak{A},{}'\alpha{::}\mathit{null})\ \mathit{Set} \Rightarrow ({}'\mathfrak{A},{}'\alpha)\ \mathit{Set}$

$OclSum$ :: $('\mathfrak{A},'\alpha::null)$ $Set \Rightarrow$ $'\mathfrak{A}$ $Integer$
$OclCount$ :: $[('\mathfrak{A},'\alpha::null)$ $Set,('\mathfrak{A},'\alpha)$ $Set] \Rightarrow$ $'\mathfrak{A}$ $Integer$

notation
OclSize (-->*size*'(') [*66*])
and
OclCount (-->*count*'(-') [*66*,*65*]*65*)
and
OclIncludes (-->*includes*'(-') [*66*,*65*]*65*)
and
OclExcludes (-->*excludes*'(-') [*66*,*65*]*65*)
and
OclSum (-->*sum*'(') [*66*])
and
OclIncludesAll (-->*includesAll*'(-') [*66*,*65*]*65*)
and
OclExcludesAll (-->*excludesAll*'(-') [*66*,*65*]*65*)
and
OclIsEmpty (-->*isEmpty*'(') [*66*])
and
OclNotEmpty (-->*notEmpty*'(') [*66*])
and
OclIncluding (-->*including*'(-'))
and
OclExcluding (-->*excluding*'(-'))
and
OclComplement (-->*complement*'('))
and
OclUnion (-->*union*'(-') [*66*,*65*]*65*)
and
OclIntersection(-->*intersection*'(-') [*71*,*70*]*70*)

lemma *cp-OclIncluding*:
$(X{-}{>}including(x))\ \tau = ((\lambda\ \text{-}.\ X\ \tau){-}{>}including(\lambda\ \text{-}.\ x\ \tau))\ \tau$
by(*auto simp*: *OclIncluding-def StrongEq-def invalid-def*
cp-defined[*symmetric*] *cp-valid*[*symmetric*])

lemma *cp-OclExcluding*:
$(X{-}{>}excluding(x))\ \tau = ((\lambda\ \text{-}.\ X\ \tau){-}{>}excluding(\lambda\ \text{-}.\ x\ \tau))\ \tau$
by(*auto simp*: *OclExcluding-def StrongEq-def invalid-def*
cp-defined[*symmetric*] *cp-valid*[*symmetric*])

lemma *cp-OclIncludes*:
$(X{-}{>}includes(x))\ \tau = (OclIncludes\ (\lambda\ \text{-}.\ X\ \tau)\ (\lambda\ \text{-}.\ x\ \tau)\ \tau)$
by(*auto simp*: *OclIncludes-def StrongEq-def invalid-def*
cp-defined[*symmetric*] *cp-valid*[*symmetric*])

lemma *including-strict1* [*simp*,*code-unfold*]:(*invalid*$->$*including*(x)) = *invalid*
by(*simp add*: *bot-fun-def OclIncluding-def invalid-def defined-def valid-def false-def true-def*)

lemma *including-strict2* [*simp*,*code-unfold*]:($X->$*including*(*invalid*)) = *invalid*
by(*simp add*: *OclIncluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *including-strict3* [*simp*,*code-unfold*]:(*null*$->$*including*(x)) = *invalid*
by(*simp add*: *OclIncluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *excluding-strict1* [*simp*,*code-unfold*]:(*invalid*$->$*excluding*(x)) = *invalid*
by(*simp add*: *bot-fun-def OclExcluding-def invalid-def defined-def valid-def false-def true-def*)

lemma *excluding-strict2* [*simp*,*code-unfold*]:($X->$*excluding*(*invalid*)) = *invalid*
by(*simp add*: *OclExcluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *excluding-strict3* [*simp*,*code-unfold*]:(*null*$->$*excluding*(x)) = *invalid*
by(*simp add*: *OclExcluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *includes-strict1* [*simp*,*code-unfold*]:(*invalid*$->$*includes*(x)) = *invalid*
by(*simp add*: *bot-fun-def OclIncludes-def invalid-def defined-def valid-def false-def true-def*)

lemma *includes-strict2* [*simp*,*code-unfold*]:($X->$*includes*(*invalid*)) = *invalid*
by(*simp add*: *OclIncludes-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *includes-strict3* [*simp*,*code-unfold*]:(*null*$->$*includes*(x)) = *invalid*
by(*simp add*: *OclIncludes-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *including-defined-args-valid*:
$(\tau \models \delta(X{-}{>}including(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof $-$
 have $A : \bot \in$ *Set-0* **by**(*simp add*: *Set-0-def bot-option-def*)

have B : $\lfloor\bot\rfloor \in$ *Set-0* **by**(*simp add*: *Set-0-def null-option-def bot-option-def*)
have C : $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor$*insert* $(x\ \tau)$ $\lceil\lceil$*Rep-Set-0* $(X\ \tau)\rceil\rceil\rfloor\rfloor$ $\in$ *Set-0*
apply(*frule Set-inv-lemma*)
apply(*simp add*: *Set-0-def bot-option-def null-Set-0-def null-fun-def foundation18 foundation16 invalid-def*)
done
have D: $(\tau \models \delta(X{-}{>}including(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclIncluding-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def split*: *bool.split-asm HOL.split-if-asm option.split*)
have E: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \delta(X{-}{>}including(x)))$
apply(*frule C*, *simp*)
apply(*auto simp*: *OclIncluding-def OclValid-def true-def false-def StrongEq-def*

defined-def invalid-def valid-def bot-fun-def null-fun-def split: *bool.split-asm HOL.split-if-asm option.split*)
apply(*simp-all add*: *null-Set-0-def bot-Set-0-def bot-option-def*)
apply(*simp-all add*: *Abs-Set-0-inject A B bot-option-def*[*symmetric*], *simp-all add*: *bot-option-def*)
done
show *?thesis* **by**(*auto dest*:*D intro*:*E*)
qed

lemma *including-valid-args-valid*:
$(\tau \models \upsilon(X{-}{>}including(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof −
have A : $\bot \in$ *Set-0* **by**(*simp add*: *Set-0-def bot-option-def*)
have B : $\lfloor\bot\rfloor \in$ *Set-0* **by**(*simp add*: *Set-0-def null-option-def bot-option-def*)
have C : $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor$*insert* $(x\ \tau)$ $\lceil\lceil$*Rep-Set-0* $(X\ \tau)\rceil\rceil\rfloor\rfloor$ $\in$ *Set-0*
apply(*frule Set-inv-lemma*)
apply(*simp add*: *Set-0-def bot-option-def null-Set-0-def null-fun-def foundation18 foundation16 invalid-def*)
done
have D: $(\tau \models \upsilon(X{-}{>}including(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclIncluding-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def split*: *bool.split-asm HOL.split-if-asm option.split*)
have E: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \upsilon(X{-}{>}including(x)))$
apply(*frule C*, *simp*)
apply(*auto simp*: *OclIncluding-def OclValid-def true-def false-def StrongEq-def*

defined-def invalid-def valid-def bot-fun-def null-fun-def split: *bool.split-asm HOL.split-if-asm option.split*)

apply(*simp-all add*: *null-Set-0-def bot-Set-0-def bot-option-def*)
apply(*simp-all add*: *Abs-Set-0-inject A B bot-option-def*[*symmetric*],
simp-all add: *bot-option-def*)
done
show *?thesis* **by**(*auto dest*:*D intro*:*E*)
qed

lemma *including-defined-args-valid'*[*simp*,*code-unfold*]:
$\delta(X{-}{>}including(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*including-defined-args-valid foundation10 defined-and-I*)

lemma *including-valid-args-valid''*[*simp*,*code-unfold*]:
$\upsilon(X{-}{>}including(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*including-valid-args-valid foundation10 defined-and-I*)

lemma *excluding-valid-args-valid'*[*simp*,*code-unfold*]:
$\delta(X{-}{>}excluding(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
sorry

lemma *excluding-valid-args-valid''*[*simp*,*code-unfold*]:
$\upsilon(X{-}{>}excluding(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
sorry

lemma *includes-valid-args-valid'*[*simp*,*code-unfold*]:
$\delta(X{-}{>}includes(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
sorry

lemma *includes-valid-args-valid''*[*simp*,*code-unfold*]:
$\upsilon(X{-}{>}includes(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
sorry

9.2 Some computational laws:

lemma *including-charn0*[*simp*]:
assumes *val-x*:$\tau \models (\upsilon\ x)$
shows $\tau \models not(Set\{\}{-}{>}includes(x))$
using *val-x*
apply(*auto simp*: *OclValid-def OclIncludes-def not-def false-def true-def*)
apply(*auto simp*: *mtSet-def OCL-lib.Set-0.Abs-Set-0-inverse Set-0-def*)
done

lemma *including-charn0'*[*simp*,*code-unfold*]:
$Set\{\}{-}{>}includes(x) = (if\ \upsilon\ x\ then\ false\ else\ invalid\ endif)$
proof −
have *A*: $\bigwedge\ \tau.\ (Set\{\}{-}{>}includes(invalid))\ \tau = (if\ (\upsilon\ invalid)\ then\ false\ else\ invalid\ endif)\ \tau$

by *simp*
have B: $\bigwedge \tau\ x.\ \tau \models (\upsilon\ x) \Longrightarrow (Set\{\}{-}{>}includes(x))\ \tau = (if\ \upsilon\ x\ then\ false\ else\ invalid\ endif)\ \tau$
apply(*frule including-charn0*, *simp add*: *OclValid-def*, *subst cp-if-ocl*, *simp*, *simp only*:*cp-if-ocl*[*symmetric*], *simp add*: *StrongEq-def*)
apply(*rule foundation21* [*THEN fun-cong*, *simplified StrongEq-def*,*simplified*,

THEN iffD1, *of - - false*])
by *simp*
show *?thesis*
apply(*rule ext*)
apply(*case-tac xa* $\models (\upsilon\ x)$)
apply(*simp add*: *B*)
apply(*simp add*: *foundation18*)
apply(*subst cp-if-ocl*, *subst cp-OclIncludes*, *subst cp-valid*, *simp*)
apply(*simp add*: *cp-if-ocl*[*symmetric*] *cp-OclIncludes*[*symmetric*] *cp-valid*[*symmetric*] *A*)
done
qed

lemma *including-charn1*:
assumes *def-X*:$\tau \models (\delta\ X)$
assumes *val-x*:$\tau \models (\upsilon\ x)$
shows $\tau \models (X{-}{>}including(x){-}{>}includes(x))$
proof −
have A : $\perp \in$ *Set-0* **by**(*simp add*: *Set-0-def bot-option-def*)
have B : $\lfloor\perp\rfloor \in$ *Set-0* **by**(*simp add*: *Set-0-def null-option-def bot-option-def*)
have C : $\lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \in$ *Set-0*
apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*] *Set-inv-lemma*[*OF def-X*])
apply(*simp add*: *Set-0-def bot-option-def null-Set-0-def null-fun-def invalid-def*)
done
show *?thesis*
apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*])
apply(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def*
invalid-def defined-def valid-def
bot-Set-0-def null-fun-def null-Set-0-def bot-option-def)
apply(*simp-all add*: *Abs-Set-0-inject A B C bot-option-def*[*symmetric*], *simp-all add*: *bot-option-def Abs-Set-0-inverse C*)
done
qed

lemma *including-charn2*:
assumes *def-X*:$\tau \models (\delta\ X)$

and *val-x*:$\tau \models (\upsilon\ x)$
and *val-y*:$\tau \models (\upsilon\ y)$
and *neq* :$\tau \models$ *not*$(x \triangleq y)$
shows $\tau \models (X\text{->}including(x)\text{->}includes(y)) \triangleq (X\text{->}includes(y))$
proof –
have *A* : $\bot \in$ *Set-0* **by**(*simp add*: *Set-0-def bot-option-def*)
have *B* : $\lfloor\bot\rfloor \in$ *Set-0* **by**(*simp add*: *Set-0-def null-option-def bot-option-def*)
have *C* : $\lfloor\lfloor$*insert* $(x\ \tau)$ $\lceil\lceil$*Rep-Set-0* $(X\ \tau)\rceil\rceil\rfloor\rfloor \in$ *Set-0*
apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*] *Set-inv-lemma*[*OF def-X*])
apply(*simp add*: *Set-0-def bot-option-def null-Set-0-def null-fun-def invalid-def*)
done
have *D* : $y\ \tau \neq x\ \tau$
apply(*insert neq*)
by(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def defined-def valid-def bot-Set-0-def null-fun-def null-Set-0-def StrongEq-def not-def*)
show *?thesis*
apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*])
apply(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def invalid-def defined-def valid-def bot-Set-0-def null-fun-def null-Set-0-def StrongEq-def*)
apply(*simp-all add*: *Abs-Set-0-inject Abs-Set-0-inverse A B C D*)
apply(*simp-all add*: *Abs-Set-0-inject A B C bot-option-def*[*symmetric*], *simp-all add*: *bot-option-def Abs-Set-0-inverse C*)
done
qed

lemma *includes-execute*[*code-unfold*]:
$(X\text{->}including(x)\text{->}includes(y)) = (if\ \delta\ X\ then\ if\ x \doteq y$
then true
else $X\text{->}includes(y)$
endif
else invalid endif)
sorry

lemma *excluding-charn0*[*simp*]:
assumes *val-x*:$\tau \models (\upsilon\ x)$
shows $\tau \models ((Set\{\}\text{->}excluding(x)) \triangleq Set\{\})$
proof –
have *A* : $\lfloor None\rfloor \in$ *Set-0* **by**(*simp add*: *Set-0-def null-option-def bot-option-def*)
have *B* : $\lfloor\lfloor\{\}\rfloor\rfloor \in$ *Set-0* **by**(*simp add*: *Set-0-def bot-option-def*)
show *?thesis* **using** *val-x*
apply(*auto simp*: *OclValid-def OclIncludes-def not-def false-def true-def StrongEq-def*

OclExcluding-def mtSet-def defined-def bot-fun-def null-fun-def null-Set-0-def)
  apply(*auto simp*: *mtSet-def Set-0-def OCL-lib.Set-0.Abs-Set-0-inverse OCL-lib.Set-0.Abs-Set-0-inject*[*OF B*, *OF A*])
 done
qed

lemma *excluding-charn0-exec*[*code-unfold*]:
$(Set\{\}{-}{>}excluding(x)) = (if\ (\upsilon\ x)\ then\ Set\{\}\ else\ invalid\ endif)$
proof −
 have *A*: $\bigwedge \tau.\ (Set\{\}{-}{>}excluding(invalid))\ \tau = (if\ (\upsilon\ invalid)\ then\ Set\{\}\ else\ invalid\ endif)\ \tau$
  by *simp*
 have *B*: $\bigwedge \tau\ x.\ \tau \models (\upsilon\ x) \Longrightarrow (Set\{\}{-}{>}excluding(x))\ \tau = (if\ (\upsilon\ x)\ then\ Set\{\}\ else\ invalid\ endif)\ \tau$
  apply(*frule excluding-charn0*, *simp add*: *OclValid-def*, *subst cp-if-ocl*, *simp*, *simp only*:*cp-if-ocl*[*symmetric*], *simp add*: *StrongEq-def*)
  by(*simp add*: *true-def*)
 show *?thesis*
  apply(*rule ext*)
  apply(*case-tac xa* $\models (\upsilon\ x)$)
   apply(*simp add*: *B*)
   apply(*simp add*: *foundation18*)
   apply(*subst cp-if-ocl*, *subst cp-OclExcluding*, *subst cp-valid*, *simp*)
  apply(*simp add*: *cp-if-ocl*[*symmetric*] *cp-OclExcluding*[*symmetric*] *cp-valid*[*symmetric*] *A*)
 done
qed

lemma *excluding-charn1*:
assumes *def-X*:$\tau \models (\delta\ X)$
and *val-x*:$\tau \models (\upsilon\ x)$
and *val-y*:$\tau \models (\upsilon\ y)$
and *neq* :$\tau \models not(x \triangleq y)$
shows $\tau \models ((X{-}{>}including(x)){-}{>}excluding(y)) \triangleq ((X{-}{>}excluding(x)){-}{>}including(y))$
proof −
 have *A* : $\bot \in Set\text{-}0$ **by**(*simp add*: *Set-0-def bot-option-def*)
 have *B* : $\lfloor\bot\rfloor \in Set\text{-}0$ **by**(*simp add*: *Set-0-def null-option-def bot-option-def*)
 have *C* : $\lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \in Set\text{-}0$
  apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*] *Set-inv-lemma*[*OF def-X*])
  apply(*simp add*: *Set-0-def bot-option-def null-Set-0-def null-fun-def invalid-def*)
  done
 have *D* : $y\ \tau \neq x\ \tau$
  apply(*insert neq*)
  by(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def defined-def valid-def bot-Set-0-def*

null-fun-def null-Set-0-def StrongEq-def not-def)
show *?thesis*
apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*])
apply(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def*
defined-def valid-def bot-Set-0-def null-fun-def null-Set-0-def StrongEq-def)
apply(*subst cp-OclExcluding*,*simp add*:*true-def*)
sorry
qed

lemma *excluding-charn2*:
assumes *def-X*:$\tau \models (\delta\ X)$
and *val-x*:$\tau \models (\upsilon\ x)$
shows $\tau \models (((X{-}{>}including(x)){-}{>}excluding(x)) \triangleq (X{-}{>}excluding(x)))$
proof −
have A : $\bot \in$ *Set-0* **by**(*simp add*: *Set-0-def bot-option-def*)
have B : $\lfloor\bot\rfloor \in$ *Set-0* **by**(*simp add*: *Set-0-def null-option-def bot-option-def*)
have C : $\lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \in$ *Set-0*
apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*] *Set-inv-lemma*[*OF def-X*])
apply(*simp add*: *Set-0-def bot-option-def null-Set-0-def null-fun-def invalid-def*)
done
show *?thesis*
apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*])
apply(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def*
invalid-def defined-def valid-def bot-Set-0-def null-fun-def null-Set-0-def

StrongEq-def)
apply(*subst cp-OclExcluding*) **back**
apply(*simp add*:*true-def*)
apply(*auto simp*:*OclExcluding-def*)
apply(*simp add*: *Abs-Set-0-inverse*[*OF C*])
apply(*simp-all add*: *false-def true-def defined-def valid-def*
null-fun-def bot-fun-def null-Set-0-def bot-Set-0-def
split: *bool.split-asm HOL.split-if-asm option.split*)
apply(*simp-all add*: *Abs-Set-0-inject A B C bot-option-def*[*symmetric*],
simp-all add: *bot-option-def Abs-Set-0-inverse C*)
done
qed

lemma *excluding-charn-exec*[*code-unfold*]:
$(X{-}{>}including(x){-}{>}excluding(y)) = (if\ \ \delta\ X\ then\ if\ x \doteq y$
$then\ X{-}{>}excluding(y)$
$else\ X{-}{>}excluding(y){-}{>}including(x)$
$endif$
$else\ invalid\ endif)$

sorry

syntax
  -OclFinset :: *args* => ('$\mathfrak{A}$,'*a*::*null*) *Set*   (*Set*{(-)})
translations
  Set{*x*, *xs*} == *CONST OclIncluding* (*Set*{*xs*}) *x*
  Set{*x*}   == *CONST OclIncluding* (*Set*{}) *x*

lemma *syntax-test*: *Set*{**2**,**1**} = (*Set*{}−>*including*(**1**)−>*including*(**2**))
by (*rule refl*)

lemma *set-test1*: τ ⊨ (*Set*{**2**,*null*}−>*includes*(*null*))
by(*simp add*: *includes-execute*)

lemma *set-test2*: ¬(τ ⊨ (*Set*{**2**,**1**}−>*includes*(*null*)))
by(*simp add*: *includes-execute*)

Here is an example of a nested collection. Note that we have to use the abstract null (since we did not (yet) define a concrete constant *null* for the non-existing Sets) :

lemma *semantic-test*: τ ⊨ (*Set*{*Set*{**2**},*null*}−>*includes*(*null*))
apply(*simp add*: *includes-execute*)
oops

lemma *set-test3*: τ ⊨ (*Set*{*null*,**2**}−>*includes*(*null*))
by(*simp-all add*: *including-charn1 including-defined-args-valid*)

find-theorems *name*:*corev* -

lemma *StrictRefEq-set-exec*[*simp*,*code-unfold*] :
((*x*::('$\mathfrak{A}$,'α::*null*)*Set*) $\doteq$ *y*) =
  (*if* δ *x then* (*if* δ *y*
      then ((*x*−>*forall*(*z*| *y*−>*includes*(*z*)) *and* (*y*−>*forall*(*z*| *x*−>*includes*(*z*)))))
        else if υ *y*
          then false (∗ *x*′−>*includes* = *null* ∗)
          else invalid
          endif
        endif)
    else if υ *x* (∗ *null* = *???* ∗)
      then if υ *y then not*(δ *y*) *else invalid endif*

else invalid
endif
endif)
sorry

lemma *forall-set-null-exec*[*simp*,*code-unfold*] :
$(null{-}{>}forall(z|\ P(z))) = invalid$
sorry

lemma *forall-set-mt-exec*[*simp*,*code-unfold*] :
$((Set\{\}){-}{>}forall(z|\ P(z))) = true$
sorry

lemma *exists-set-null-exec*[*simp*,*code-unfold*] :
$(null{-}{>}exists(z \mid P(z))) = invalid$
sorry

lemma *exists-set-mt-exec*[*simp*,*code-unfold*] :
$((Set\{\}){-}{>}exists(z \mid P(z))) = false$
sorry

lemma *forall-set-including-exec*[*simp*,*code-unfold*] :
$((S{-}{>}including(x)){-}{>}forall(z \mid P(z))) = (if\ (\delta\ S)\ and\ (\upsilon\ x)$
$then\ P(x)\ and\ S{-}{>}forall(z \mid P(z))$
else invalid
endif)
sorry

lemma *exists-set-including-exec*[*simp*,*code-unfold*] :
$((S{-}{>}including(x)){-}{>}exists(z \mid P(z))) = (if\ (\delta\ S)\ and\ (\upsilon\ x)$
$then\ P(x)\ or\ S{-}{>}exists(z \mid P(z))$
else invalid
endif)
sorry

lemma *set-test4* : $\tau \models (Set\{\mathbf{2},null,\mathbf{2}\} \doteq Set\{null,\mathbf{2}\})$
by(*simp add:includes-execute*)

definition $OclIterate_{Set}$ :: $[('\mathfrak{A},'\alpha::null)\ Set,('\mathfrak{A},'\beta::null)val,$
$('\mathfrak{A},'\alpha)val \Rightarrow ('\mathfrak{A},'\beta)val \Rightarrow ('\mathfrak{A},'\beta)val] \Rightarrow ('\mathfrak{A},'\beta)val$
where $OclIterate_{Set}\ S\ A\ F = (\lambda\ \tau.\ if\ (\delta\ S)\ \tau = true\ \tau \wedge (\upsilon\ A)\ \tau = true\ \tau \wedge$
$finite \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil$
$then\ (Finite\text{-}Set.fold\ (F)\ (A)\ ((\lambda a\ \tau.\ a)\ `\ \lceil\lceil Rep\text{-}Set\text{-}0$
$(S\ \tau)\rceil\rceil))\tau$
$else\ \bot)$

syntax
-*OclIterate* :: $[('\mathfrak{A},'\alpha{::}null)$ *Set*, *idt*, *idt*, $'\alpha$, $'\beta] => ('\mathfrak{A},'\gamma)val$
(- $->$*iterate*$'$(-;-=- | -$'$) [71,100,70]50)
translations
$X->iterate(a;\ x = A \mid P) == CONST\ OclIterate_{Set}\ X\ A\ (\%a.\ (\%\ x.\ P))$

lemma $OclIterate_{Set}$-*strict1*[*simp*]:$invalid->iterate(a;\ x = A \mid P\ a\ x) = invalid$
by(*simp add*: *bot-fun-def invalid-def* $OclIterate_{Set}$-*def defined-def valid-def false-def true-def*)

lemma $OclIterate_{Set}$-*null1*[*simp*]:$null->iterate(a;\ x = A \mid P\ a\ x) = invalid$
by(*simp add*: *bot-fun-def invalid-def* $OclIterate_{Set}$-*def defined-def valid-def false-def true-def*)

lemma $OclIterate_{Set}$-*strict2*[*simp*]:$S->iterate(a;\ x = invalid \mid P\ a\ x) = invalid$
by(*simp add*: *bot-fun-def invalid-def* $OclIterate_{Set}$-*def defined-def valid-def false-def true-def*)

An open question is this ...

lemma $OclIterate_{Set}$-*null2*[*simp*]:$S->iterate(a;\ x = null \mid P\ a\ x) = invalid$
oops

In the definition above, this does not hold in general. And I believe, this is how it should be ...

lemma $OclIterate_{Set}$-*infinite*:
assumes *non-finite*: $\tau \models not(\delta(S->size()))$
shows $(OclIterate_{Set}\ S\ A\ F)\ \tau = invalid\ \tau$
sorry

lemma $OclIterate_{Set}$-*empty*[*simp*]: $((Set\{\})->iterate(a;\ x = A \mid P\ a\ x)) = A$
sorry

In particular, this does hold for A = null.

lemma $OclIterate_{Set}$-*including*:
assumes *S-finite*: $\tau \models \delta(S->size())$

shows $((S->including(a))->iterate(a;\ x = A \mid F\ a\ x))\ \tau =$
$(\ ((S->excluding(a))->iterate(a;\ x = F\ a\ A \mid F\ a\ x)))\ \tau$
sorry

lemma *short-cut*[*simp*]: $x \models \delta\ S->size()$
sorry

lemma *short-cut'*[*simp*]: $(\mathbf{8} \doteq \mathbf{6}) \ = \mathit{false}$
sorry

lemma [*simp*]: $\upsilon\ \mathbf{6} = \mathit{true}$ **sorry**
lemma [*simp*]: $\upsilon\ \mathbf{8} = \mathit{true}$ **sorry**
lemma [*simp*]: $\upsilon\ \mathbf{9} = \mathit{true}$ **sorry**

lemma *GogollasChallenge-on-sets*:
$(\mathit{Set}\{\ \mathbf{6},\mathbf{8}\ \}{-}{>}\mathit{iterate}(i;r1{=}\mathit{Set}\{\mathbf{9}\}|$
$r1{-}{>}\mathit{iterate}(j;r2{=}r1|$
$r2{-}{>}\mathit{including}(\mathbf{0}){-}{>}\mathit{including}(i){-}{>}\mathit{including}(j))) =$
$\mathit{Set}\{\mathbf{0},\ \ \mathbf{6},\ \mathbf{9}\})$
apply(*rule ext*,
simp add: *excluding-charn-exec* $\mathit{OclIterate}_{\mathit{Set}}$*-including excluding-charn0-exec*)
sorry

Elementary computations on Sets.

value $\neg\ (\tau_0 \models \upsilon(\mathit{invalid}::(\mathfrak{'A},'\alpha::\mathit{null})\ \mathit{Set}))$
value $\quad \tau_0 \models \upsilon(\mathit{null}::(\mathfrak{'A},'\alpha::\mathit{null})\ \mathit{Set})$
value $\neg\ (\tau_0 \models \delta(\mathit{null}::(\mathfrak{'A},'\alpha::\mathit{null})\ \mathit{Set}))$
value $\quad \tau_0 \models \upsilon(\mathit{Set}\{\})$
value $\quad \tau_0 \models \upsilon(\mathit{Set}\{\mathit{Set}\{\mathbf{2}\},\mathit{null}\})$
value $\quad \tau_0 \models \delta(\mathit{Set}\{\mathit{Set}\{\mathbf{2}\},\mathit{null}\})$
value $\quad \tau_0 \models (\mathit{Set}\{\mathbf{2},\mathbf{1}\}{-}{>}\mathit{includes}(\mathbf{1}))$
value $\neg\ (\tau_0 \models (\mathit{Set}\{\mathbf{2}\}{-}{>}\mathit{includes}(\mathbf{1})))$
value $\neg\ (\tau_0 \models (\mathit{Set}\{\mathbf{2},\mathbf{1}\}{-}{>}\mathit{includes}(\mathit{null})))$
value $\quad \tau_0 \models (\mathit{Set}\{\mathbf{2},\mathit{null}\}{-}{>}\mathit{includes}(\mathit{null}))$
value $\quad \tau \models ((\mathit{Set}\{\mathbf{2},\mathbf{1}\}){-}{>}\mathit{forall}(z \mid \mathbf{0} \prec z))$
value $\neg\ (\tau \models ((\mathit{Set}\{\mathbf{2},\mathbf{1}\}){-}{>}\mathit{exists}(z \mid z \prec \mathbf{0}\ )))$

value $\neg\ (\tau \models ((\mathit{Set}\{\mathbf{2},\mathit{null}\}){-}{>}\mathit{forall}(z \mid \mathbf{0} \prec z)))$
value $\quad \tau \models ((\mathit{Set}\{\mathbf{2},\mathit{null}\}){-}{>}\mathit{exists}(z \mid \mathbf{0} \prec z))$

value $\quad \tau \models (\mathit{Set}\{\mathbf{2},\mathit{null},\mathbf{2}\} \doteq \mathit{Set}\{\mathit{null},\mathbf{2}\})$
value $\quad \tau \models (\mathit{Set}\{\mathbf{1},\mathit{null},\mathbf{2}\} <> \mathit{Set}\{\mathit{null},\mathbf{2}\})$

value $\quad \tau \models (\mathit{Set}\{\mathit{Set}\{\mathbf{2},\mathit{null}\}\} \doteq \mathit{Set}\{\mathit{Set}\{\mathit{null},\mathbf{2}\}\})$
value $\quad \tau \models (\mathit{Set}\{\mathit{Set}\{\mathbf{2},\mathit{null}\}\} <> \mathit{Set}\{\mathit{Set}\{\mathit{null},\mathbf{2}\},\mathit{null}\})$

end

theory *OCL-state*
imports *OCL-lib*
begin

10 Recall: The generic structure of States

Next we will introduce the foundational concept of an object id (oid), which is just some infinite set.

type-synonym $oid = ind$

States are just a partial map from oid's to elements of an object universe $'\mathfrak{A}$, and state transitions pairs of states...

type-synonym $('\mathfrak{A})state = oid \rightharpoonup {'\mathfrak{A}}$

type-synonym $('\mathfrak{A})st = {'\mathfrak{A}}\ state \times {'\mathfrak{A}}\ state$

Now we refine our state-interface. In certain contexts, we will require that the elements of the object universe have a particular structure; more precisely, we will require that there is a function that reconstructs the oid of an object in the state (we will settle the question how to define this function later).

class $object =$ **fixes** $oid\text{-}of :: {'a} \Rightarrow oid$

Thus, if needed, we can constrain the object universe to objects by adding the following type class constraint:

typ $'\mathfrak{A} :: object$

11 Referential Object Equality in States

Generic referential equality - to be used for instantiations with concrete object types ...

definition $gen\text{-}ref\text{-}eq :: ('\mathfrak{A},'a{::}\{object,null\})val \Rightarrow ('\mathfrak{A},'a)val \Rightarrow ('\mathfrak{A})Boolean$
where $gen\text{-}ref\text{-}eq\ x\ y$
$\equiv \lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\delta\ y)\ \tau = true\ \tau$
$then\ if\ x\ \tau = null \vee y\ \tau = null$
$then\ \lfloor\lfloor x\ \tau = null \wedge y\ \tau = null\rfloor\rfloor$
$else\ \lfloor\lfloor(oid\text{-}of\ (x\ \tau)) = (oid\text{-}of\ (y\ \tau))\ \rfloor\rfloor$
$else\ invalid\ \tau$

lemma $gen\text{-}ref\text{-}eq\text{-}object\text{-}strict1[simp]$:
$(gen\text{-}ref\text{-}eq\ x\ invalid) = invalid$
by(*rule ext*, *simp add*: *gen-ref-eq-def true-def false-def*)

lemma $gen\text{-}ref\text{-}eq\text{-}object\text{-}strict2[simp]$:
$(gen\text{-}ref\text{-}eq\ invalid\ x) = invalid$
by(*rule ext*, *simp add*: *gen-ref-eq-def true-def false-def*)

lemma $gen\text{-}ref\text{-}eq\text{-}object\text{-}strict3[simp]$:
$(gen\text{-}ref\text{-}eq\ x\ null) = invalid$

by(*rule ext*, *simp add*: *gen-ref-eq-def true-def false-def*)

lemma *gen-ref-eq-object-strict4*[*simp*] :
(*gen-ref-eq null x*) = *invalid*
by(*rule ext*, *simp add*: *gen-ref-eq-def true-def false-def*)

lemma *cp-gen-ref-eq-object*:
(*gen-ref-eq x y* τ) = (*gen-ref-eq* (λ-. *x* τ) (λ-. *y* τ)) τ
by(*auto simp*: *gen-ref-eq-def StrongEq-def invalid-def cp-defined*[*symmetric*])

lemmas *cp-intro*[*simp*,*intro*!] =
    OCL-core.cp-intro
    cp-gen-ref-eq-object[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]],
        of gen-ref-eq]]

Finally, we derive the usual laws on definedness for (generic) object equality:

lemma *gen-ref-eq-defargs*:
$\tau \models$ (*gen-ref-eq x* (*y*::($'\mathfrak{A}$,$'a$::{*null*,*object*})*val*))$\Longrightarrow$ ($\tau \models$(δ *x*)) $\wedge$ ($\tau \models$(δ *y*))
by(*simp add*: *gen-ref-eq-def OclValid-def true-def invalid-def*
        defined-def invalid-def bot-fun-def bot-option-def
    split: *bool.split-asm HOL.split-if-asm*)

12 Further requirements on States

A key-concept for linking strict referential equality to logical equality: in well-formed states (i.e. those states where the self (oid-of) field contains the pointer to which the object is associated to in the state), referential equality coincides with logical equality.

definition *WFF* :: ($'\mathfrak{A}$::*object*)*st* $\Rightarrow$ *bool*
where *WFF* τ = (($\forall$ *x* $\in$ *ran*(*fst* τ). $\lceil$*fst* τ (*oid-of x*)$\rceil$ = *x*) $\wedge$
        ($\forall$ *x* $\in$ *ran*(*snd* τ). $\lceil$*snd* τ (*oid-of x*)$\rceil$ = *x*))

This is a generic definition of referential equality: Equality on objects in a state is reduced to equality on the references to these objects. As in HOL-OCL, we will store the reference of an object inside the object in a (ghost) field. By establishing certain invariants ("consistent state"), it can be assured that there is a "one-to-one-correspondance" of objects to their references — and therefore the definition below behaves as we expect.

Generic Referential Equality enjoys the usual properties: (quasi) reflexivity, symmetry, transitivity, substitutivity for defined values. For type-technical reasons, for each concrete object type, the equality $\doteq$ is defined by generic referential equality.

theorem *strictEqGen-vs-strongEq*:
WFF $\tau \Longrightarrow \tau \models$($\delta$ *x*) $\Longrightarrow \tau \models$($\delta$ *y*) $\Longrightarrow$
        (*x* τ $\in$ *ran* (*fst* τ) $\wedge$ *y* τ $\in$ *ran* (*fst* τ)) $\wedge$

$(x\ \tau \in ran\ (snd\ \tau) \wedge y\ \tau \in ran\ (snd\ \tau)) \Longrightarrow$ (* *x and y must be object representations*

that exist in either the pre or post state *)

$(\tau \models (gen\text{-}ref\text{-}eq\ x\ y)) = (\tau \models (x \triangleq y))$

apply(*auto simp*: *gen-ref-eq-def OclValid-def WFF-def StrongEq-def true-def Ball-def*)
apply(*erule-tac x=x* τ **in** *allE'*, *simp-all*)
done

So, if two object descriptions live in the same state (both pre or post), the referential equality on objects implies in a WFF state the logical equality. Uffz.

13 Miscellaneous: Initial States (for Testing and Code Generation)

definition τ_0 :: $('\mathfrak{A})st$
where $\tau_0 \equiv (Map.empty, Map.empty)$

14 Generic Operations on States

In order to denote OCL-types occuring in OCL expressions syntactically — as, for example, as "argument" of allInstances — we use the inverses of the injection functions into the object universes; we show that this is sufficient "characterization".

definition *allinstances* :: $('\mathfrak{A} \Rightarrow '\alpha) \Rightarrow ('\mathfrak{A}::object,'\alpha\ option\ option)\ Set$
(- .*oclAllInstances'*(*'*))
where $((H).oclAllInstances())\ \tau =$
$Abs\text{-}Set\text{-}0\ \lfloor\lfloor(Some\ o\ Some\ o\ H)\ `\ (ran(snd\ \tau) \cap \{x.\ \exists\ y.\ y=H\ x\})$
$\rfloor\rfloor$

definition *allinstancesATpre* :: $('\mathfrak{A} \Rightarrow '\alpha) \Rightarrow ('\mathfrak{A}::object,'\alpha\ option\ option)\ Set$
(- .*oclAllInstances@pre'*(*'*))
where $((H).oclAllInstances@pre())\ \tau =$
$Abs\text{-}Set\text{-}0\ \lfloor\lfloor(Some\ o\ Some\ o\ H)\ `\ (ran(fst\ \tau) \cap \{x.\ \exists\ y.\ y=H\ x\})$
$\rfloor\rfloor$

lemma $\tau_0 \models H\ .oclAllInstances() \triangleq Set\{\}$
sorry

lemma $\tau_0 \models H\ .oclAllInstances@pre() \triangleq Set\{\}$
sorry

theorem *state-update-vs-allInstances*:
assumes $oid \notin dom\ \sigma'$

and $cp\ P$
shows $((\sigma, \sigma'(oid \mapsto Object)) \models (P(Type\ .oclAllInstances()))) =$
$((\sigma, \sigma') \models (P((Type\ .oclAllInstances()){-}{>}including(\lambda\ \text{-}.\ Some(Some((the\text{-}inv\ Type)\ Object))))))$
sorry

theorem *state-update-vs-allInstancesATpre*:
assumes $oid \notin dom\ \sigma$
and $cp\ P$
shows $((\sigma(oid \mapsto Object), \sigma') \models (P(Type\ .oclAllInstances@pre()))) =$
$((\sigma, \sigma') \models (P((Type\ .oclAllInstances@pre()){-}{>}including(\lambda\ \text{-}.\ Some(Some((the\text{-}inv\ Type)\ Object))))))$
sorry

definition $oclisnew{::}\ ('\mathfrak{A}, '\alpha{::}\{null,object\})val \Rightarrow ('\mathfrak{A})Boolean \quad ((\text{-}).oclIsNew'('))$
where $X\ .oclIsNew() \equiv (\lambda\tau\ .\ if\ (\delta\ X)\ \tau = true\ \tau$
$then\ \lfloor\lfloor oid\text{-}of\ (X\ \tau) \notin dom(fst\ \tau) \wedge oid\text{-}of\ (X\ \tau) \in dom(snd\ \tau)\rfloor\rfloor$
$else\ invalid\ \tau)$

The following predicate — which is not part of the OCL standard descriptions — provides a simple, but powerful means to describe framing conditions. For any formal approach, be it animation of OCL contracts, test-case generation or die-hard theorem proving, the specification of the part of a system transistion that DOES NOT CHANGE is of premordial importance. The following operator establishes the equality between old and new objects in the state (provided that they exist in both states), with the exception of those objects

definition $oclismodified\ {::}('\mathfrak{A}{::}object,'\alpha{::}\{null,object\})Set \Rightarrow '\mathfrak{A}\ Boolean$
$(\text{-}{-}{>}oclIsModifiedOnly'('))$
where $X{-}{>}oclIsModifiedOnly() \equiv (\lambda(\sigma,\sigma').\ let\ X' = (oid\text{-}of\ `\ \lceil\lceil Rep\text{-}Set\text{-}0(X(\sigma,\sigma'))\rceil\rceil);$
$S = ((dom\ \sigma \cap dom\ \sigma') - X')$
$in\ if\ (\delta\ X)\ (\sigma,\sigma') = true\ (\sigma,\sigma')$
$then\ \lfloor\lfloor\forall\ x \in S.\ \sigma\ x = \sigma'\ x\rfloor\rfloor$
$else\ invalid\ (\sigma,\sigma'))$

definition $atSelf\ {::}\ ('\mathfrak{A}{::}object,'\alpha{::}\{null,object\})val \Rightarrow$
$('\mathfrak{A} \Rightarrow '\alpha) \Rightarrow$
$('\mathfrak{A}{::}object,'\alpha{::}\{null,object\})val\ ((\text{-})@pre(\text{-}))$
where $x\ @pre\ H = (\lambda\tau\ .\ if\ (\delta\ x)\ \tau = true\ \tau$
$then\ if\ oid\text{-}of\ (x\ \tau) \in dom(fst\ \tau) \wedge oid\text{-}of\ (x\ \tau) \in dom(snd\ \tau)$
$then\ \ H\ \lceil(fst\ \tau)(oid\text{-}of\ (x\ \tau))\rceil$
$else\ invalid\ \tau$
$else\ invalid\ \tau)$

theorem *framing*:
  assumes *modifiesclause*:$\tau \models (X{-}{>}excluding(x)){-}{>}oclIsModifiedOnly()$
  and *represented-x*: $\tau \models \delta(x \ @pre\ H)$
  and *H-is-typerepr*: *inj H*
  shows $\tau \models (x \ \triangleq \ (x \ @pre\ H))$
sorry

end

theory *OCL-tools*
imports *OCL-core*
begin

end

theory *OCL-main*
imports *OCL-lib OCL-state OCL-tools*
begin

end

theory
OCL-linked-list
imports
../OCL-main
begin

15 Introduction

For certain concepts like Classes and Class-types, only a generic definition for its resulting semantics can be given. Generic means, there is a function outside HOL that "compiles" a concrete, closed-world class diagram into a "theory" of this data model, consisting of a bunch of definitions for classes, accessors, method, casts, and tests for actual types, as well as proofs for the fundamental properties of these operations in this concrete data model.

Such generic function or "compiler" can be implemented in Isabelle on the ML level. This has been done, for a semantics following the open-world assumption, for UML 2.0 in [**?**]. In this paper, we follow another approach for UML 2.4: we define the concepts of the compilation informally, an present a concrete example which is verified in Isabelle/HOL.

16 Outlining the Example

17 Example Data-Universe and its Infrastructure

Should be generated entirely from a class-diagram.

Our data universe consists in the concrete class diagram just of node's, and implicitly of the class object. Each class implies the existence of a class type defined for the corresponding object representations as follows:

datatype *node* = mk_{node} *oid*
int option
oid option

datatype *object*= mk_{object} *oid*
$(int\ option \times oid\ option)\ option$

Now, we construct a concrete "universe of object types" by injection into a sum type containing the class types. This type of objects will be used as instance for all resp. type-variables ...

datatype $\mathfrak{A} = in_{node}\ node \mid in_{object}\ object$

Recall that in order to denote OCL-types occuring in OCL expressions syntactically — as, for example, as "argument" of allInstances — we use the inverses of the injection functions into the object universes; we show that this is sufficient "characterization".

definition $Node :: \mathfrak{A} \Rightarrow node$
where $Node \equiv (the\text{-}inv\ in_{node})$

definition $Object :: \mathfrak{A} \Rightarrow object$
where $Object \equiv (the\text{-}inv\ in_{object})$

Having fixed the object universe, we can introduce type synonyms that exactly correspond to OCL types. Again, we exploit that our representation of OCL is a "shallow embedding" with a one-to-one correspondance of OCL-types to types of the meta-language HOL.

type-synonym *Boolean* $= (\mathfrak{A})Boolean$
type-synonym *Integer* $= (\mathfrak{A})Integer$
type-synonym *Void* $= (\mathfrak{A})Void$
type-synonym *Object* $= (\mathfrak{A}, object\ option\ option)\ val$
type-synonym *Node* $= (\mathfrak{A},\ node\ option\ option)val$
type-synonym *Set-Integer* $= (\mathfrak{A},\ int\ option\ option)Set$
type-synonym *Set-Node* $= (\mathfrak{A},\ node\ option\ option)Set$

Just a little check:

typ *Boolean*

In order to reuse key-elements of the library like referential equality, we have to show that the object universe belongs to the type class "object", i.e. each class type has to provide a function *oid-of* yielding the object id (oid) of the object.

instantiation *node* :: *object*
begin
  definition *oid-of-node-def*: *oid-of* x = (*case* x *of* mk_{node} *oid* - - ⇒ *oid*)
  instance ..
end

instantiation *object* :: *object*
begin
  definition *oid-of-object-def*: *oid-of* x = (*case* x *of* mk_{object} *oid* - ⇒ *oid*)
  instance ..
end

instantiation $\mathfrak{A}$ :: *object*
begin
  definition *oid-of-*$\mathfrak{A}$*-def*: *oid-of* x = (*case* x *of*
    in_{node} *node* ⇒ *oid-of node*
    | in_{object} *obj* ⇒ *oid-of obj*)
  instance ..
end

instantiation *option* :: (*object*)*object*
begin
  definition *oid-of-option-def*: *oid-of* x = *oid-of* (*the* x)
  instance ..
end

18 Instantiation of the generic strict equality. We instantiate the referential equality on *Node* and *Object*

defs(**overloaded**) $StrictRefEq_{node}$: (x::*Node*) $\doteq y$ ≡ *gen-ref-eq* x y
defs(**overloaded**) $StrictRefEq_{object}$: (x::*Object*) $\doteq y$ ≡ *gen-ref-eq* x y

lemmas *strict-eq-node* =
  cp-gen-ref-eq-object[*of* x::*Node* y::*Node* τ,
    simplified $StrictRefEq_{node}$[*symmetric*]]
  cp-intro(*9*) [*of* P::*Node* ⇒*NodeQ*::*Node* ⇒*Node*,
    simplified $StrictRefEq_{node}$[*symmetric*]]
  gen-ref-eq-def [*of* x::*Node* y::*Node*,
    simplified $StrictRefEq_{node}$[*symmetric*]]
  gen-ref-eq-defargs [*of* - x::*Node* y::*Node*,
    simplified $StrictRefEq_{node}$[*symmetric*]]
  gen-ref-eq-object-strict1

$[$*of x::Node*,
simplified $StrictRefEq_{node}$[*symmetric*]]
gen-ref-eq-object-strict2
$[$*of x::Node*,
simplified $StrictRefEq_{node}$[*symmetric*]]
gen-ref-eq-object-strict3
$[$*of x::Node*,
simplified $StrictRefEq_{node}$[*symmetric*]]
gen-ref-eq-object-strict3
$[$*of x::Node*,
simplified $StrictRefEq_{node}$[*symmetric*]]
gen-ref-eq-object-strict4
$[$*of x::Node*,
simplified $StrictRefEq_{node}$[*symmetric*]]

19 AllInstances

lemma (*Node .oclAllInstances*()) =
$(\lambda\tau.\ \mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor(\mathit{Some} \circ \mathit{Some} \circ (\mathit{the\text{-}inv}\ \mathit{in}_{node}))\text{`}(\mathit{ran}(\mathit{snd}\ \tau))\ \rfloor\rfloor)$
by(*rule ext*, *simp add*:*allinstances-def Node-def*)

lemma (*Object .oclAllInstances@pre*()) =
$(\lambda\tau.\ \mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor(\mathit{Some} \circ \mathit{Some} \circ (\mathit{the\text{-}inv}\ \mathit{in}_{object}))\text{`}(\mathit{ran}(\mathit{fst}\ \tau))\ \rfloor\rfloor)$
by(*rule ext*, *simp add*:*allinstancesATpre-def Object-def*)

For each Class C, we will have an casting operation `.oclAsType(`C`)`, a test on the actual type `.oclIsTypeOf(`C`)` as well as its relaxed form `.oclIsKindOf(`C`)` (corresponding exactly to Java's `instanceof`-operator.

Thus, since we have two class-types in our concrete class hierarchy, we have two operations to declare and and to provide two overloading definitions for the two static types.

20 Selector Definition

Should be generated entirely from a class-diagram.

typ *Node* ⇒ *Node*
fun *dot-next*:: *Node* ⇒ *Node* ((*1*(-).*next*) *50*)
where (*X*).*next* = (λ τ. *case X* τ *of*
$\bot \Rightarrow$ *invalid* τ (∗ *undefined pointer* ∗)
| $\lfloor\ \bot\ \rfloor \Rightarrow$ *invalid* τ (∗ *dereferencing null pointer* ∗)
| $\lfloor\lfloor\ mk_{node}$ *oid i* $\bot\ \rfloor\rfloor \Rightarrow$ *null* τ(∗ *object contains null pointer* ∗)
| $\lfloor\lfloor\ mk_{node}$ *oid i* $\lfloor next \rfloor\ \rfloor\rfloor \Rightarrow$ (∗ *We assume here that oid is indeed 'the' oid of the Node*,
ie. we assume that τ *is well−formed.* ∗)
case (*snd* τ) *next of*
$\bot \Rightarrow$ *invalid* τ

$| \lfloor in_{node}\ (mk_{node}\ a\ b\ c) \rfloor \Rightarrow \lfloor \lfloor mk_{node}\ a\ b\ c\ \rfloor \rfloor$
$| \lfloor\ \text{-}\ \rfloor \Rightarrow invalid\ \tau)$

fun *dot-i*:: *Node* ⇒ *Integer* ((*1*(-).*i*) *50*)
where $(X).i = (\lambda\ \tau.\ case\ X\ \tau\ of$
$\perp \Rightarrow invalid\ \tau$
$| \lfloor\ \perp\ \rfloor \Rightarrow invalid\ \tau$
$| \lfloor \lfloor\ mk_{node}\ oid\ \perp\ \text{-}\ \rfloor \rfloor \Rightarrow\ null\ \tau$
$| \lfloor \lfloor\ mk_{node}\ oid\ \lfloor i \rfloor\ \text{-}\ \rfloor \rfloor \Rightarrow\ \lfloor \lfloor\ i\ \rfloor \rfloor)$

fun *dot-next-at-pre*:: *Node* ⇒ *Node* ((*1*(-).*next@pre*) *50*)
where (*X*).*next@pre* = (λ τ. *case X* τ *of*
$\perp \Rightarrow invalid\ \tau$
$| \lfloor\ \perp\ \rfloor \Rightarrow invalid\ \tau$
$| \lfloor \lfloor\ mk_{node}\ oid\ i\ \perp\ \rfloor \rfloor \Rightarrow null\ \tau$(∗ *object contains null pointer. REALLY ?*
And if this pointer was defined in the pre−state ?∗)
$| \lfloor \lfloor\ mk_{node}\ oid\ i\ \lfloor next \rfloor\ \rfloor \rfloor \Rightarrow$ (∗ *We assume here that oid is indeed 'the' oid of the Node,*
ie. we assume that τ *is well−formed.* ∗)
(*case* (*fst* τ) *next of*
$\perp \Rightarrow invalid\ \tau$
$| \lfloor in_{node}\ (mk_{node}\ a\ b\ c) \rfloor \Rightarrow \lfloor \lfloor mk_{node}\ a\ b\ c\ \rfloor \rfloor$
$| \lfloor\ \text{-}\ \rfloor \Rightarrow invalid\ \tau))$

fun *dot-i-at-pre*:: *Node* ⇒ *Integer* ((*1*(-).*i@pre*) *50*)
where (*X*).*i@pre* = (λ τ. *case X* τ *of*
$\perp \Rightarrow invalid\ \tau$
$| \lfloor\ \perp\ \rfloor \Rightarrow invalid\ \tau$
$| \lfloor \lfloor\ mk_{node}\ oid\ \text{-}\ \text{-}\ \rfloor \rfloor \Rightarrow$
$if\ oid \in dom\ (fst\ \tau)$
$then\ (case\ (fst\ \tau)\ oid\ of$
$\perp \Rightarrow invalid\ \tau$
$| \lfloor in_{node}\ (mk_{node}\ oid\ \perp\ next)\ \rfloor \Rightarrow null\ \tau$
$| \lfloor in_{node}\ (mk_{node}\ oid\ \lfloor i \rfloor next)\ \rfloor \Rightarrow \lfloor \lfloor\ i\ \rfloor \rfloor$
$| \lfloor\ \text{-}\ \rfloor \Rightarrow invalid\ \tau)$
$else\ invalid\ \tau)$

lemma *cp-dot-next*: ((*X*).*next*) τ = ((λ-. *X* τ).*next*) τ **by**(*simp*)

lemma *cp-dot-i*: ((*X*).*i*) τ = ((λ-. *X* τ).*i*) τ **by**(*simp*)

lemma *cp-dot-next-at-pre*: ((*X*).*next@pre*) τ = ((λ-. *X* τ).*next@pre*) τ **by**(*simp*)

lemma *cp-dot-i-pre*: ((*X*).*i@pre*) τ = ((λ-. *X* τ).*i@pre*) τ **by**(*simp*)

lemmas *cp-dot-nextI* [*simp*, *intro*!]=
cp-dot-next[*THEN allI*[*THEN allI*], *of* λ *X* -. *X* λ - τ. τ, *THEN cpI1*]

lemmas *cp-dot-nextI-at-pre* [*simp*, *intro*!]=
cp-dot-next-at-pre[*THEN allI*[*THEN allI*],
of λ *X* -. *X* λ - τ. τ, *THEN cpI1*]

lemma *dot-next-nullstrict* [*simp*]: (*null*).*next* = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def*)

lemma *dot-next-at-pre-nullstrict* [*simp*] : (*null*).*next@pre* = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def*)

lemma *dot-next-strict*[*simp*] : (*invalid*).*next* = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def*)

lemma *dot-next-strict'*[*simp*] : (*null*).*next* = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def*)

lemma *dot-nextATpre-strict*[*simp*] : (*invalid*).*next@pre* = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def*)

lemma *dot-nextATpre-strict'*[*simp*] : (*null*).*next@pre* = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def*)

21 Casts

consts $oclastype_{object}$ :: $'\alpha \Rightarrow$ *Object* ((-).*oclAsType'*(*Object'*))
consts $oclastype_{node}$ :: $'\alpha \Rightarrow$ *Node* ((-).*oclAsType'*(*Node'*))

defs (**overloaded**) $oclastype_{object}$-*Object*:
(*X*::*Object*) .*oclAsType*(*Object*) $\equiv$
($\lambda\tau$. *case X* τ *of*
$\bot$ $\Rightarrow$ *invalid* τ
| $\lfloor\bot\rfloor$ $\Rightarrow$ *invalid* τ (* *to avoid*: *null* .*oclAsType*(*Object*) =
null ? *)
| $\lfloor\lfloor mk_{object}$ *oid a* $\rfloor\rfloor$ $\Rightarrow$ $\lfloor\lfloor mk_{object}$ *oid a* $\rfloor\rfloor$)

defs (**overloaded**) $oclastype_{object}$-*Node*:
(*X*::*Node*) .*oclAsType*(*Object*) $\equiv$
($\lambda\tau$. *case X* τ *of*
$\bot$ $\Rightarrow$ *invalid* τ
| $\lfloor\bot\rfloor$ $\Rightarrow$ *invalid* τ (* *OTHER POSSIBILITY* : *null ???*
Really excluded
by standard *)
| $\lfloor\lfloor mk_{node}$ *oid a b* $\rfloor\rfloor$ $\Rightarrow$ $\lfloor\lfloor$ (mk_{object} *oid* $\lfloor(a,b)\rfloor$) $\rfloor\rfloor$)

defs (**overloaded**) $oclastype_{node}$-*Object*:
(*X*::*Object*) .*oclAsType*(*Node*) $\equiv$
($\lambda\tau$. *case X* τ *of*
$\bot$ $\Rightarrow$ *invalid* τ

| $\lfloor\perp\rfloor \Rightarrow$ *invalid* τ
| $\lfloor\lfloor mk_{object}$ *oid* $\perp \rfloor\rfloor \Rightarrow$ *invalid* τ (* *down−cast exception* *)
| $\lfloor\lfloor mk_{object}$ *oid* $\lfloor(a,b)\rfloor \rfloor\rfloor \Rightarrow \lfloor\lfloor mk_{node}$ *oid a b* $\rfloor\rfloor$)

defs (**overloaded**) *oclastype*$_{node}$*-Node*:
(*X*::*Node*) .*oclAsType*(*Node*) $\equiv$
($\lambda\tau$. *case X* τ *of*
$\perp \Rightarrow$ *invalid* τ
| $\lfloor\perp\rfloor \Rightarrow$ *invalid* τ (* *to avoid*: *null* .*oclAsType*(*Object*) = *null* ? *)
| $\lfloor\lfloor mk_{node}$ *oid a b* $\rfloor\rfloor \Rightarrow \lfloor\lfloor mk_{node}$ *oid a b*$\rfloor\rfloor$)

lemma *oclastype*$_{object}$*-Object-strict*[*simp*] : (*invalid*::*Object*) .*oclAsType*(*Object*) = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def oclastype*$_{object}$*-Object*)

lemma *oclastype*$_{object}$*-Object-nullstrict*[*simp*] : (*null*::*Object*) .*oclAsType*(*Object*) = *invalid*
by(*rule ext*, *simp add*: *null-fun-def null-option-def bot-option-def null-def invalid-def oclastype*$_{object}$*-Object*)

22 Tests for Actual Types

consts *oclistypeof*$_{object}$ :: $'\alpha \Rightarrow$ *Boolean* ((-).*oclIsTypeOf*′(*Object*′))
consts *oclistypeof*$_{node}$ :: $'\alpha \Rightarrow$ *Boolean* ((-).*oclIsTypeOf*′(*Node*′))

defs (**overloaded**) *oclistypeof*$_{object}$*-Object*:
(*X*::*Object*) .*oclIsTypeOf*(*Object*) $\equiv$
($\lambda\tau$. *case X* τ *of*
$\perp \Rightarrow$ *invalid* τ
| $\lfloor\perp\rfloor \Rightarrow$ *invalid* τ
| $\lfloor\lfloor mk_{object}$ *oid* $\perp \rfloor\rfloor \Rightarrow$ *true* τ
| $\lfloor\lfloor mk_{object}$ *oid* $\lfloor$-$\rfloor \rfloor\rfloor \Rightarrow$ *false* τ)

defs (**overloaded**) *oclistypeof*$_{object}$*-Node*:
(*X*::*Node*) .*oclIsTypeOf*(*Object*) $\equiv$
($\lambda\tau$. *case X* τ *of*
$\perp \Rightarrow$ *invalid* τ
| $\lfloor\perp\rfloor \Rightarrow$ *invalid* τ
| $\lfloor\lfloor$ - $\rfloor\rfloor \Rightarrow$ *false* τ)

defs (**overloaded**) *oclistypeof*$_{node}$*-Object*:
(*X*::*Object*) .*oclIsTypeOf*(*Node*) $\equiv$
($\lambda\tau$. *case X* τ *of*
$\perp \Rightarrow$ *invalid* τ
| $\lfloor\perp\rfloor \Rightarrow$ *invalid* τ
| $\lfloor\lfloor mk_{object}$ *oid* $\perp \rfloor\rfloor \Rightarrow$ *false* τ

$| \lfloor\lfloor mk_{object}\ oid\ \lfloor\text{-}\rfloor\ \rfloor\rfloor \Rightarrow true\ \tau)$

defs (**overloaded**) $oclistypeof_{node}\text{-}Node$:
$(X::Node)\ .oclIsTypeOf(Node) \equiv$
$(\lambda\tau.\ case\ X\ \tau\ of$
$\bot \Rightarrow invalid\ \tau$
$| \lfloor\bot\rfloor \Rightarrow invalid\ \tau$
$| \lfloor\lfloor\ \text{-}\ \rfloor\rfloor \Rightarrow true\ \tau)$

23 Standard State Infrastructure

These definitions should be generated — again — from the class diagram.

24 Invariant

These recursive predicates can be defined conservatively by greatest fix-point constructions - automatically. See HOL-OCL Book for details. For the purpose of this example, we state them as axioms here.

axiomatization $inv\text{-}Node :: Node \Rightarrow Boolean$
where $A : (\tau \models (\delta\ self)) \longrightarrow$
$(\tau \models inv\text{-}Node(self)) =$
$((\tau \models (self\ .next \doteq null)) \lor$
$(\tau \models (self\ .next <> null) \land (\tau \models (self\ .next\ .i \prec self\ .i)) \land$
$(\tau \models (inv\text{-}Node(self\ .next)))))$

axiomatization $inv\text{-}Node\text{-}at\text{-}pre :: Node \Rightarrow Boolean$
where $B : (\tau \models (\delta\ self)) \longrightarrow$
$(\tau \models inv\text{-}Node\text{-}at\text{-}pre(self)) =$
$((\tau \models (self\ .next@pre \doteq null)) \lor$
$(\tau \models (self\ .next@pre <> null) \land (\tau \models (self\ .next@pre\ .i@pre \prec$
$self\ .i@pre)) \land$
$(\tau \models (inv\text{-}Node\text{-}at\text{-}pre(self\ .next@pre)))))$

A very first attempt to characterize the axiomatization by an inductive definition - this can not be the last word since too weak (should be equality!)

coinductive $inv :: Node \Rightarrow (\mathfrak{A})st \Rightarrow bool$ **where**
$(\tau \models (\delta\ self)) \Longrightarrow ((\tau \models (self\ .next \doteq null)) \lor$
$(\tau \models (self\ .next <> null) \land (\tau \models (self\ .next\ .i \prec self\ .i)) \land$
$(\ (inv(self\ .next))\tau\ )))$
$\Longrightarrow (\ inv\ self\ \tau)$

25 The contract of a recursive query :

The original specification of a recursive query :

```
context Node::contents():Set(Integer)
post:  result = if self.next = null
                then Set{i}
                else self.next.contents()->including(i)
                endif
```

consts *dot-contents* :: *Node* ⇒ *Set-Integer* ((*1*(-).*contents*′(′)) *50*)

axiomatization *dot-contents-def* **where**

$(\tau \models ((self).contents() \triangleq result)) =$
$\ (if\ (\delta\ self)\ \tau = true\ \tau$
$\ \ then\ ((\tau \models true) \wedge$
$\qquad (\tau \models (result \triangleq if\ (self\ .next \doteq null)$
$\qquad\qquad then\ (Set\{self\ .i\})$
$\qquad\qquad else\ (self\ .next\ .contents()\text{−>}including(self\ .i))$
$\qquad\qquad endif)))$
$\ \ else\ \tau \models result \triangleq invalid)$

consts *dot-contents-AT-pre* :: *Node* ⇒ *Set-Integer* ((*1*(-).*contents@pre*′(′)) *50*)

axiomatization where *dot-contents-AT-pre-def*:

$(\tau \models (self).contents@pre() \triangleq result) =$
$\ (if\ (\delta\ self)\ \tau = true\ \tau$
$\ \ then\ \tau \models true\ \wedge \qquad\qquad (*\ pre\ *)$
$\qquad \tau \models (result \triangleq if\ (self).next@pre \doteq null\ \ (*\ post\ *)$
$\qquad\qquad then\ Set\{(self).i@pre\}$
$\qquad\qquad else\ (self).next@pre\ .contents@pre()\text{−>}including(self\ .i@pre)$
$\qquad\qquad endif)$
$\ \ else\ \tau \models result \triangleq invalid)$

Note that these @pre variants on methods are only available on queries, i.e. operations without side-effect.

26 The contract of a method.

The specification in high-level OCL input syntax reads as follows:

```
context Node::insert(x:Integer)
post: contents():Set(Integer)
contents() = contents@pre()->including(x)
```

consts *dot-insert* :: *Node* ⇒ *Integer* ⇒ *Void* ((*1*(-).*insert*′(-′)) *50*)

axiomatization where *dot-insert-def*:

$(\tau \models (self).insert(x) \triangleq result) =$
$\ (if\ (\delta\ self)\ \tau = true\ \tau \wedge (\upsilon\ x)\ \tau = true\ \tau$

then $\tau \models$ *true* $\wedge$
$\tau \models (self).contents() \triangleq (self).contents@pre()\text{−>}including(x)$
else $\tau \models (self).insert(x) \triangleq invalid)$

lemma H : $(\tau \models (self).insert(x) \triangleq result)$
nitpick
thm *dot-insert-def*
oops

end