Extended Version

Featherweight OCL

A Study for a Consistent Semantics of UML/OCL 2.3 in HOL

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Abstract

UML/OCL is one of the few modeling languages that is widely used in industry. Besides numerous diagrams describing various aspects of models, the core of the UML, the language OCL, is a textual annotation language that turns it into a formal language. Unfortunately the semantics of this specification language, captured in the "Appendix A" of the OCL standard lead to different interpretations of corner cases and had been subject to formal analysis earlier. The situation complicated when with version 2.3 the OCL was aligned with the UML; this lead to the extension of the 3 valued logic by a second exception element, called "null". While the first exception element, "undefined", has a strict semantics, "null" has a non strict semantic interpretation. This semantic difficulties lead to remarkable confusion for implementors of OCL compilers and interpreters.

In this paper, we provide a formalization of the core of OCL in higher-order logic (HOL). It provides denotational definitions, a logical calculus and operational rules that allows for the execution of OCL expressions by a mixture of term rewriting and code compilation. Our formalization revealed several inconsistencies and contradictions in the current version of the OCL standard. They reflect a challenge to define and implement OCL tools in a uniform manner. This document is intended to provide the basis for a machine-checked text "Appendix A" of the UML standard targeting at tool implementors.

Further readings: This theory extends the paper "Featherweight OCL: A study for the consistent semantics of OCL 2.3 in HOL" [12] that is published as part of the proceedings of the OCL workshop 2012.

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Part I.

Introduction

1. Motivation

At its origins [18, 22], OCL was conceived as a strict semantics for undefinedness, with the exception of the logical connectives of type `Boolean` that constitute a three-valued propositional logic. Recent versions of the OCL standard [20, 21] added a second exception element, which is given a non-strict semantics. Unfortunately, this extension results in several inconsistencies and contradictions. These problems are reflected in difficulties to define interpreters, code-generators, specification animators or theorem provers for OCL in a uniform manner and resulting incompatibilities of various tools. For the OCL community, this results in the challenge to define a new formal semantics definition OCL that could replace the "Annex A" of the OCL standard [21].

In the paper "Extending OCL with Null-References" [6] we explored—based on mathematical arguments and paper and pencil proofs—a consistent formal semantics that comprises two exception elements: `invalid` ("bottom" in semantics terminology) and `null` (for "non-existing element").

This short paper is based on a formalization of [6], called "Featherweight OCL," in Isabelle/HOL [17]. This formalization is in its present form merely a semantical study and a proof of technology than a real tool. It focuses on the formalization of the key semantical constructions, i. e., the type `Boolean` and the logic, the type `Integer` and a standard strict operator library, and the collection type `Set(A)` with quantifiers, iterators and key operators.

2. Background

2.1. Formal Foundation

2.1.1. Isabelle

Isabelle [17] is a *generic* theorem prover. New object logics can be introduced by specifying their syntax and natural deduction inference rules. Among other logics, Isabelle supports first-order logic, Zermelo-Fraenkel set theory and the instance for Church's higher-order logic HOL, which we choose as basis for HOL-TestGen and which is introduced in the subsequent section.

Isabelle's inference rules are based on the built-in meta-level implication $\_ \Longrightarrow \_$ allowing to form constructs like $A_1 \Longrightarrow \cdots \Longrightarrow A_n \Longrightarrow A_{n+1}$, which are viewed as a *rule* of the form "from assumptions A_1 to A_n, infer conclusion A_{n+1}" and which is written in Isabelle as

$$[\![A_1; \ldots; A_n]\!] \Longrightarrow A_{n+1} \qquad \text{or, in mathematical notation,} \qquad \frac{A_1 \qquad \cdots \qquad A_n}{A_{n+1}} \,. \tag{2.1}$$

The built-in meta-level quantification $\bigwedge x.\ x$ captures the usual side-constraints "x must not occur free in the assumptions" for quantifier rules; meta-quantified variables can be considered as "fresh" free variables. Meta-level quantification leads to a generalization of Horn-clauses of the form:

$$\bigwedge x_1, \ldots, x_m.\ [\![A_1; \ldots; A_n]\!] \Longrightarrow A_{n+1} \,. \tag{2.2}$$

Isabelle supports forward- and backward reasoning on rules. For backward-reasoning, a *proof-state* can be initialized and further transformed into others. For example, a proof of ϕ, using the Isar [25] language, will look as follows in Isabelle:

```
lemma label:   φ
  apply(case_tac)
  apply(simp_all)
done
```
(2.3)

This proof script instructs Isabelle to prove ϕ by case distinction followed by a simplification of the resulting proof state. Such a proof state is an implicitly conjoint sequence of generalized Horn-clauses (called *subgoals*) $\phi_1, \ldots, \phi_n$ and a *goal* ϕ. Proof states were usually denoted by:

$$
\begin{array}{rl}
\text{label}: & \phi \\
1. & \phi_1 \\
 & \vdots \\
n. & \phi_n
\end{array}
\tag{2.4}
$$

Subgoals and goals may be extracted from the proof state into theorems of the form $[\![\phi_1; \ldots; \phi_n]\!] \Longrightarrow \phi$ at any time; this mechanism helps to generate test theorems. Further, Isabelle supports meta-variables (written $?x, ?y, \ldots$), which can be seen as "holes in a term" that can still be substituted. Meta-variables are instantiated by Isabelle's built-in higher-order unification.

2.1.2. Higher-order logic

Higher-order logic (HOL) [1, 13] is a classical logic based on a simple type system. It provides the usual logical connectives like $\_ \wedge \_$, $\_ \rightarrow \_$, $\neg \_$ as well as the object-logical quantifiers $\forall x.\ P\ x$ and $\exists x.\ P\ x$; in contrast to first-order logic, quantifiers may range over arbitrary types, including total functions $f :: \alpha \Rightarrow \beta$. HOL is centered around extensional equality $\_ = \_ :: \alpha \Rightarrow \alpha \Rightarrow \text{bool}$. HOL is more expressive than first-order logic, since, e. g., induction schemes can be expressed inside the logic. Being based on some polymorphically typed λ-calculus, HOL can be viewed as a combination of a programming language like SML or Haskell and a specification language providing powerful logical quantifiers ranging over elementary and function types.

Isabelle/HOL is a logical embedding of HOL into Isabelle. The (original) simple-type system underlying HOL has been extended by Hindley-Milner style polymorphism with type-classes similar to Haskell. While Isabelle/HOL is usually seen as proof assistant, we use it as symbolic computation environment. Implementations on top of Isabelle/HOL can re-use existing powerful deduction mechanisms such as higher-order resolution, tableaux-based reasoners, rewriting procedures, Presburger arithmetic, and via various integration mechanisms, also external provers such as Vampire and the SMT-solver Z3.

Isabelle/HOL offers support for a particular methodology to extend given theories in a logically safe way: A theory-extension is *conservative* if the extended theory is consistent provided that the original theory was consistent. Conservative extensions can be *constant definitions*, *type definitions*, *datatype definitions*, *primitive recursive definitions* and *wellfounded recursive definitions*.

For instance, the library includes the type constructor $\tau_\bot := \bot \mid \lfloor\_\rfloor : \alpha$ that assigns to each type τ a type $\tau_\bot$ *disjointly extended* by the exceptional element $\bot$. The function $\lceil\_\rceil : \alpha_\bot \Rightarrow \alpha$ is the inverse of $\lfloor\_\rfloor$ (unspecified for $\bot$). Partial functions $\alpha \rightharpoonup \beta$ are defined as functions $\alpha \Rightarrow \beta_\bot$ supporting the usual concepts of domain ($\text{dom}\ \_$) and range ($\text{ran}\ \_$).

As another example of a conservative extension, typed sets were built in the Isabelle libraries conservatively on top of the kernel of HOL as functions to bool; consequently, the constant definitions for membership is as follows:[1]

$$\begin{array}{llll}
\textbf{types} & \alpha\ \mathrm{set} & = \alpha \Rightarrow \mathrm{bool} & \\
\textbf{definition} & \mathrm{Collect} & ::(\alpha \Rightarrow \mathrm{bool}) \Rightarrow \alpha\ \mathrm{set} & \text{— set comprehension} \\
\textbf{where} & \mathrm{Collect}\ S & \equiv S & \\
\textbf{definition} & \mathrm{member} & ::\alpha \Rightarrow \alpha \Rightarrow \mathrm{bool} & \text{— membership test} \\
\textbf{where} & \mathrm{member}\ s\ S & \equiv S s &
\end{array} \tag{2.5}$$

Isabelle's powerful syntax engine is instructed to accept the notation $\{x \mid P\}$ for Collect $\lambda\, x.\ P$ and the notation $s \in S$ for member $s\,S$. As can be inferred from the example, constant definitions are axioms that introduce a fresh constant symbol by some closed, non-recursive expressions; this type of axiom is logically safe since it works like an abbreviation. The syntactic side conditions of this axiom are mechanically checked, of course. It is straightforward to express the usual operations on sets like $\_\cup\_$, $\_\cap\_::\alpha\mathrm{set} \Rightarrow \alpha\mathrm{set} \Rightarrow \alpha\mathrm{set}$ as conservative extensions, too, while the rules of typed set theory were derived by proofs from these definitions.

Similarly, a logical compiler is invoked for the following statements introducing the types option and list:

$$\begin{array}{lll}
\textbf{datatype} & \mathrm{option} & = \mathrm{None} \mid \mathrm{Some}\,\alpha \\
\textbf{datatype} & \alpha\ \mathrm{list} & = \mathrm{Nil} \mid \mathrm{Cons}\ a\ l
\end{array} \tag{2.6}$$

Here, $[]$ or $a\#l$ are an alternative syntax for Nil or Cons a l; moreover, $[a, b, c]$ is defined as alternative syntax for $a\#b\#c\#[]$. These (recursive) statements were internally represented in by internal type and constant definitions. Besides the *constructors* None, Some, $[]$ and Cons, there is the match operation

$$\mathrm{case}\ x\ \mathrm{of}\ \ \mathrm{None} \Rightarrow F\ \mid \mathrm{Some}\, a \Rightarrow G\, a \tag{2.7}$$

respectively

$$\mathrm{case}\ x\ \mathrm{of}\ \ [] \Rightarrow F\ \mid \mathrm{Cons}\ a\, r \Rightarrow G\, a\, r\,. \tag{2.8}$$

From the internal definitions (not shown here) a number of properties were automatically derived. We show only the case for lists:

$$\begin{array}{ll}
(\mathrm{case}\ []\ \mathrm{of}\ [] \Rightarrow F \mid (a\#r) \Rightarrow G\ a\ r) = F & \\
(\mathrm{case}\ b\#t\ \mathrm{of}\ [] \Rightarrow F \mid (a\#r) \Rightarrow G\ a\ r) = G\ b\ t & \\
[] \neq a\#t & \text{– distinctness} \\
[\![a = [] \to P; \exists\ x\ t.\ a = x\#t \to P]\!] \Longrightarrow P & \text{– exhaust} \\
[\![P[]; \forall\ at.\ Pt \to P(a\#t)]\!] \Longrightarrow Px & \text{– induct}
\end{array} \tag{2.9}$$

[1]To increase readability, we use a slightly simplified presentation.

Finally, there is a compiler for primitive and wellfounded recursive function definitions. For example, we may define the sort operation of our running test example by:

$$\begin{array}{lll} \textbf{fun} & \text{ins} & ::[\alpha :: \text{linorder}, \alpha \text{ list}] \Rightarrow \alpha \text{ list} \\ \textbf{where} & \text{ins } x\, [\,] & = [x] \\ & \text{ins } x\, (y\#ys) & = \text{if } x < y \text{ then } x\#y\#ys \text{ else } y\#(\text{ins } x\ ys) \end{array} \tag{2.10}$$

$$\begin{array}{lll} \textbf{fun} & \text{sort} & ::(\alpha :: \text{linorder}) \text{ list} \Rightarrow \alpha \text{ list} \\ \textbf{where} & \text{sort } [\,] & = [\,] \\ & \text{sort}(x\#xs) & = \text{ins } x\ (\text{sort } xs) \end{array} \tag{2.11}$$

The internal (non-recursive) constant definition for these operations is quite involved; however, the logical compiler will finally derive all the equations in the statements above from this definition and make them available for automated simplification.

Thus, Isabelle/HOL also provides a large collection of theories like sets, lists, multisets, orderings, and various arithmetic theories which only contain rules derived from conservative definitions. In particular, Isabelle manages a set of *executable types and operators*, i. e., types and operators for which a compilation to SML, OCaml or Haskell is possible. Setups for arithmetic types such as int have been done; moreover any datatype and any recursive function were included in this executable set (providing that they only consist of executable operators). Similarly, Isabelle manages a large set of (higher-order) rewrite rules into which recursive function definitions were included. Provided that this rule set represents a terminating and confluent rewrite system, the Isabelle simplifier provides also a highly potent decision procedure for many fragments of theories underlying the constraints to be processed when constructing test theorems.

2.1.3. Specification Constructs in Isabelle/HOL

2.2. Featherweight OCL: Design Goals

Featherweight OCL is a formalization of the core of OCL aiming at formally investigating the relationship between the different notions of "undefinedness," i. e., `invalid` and `null`. As such, it does not attempt to define the complete OCL library. Instead, it concentrates on the core concepts of OCL as well as the types `Boolean`, `Integer`, and typed sets (`Set(T)`). Following the tradition of HOL-OCL [7, 8], Featherweight OCL is based on the following principles:

1. It is an embedding into a powerful semantic meta-language and environment, namely Isabelle/HOL [17].
2. It is a *shallow embedding* in HOL; types in OCL were injectively mapped to types in Featherweight OCL. Ill-typed OCL specifications cannot be represented in Featherweight OCL and a type in Featherweight OCL contains exactly the values that are possible in OCL. Thus, sets may contain `null` (`Set{null}` is a defined set) but not `invalid` (`Set{invalid}` is just `invalid`).
3. Any Featherweight OCL type contains at least `invalid` and `null` (the type `Void`

contains only these instances). The logic is consequently four-valued, and there is a `null`-element in the type `Set(A)`.

4. It is a strongly typed language in the Hindley-Milner tradition. We assume that a pre-process eliminates all implicit conversions due to subtyping by introducing explicit casts (e. g., `oclAsType()`). The details of such a pre-processing are described in [4]. Casts are semantic functions, typically injections, that may convert data between the different Featherweight OCL types.
5. All objects are represented in an object universe in the HOL-OCL tradition [9]. The universe construction also gives semantics to type casts, dynamic type tests, as well as functions such as `oclAllInstances()`, or `isNewInState()`.
6. Featherweight OCL types may be arbitrarily nested: `Set{Set{1,2}} = Set{Set{2,1}}` is legal and true.
7. For demonstration purposes, the set type in Featherweight OCL may be infinite, allowing infinite quantification and a constant that contains the set of all Integers. Arithmetic laws like commutativity may therefore be expressed in OCL itself. The iterator is only defined on finite sets.
8. It supports equational reasoning and congruence reasoning, but this requires a differentiation of the different equalities like strict equality, strong equality, meta-equality (HOL). Strict equality and strong equality require a subcalculus, "cp" (a detailed discussion of the different equalities as well as the subcalculus "cp"—for three-valued OCL 2.0—is given in [11]), which is nasty but can be hidden from the user inside tools.

2.3. The Theory Organization

The semantic theory is organized in a quite conventional manner in three layers. The first layer, called the *denotational semantics* comprises a set of definitions of the operators of the language. Presented as *definitional axioms* inside Isabelle/HOL, this part assures the logically consistency of the overall construction. The second layer, called *logical layer*, is derived from the former and centered around the notion of validity of an OCL formula P for a state-transition from pre-state σ to post-state σ', validity statements were written $(\sigma, \sigma') \models P$. The third layer, called *algebraic layer*, also derived from the former layers, tries to establish a number of algebraic laws of the form $P = P'$; such laws are amenable to equational reasoning and also help for automated reasoning and code-generation.

For space reasons, we will restrict ourselves in this paper to a few operators and make a traversal through all three layers in order to give a high-level description of our formalization. Especially, the details of the semantic construction for sets and the handling of objects and object universes were excluded from a presentation here.

2.3.1. Denotational Semantics

OCL is composed of 1) operators on built-in data structures such as Boolean, Integer or Set(A), 2) operators of the user-defined data-model such as accessors, type-casts and tests, and 3) user-defined, side-effect-free methods. Conceptually, an OCL expression in general and Boolean expressions in particular (i. e., *formulae*) that depends on the pair (σ, σ') of pre-and post-state. The precise form of states is irrelevant for this paper (compare [6]) and will be left abstract in this presentation. We construct in Isabelle a type-class null that contains two distinguishable elements bot and null. Any type of the form $(\alpha_\perp)_\perp$ is an instance of this type-class with bot $\equiv \perp$ and null $\equiv \lfloor\perp\rfloor$. Now, any OCL type can be represented by an HOL type of the form:

$$V(\alpha) := \mathrm{state} \times \mathrm{state} \Rightarrow \alpha :: \mathrm{null}\ .$$

On this basis, we define $V((\mathrm{bool}_\perp)_\perp)$ as the HOL type for the OCL type `Boolean` by and define:

$$I[\![\mathtt{invalid} :: V(\alpha)]\!]\tau \equiv \mathrm{bot} \qquad I[\![\mathtt{null} :: V(\alpha)]\!]\tau \equiv \mathrm{null}$$
$$I[\![\mathtt{true} :: \mathtt{Boolean}]\!]\tau = \lfloor\lfloor \mathrm{true} \rfloor\rfloor \qquad I[\![\mathtt{false}]\!]\tau = \lfloor\lfloor \mathrm{false} \rfloor\rfloor$$

$$I[\![X\mathtt{.oclIsUndefined()}]\!]\tau = \\ (\mathrm{if}\ I[\![X]\!]\tau \in \{\mathrm{bot}, \mathrm{null}\}\ \mathrm{then}\ I[\![\mathtt{true}]\!]\tau\ \mathrm{else}\ I[\![\mathtt{false}]\!]\tau)$$

$$I[\![X\mathtt{.oclIsInvalid()}]\!]\tau = \\ (\mathrm{if}\ I[\![X]\!]\tau = \mathrm{bot}\ \mathrm{then}\ I[\![\mathtt{true}]\!]\tau\ \mathrm{else}\ I[\![\mathtt{false}]\!]\tau)$$

where $I[\![E]\!]$ is the semantic interpretation function commonly used in mathematical textbooks and τ stands for pairs of pre- and post state (σ, σ'). Due to the used style of semantic representation (a shallow embedding) I is in fact superfluous and defined semantically as the identity; in Isabelle theories, it is usually left out in definitions to pave the way for Isabelle to checks that the underlying equations are axiomatic definitions and therefore logically safe. For reasons of conciseness, we will write $\delta\ X$ for `not` X`.oclIsUndefined()` and $\upsilon\ X$ for `not` X`.oclIsInvalid()` throughout this paper.

On this basis, one can define the core logical operators `not` and `and` as follows:

$$
\begin{array}{lll}
I[\![\texttt{not}\ X]\!]\tau & = & (\text{case}\ I[\![X]\!]\tau\ \text{of} \\
 & & \quad \bot \qquad\quad \Rightarrow \bot \\
 & & \quad |\lfloor\bot\rfloor \qquad \Rightarrow \lfloor\bot\rfloor \\
 & & \quad |\lfloor\lfloor x\rfloor\rfloor \quad \Rightarrow \lfloor\lfloor\neg x\rfloor\rfloor) \\
I[\![X\ \texttt{and}\ Y]\!]\tau & = & (\text{case}\ I[\![X]\!]\tau\ \text{of} \\
 & & \quad \bot \qquad\quad \Rightarrow (\text{case}\ I[\![Y]\!]\tau\ \text{of} \\
 & & \qquad\qquad\qquad\quad \bot \qquad\qquad \Rightarrow \bot \\
 & & \qquad\qquad\qquad\quad |\lfloor\bot\rfloor \qquad\quad \Rightarrow \bot \\
 & & \qquad\qquad\qquad\quad |\lfloor\lfloor\text{true}\rfloor\rfloor \quad \Rightarrow \bot \\
 & & \qquad\qquad\qquad\quad |\lfloor\lfloor\text{false}\rfloor\rfloor \quad \Rightarrow \lfloor\lfloor\text{false}\rfloor\rfloor) \\
 & & \quad |\lfloor\bot\rfloor \qquad \Rightarrow (\text{case}\ I[\![Y]\!]\tau\ \text{of} \\
 & & \qquad\qquad\qquad\quad \bot \qquad\qquad \Rightarrow \bot \\
 & & \qquad\qquad\qquad\quad |\lfloor\bot\rfloor \qquad\quad \Rightarrow \lfloor\bot\rfloor \\
 & & \qquad\qquad\qquad\quad |\lfloor\lfloor\text{true}\rfloor\rfloor \quad \Rightarrow \lfloor\bot\rfloor \\
 & & \qquad\qquad\qquad\quad |\lfloor\lfloor\text{false}\rfloor\rfloor \quad \Rightarrow \lfloor\lfloor\text{false}\rfloor\rfloor) \\
 & & \quad |\lfloor\lfloor\text{true}\rfloor\rfloor \quad \Rightarrow (\text{case}\ I[\![Y]\!]\tau\ \text{of} \\
 & & \qquad\qquad\qquad\quad \bot \qquad\quad \Rightarrow \bot \\
 & & \qquad\qquad\qquad\quad |\lfloor\bot\rfloor \qquad \Rightarrow \lfloor\bot\rfloor \\
 & & \qquad\qquad\qquad\quad |\lfloor\lfloor y\rfloor\rfloor \quad \Rightarrow \lfloor\lfloor y\rfloor\rfloor) \\
 & & \quad |\lfloor\lfloor\text{false}\rfloor\rfloor \quad \Rightarrow \lfloor\lfloor\text{false}\rfloor\rfloor)
\end{array}
$$

These non-strict operations were used to define the other logical connectives in the usual classical way: X `or` $Y \equiv$ (`not` X) `and` (`not` Y) or X `implies` $Y \equiv$ (`not` X) `or` Y.

The default semantics for an OCL library operator is strict semantics; this means that the result of an operation f is invalid if one of its arguments is invalid. For a semantics comprising null, we suggest to stay conform to the standard and define the addition for integers as follows:

$$
\begin{array}{ll}
I[\![x\texttt{+}y]\!]\tau = & \text{if}\ I[\![\delta\ x]\!]\tau = \lfloor\lfloor\text{true}\rfloor\rfloor \wedge I[\![\delta\ y]\!]\tau = \lfloor\lfloor\text{true}\rfloor\rfloor \\
 & \text{then} \lfloor\lfloor\lceil\lceil I[\![x]\!]\tau\rceil\rceil + \lceil\lceil I[\![y]\!]\tau\rceil\rceil\rfloor\rfloor \\
 & \text{else}\ \bot
\end{array}
$$

where the operator "`+`" on the left-hand side of the equation denotes the OCL addition of type $[V((\text{int}_\bot)_\bot), V((\text{int}_\bot)_\bot)] \Rightarrow V((\text{int}_\bot)_\bot)$ while the "+" on the right-hand side of the equation of type $[\text{int}, \text{int}] \Rightarrow \text{int}$ denotes the integer-addition from the HOL library.

2.3.2. Logical Layer

The topmost goal of the logic for OCL is to define the *validity statement*:

$$(\sigma, \sigma') \models P\,,$$

where σ is the pre-state and σ' the post-state of the underlying system and P is a formula. Informally, a formula P is valid if and only if its evaluation in (σ, σ') (i. e., τ

for short) yields true. Formally this means:

$$\tau \models P \equiv \big(I[\![P]\!]\tau = \lfloor\lfloor \mathrm{true} \rfloor\rfloor\big)\,.$$

On this basis, classical, two-valued inference rules can be established for reasoning over the logical connective, the different notions of equality, definedness and validity. Generally speaking, rules over logical validity can relate bits and pieces in various OCL terms and allow—via strong logical equality discussed below—the replacement of semantically equivalent sub-expressions. The core inference rules are:

$$\tau \models \texttt{true} \quad \neg(\tau \models \texttt{false}) \quad \neg(\tau \models \texttt{invalid}) \quad \neg(\tau \models \texttt{null})$$
$$\tau \models \texttt{not}\ P \Longrightarrow \tau \neg \models P$$
$$\tau \models P\ \texttt{and}\ Q \Longrightarrow \tau \models P \qquad \tau \models P\ \texttt{and}\ Q \Longrightarrow \tau \models Q$$
$$\tau \models P \Longrightarrow (\texttt{if}\ P\ \texttt{then}\ B_1\ \texttt{else}\ B_2\ \texttt{endif})\tau = B_1\tau$$
$$\tau \models \texttt{not}\ P \Longrightarrow (\texttt{if}\ P\ \texttt{then}\ B_1\ \texttt{else}\ B_2\ \texttt{endif})\tau = B_2\tau$$
$$\tau \models P \Longrightarrow \tau \models \delta P \qquad \tau \models (\delta X) \Longrightarrow \tau \models \upsilon X$$

By the latter two properties it can be inferred that any valid property P (so for example: a valid invariant) is actually defined, which allows to infer for terms composed by strict operations that their arguments and finally the variables occurring in it are valid or defined.

We propose to distinguish the *strong logical equality* (written $\_ \triangleq \_$), which follows the general principle that "equals can be replaced by equals," from the *strict referential equality* (written $\_ \doteq \_$), which is an object-oriented concept that attempts to approximate and to implement the former. Strict referential equality, which is the default in the OCL language and is written simply `_ = _` in the standard, is an overloaded concept and has to be defined for each OCL type individually; for objects resulting from class definitions, it is implemented by simply comparing the references to the objects. In contrast, strong logical equality is a polymorphic concept which is defined once and for all by:

$$I[\![X \triangleq Y]\!]\tau \equiv \lfloor\lfloor I[\![X]\!]\tau = I[\![Y]\!]\tau \rfloor\rfloor$$

It enjoys nearly the laws of a congruence:

$$\tau \models (x \triangleq x)$$
$$\tau \models (x \triangleq y) \Longrightarrow \tau \models (y \triangleq x)$$
$$\tau \models (x \triangleq y) \Longrightarrow \tau \models (y \triangleq z) \Longrightarrow \tau \models (x \triangleq z)$$
$$\mathrm{cp}\, P \Longrightarrow \tau \models (x \triangleq y) \Longrightarrow \tau \models (P\,x) \Longrightarrow \tau \models (P\,y)$$

where the predicate cp stands for *context-passing*, a property that is characterized by $P(X)$ equals $\lambda\,\tau.\ P(\lambda\,\_.\ X\tau)\tau$. It means that the state tuple $\tau = (\sigma, \sigma')$ is passed unchanged from surrounding expressions to sub-expressions. it is true for all pure OCL expressions (but not arbitrary mixtures of OCL and HOL) in Featherweight OCL. The necessary side-calculus for establishing cp can be fully automated.

The logical layer of the Featherweight OCL rules gives also a means to convert an OCL formula living in its for-valued world into a representation that is classically two-valued and can be processed by standard SMT solvers such as **cvc3!** [?] or Z3 [14]. Delta-closure rules for all logical connectives have the following format, e. g.:

$$\tau \models \delta x \Longrightarrow (\tau \models \texttt{not}\, x) = (\neg(\tau \models x))$$

$$\tau \models \delta x \Longrightarrow \tau \models \delta y \Longrightarrow (\tau \models x\, \texttt{and}\, y) = (\tau \models x \wedge \tau \models y)$$

$$\begin{aligned} &\tau \models \delta x \Longrightarrow \tau \models \delta y \\ &\quad \Longrightarrow (\tau \models (x\, \texttt{implies}\, y)) = ((\tau \models x) \longrightarrow (\tau \models y)) \end{aligned}$$

Together with the general case-distinction

$$\tau \models \delta x \vee \tau \models x \triangleq \texttt{invalid} \vee \tau \models x \triangleq \texttt{null}\,,$$

which is possible for any OCL type, a case distinction on the variables in a formula can be performed; due to strictness rules, formulae containing somewhere a variable x that is known to be `invalid` or `null` reduce usually quickly to contradictions. For example, we can infer from an invariant $\tau \models x \doteq y\text{-}3$ that we have actually $\tau \models x \doteq y\text{-}3 \wedge \tau \models \delta x \wedge \tau \models \delta y$. We call the latter formula the δ-closure of the former. Now, we can convert a formula like $\tau \models x\text{>}0\ \text{or} 3{*}y\text{>}x{*}x$ into the equivalent formula $\tau \models x > 0 \vee \tau \models 3{*}y\text{>}x{*}x$ and thus internalize the OCL-logic into a classical (and more tool-conform) logic. This works—for the price of a potential, but due to the usually "rich" δ-closures of invariants rare—exponential blow-up of the formula for all OCL formulas.

2.3.3. Algebraic Layer

Based on the logical layer, we build a system with simpler rules which are amenable to automated reasoning. We restrict ourselves to pure equations on OCL expressions, where the used equality is the meta-(HOL-)equality.

Our denotational definitions on `not` and `and` can be re-formulated in the following ground equations:

$$
\begin{array}{cc}
\upsilon\ \texttt{invalid} = \texttt{false} & \upsilon\ \texttt{null} = \texttt{true} \\
\upsilon\ \texttt{true} = \texttt{true} & \upsilon\ \texttt{false} = \texttt{true} \\
\delta\ \texttt{invalid} = \texttt{false} & \delta\ \texttt{null} = \texttt{false} \\
\delta\ \texttt{true} = \texttt{true} & \delta\ \texttt{false} = \texttt{true} \\
\texttt{not invalid} = \texttt{invalid} & \texttt{not null} = \texttt{null} \\
\texttt{not true} = \texttt{false} & \texttt{not false} = \texttt{true} \\
(\texttt{null and true}) = \texttt{null} & (\texttt{null and false}) = \texttt{false} \\
(\texttt{null and null}) = \texttt{null} & (\texttt{null and invalid}) = \texttt{invalid} \\
(\texttt{false and true}) = \texttt{false} & (\texttt{false and false}) = \texttt{false} \\
(\texttt{false and null}) = \texttt{false} & (\texttt{false and invalid}) = \texttt{false} \\
(\texttt{true and true}) = \texttt{true} & (\texttt{true and false}) = \texttt{false} \\
(\texttt{true and null}) = \texttt{null} & (\texttt{true and invalid}) = \texttt{invalid}
\end{array}
$$

$$
\begin{array}{c}
(\texttt{invalid and true}) = \texttt{invalid} \\
(\texttt{invalid and false}) = \texttt{false} \\
(\texttt{invalid and null}) = \texttt{invalid} \\
(\texttt{invalid and invalid}) = \texttt{invalid}
\end{array}
$$

On this core, the structure of a conventional lattice arises:

$$
\begin{array}{cc}
X\ \texttt{and}\ X = X & X\ \texttt{and}\ Y = Y\ \texttt{and}\ X \\
\texttt{false and}\ X = \texttt{false} & X\ \texttt{and false} = \texttt{false} \\
\texttt{true and}\ X = X & X\ \texttt{and true} = X
\end{array}
$$

$$
X\ \texttt{and}\ (Y\ \texttt{and}\ Z) = X\ \texttt{and}\ Y\ \texttt{and}\ Z
$$

as well as the dual equalities for `or` and the De Morgan rules. This wealth of algebraic properties makes the understanding of the logic easier as well as automated analysis possible: it allows for, for example, computing a DNF of invariant systems (by clever term-rewriting techniques) which are a prerequisite for δ-closures.

The above equations explain the behavior for the most-important non-strict operations. The clarification of the exceptional behaviors is of key-importance for a semantic definition the standard and the major deviation point from HOL-OCL [7, 8], to Featherweight OCL as presented here. The standard expresses at many places that most operations are strict, i. e., enjoy the properties (exemplary for `_ + _`):

$$
\begin{array}{cc}
\texttt{invalid + } x = \texttt{invalid} & \texttt{x + invalid} = \texttt{invalid} \\
x\ \texttt{+ null} = \texttt{invalid} & \texttt{null + } x = \texttt{invalid}
\end{array}
$$

$$
\texttt{null.asType(}X\texttt{)} = \texttt{invalid}
$$

besides “classical” exceptional behavior:

$$\texttt{1 / 0} = \texttt{invalid} \qquad \texttt{1 / null} = \texttt{invalid}$$
$$\texttt{null->isEmpty()} = \texttt{true}$$

Moreover, there is also the proposal to use `null` as a kind of “don’t know” value for all strict operations, not only in the semantics of the logical connectives. Expressed in algebraic equations, this semantic alternative (this is *not* Featherweight OCL at present) would boil down to:

$$\begin{aligned} \texttt{invalid + } x &= \texttt{invalid} & x \texttt{ + invalid} &= \texttt{invalid} \\ x \texttt{ + null} &= \texttt{null} & \texttt{null + } x &= \texttt{null} \\ 1/0 &= \texttt{invalid} & 1/\texttt{null} &= \texttt{null} \\ \texttt{null->isEmpty()} &= \texttt{null} & \texttt{null.asType(}X\texttt{)} &= \texttt{null} \end{aligned}$$

While this is logically perfectly possible, while it can be argued that this semantics is “intuitive,” and although we do not expect a too heavy cost in deduction when computing δ-closures, we object that there are other, also “intuitive” interpretations that are even more wide-spread: In classical spreadsheet programs, for example, the semantics tend to interpret `null` (representing empty cells in a sheet) as the neutral element of the type, so `0` or the empty string, for example.[2] This semantic alternative (this is *not* Featherweight OCL at present) would yield:

$$\begin{aligned} \texttt{invalid + } x &= \texttt{invalid} & x \texttt{ + invalid} &= \texttt{invalid} \\ x \texttt{ + null} &= x & \texttt{null + } x &= x \\ 1/0 &= \texttt{invalid} & 1/\texttt{null} &= \texttt{invalid} \\ \texttt{null->isEmpty()} &= \texttt{true} & \texttt{null.asType(}X\texttt{)} &= \texttt{invalid} \end{aligned}$$

Algebraic rules are also the key for execution and compilation of Featherweight OCL

[2] In spreadsheet programs the interpretation of null varies from operation to operation; e. g., the `average` function treats `null` as non-existing value and not as `0`.

expressions. We derived, e. g.:

$$
\begin{aligned}
\delta\,\texttt{Set\{\}} &= \texttt{true}\\
\delta\,(X\texttt{->including(}x\texttt{)}) &= \delta X \texttt{ and } \delta x\\
\texttt{Set\{\}->includes(}x\texttt{)} &= (\texttt{if } \upsilon\, x \texttt{ then false}\\
&\qquad\qquad \texttt{else invalid endif})\\
(X\texttt{->including(}x\texttt{)->includes(}y\texttt{)}) &=\\
&(\texttt{if } \delta\, X\\
&\ \texttt{then}\ \ \texttt{if } x \doteq y\\
&\qquad\quad \texttt{then true}\\
&\qquad\quad \texttt{else}X\texttt{->includes(}y\texttt{)}\\
&\qquad\quad \texttt{endif}\\
&\ \texttt{else invalid}\\
&\ \texttt{endif})
\end{aligned}
$$

As `Set{1,2}` is only syntactic sugar for

```
Set{}->including(1)->including(2)
```

an expression like `Set{1,2}->includes(null)` becomes automatically decidable in Featherweight OCL by a combination of rewriting and code-generation and execution. The generated documentation from the theory files can thus be enriched by numerous "test-statements" like:

value ”$\tau \models (\texttt{Set}\{\texttt{Set}\{2, \texttt{null}\}\} \doteq \texttt{Set}\{\texttt{Set}\{\texttt{null}, 2\}\})$”

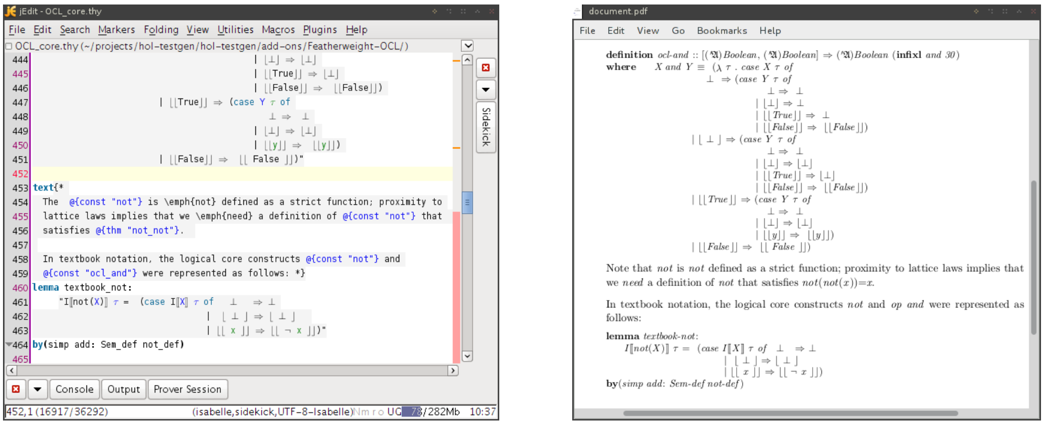
which have been machine-checked and which present a high-level and in our opinion fairly readable information for OCL tool manufactures and users.

2.4. A Machine-checked Annex A

Isabelle, as a framework for building formal tools [24], provides the means for generating *formal documents*. With formal documents we refer to documents that are machine-generated and ensure certain formal guarantees. In particular, all formal content (e. g., definitions, formulae, types) are checked for consistency during the document generation.

For writing documents, Isabelle supports the embedding of informal texts using a LaTeX-based markup language within the theory files. To ensure the consistency, Isabelle supports to use, within these informal texts, *antiquotations* that refer to the formal parts and that are checked while generating the actual document as **pdf!**. For example, in an informal text, the antiquotation @{thm ”not_not”} will instruct Isabelle to lock-up the (formally proven) theorem of name ocl_not_not and to replace the antiquotation with the actual theorem, i. e., `not (not x) = x`.

Figure 2.1 illustrates this approach: 2.1a shows the jEdit-based development environment of Isabelle with an excerpt of one of the core theories of Featherweight OCL. 2.1b



(a) The Isabelle jEdit environment.

(b) The generated formal document.

Figure 2.1.: Generating documents with guaranteed syntactical and semantical consistency.

shows the generated **pdf!** document where all antiquotations are replaced. Moreover, the document generation tools allows for defining syntactic sugar as well as skipping technical details of the formalization.

Thus, applying the Featherweight OCL approach to writing an updated Annex A that provides a formal semantics of the most fundamental concepts of OCL would ensure 1. that all formal context is syntactically correct and well-typed, and 2. all formal definitions and the derived logical rules are semantically consistent.

Part II.

A Formal Semantics of OCL 2.3 in Isabelle/HOL

2.5. Formal and Technical Background

2.5.1. Validity and Evaluations

The topmost goal of the formal semantics is to define the *validity statement*:

$$(\sigma, \sigma') \vDash P\,,$$

where σ is the pre-state and σ' the post-state of the underlying system and P is a Boolean expression (a *formula*). The assertion language of P is composed of 1) operators on built-in data structures such as Boolean or set, 2) operators of the user-defined data-model such as accessors, type-casts and tests, and 3) user-defined, side-effect-free methods. Informally, a formula P is valid if and only if its evaluation in the context (σ, σ') yields true. As all types in HOL-OCL are extended by the special element $\bot$ denoting undefinedness, we define formally:

$$(\sigma, \sigma') \vDash P \equiv \big(P(\sigma, \sigma') = \llcorner\mathrm{true}\lrcorner\big)\,.$$

Since all operators of the assertion language depend on the context (σ, σ') and result in values that can be $\bot$, all expressions can be viewed as *evaluations* from (σ, σ') to a type $\tau_\bot$. All types of expressions are of a form captured by

$$V(\alpha) := \mathrm{state} \times \mathrm{state} \Rightarrow \alpha_\bot\,,$$

where state stands for the system state and state $\times$ state describes the pair of pre-state and post-state and $\_ := \_$ denotes the type abbreviation.

The OCL semantics [19, Annex A] uses different interpretation functions for invariants and pre-conditions; we achieve their semantic effect by a syntactic transformation $\_{}_{\mathrm{pre}}$ which replaces all accessor functions $\_ .\,\mathrm{a}$ by their counterparts $\_ .\,\mathrm{a}$ `@pre`. For example, $(\mathit{self}\,.\,\mathrm{a} > 5)_{\mathrm{pre}}$ is just $(\mathit{self}\,.\,\mathrm{a}\ \texttt{@pre} > 5)$.

2.5.2. Strict Operations

An operation is called strict if it returns $\bot$ if one of its arguments is $\bot$. Most OCL operations are strict, e. g., the Boolean negation is formally presented as:

$$I[\![\texttt{not}\ X]\!]\tau \equiv \begin{cases} \llcorner\neg\ulcorner I[\![X]\!]\tau\urcorner\lrcorner & \text{if } I[\![X]\!]\tau \neq \bot, \\ \bot & \text{otherwise}\,. \end{cases}$$

where $\tau = (\sigma, \sigma')$ and $I[\![\_]\!]$ is a notation marking the HOL-OCL constructs to be defined. This notation is motivated by the definitions in the OCL standard [19]. In our case, $I[\![\_]\!]$ is just the identity, i. e., $I[\![X]\!] \equiv X$. These constructs, i. e., `not` _ are HOL functions (in this case of HOL type $V(\mathrm{bool}) \Rightarrow V(\mathrm{bool})$) that can be viewed as *transformers on evaluations*.

The binary case of the integer addition is analogous:

$$I[\![X + Y]\!]\,\tau \equiv \begin{cases} \llcorner\ulcorner I[\![X]\!]\,\tau\urcorner + \ulcorner I[\![Y]\!]\,\tau\urcorner\lrcorner & \text{if } I[\![X]\!]\,\tau \neq \bot \text{ and } I[\![Y]\!]\,\tau \neq \bot, \\ \bot & \text{otherwise}\,. \end{cases}$$

Here, the operator $\_ + \_$ on the right refers to the integer HOL operation with type $[\text{int}, \text{int}] \Rightarrow \text{int}$. The type of the corresponding strict HOL-OCL operator $\_+\_$ is $[V(\text{int}), V(\text{int})] \Rightarrow V(\text{int})$. A slight variation of this definition scheme is used for the operators on collection types such as HOL-OCL sets or sequences:

$$I[\![X\texttt{->union(}Y\texttt{)}]\!]\tau \equiv \begin{cases} S_{\llcorner}\ulcorner I[\![X]\!]\tau\urcorner \cup \ulcorner I[\![Y]\!]\tau\urcorner{}_{\lrcorner} & \text{if } I[\![X]\!]\tau{\neq}\bot \text{ and } I[\![Y]\!]\tau{\neq}\bot, \\ \bot & \text{otherwise.} \end{cases}$$

Here, S ("smash") is a function that maps a lifted set $\llcorner X \lrcorner$ to $\bot$ if and only if $\bot \in X$ and to the identity otherwise. Smashedness of collection types is the natural extension of the strictness principle for data structures.

Intuitively, the type expression $V(\tau)$ is a representation of the type that corresponds to the HOL-OCL type τ. We introduce the following type abbreviations:

$$\texttt{Boolean} := V(\text{bool})\,, \qquad \alpha\,\texttt{Set} := V(\alpha\,\text{set})\,,$$
$$\texttt{Integer} := V(\text{int})\,, \text{ and} \qquad \alpha\,\texttt{Sequence} := V(\alpha\,\text{list})\,.$$

The mapping of an expression `E` of HOL-OCL type `T` to a HOL expression E of HOL type T is injective and preserves well-typedness.

2.5.3. Boolean Operators

There is a small number of explicitly stated exceptions from the general rule that HOL-OCL operators are strict: the strong equality, the definedness operator and the logical connectives. As a prerequisite, we define the logical constants for truth, absurdity and undefinedness. We write these definitions as follows:

$$I[\![\texttt{true}]\!]\tau \equiv \llcorner\text{true}\lrcorner\,, \qquad I[\![\texttt{false}]\!]\tau \equiv \llcorner\text{false}\lrcorner\,, \text{ and} \qquad I[\![\texttt{invalid}]\!]\tau \equiv \bot\,.$$

HOL-OCL has a *strict equality* $\_ \doteq \_$. On the primitive types, it is defined similarly to the integer addition; the case for objects is discussed later. For logical purposes, we introduce also a *strong equality* $\_ \triangleq \_$ which is defined as follows:

$$I[\![X \triangleq Y]\!]\,\tau \equiv (I[\![X]\!]\,\tau = I[\![Y]\!]\,\tau)\,,$$

where the $\_ = \_$ operator on the right denotes the logical equality of HOL. The undefinedness test is defined by $X\,\texttt{.oclIsInvalid()} \equiv (X \triangleq \texttt{invalid})$. The strong equality can be used to state reduction rules like: $\tau \vDash (\texttt{invalid} \doteq X) \triangleq \texttt{invalid}$. The OCL standard requires a Strong Kleene Logic. In particular:

$$I[\![X \texttt{ and } Y]\!]\tau \equiv \begin{cases} \llcorner\ulcorner x\urcorner \wedge \ulcorner y\urcorner\lrcorner & \text{if } x{\neq}\bot \text{ and } y{\neq}\bot, \\ \llcorner\text{false}\lrcorner & \text{if } x = \llcorner\text{false}\lrcorner \text{ or } y = \llcorner\text{false}\lrcorner, \\ \bot & \text{otherwise}\,. \end{cases}$$

where $x = I[\![X]\!]\tau$ and $y = I[\![Y]\!]\tau$. The other Boolean connectives were just shortcuts: $X \texttt{ or } Y \equiv \texttt{not}\,(\texttt{not } X \texttt{ and not } Y)$ and $X \texttt{ implies } Y \equiv \texttt{not } X \texttt{ or } Y$.

2.5.4. Object-oriented Data Structures

Now we turn to several families of operations that the user implicitly defines when stating a class model as logical context of a specification. This is the part of the language where object-oriented features such as type casts, accessor functions, and tests for dynamic types come into play. Syntactically, a class model provides a collection of classes C, an inheritance relation $\_ < \_$ on classes and a collection of attributes A associated to classes. Semantically, a class model means a collection of accessor functions (denoted $\_.\mathrm{a} :: A \to B$ and $\_.\mathrm{a}$ `@pre` $:: A \to B$ for $\mathrm{a} \in A$ and $A, B \in C$), type casts that can change the static type of an object of a class (denoted $\_{}_{[\mathrm{C}]}$ of type $A \to C$) and dynamic type tests (denoted $\mathrm{isType}_{\mathrm{C}}\ \_$). A precise formal definition can be found in [11].

Class models: A simplified semantics.

In this section, we will have to clarify the notions of *object identifiers*, *object representations*, *class types* and *state*. We will give a formal model for this, that will satisfy all properties discussed in the subsequent section except one (see [9] for the complete model).

First, object identifiers are captured by an abstract type oid comprising countably many elements and a special element `nullid`. Second, object representations model "a piece of typed memory," i. e., a kind of record comprising administration information and the information for all attributes of an object; here, the primitive types as well as collections over them are stored directly in the object representations, class types and collections over them are represented by oid's (respectively lifted collections over them). Third, the class type C will be the type of such an object representation: $C := (\mathrm{oid} \times \mathrm{C_t} \times A_1 \times \cdots \times A_k)$ where a unique tag-type $\mathrm{C_t}$ (ensuring type-safety) is created for each class type, where the types $A_1, \ldots, A_k$ are the attribute types (including inherited attributes) with class types substituted by oid. The function OidOf projects the first component, the oid, out of an object representation. Fourth, for a class model M with the classes $C_1, \ldots, C_n$, we define states as partial functions from oid's to object representations satisfying a *state invariant* inv_σ:

$$\mathrm{state} := \{f :: \mathrm{oid} \rightharpoonup (C_1 + \ldots + C_n) \mid \mathrm{inv}_\sigma(f)\}$$

where $\mathrm{inv}_\sigma(f)$ states two conditions: 1) there is no object representation for `nullid`: `nullid` $\notin (\mathrm{dom}\ f)$. 2) there is a"one-to-one" correspondence between object representations and oid's: $\forall oid \in \mathrm{dom}\ f.\ oid = \mathrm{OidOf}\ \ulcorner f(oid) \urcorner$. The latter condition is also mentioned in [19, Annex A] and goes back to Mark Richters [22].

2.5.5. The Accessors

On states built over object universes, we can now define accessors, casts, and type tests of an object model. We consider the case of an attribute a of class C which has the simple class type D (not a primitive type, not a collection):

$$I[\![\mathit{self}\,.\,\mathrm{a}]\!](\sigma,\sigma') \equiv \begin{cases} \bot & \text{if } O = \bot \vee \mathrm{OidOf}\ulcorner O\urcorner \notin \mathrm{dom}\ \sigma' \\ \mathrm{get_D}\ u & \text{if } \sigma'(\mathrm{get_C}\ulcorner\sigma'(\mathrm{OidOf}\ulcorner O\urcorner)\urcorner.\,\mathrm{a}^{(0)}) = \llcorner u\lrcorner, \\ \bot & \text{otherwise.} \end{cases}$$

$$I[\![\mathit{self}\,.\,\mathrm{a@pre}]\!](\sigma,\sigma') \equiv \begin{cases} \bot & \text{if } O = \bot \vee \mathrm{OidOf}\ulcorner O\urcorner \notin \mathrm{dom}\ \sigma \\ \mathrm{get_D}\ u & \text{if } \sigma(\mathrm{get_C}\ulcorner\sigma(\mathrm{OidOf}\ulcorner O\urcorner)\urcorner.\,\mathrm{a}) = \llcorner u\lrcorner, \\ \bot & \text{otherwise.} \end{cases}$$

where $O = I[\![\mathit{self}]\!](\sigma,\sigma')$. Here, $\mathrm{get_D}$ is the projection function from the object universe to $D_\bot$, and $x.\mathrm{a}$ is the projection of the attribute from the class type (the Cartesian product). For simple class types, we have to evaluate expression *self*, get an object representation (or undefined), project the attribute, de-reference it in the pre or post state and project the class object from the object universe ($\mathrm{get_D}$ may yield $\bot$ if the element in the universe does not correspond to a D object representation.) In the case for a primitive type attribute, the de-referentiation step is left out, and in the case of a collection over class types, the elements of the collection have to be point-wise de-referenced and smashed.

In our model accessors always yield (type-safe) object representations; not oid's. Thus, a dangling reference, i. e., one that is *not* in dom σ, results in `invalid` (this is a subtle difference to [19, Annex A] where the undefinedness is detected one de-referentiation step later). The strict equality $\_ \doteq \_$ must be defined via OidOf when applied to objects. It satisfies ($\texttt{invalid} \doteq X$) $\triangleq$ `invalid`.

The definitions of casts and type tests can be found in [9], together with other details of the construction above and its automation in HOL-OCL.

2.6. A Proposal for an OCL 2.1 Semantics

In this section, we describe our OCL 2.1 semantics proposal as an increment to the OCL 2.0 semantics (underlying HOL-OCL and essentially formalizing [19, Annex A]). In later versions of the standard [20] the formal semantics appendix reappears although being incompatible with the normative parts of the standard. Not all rules shown here are formally proven; technically, these are informal proofs "with a glance" on the formal proofs shown in the previous section.

2.6.1. Revised Operations on Primitive Types

In UML, and since [20] in OCL, all primitive types comprise the `null`-element, modeling the possibility to be non-existent. From a functional language perspective, this corresponds to the view that each basic value is a type like `int option` as in SML. Technically, this results in lifting any primitive type twice:

$$\texttt{Integer} := V(\mathrm{int}_\bot)\,, \text{ etc.}$$

and basic operations have to take the null elements into account. The distinguishable undefined and null-elements were defined as follows:

$$I[\![\texttt{invalid}]\!]\tau \equiv \bot \quad\text{and}\quad I[\![\texttt{null}_{\texttt{Integer}}]\!]\tau \equiv \llcorner\bot\lrcorner\,.$$

An interpretation (consistent with [20]) is that $\texttt{null}_{\texttt{Integer}} + 3 = \texttt{invalid}$, and due to commutativity, we postulate $3+\texttt{null}_{\texttt{Integer}} = \texttt{invalid}$, too. The necessary modification of the semantic interpretation looks as follows:

$$I[\![X + Y]\!]\,\tau \equiv \begin{cases} \llcorner\llcorner\ulcorner\ulcorner x\urcorner\urcorner + \ulcorner\ulcorner y\urcorner\urcorner\lrcorner\lrcorner & \text{if } x{\neq}\bot,\ y{\neq}\bot,\ \ulcorner x\urcorner{\neq}\bot \text{ and } \ulcorner y\urcorner{\neq}\bot\\ \bot & \text{otherwise}\,. \end{cases}$$

where $x = I[\![X]\!]\,\tau$ and $y = I[\![Y]\!]\,\tau$. The resulting principle here is that operations on the primitive types Boolean, Integer, Real, and String treat null as invalid (except $\_ \doteq \_$, $\_$`.oclIsInvalid()`, $\_$`.oclIsUndefined()`, casts between the different representations of `null`, and type-tests).

This principle is motivated by our intuition that invalid represents known errors, and null-arguments of operations for Boolean, Integer, Real, and String belong to this class. Thus, we must also modify the logical operators such that $\texttt{null}_{\texttt{Boolean}}$ `and false` $\triangleq$ `false` and $\texttt{null}_{\texttt{Boolean}}$ `and true` $\triangleq \bot$.

With respect to definedness reasoning, there is a price to pay. For most basic operations we have the rule:

$$\texttt{not}\ (X + Y)\,\texttt{.oclIsInvalid()} \triangleq (\texttt{not}\ X\,\texttt{.oclIsUndefined()})\\ \texttt{and}\ (\texttt{not}\ Y\,\texttt{.oclIsUndefined()})$$

where the test x`.oclIsUndefined()` covers two cases: x`.oclIsInvalid()` and $x \doteq$ `null` (i. e., x is `invalid` or `null`). As a consequence, for the inverse case $(X{+}Y)$`.oclIsInvalid()`[3] there are four possible cases for the failure instead of two in the semantics described in [19]: each expression can be an erroneous `null`, or report an error. However, since all built-in OCL operations yield non-null elements (e. g., we have the rule `not` $(X + Y \doteq \texttt{null}_{\texttt{Integer}})$), a pre-computation can drastically reduce the number of cases occurring in expressions except for the base case of variables (e. g., parameters of operations and *self* in invariants). For these cases, it is desirable that implicit pre-conditions were generated as default, ruling out the `null` case. A convenient place for this are the multiplicities, which can be set to `1` (i. e., `1..1`) and will be interpreted as being non-null (see discussion in section 2.7 for more details).

Besides, the case for collection types is analogous: in addition to the invalid collection, there is a $\texttt{null}_{\mathrm{Set(T)}}$ collection as well as collections that contain null values (such as $\mathrm{Set}\{\texttt{null}_{\mathrm{T}}\}$) but never `invalid`.

[3] The same holds for $(X + Y)$`.oclIsUndefined()`.

2.6.2. Null in Class Types

It is a viable option to rule out undefinedness in object-graphs *as such*. The essential source for such undefinedness are oid's which do not occur in the state, i. e., which represent "dangling references." Ruling out undefinedness as result of object accessors would correspond to a world where an accessor is always set explicitly to `null` or to a defined object; in a programming language without explicit deletion and where constructors always initialize their arguments (e. g., Spec# [2]), this may suffice. Semantically, this can be modeled by strengthening the state invariant inv_σ by adding clauses that state that in each object representation all oid's are either `nullid` or element of the domain of the state.

We deliberately decided against this option for the following reasons:

1. *methodologically* we do not like to constrain the semantics of OCL without clear reason; in particular, "dangling references" exist in C and C++ programs and it might be necessary to write contracts for them, and
2. *semantically*, the condition "no dangling references" can only be formulated with the complete knowledge of all classes and their layout in form of object representations. This restricts the OCL semantics to a closed world model.[4]

We can model `null`-elements as object-representations with `nullid` as their oid:

1 (Representation of `null`-Elements) *Let C_i be a class type with the attributes $A_1, \ldots, A_n$. Then we define its null object representation by:*

$$I[\![\mathtt{null}_{Ci}]\!]\tau \equiv \llcorner(\mathtt{nullid}, \mathrm{arb}_t, a_1, \ldots, a_n)\lrcorner$$

where the a_i are $\perp$ for primitive types and collection types, and `nullid` *for simple class types.* arb_t *is an arbitrary underspecified constant of the tag-type.*

Due to the outermost lifting, the null object representation is a defined value, and due to its special reference `nullid` and the state invariant, it is a typed value not "living" in the state. The $\mathtt{null}_\mathrm{T}$-elements are not equal, but isomorphic: Each type, has its own unique $\mathtt{null}_\mathrm{T}$-element; they can be mapped, i. e., casted, isomorphic to each other. In HOL-OCL, we can overload constants by parametrized polymorphism which allows us to drop the index in this environment.

The referential strict equality allows us to write *self* $\doteq$ `null` in OCL. Recall that $\_ \doteq \_$ is based on the projection OidOf from object-representations.

[4]In our presentation, the definition of `state` in ?? assumes a closed world. This limitation can be easily overcome by leaving "polymorphic holes" in our object representation universe, i. e., by extending the type sum in the state definition to $C_1 + \cdots + C_n + \alpha$. The details of the management of universe extensions are involved, but implemented in HOL-OCL (see [9] for details). However, these constructions exclude knowing the set of sub-oid's in advance.

2.6.3. Revised Accessors

The modification of the accessor functions is now straight-forward:

$$I[\![obj.\,\mathrm{a}]\!](\sigma,\sigma') \equiv \begin{cases} \bot & \text{if } I[\![obj]\!](\sigma,\sigma') = \bot \vee \mathrm{OidOf}\ulcorner I[\![obj]\!](\sigma,\sigma')\urcorner \notin \mathrm{dom}\ \sigma' \\ \texttt{null}_{\mathrm{D}} & \text{if } \mathrm{get}_{\mathrm{C}}\ulcorner\sigma'(\mathrm{OidOf}\ulcorner I[\![obj]\!](\sigma,\sigma')\urcorner)\urcorner.\,\mathrm{a}^{(0)} = \texttt{nullid} \\ \mathrm{get}_{\mathrm{D}}\ u & \text{if } \sigma'(\mathrm{get}_{\mathrm{C}}\ulcorner\sigma'(\mathrm{OidOf}\ulcorner I[\![obj]\!](\sigma,\sigma')\urcorner)\urcorner.\,\mathrm{a}^{(0)}) = \llcorner u \lrcorner, \\ \bot & \text{otherwise.} \end{cases}$$

The definitions for type-cast and dynamic type test—which are not explicitly shown in this paper, see [9] for details—can be generalized accordingly. In the sequel, we will discuss the resulting properties of these modified accessors.

All functions of the induced signature are strict. This means that this holds for accessors, casts and tests, too:

$$\texttt{invalid.}\,\mathrm{a} \triangleq \texttt{invalid} \qquad \texttt{invalid}_{[\mathrm{C}]} \triangleq \texttt{invalid}$$
$$\mathrm{isType}_{\mathrm{C}}\,\texttt{invalid} \triangleq \texttt{invalid}$$

Casts on `null` are always valid, since they have an individual dynamic type and can be casted to any other null-element due to their isomorphism.

$$\texttt{null}_{\mathrm{A}}.\,\mathrm{a} \triangleq \texttt{invalid} \qquad \texttt{null}_{\mathrm{A}[\mathrm{B}]} \triangleq \texttt{null}_{\mathrm{B}}$$
$$\mathrm{isType}_{\mathrm{A}}\,\texttt{null}_{\mathrm{A}} \triangleq \texttt{true}$$

for all attributes a and classes A, B, C where $C < B < A$. These rules are further exceptions from the standard's general rule that `null` may never be passed as first ("*self*") argument.

2.6.4. Other Operations on States

Defining `_.allInstances()` is straight-forward; the only difference is the property T `.allInstances()->excludes(null)` which is a consequence of the fact that `null`'s are values and do not "live" in the state. In our semantics which admits states with "dangling references," it is possible to define a counterpart to `_.oclIsNew()` called `_.oclIsDeleted()` which asks if an object id (represented by an object representation) is contained in the pre-state, but not the post-state.

OCL does not guarantee that an operation only modifies the path-expressions mentioned in the postcondition, i. e., it allows arbitrary relations from pre-states to post-states. This framing problem is well-known (one of the suggested solutions is [15]). We define

```
(S:Set(OclAny))->modifiedOnly():Boolean
```

where `S` is a set of object representations, encoding a set of oid's. The semantics of this operator is defined such that for any object whose oid is *not* represented in `S` and that is defined in pre and post state, the corresponding object representation will not change in the state transition:

$$I[\![X\texttt{->modifiedOnly()}]\!](\sigma,\sigma') \equiv \begin{cases} \bot & \text{if } X' = \bot \\ \llcorner \forall i{\in}M.\ \sigma\ i = \sigma'\ i \lrcorner & \text{otherwise}\,. \end{cases}$$

where $X' = I[\![X]\!](\sigma,\sigma')$ and $M = (\mathrm{dom}\ \sigma \cap \mathrm{dom}\ \sigma') - \{\mathrm{OidOf}\ x |\ x \in \ulcorner X' \urcorner\}$. Thus, if we require in a postcondition `Set{}->modifiedOnly()` and exclude via `_.oclIsNew()` and `_.oclIsDeleted()` the existence of new or deleted objects, the operation is a query in the sense of the OCL standard, i. e., the `isQuery` property is true. So, whenever we have $\tau \models X$`->modifiedOnly()` and $\tau \models X$`->excludes`$(s.a)$, we can infer that $\tau \models s.a = s.a$ `@pre` (if they are valid).

2.7. Attribute Values

Depending on the specified multiplicity, the evaluation of an attribute can yield a value or a collection of values. A multiplicity defines a lower bound as well as a possibly infinite upper bound on the cardinality of the attribute's values.

2.7.1. Single-Valued Attributes

If the upper bound specified by the attribute's multiplicity is one, then an evaluation of the attribute yields a single value. Thus, the evaluation result is not a collection. If the lower bound specified by the multiplicity is zero, the evaluation is not required to yield a non-null value. In this case an evaluation of the attribute can return `null` to indicate an absence of value.

To facilitate accessing attributes with multiplicity `0..1`, the OCL standard states that single values can be used as sets by calling collection operations on them. This implicit conversion of a value to a `Set` is not defined by the standard. We argue that the resulting set cannot be constructed the same way as when evaluating a `Set` literal. Otherwise, `null` would be mapped to the singleton set containing `null`, but the standard demands that the resulting set is empty in this case. The conversion should instead be defined as follows:

```
context OclAny::asSet():T
  post: if self ≐ null then result ≐ Set{}
        else result ≐ Set{self} endif
```

2.7.2. Collection-Valued Attributes

If the upper bound specified by the attribute's multiplicity is larger than one, then an evaluation of the attribute yields a collection of values. This raises the question whether `null` can belong to this collection. The OCL standard states that `null` can be owned by collections. However, if an attribute can evaluate to a collection containing `null`, it is not clear how multiplicity constraints should be interpreted for this attribute. The question arises whether the `null` element should be counted or not when determining the cardinality of the collection. Recall that `null` denotes the absence of value in the case of a cardinality upper bound of one, so we would assume that `null` is not counted. On the other hand, the operation `size` defined for collections in OCL does count `null`.

We propose to resolve this dilemma by regarding multiplicities as optional. This point of view complies with the UML standard, that does not require lower and upper bounds to be defined for multiplicities.[5] In case a multiplicity is specified for an attribute, i. e., a lower and an upper bound are provided, we require any collection the attribute evaluates to to not contain `null`. This allows for a straightforward interpretation of the multiplicity

[5] We are however aware that a well-formedness rule of the UML standard does define a default bound of one in case a lower or upper bound is not specified.

constraint. If bounds are not provided for an attribute, we consider the attribute values to not be restricted in any way. Because in particular the cardinality of the attribute's values is not bounded, the result of an evaluation of the attribute is of collection type. As the range of values that the attribute can assume is not restricted, the attribute can evaluate to a collection containing `null`. The attribute can also evaluate to `invalid`. Allowing multiplicities to be optional in this way gives the modeler the freedom to define attributes that can assume the full ranges of values provided by their types. However, we do not permit the omission of multiplicities for association ends, since the values of association ends are not only restricted by multiplicities, but also by other constraints enforcing the semantics of associations. Hence, the values of association ends cannot be completely unrestricted.

2.7.3. The Precise Meaning of Multiplicity Constraints

We are now ready to define the meaning of multiplicity constraints by giving equivalent invariants written in OCL. Let `a` be an attribute of a class `C` with a multiplicity specifying a lower bound m and an upper bound n. Then we can define the multiplicity constraint on the values of attribute `a` to be equivalent to the following invariants written in OCL:

```
context C inv lowerBound: a->size() >= m
          inv upperBound: a->size() <= n
          inv notNull: not a->includes(null)
```

If the upper bound n is infinite, the second invariant is omitted. For the definition of these invariants we are making use of the conversion of single values to sets described in subsection 2.7.1. If $n \leq 1$, the attribute `a` evaluates to a single value, which is then converted to a `Set` on which the `size` operation is called.

If a value of the attribute `a` includes a reference to a non-existent object, the attribute call evaluates to `invalid`. As a result, the entire expressions evaluate to `invalid`, and the invariants are not satisfied. Thus, references to non-existent objects are ruled out by these invariants. We believe that this result is appropriate, since we argue that the presence of such references in a system state is usually not intended and likely to be the result of an error. If the modeler wishes to allow references to non-existent objects, she can make use of the possibility described above to omit the multiplicity.

3. Part I: Core Definitions

theory
  OCL-core
imports
  Main
begin

3.1. Preliminaries

3.1.1. Notations for the option type

First of all, we will use a more compact notation for the library option type which occur all over in our definitions and which will make the presentation more "textbook"-like:

notation *Some* $(\lfloor(\text{-})\rfloor)$
notation *None* $(\bot)$

The following function (corresponding to *the* in the Isabelle/HOL library) is defined as the inverse of the injection *Some*.

fun $\quad$ *drop* :: $'\alpha$ *option* $\Rightarrow$ $'\alpha$ $(\lceil(\text{-})\rceil)$
where *drop-lift*[*simp*]: $\lceil\lfloor v\rfloor\rceil = v$

3.1.2. Minimal Notions of State and State Transitions

Next we will introduce the foundational concept of an object id (oid), which is just some infinite set.

In order to assure executability of as much as possible formulas, we fixed the type of object id's to just natural numbers.

type-synonym *oid* = *nat*

We refrained from the alternative:

type_synonym oid = ind

which is slightly more abstract but non-executable.

States are just a partial map from oid's to elements of an object universe $'\mathfrak{A}$, and state transitions pairs of states...

record $('\mathfrak{A})$*state* =
  heap :: *oid* $\rightharpoonup$ $'\mathfrak{A}$
  assocs :: *oid* $\rightharpoonup$ (*oid* $\times$ *oid*) *list*

type-synonym $('\mathfrak{A})$*st* = $'\mathfrak{A}$ *state* $\times$ $'\mathfrak{A}$ *state*

3.1.3. Prerequisite: An Abstract Interface for OCL Types

In order to have the possibility to nest collection types, such that we can give semantics to expressions like $Set\{Set\{\mathbf{2}\},null\}$, it is necessary to introduce a uniform interface for types having the *invalid* (= bottom) element. The reason is that we impose a data-invariant on raw-collection `types_code` which assures that the *invalid* element is not allowed inside the collection; all raw-collections of this form were identified with the *invalid* element itself. The construction requires that the new collection type is uncomparable with the raw-types (consisting of nested option type constructions), such that the data-invariant must be expressed in terms of the interface. In a second step, our base-types will be shown to be instances of this interface.

This uniform interface consists in a type class requiring the existence of a bot and a null element. The construction proceeds by abstracting the null (which is defined by $\lfloor \perp \rfloor$ on $'a\ option\ option$ to a null - element, which may have an abritrary semantic structure, and an undefinedness element $\perp$ to an abstract undefinedness element *bot* (also written $\perp$ whenever no confusion arises). As a consequence, it is necessary to redefine the notions of invalid, defined, valuation etc. on top of this interface.

This interface consists in two abstract type classes *bot* and *null* for the class of all types comprising a bot and a distinct null element.

instance *option* :: (*plus*) *plus* **by** *intro-classes*
instance *fun* :: (*type*, *plus*) *plus* **by** *intro-classes*

class *bot* =
  fixes $bot :: 'a$
  assumes *nonEmpty* : $\exists\ x.\ x \neq bot$

class *null* = *bot* +
  fixes $null :: 'a$
  assumes *null-is-valid* : $null \neq bot$

3.1.4. Accomodation of Basic Types to the Abstract Interface

In the following it is shown that the option-option type type is in fact in the *null* class and that function spaces over these classes again "live" in these classes. This motivates the default construction of the semantic domain for the basic types (Boolean, Integer, Reals, ...).

instantiation *option* :: (*type*)*bot*
begin
  definition *bot-option-def*: $(bot{::}'a\ option) \equiv (None{::}'a\ option)$
  instance proof show $\exists x{::}'a\ option.\ x \neq bot$
      by(*rule-tac x=Some x* **in** *exI*, *simp add*:*bot-option-def*)
    qed
end

instantiation *option* :: (*bot*)*null*
begin
definition *null-option-def*: ($null::'a::bot\ option$) $\equiv$ $\lfloor$ *bot* $\rfloor$
instance proof **show** ($null::'a::bot\ option$) $\neq$ *bot*
by(*simp add*:*null-option-def bot-option-def*)
qed
end

instantiation *fun* :: (*type*,*bot*) *bot*
begin
definition *bot-fun-def*: *bot* $\equiv$ (λ *x*. *bot*)

instance proof **show** $\exists\,(x::'a \Rightarrow 'b).\ x \neq bot$
apply(*rule-tac x*=λ -. (*SOME y*. *y* $\neq$ *bot*) **in** *exI*, *auto*)
apply(*drule-tac x*=*x* **in** *fun-cong*,*auto simp*:*bot-fun-def*)
apply(*erule contrapos-pp*, *simp*)
apply(*rule some-eq-ex*[*THEN iffD2*])
apply(*simp add*: *nonEmpty*)
done
qed
end

instantiation *fun* :: (*type*,*null*) *null*
begin
definition *null-fun-def*: ($null::'a \Rightarrow 'b::null$) $\equiv$ (λ *x*. *null*)

instance proof
show ($null::'a \Rightarrow 'b::null$) $\neq$ *bot*
apply(*auto simp*: *null-fun-def bot-fun-def*)
apply(*drule-tac x*=*x* **in** *fun-cong*)
apply(*erule contrapos-pp*, *simp add*: *null-is-valid*)
done
qed
end

A trivial consequence of this adaption of the interface is that abstract and concrete versions of null are the same on base types (as could be expected).

3.1.5. The Semantic Space of OCL Types: Valuations.

Valuations are now functions from a state pair (built upon data universe $'\mathfrak{A}$) to an arbitrary null-type (i.e. containing at least a destinguished *null* and *invalid* element.

type-synonym $('\mathfrak{A},'\alpha)\ val = '\mathfrak{A}\ st \Rightarrow '\alpha::null$

The definitions for the constants and operations based on valuations will be geared towards a format that Isabelle can check to be a "conservative" (i.e. logically safe) axiomatic definition. By introducing an explicit interpretation function (which happens to be defined just as the identity since we are using a shallow embedding of OCL into HOL), all these definions can be rewritten into the conventional semantic "textbook" format as follows:

definition *Sem* :: $'a \Rightarrow 'a$ ($I[\![\text{-}]\!]$)
where $I[\![x]\!] \equiv x$

As a consequence of semantic domain definition, any OCL type will have the two semantic constants *invalid* (for exceptional, aborted computation) and *null*; the latter, however is either defined

definition *invalid* :: $('\mathfrak{A},'\alpha::bot)\ val$
where $invalid \equiv \lambda\ \tau.\ bot$

This conservative Isabelle definition of the polymorphic constant *invalid* is equivalent with the textbook definition:

lemma *invalid-def-textbook*: $I[\![invalid]\!]\tau = bot$
by(*simp add*: *invalid-def Sem-def*)

Note that the definition :

```
definition null    :: "('\<AA>,'\<alpha>::null) val"
where     "null    \<equiv> \<lambda> \<tau>. null"
```

is not necessary since we defined the entire function space over null types again as null-types; the crucial definition is $null \equiv \lambda x.\ null$. Thus, the polymporhic constant *null* is simply the result of a general type class construction. Nevertheless, we can derive the semantic textbook definition for the OCL null constant based on the abstract null:

lemma *null-def-textbook*: $I[\![null::('\mathfrak{A},'\alpha::null)\ val]\!]\ \tau = (null::'\alpha::null)$
by(*simp add*: *null-fun-def Sem-def*)

3.2. Boolean Type and Logic

The semantic domain of the (basic) boolean type is now defined as standard: the space of valuation to *bool option option*:

type-synonym $('\mathfrak{A})Boolean = ('\mathfrak{A},bool\ option\ option)\ val$

3.2.1. Basic Constants

lemma *bot-Boolean-def* : $(bot::('\mathfrak{A})Boolean) = (\lambda\ \tau.\ \bot)$
by(*simp add*: *bot-fun-def bot-option-def*)

lemma *null-Boolean-def* : $(null::('\mathfrak{A})Boolean) = (\lambda\ \tau.\ \lfloor\bot\rfloor)$
by(*simp add*: *null-fun-def null-option-def bot-option-def*)

definition *true* :: $('\mathfrak{A})Boolean$

where $true \equiv \lambda\ \tau.\ \lfloor\lfloor True \rfloor\rfloor$

definition $false :: ('\mathfrak{A})Boolean$
where $false \equiv\ \lambda\ \tau.\ \lfloor\lfloor False \rfloor\rfloor$

lemma *bool-split*: $X\ \tau = invalid\ \tau \lor X\ \tau = null\ \tau \lor$
$X\ \tau = true\ \tau \quad \lor X\ \tau = false\ \tau$
apply(*simp add*: *invalid-def null-def true-def false-def*)
apply(*case-tac X* τ,*simp-all add*: *null-fun-def null-option-def bot-option-def*)
apply(*case-tac a*,*simp*)
apply(*case-tac aa*,*simp*)
apply *auto*
done

lemma [*simp*]: $false\ (a,\ b) = \lfloor\lfloor False \rfloor\rfloor$
by(*simp add*:*false-def*)

lemma [*simp*]: $true\ (a,\ b) = \lfloor\lfloor True \rfloor\rfloor$
by(*simp add*:*true-def*)

lemma *true-def-textbook*: $I[\![true]\!]\ \tau = \lfloor\lfloor True \rfloor\rfloor$
by(*simp add*: *Sem-def true-def*)

lemma *false-def-textbook*: $I[\![false]\!]\ \tau = \lfloor\lfloor False \rfloor\rfloor$
by(*simp add*: *Sem-def false-def*)

Summary:

Name	Theorem
invalid-def-textbook	$I[\![invalid]\!]\ ?\tau = OCL\text{-}core.bot\text{-}class.bot$
null-def-textbook	$I[\![null]\!]\ ?\tau = null$
true-def-textbook	
	$I[\![true]\!]\ ?\tau = \lfloor\lfloor True \rfloor\rfloor$
false-def-textbook	$I[\![false]\!]\ ?\tau = \lfloor\lfloor False \rfloor\rfloor$

Table 3.1.: Basic semantic constant definitions of the logic (except *null*)

3.2.2. Fundamental Predicates I: Validity and Definedness

However, this has also the consequence that core concepts like definedness, validness and even cp have to be redefined on this type class:

definition *valid* :: $('\mathfrak{A},'a::null)val \Rightarrow ('\mathfrak{A})Boolean$ (υ - [*100*]*100*)
where $\upsilon\ X \equiv\ \lambda\ \tau\ .$ *if* $X\ \tau = bot\ \tau$ *then false* τ *else true* τ

lemma *valid1*[*simp*]: υ *invalid* = *false*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def*
invalid-def true-def false-def)

lemma *valid2*[*simp*]: υ *null* = *true*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def null-is-valid*
null-fun-def invalid-def true-def false-def)

lemma *valid3*[*simp*]: υ *true* = *true*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def null-is-valid*
null-fun-def invalid-def true-def false-def)

lemma *valid4*[*simp*]: υ *false* = *true*
by(*rule ext*,*simp add*: *valid-def bot-fun-def bot-option-def null-is-valid*
null-fun-def invalid-def true-def false-def)

lemma *cp-valid*: $(\upsilon\ X)\ \tau = (\upsilon\ (\lambda\ \text{-}.\ X\ \tau))\ \tau$
by(*simp add*: *valid-def*)

definition *defined* :: $('\mathfrak{A},'a::null)val \Rightarrow ('\mathfrak{A})Boolean$ (δ - [*100*]*100*)
where $\delta\ X \equiv\ \lambda\ \tau\ .$ *if* $X\ \tau = bot\ \tau\ \vee X\ \tau = null\ \tau$ *then false* τ *else true* τ

The generalized definitions of invalid and definedness have the same properties as the old ones :

lemma *defined1*[*simp*]: δ *invalid* = *false*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def*
null-def invalid-def true-def false-def)

lemma *defined2*[*simp*]: δ *null* = *false*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def*
null-def null-option-def null-fun-def invalid-def true-def false-def)

lemma *defined3*[*simp*]: δ *true* = *true*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def null-is-valid null-option-def*
null-fun-def invalid-def true-def false-def)

lemma *defined4*[*simp*]: δ *false* = *true*
by(*rule ext*,*simp add*: *defined-def bot-fun-def bot-option-def null-is-valid null-option-def*
null-fun-def invalid-def true-def false-def)

lemma *defined5*[*simp*]: $\delta\ \delta\ X$ = *true*

by(*rule ext*,
  auto simp: *defined-def true-def false-def*
    bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *defined6*[*simp*]: $\delta\ \upsilon\ X = true$
  by(*rule ext*,
  auto simp: *valid-def defined-def true-def false-def*
    bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *valid5*[*simp*]: $\upsilon\ \upsilon\ X = true$
  by(*rule ext*,
  auto simp: *valid-def true-def false-def*
    bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *valid6*[*simp*]: $\upsilon\ \delta\ X = true$
  by(*rule ext*,
  auto simp: *valid-def defined-def true-def false-def*
    bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *cp-defined*:$(\delta\ X)\tau = (\delta\ (\lambda\ \text{-.}\ X\ \tau))\ \tau$
by(*simp add*: *defined-def*)

The definitions above for the constants *defined* and *valid* can be rewritten into the conventional semantic "textbook" format as follows:

lemma *defined-def-textbook*: $I[\![\delta(X)]\!]\ \tau = (if\ I[\![X]\!]\ \tau = I[\![bot]\!]\ \tau\ \lor I[\![X]\!]\ \tau = I[\![null]\!]\ \tau$
  $then\ I[\![false]\!]\ \tau$
  $else\ I[\![true]\!]\ \tau)$
by(*simp add*: *Sem-def defined-def*)

lemma *valid-def-textbook*: $I[\![\upsilon(X)]\!]\ \tau = (if\ I[\![X]\!]\ \tau = I[\![bot]\!]\ \tau$
  $then\ I[\![false]\!]\ \tau$
  $else\ I[\![true]\!]\ \tau)$
by(*simp add*: *Sem-def valid-def*)

Summary: These definitions lead quite directly to the algebraic laws on these predicates:

Name	Theorem
defined-def-textbook	$I[\![\delta\ X]\!]\ \tau = (if\ I[\![X]\!]\ \tau = I[\![OCL\text{-}core.bot\text{-}class.bot]\!]\ \tau \lor I[\![X]\!]\ \tau = I[\![null]\!]\ \tau\ then\ I[\![fals$
valid-def-textbook	$I[\![\upsilon\ X]\!]\ \tau = (if\ I[\![X]\!]\ \tau = I[\![OCL\text{-}core.bot\text{-}class.bot]\!]\ \tau\ then\ I[\![false]\!]\ \tau\ else\ I[\![true]\!]\ \tau)$

Table 3.2.: Basic predicate definitions of the logic.)

Name	Theorem
defined1	δ *invalid* $=$ *false*
defined2	δ *null* $=$ *false*
defined3	δ *true* $=$ *true*
defined4	δ *false* $=$ *true*
defined5	$\delta\ \delta\ ?X = true$
defined6	$\delta\ \upsilon\ ?X = true$

Table 3.3.: Laws of the basic predicates of the logic.)

3.2.3. Fundamental Predicates II: Logical (Strong) Equality

Note that we define strong equality extremely generic, even for types that contain an *null* or $\perp$ element:

definition *StrongEq*::$['\mathfrak{A}\ st \Rightarrow '\alpha, '\mathfrak{A}\ st \Rightarrow '\alpha] \Rightarrow ('\mathfrak{A})Boolean$ (**infixl** $\triangleq$ *30*)
where $X \triangleq Y \equiv \lambda\,\tau.\ \lfloor\lfloor X\ \tau = Y\ \tau\ \rfloor\rfloor$

Equality reasoning in OCL is not humpty dumpty. While strong equality is clearly an equivalence:

lemma *StrongEq-refl* [*simp*]: $(X \triangleq X) = true$
by(*rule ext*, *simp add*: *null-def invalid-def true-def false-def StrongEq-def*)

lemma *StrongEq-sym*: $(X \triangleq Y) = (Y \triangleq X)$
by(*rule ext*, *simp add*: *eq-sym-conv invalid-def true-def false-def StrongEq-def*)

lemma *StrongEq-trans-strong* [*simp*]:
assumes *A*: $(X \triangleq Y) = true$
and *B*: $(Y \triangleq Z) = true$
shows $(X \triangleq Z) = true$
apply(*insert A B*) **apply**(*rule ext*)
apply(*simp add*: *null-def invalid-def true-def false-def StrongEq-def*)
apply(*drule-tac x=x* **in** *fun-cong*)+
by *auto*

... it is only in a limited sense a congruence, at least from the point of view of this semantic theory. The point is that it is only a congruence on OCL- expressions, not arbitrary HOL expressions (with which we can mix Essential OCL expressions. A semantic — not syntactic — characterization of OCL-expressions is that they are *context-passing* or *context-invariant*, i.e. the context of an entire OCL expression, i.e. the pre-and post-state it referes to, is passed constantly and unmodified to the sub-expressions, i.e. all sub-expressions inside an OCL expression refer to the same context. Expressed formally, this boils down to:

lemma *StrongEq-subst* :
assumes *cp*: $\bigwedge X.\ P(X)\tau = P(\lambda\ \text{-}.\ X\ \tau)\tau$
and *eq*: $(X \triangleq Y)\tau = true\ \tau$

shows $(P\ X \triangleq P\ Y)\tau = \mathit{true}\ \tau$
apply(*insert cp eq*)
apply(*simp add*: *null-def invalid-def true-def false-def StrongEq-def*)
apply(*subst cp*[*of X*])
apply(*subst cp*[*of Y*])
by *simp*

3.2.4. Fundamental Predicates III

And, last but not least,

lemma *defined7*[*simp*]: $\delta\ (X \triangleq Y) = \mathit{true}$
by(*rule ext*,
auto simp: *defined-def* *true-def false-def StrongEq-def*
bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *valid7*[*simp*]: $\upsilon\ (X \triangleq Y) = \mathit{true}$
by(*rule ext*,
auto simp: *valid-def true-def false-def StrongEq-def*
bot-fun-def bot-option-def null-option-def null-fun-def)

lemma *cp-StrongEq*: $(X \triangleq Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau) \triangleq (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
by(*simp add*: *StrongEq-def*)

The semantics of strict equality of OCL is constructed by overloading: for each base type, there is an equality.

3.2.5. Logical Connectives and their Universal Properties

It is a design goal to give OCL a semantics that is as closely as possible to a "logical system" in a known sense; a specification logic where the logical connectives can not be understood other that having the truth-table aside when reading fails its purpose in our view.

Practically, this means that we want to give a definition to the core operations to be as close as possible to the lattice laws; this makes also powerful symbolic normalizations of OCL specifications possible as a pre-requisite for automated theorem provers. For example, it is still possible to compute without any definedness- and validity reasoning the DNF of an OCL specification; be it for test-case generations or for a smooth transition to a two-valued representation of the specification amenable to fast standard SMT-solvers, for example.

Thus, our representation of the OCL is merely a 4-valued Kleene-Logics with *invalid* as least, *null* as middle and *true* resp. *false* as unrelated top-elements.

definition $\mathit{not} :: ('\mathfrak{A})\mathit{Boolean} \Rightarrow ('\mathfrak{A})\mathit{Boolean}$
where $\mathit{not}\ X \equiv\ \lambda\ \tau\ .\ \mathit{case}\ X\ \tau\ \mathit{of}$

$$\begin{array}{ll} \quad\bot & \Rightarrow \bot \\ \mid \lfloor \bot \rfloor & \Rightarrow \lfloor \bot \rfloor \\ \mid \lfloor\lfloor x \rfloor\rfloor & \Rightarrow \lfloor\lfloor \neg\ x \rfloor\rfloor \end{array}$$

lemma *cp-not*: $(not\ X)\tau = (not\ (\lambda\ \text{-}.\ X\ \tau))\ \tau$
by(*simp add*: *not-def*)

lemma *not1*[*simp*]: *not invalid* = *invalid*
by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def bot-option-def*)

lemma *not2*[*simp*]: *not null* = *null*
by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)

lemma *not3*[*simp*]: *not true* = *false*
by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def*)

lemma *not4*[*simp*]: *not false* = *true*
by(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def*)

lemma *not-not*[*simp*]: *not* (*not* X) = X
apply(*rule ext*,*simp add*: *not-def null-def invalid-def true-def false-def*)
apply(*case-tac X x*, *simp-all*)
apply(*case-tac a*, *simp-all*)
done

lemma *not-inject*: $\bigwedge\ x\ y.\ not\ x = not\ y \Longrightarrow x = y$
by(*subst not-not*[*THEN sym*], *simp*)

definition *ocl-and* :: $[('\mathfrak{A})Boolean,\ ('\mathfrak{A})Boolean] \Rightarrow ('\mathfrak{A})Boolean$ (**infixl** *and 30*)
where $X\ and\ Y \equiv\ (\lambda\ \tau\ .\ case\ X\ \tau\ of$

$$
\begin{array}{lll}
\lfloor\lfloor False\rfloor\rfloor & \Rightarrow & \lfloor\lfloor False\rfloor\rfloor \\
\mid \bot & \Rightarrow & (case\ Y\ \tau\ of \\
& & \quad \lfloor\lfloor False\rfloor\rfloor \Rightarrow \lfloor\lfloor False\rfloor\rfloor \\
& & \quad \mid \text{-} \Rightarrow \bot) \\
\mid \lfloor\bot\rfloor & \Rightarrow & (case\ Y\ \tau\ of \\
& & \quad \lfloor\lfloor False\rfloor\rfloor \Rightarrow \lfloor\lfloor False\rfloor\rfloor \\
& & \quad \mid \bot \Rightarrow \bot \\
& & \quad \mid \text{-} \Rightarrow \lfloor\bot\rfloor) \\
\mid \lfloor\lfloor True\rfloor\rfloor & \Rightarrow & Y\ \tau)
\end{array}
$$

Note that *not* is *not* defined as a strict function; proximity to lattice laws implies that we *need* a definition of *not* that satisfies $not(not(x))=x$.

In textbook notation, the logical core constructs *not* and *op and* were represented as follows:

lemma *textbook-not*:

$$
I[\![not(X)]\!]\ \tau = \ (case\ I[\![X]\!]\ \tau\ of\ \ \bot\ \Rightarrow \bot \\
\mid\ \lfloor\ \bot\ \rfloor \Rightarrow \lfloor\ \bot\ \rfloor \\
\mid\ \lfloor\lfloor\ x\ \rfloor\rfloor \Rightarrow \lfloor\lfloor\ \neg\ x\ \rfloor\rfloor)
$$

by(*simp add*: *Sem-def not-def*)

lemma *textbook-and*:

$$
\begin{array}{l}
I[\![X \; and \; Y]\!] \; \tau = (case \; I[\![X]\!] \; \tau \; of \\
\qquad\qquad \bot \;\Rightarrow (case \; I[\![Y]\!] \; \tau \; of \\
\qquad\qquad\qquad\qquad \bot \Rightarrow \; \bot \\
\qquad\qquad\qquad\qquad | \; \lfloor\bot\rfloor \Rightarrow \bot \\
\qquad\qquad\qquad\qquad | \; \lfloor\lfloor True\rfloor\rfloor \Rightarrow \; \bot \\
\qquad\qquad\qquad\qquad | \; \lfloor\lfloor False\rfloor\rfloor \Rightarrow \; \lfloor\lfloor False\rfloor\rfloor) \\
\qquad\qquad | \; \lfloor \; \bot \; \rfloor \Rightarrow (case \; I[\![Y]\!] \; \tau \; of \\
\qquad\qquad\qquad\qquad \bot \Rightarrow \; \bot \\
\qquad\qquad\qquad\qquad | \; \lfloor\bot\rfloor \Rightarrow \lfloor\bot\rfloor \\
\qquad\qquad\qquad\qquad | \; \lfloor\lfloor True\rfloor\rfloor \Rightarrow \lfloor\bot\rfloor \\
\qquad\qquad\qquad\qquad | \; \lfloor\lfloor False\rfloor\rfloor \Rightarrow \; \lfloor\lfloor False\rfloor\rfloor) \\
\qquad\qquad | \; \lfloor\lfloor True\rfloor\rfloor \Rightarrow (case \; I[\![Y]\!] \; \tau \; of \\
\qquad\qquad\qquad\qquad \bot \Rightarrow \; \bot \\
\qquad\qquad\qquad\qquad | \; \lfloor\bot\rfloor \Rightarrow \lfloor\bot\rfloor \\
\qquad\qquad\qquad\qquad | \; \lfloor\lfloor y\rfloor\rfloor \Rightarrow \; \lfloor\lfloor y\rfloor\rfloor) \\
\qquad\qquad | \; \lfloor\lfloor False\rfloor\rfloor \Rightarrow \; \lfloor\lfloor \; False \; \rfloor\rfloor)
\end{array}
$$

by(*simp add*: *ocl-and-def Sem-def split*: *option.split bool.split*)

definition *ocl-or* :: $[('\mathfrak{A})Boolean, \; ('\mathfrak{A})Boolean] \Rightarrow ('\mathfrak{A})Boolean$ (**infixl** *or 25*)
where $X \; or \; Y \equiv not(not \; X \; and \; not \; Y)$

definition *ocl-implies* :: $[('\mathfrak{A})Boolean, \; ('\mathfrak{A})Boolean] \Rightarrow ('\mathfrak{A})Boolean$ (**infixl** *implies 25*)
where $X \; implies \; Y \equiv not \; X \; or \; Y$

lemma *cp-ocl-and*:$(X \; and \; Y) \; \tau = ((\lambda \; \text{-}. \; X \; \tau) \; and \; (\lambda \; \text{-}. \; Y \; \tau)) \; \tau$
by(*simp add*: *ocl-and-def*)

lemma *cp-ocl-or*:$((X{::}('\mathfrak{A})Boolean) \; or \; Y) \; \tau = ((\lambda \; \text{-}. \; X \; \tau) \; or \; (\lambda \; \text{-}. \; Y \; \tau)) \; \tau$
apply(*simp add*: *ocl-or-def*)
apply(*subst cp-not*[*of not* $(\lambda\text{-}. \; X \; \tau)$ *and not* $(\lambda\text{-}. \; Y \; \tau)$])
apply(*subst cp-ocl-and*[*of not* $(\lambda\text{-}. \; X \; \tau)$ *not* $(\lambda\text{-}. \; Y \; \tau)$])
by(*simp add*: *cp-not*[*symmetric*] *cp-ocl-and*[*symmetric*])

lemma *cp-ocl-implies*:$(X \; implies \; Y) \; \tau = ((\lambda \; \text{-}. \; X \; \tau) \; implies \; (\lambda \; \text{-}. \; Y \; \tau)) \; \tau$
apply(*simp add*: *ocl-implies-def*)
apply(*subst cp-ocl-or*[*of not* $(\lambda\text{-}. \; X \; \tau)$ $(\lambda\text{-}. \; Y \; \tau)$])
by(*simp add*: *cp-not*[*symmetric*] *cp-ocl-or*[*symmetric*])

lemma *ocl-and1*[*simp*]: (*invalid and true*) = *invalid*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def*)
lemma *ocl-and2*[*simp*]: (*invalid and false*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def*)
lemma *ocl-and3*[*simp*]: (*invalid and null*) = *invalid*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)

lemma *ocl-and4*[*simp*]: (*invalid and invalid*) = *invalid*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def*)

lemma *ocl-and5*[*simp*]: (*null and true*) = *null*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)
lemma *ocl-and6*[*simp*]: (*null and false*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)
lemma *ocl-and7*[*simp*]: (*null and null*) = *null*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)
lemma *ocl-and8*[*simp*]: (*null and invalid*) = *invalid*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)

lemma *ocl-and9*[*simp*]: (*false and true*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def*)
lemma *ocl-and10*[*simp*]: (*false and false*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def*)
lemma *ocl-and11*[*simp*]: (*false and null*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def*)
lemma *ocl-and12*[*simp*]: (*false and invalid*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def*)

lemma *ocl-and13*[*simp*]: (*true and true*) = *true*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def*)
lemma *ocl-and14*[*simp*]: (*true and false*) = *false*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def*)
lemma *ocl-and15*[*simp*]: (*true and null*) = *null*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)
lemma *ocl-and16*[*simp*]: (*true and invalid*) = *invalid*
by(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def bot-option-def null-fun-def null-option-def*)

lemma *ocl-and-idem*[*simp*]: (*X and X*) = *X*
apply(*rule ext*,*simp add*: *ocl-and-def null-def invalid-def true-def false-def*)
apply(*case-tac X x*, *simp-all*)
apply(*case-tac a*, *simp-all*)
apply(*case-tac aa*, *simp-all*)
done

lemma *ocl-and-commute*: (*X and Y*) = (*Y and X*)
by(*rule ext*,*auto simp*:*true-def false-def ocl-and-def invalid-def*
split: *option.split option.split-asm*
bool.split bool.split-asm)

lemma *ocl-and-false1*[*simp*]: (*false and X*) = *false*
apply(*rule ext*, *simp add*: *ocl-and-def*)
apply(*auto simp*:*true-def false-def invalid-def*
split: *option.split option.split-asm*)
done

lemma *ocl-and-false2*[*simp*]: (*X and false*) = *false*
by(*simp add*: *ocl-and-commute*)

lemma *ocl-and-true1*[*simp*]: (*true and X*) = *X*
apply(*rule ext*, *simp add*: *ocl-and-def*)
apply(*auto simp*:*true-def false-def invalid-def*
split: *option.split option.split-asm*)
done

lemma *ocl-and-true2*[*simp*]: (*X and true*) = *X*
by(*simp add*: *ocl-and-commute*)

lemma *ocl-and-bot1*[*simp*]: $\bigwedge\tau.\ X\ \tau \neq \mathit{false}\ \tau \Longrightarrow (\mathit{bot}\ \mathit{and}\ X)\ \tau = \mathit{bot}\ \tau$
apply(*simp add*: *ocl-and-def*)
apply(*auto simp*:*true-def false-def bot-fun-def bot-option-def*
split: *option.split option.split-asm*)
done

lemma *ocl-and-bot2*[*simp*]: $\bigwedge\tau.\ X\ \tau \neq \mathit{false}\ \tau \Longrightarrow (X\ \mathit{and}\ \mathit{bot})\ \tau = \mathit{bot}\ \tau$
by(*simp add*: *ocl-and-commute*)

lemma *ocl-and-null1*[*simp*]: $\bigwedge\tau.\ X\ \tau \neq \mathit{false}\ \tau \Longrightarrow X\ \tau \neq \mathit{bot}\ \tau \Longrightarrow (\mathit{null}\ \mathit{and}\ X)\ \tau = \mathit{null}\ \tau$
apply(*simp add*: *ocl-and-def*)
apply(*auto simp*:*true-def false-def bot-fun-def bot-option-def null-fun-def null-option-def*
split: *option.split option.split-asm*)
done

lemma *ocl-and-null2*[*simp*]: $\bigwedge\tau.\ X\ \tau \neq \mathit{false}\ \tau \Longrightarrow X\ \tau \neq \mathit{bot}\ \tau \Longrightarrow (X\ \mathit{and}\ \mathit{null})\ \tau = \mathit{null}\ \tau$
by(*simp add*: *ocl-and-commute*)

lemma *ocl-and-assoc*: (*X and* (*Y and Z*)) = (*X and Y and Z*)
apply(*rule ext*, *simp add*: *ocl-and-def*)
apply(*auto simp*:*true-def false-def null-def invalid-def*
split: *option.split option.split-asm*
bool.split bool.split-asm)
done

lemma *ocl-or-idem*[*simp*]: (*X or X*) = *X*
by(*simp add*: *ocl-or-def*)

lemma *ocl-or-commute*: (*X or Y*) = (*Y or X*)

by(*simp add*: *ocl-or-def ocl-and-commute*)

lemma *ocl-or-false1*[*simp*]: (*false or* Y) = Y
by(*simp add*: *ocl-or-def*)

lemma *ocl-or-false2*[*simp*]: (Y *or false*) = Y
by(*simp add*: *ocl-or-def*)

lemma *ocl-or-true1*[*simp*]: (*true or* Y) = *true*
by(*simp add*: *ocl-or-def*)

lemma *ocl-or-true2*: (Y *or true*) = *true*
by(*simp add*: *ocl-or-def*)

lemma *ocl-or-assoc*: (X *or* (Y *or* Z)) = (X *or* Y *or* Z)
by(*simp add*: *ocl-or-def ocl-and-assoc*)

lemma *deMorgan1*: *not*(X *and* Y) = ((*not* X) *or* (*not* Y))
by(*simp add*: *ocl-or-def*)

lemma *deMorgan2*: *not*(X *or* Y) = ((*not* X) *and* (*not* Y))
by(*simp add*: *ocl-or-def*)

3.3. A Standard Logical Calculus for OCL

Besides the need for algebraic laws for OCL in order to normalize

definition *OclValid* :: $[('\mathfrak{A})st, ('\mathfrak{A})Boolean] \Rightarrow bool$ ((*1*(-)/ $\models$ (-)) *50*)
where $\tau \models P \equiv ((P\ \tau) = true\ \tau)$

3.3.1. Global vs. Local Judgements

lemma *transform1*: $P = true \Longrightarrow \tau \models P$
by(*simp add*: *OclValid-def*)

lemma *transform1-rev*: $\forall\ \tau.\ \tau \models P \Longrightarrow P = true$
by(*rule ext*, *auto simp*: *OclValid-def true-def*)

lemma *transform2*: $(P = Q) \Longrightarrow ((\tau \models P) = (\tau \models Q))$
by(*auto simp*: *OclValid-def*)

lemma *transform2-rev*: $\forall\ \tau.\ (\tau \models \delta\ P) \wedge (\tau \models \delta\ Q) \wedge (\tau \models P) = (\tau \models Q) \Longrightarrow P = Q$
apply(*rule ext*,*auto simp*: *OclValid-def true-def defined-def*)
apply(*erule-tac* x=a **in** *allE*)
apply(*erule-tac* x=b **in** *allE*)
apply(*auto simp*: *false-def true-def defined-def bot-Boolean-def null-Boolean-def*
split: *option.split-asm HOL.split-if-asm*)
done

However, certain properties (like transitivity) can not be *transformed* from the global level to the local one, they have to be re-proven on the local level.

lemma *transform3*:
assumes $H : P = true \Longrightarrow Q = true$
shows $\tau \models P \Longrightarrow \tau \models Q$
apply(*simp add*: *OclValid-def*)
apply(*rule H*[*THEN fun-cong*])
apply(*rule ext*)
oops

3.3.2. Local Validity and Meta-logic

lemma *foundation1*[*simp*]: $\tau \models true$
by(*auto simp*: *OclValid-def*)

lemma *foundation2*[*simp*]: $\neg(\tau \models false)$
by(*auto simp*: *OclValid-def true-def false-def*)

lemma *foundation3*[*simp*]: $\neg(\tau \models invalid)$
by(*auto simp*: *OclValid-def true-def false-def invalid-def bot-option-def*)

lemma *foundation4*[*simp*]: $\neg(\tau \models null)$
by(*auto simp*: *OclValid-def true-def false-def null-def null-fun-def null-option-def bot-option-def*)

lemma *bool-split-local*[*simp*]:
$(\tau \models (x \triangleq invalid)) \vee (\tau \models (x \triangleq null)) \vee (\tau \models (x \triangleq true)) \vee (\tau \models (x \triangleq false))$
apply(*insert bool-split*[*of x* τ], *auto*)
apply(*simp-all add*: *OclValid-def StrongEq-def true-def null-def invalid-def*)
done

lemma *def-split-local*:
$(\tau \models \delta\ x) = ((\neg(\tau \models (x \triangleq invalid))) \wedge (\neg\ (\tau \models (x \triangleq null))))$
by(*simp add*:*defined-def true-def false-def invalid-def null-def*
StrongEq-def OclValid-def bot-fun-def null-fun-def)

lemma *foundation5*:
$\tau \models (P\ and\ Q) \Longrightarrow (\tau \models P) \wedge (\tau \models Q)$
by(*simp add*: *ocl-and-def OclValid-def true-def false-def defined-def*
split: *option.split option.split-asm bool.split bool.split-asm*)

lemma *foundation6*:
$\tau \models P \Longrightarrow \tau \models \delta\ P$
by(*simp add*: *not-def OclValid-def true-def false-def defined-def*
null-option-def null-fun-def bot-option-def bot-fun-def
split: *option.split option.split-asm*)

lemma *foundation7*[*simp*]:
$(\tau \models not\ (\delta\ x)) = (\neg\ (\tau \models \delta\ x))$

by(*simp add*: *not-def OclValid-def true-def false-def defined-def*
split: *option.split option.split-asm*)

lemma *foundation7'*[*simp*]:
$(\tau \models \mathit{not}\ (\upsilon\ x)) = (\neg\ (\tau \models \upsilon\ x))$
by(*simp add*: *not-def OclValid-def true-def false-def valid-def*
split: *option.split option.split-asm*)

Key theorem for the Delta-closure: either an expression is defined, or it can be replaced (substituted via `StrongEq_L_subst2`; see below) by invalid or null. Strictness-reduction rules will usually reduce these substituted terms drastically.

lemma *foundation8*:
$(\tau \models \delta\ x) \lor (\tau \models (x \triangleq \mathit{invalid})) \lor (\tau \models (x \triangleq \mathit{null}))$
proof −
have *1* : $(\tau \models \delta\ x) \lor (\neg(\tau \models \delta\ x))$ **by** *auto*
have *2* : $(\neg(\tau \models \delta\ x)) = ((\tau \models (x \triangleq \mathit{invalid})) \lor (\tau \models (x \triangleq \mathit{null})))$
by(*simp only*: *def-split-local*, *simp*)
show *?thesis* **by**(*insert 1*, *simp add*:*2*)
qed

lemma *foundation9*:
$\tau \models \delta\ x \Longrightarrow (\tau \models \mathit{not}\ x) = (\neg\ (\tau \models x))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *not-def null-fun-def null-option-def bot-option-def*
OclValid-def invalid-def true-def null-def StrongEq-def)

lemma *foundation10*:
$\tau \models \delta\ x \Longrightarrow \tau \models \delta\ y \Longrightarrow (\tau \models (x\ \mathit{and}\ y)) = (\ (\tau \models x) \land (\tau \models y))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *ocl-and-def OclValid-def invalid-def*
true-def null-def StrongEq-def null-fun-def null-option-def bot-option-def
split:*bool.split-asm*)

lemma *foundation11*:
$\tau \models \delta\ x \Longrightarrow\ \tau \models \delta\ y \Longrightarrow (\tau \models (x\ \mathit{or}\ y)) = (\ (\tau \models x) \lor (\tau \models y))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *not-def ocl-or-def ocl-and-def OclValid-def invalid-def*
true-def null-def StrongEq-def null-fun-def null-option-def bot-option-def
split:*bool.split-asm bool.split*)

lemma *foundation12*:
$\tau \models \delta\ x \Longrightarrow\ \tau \models \delta\ y \Longrightarrow (\tau \models (x\ \mathit{implies}\ y)) = (\ (\tau \models x) \longrightarrow (\tau \models y))$
apply(*simp add*: *def-split-local*)
by(*auto simp*: *not-def ocl-or-def ocl-and-def ocl-implies-def bot-option-def*
OclValid-def invalid-def true-def null-def StrongEq-def null-fun-def null-option-def

split:*bool.split-asm bool.split*)

lemma *foundation13*:($\tau \models A \triangleq$ *true*) $= (\tau \models A)$
by(*auto simp*: *not-def OclValid-def invalid-def true-def null-def StrongEq-def*
split:*bool.split-asm bool.split*)

lemma *foundation14*:($\tau \models A \triangleq$ *false*) $= (\tau \models$ *not* $A)$
by(*auto simp*: *not-def OclValid-def invalid-def false-def true-def null-def StrongEq-def*
split:*bool.split-asm bool.split option.split*)

lemma *foundation15*:($\tau \models A \triangleq$ *invalid*) $= (\tau \models$ *not*$(\upsilon\ A))$
by(*auto simp*: *not-def OclValid-def valid-def invalid-def false-def true-def null-def*
StrongEq-def bot-option-def null-fun-def null-option-def bot-option-def bot-fun-def
split:*bool.split-asm bool.split option.split*)

lemma *foundation16*: $\tau \models (\delta\ X) = (X\ \tau \neq bot \wedge X\ \tau \neq null)$
by(*auto simp*: *OclValid-def defined-def false-def true-def bot-fun-def null-fun-def*
split:*split-if-asm*)

lemmas *foundation17* = *foundation16*[*THEN iffD1*,*standard*]

lemma *foundation18*: $\tau \models (\upsilon\ X) = (X\ \tau \neq invalid\ \tau)$
by(*auto simp*: *OclValid-def valid-def false-def true-def bot-fun-def invalid-def*
split:*split-if-asm*)

lemma *foundation18'*: $\tau \models (\upsilon\ X) = (X\ \tau \neq bot)$
by(*auto simp*: *OclValid-def valid-def false-def true-def bot-fun-def*
split:*split-if-asm*)

lemmas *foundation19* = *foundation18*[*THEN iffD1*,*standard*]

lemma *foundation20* : $\tau \models (\delta\ X) \Longrightarrow \tau \models \upsilon\ X$
by(*simp add*: *foundation18 foundation16 invalid-def*)

lemma *foundation21*: (*not* $A \triangleq$ *not* $B) = (A \triangleq B)$
by(*rule ext*, *auto simp*: *not-def StrongEq-def*
split: *bool.split-asm HOL.split-if-asm option.split*)

lemma *foundation22*: $(\tau \models (X \triangleq Y)) = (X\ \tau = Y\ \tau)$
by(*auto simp*: *StrongEq-def OclValid-def true-def*)

lemma *foundation23*: $(\tau \models P) = (\tau \models (\lambda\ \text{-}\ .\ P\ \tau))$
by(*auto simp*: *OclValid-def true-def*)

lemmas *cp-validity*=*foundation23*

lemma *defined-not-I* : $\tau \models \delta\ (x) \Longrightarrow \tau \models \delta\ (not\ x)$
by(*auto simp*: *not-def null-def invalid-def defined-def valid-def OclValid-def true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def split*: *option.split-asm HOL.split-if-asm*)

lemma *valid-not-I* : $\tau \models \upsilon\ (x) \Longrightarrow \tau \models \upsilon\ (not\ x)$
by(*auto simp*: *not-def null-def invalid-def defined-def valid-def OclValid-def true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def split*: *option.split-asm option.split HOL.split-if-asm*)

lemma *defined-and-I* : $\tau \models \delta\ (x) \Longrightarrow\ \tau \models \delta\ (y) \Longrightarrow \tau \models \delta\ (x\ and\ y)$
apply(*simp add*: *ocl-and-def null-def invalid-def defined-def valid-def OclValid-def true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def split*: *option.split-asm HOL.split-if-asm*)
apply(*auto simp*: *null-option-def split*: *bool.split*)
by(*case-tac ya,simp-all*)

lemma *valid-and-I* : $\tau \models \upsilon\ (x) \Longrightarrow\ \tau \models \upsilon\ (y) \Longrightarrow \tau \models \upsilon\ (x\ and\ y)$
apply(*simp add*: *ocl-and-def null-def invalid-def defined-def valid-def OclValid-def true-def false-def bot-option-def null-option-def null-fun-def bot-fun-def split*: *option.split-asm HOL.split-if-asm*)
by(*auto simp*: *null-option-def split*: *option.split bool.split*)

3.3.3. Local Judgements and Strong Equality

lemma *StrongEq-L-refl*: $\tau \models (x \triangleq x)$
by(*simp add*: *OclValid-def StrongEq-def*)

lemma *StrongEq-L-sym*: $\tau \models (x \triangleq y) \Longrightarrow \tau \models (y \triangleq x)$
by(*simp add*: *StrongEq-sym*)

lemma *StrongEq-L-trans*: $\tau \models (x \triangleq y) \Longrightarrow \tau \models (y \triangleq z) \Longrightarrow \tau \models (x \triangleq z)$
by(*simp add*: *OclValid-def StrongEq-def true-def*)

In order to establish substitutivity (which does not hold in general HOL-formulas we introduce the following predicate that allows for a calculus of the necessary side-conditions.

definition *cp* $:: (('\mathfrak{A},'\alpha)\ val \Rightarrow ('\mathfrak{A},'\beta)\ val) \Rightarrow bool$
where $cp\ P \equiv (\exists\ f.\ \forall\ X\ \tau.\ P\ X\ \tau = f\ (X\ \tau)\ \tau)$

The rule of substitutivity in HOL-OCL holds only for context-passing expressions - i.e. those, that pass the context τ without changing it. Fortunately, all operators of the OCL language satisfy this property (but not all HOL operators).

lemma *StrongEq-L-subst1*: $\bigwedge \tau.\ cp\ P \Longrightarrow \tau \models (x \triangleq y) \Longrightarrow \tau \models (P\ x \triangleq P\ y)$
by(*auto simp*: *OclValid-def StrongEq-def true-def cp-def*)

lemma *StrongEq-L-subst2*:
$\bigwedge \tau.\ \ cp\ P \Longrightarrow \tau \models (x \triangleq y) \Longrightarrow \tau \models (P\ x) \Longrightarrow \tau \models (P\ y)$

by(*auto simp*: *OclValid-def StrongEq-def true-def cp-def*)

lemma *StrongEq-L-subst2-rev*: $\tau \models y \triangleq x \Longrightarrow cp\ P \Longrightarrow \tau \models P\ x \Longrightarrow \tau \models P\ y$
apply(*erule StrongEq-L-subst2*)
apply(*erule StrongEq-L-sym*)
by *assumption*

```
ML⟪ (* just a fist sketch *)
fun ocl-subst-tac subst =
         let val foundation22-THEN-iffD1 = @{thm foundation22} RS @{thm iffD1}
             val StrongEq-L-subst2-rev- = @{thm StrongEq-L-subst2-rev}
             val the-context = @{context} (* Hack of bu : will not work in general *)
         in  EVERY[rtac foundation22-THEN-iffD1 1,
                  eres-inst-tac the-context [((P,0),subst)] StrongEq-L-subst2-rev- 1,
                  simp-tac (simpset-of the-context) 1,
                  simp-tac (simpset-of the-context) 1]
         end

⟫
```

lemma *cpI1*:
$(\forall\ X\ \tau.\ f\ X\ \tau = f(\lambda\text{-}.\ X\ \tau)\ \tau) \Longrightarrow cp\ P \Longrightarrow cp(\lambda X.\ f\ (P\ X))$
apply(*auto simp*: *true-def cp-def*)
apply(*rule exI*, (*rule allI*)+)
by(*erule-tac x=P X* **in** *allE*, *auto*)

lemma *cpI2*:
$(\forall\ X\ Y\ \tau.\ f\ X\ Y\ \tau = f(\lambda\text{-}.\ X\ \tau)(\lambda\text{-}.\ Y\ \tau)\ \tau) \Longrightarrow$
$cp\ P \Longrightarrow cp\ Q \Longrightarrow cp(\lambda X.\ f\ (P\ X)\ (Q\ X))$
apply(*auto simp*: *true-def cp-def*)
apply(*rule exI*, (*rule allI*)+)
by(*erule-tac x=P X* **in** *allE*, *auto*)

lemma *cpI3*:
$(\forall\ X\ Y\ Z\ \tau.\ f\ X\ Y\ Z\ \tau = f(\lambda\text{-}.\ X\ \tau)(\lambda\text{-}.\ Y\ \tau)(\lambda\text{-}.\ Z\ \tau)\ \tau) \Longrightarrow$
$cp\ P \Longrightarrow cp\ Q \Longrightarrow cp\ R \Longrightarrow cp(\lambda X.\ f\ (P\ X)\ (Q\ X)\ (R\ X))$
apply(*auto simp*: *true-def cp-def*)
apply(*rule exI*, (*rule allI*)+)
by(*erule-tac x=P X* **in** *allE*, *auto*)

lemma *cpI4*:
$(\forall\ W\ X\ Y\ Z\ \tau.\ f\ W\ X\ Y\ Z\ \tau = f(\lambda\text{-}.\ W\ \tau)(\lambda\text{-}.\ X\ \tau)(\lambda\text{-}.\ Y\ \tau)(\lambda\text{-}.\ Z\ \tau)\ \tau) \Longrightarrow$
$cp\ P \Longrightarrow cp\ Q \Longrightarrow cp\ R \Longrightarrow cp\ S \Longrightarrow cp(\lambda X.\ f\ (P\ X)\ (Q\ X)\ (R\ X)\ (S\ X))$
apply(*auto simp*: *true-def cp-def*)
apply(*rule exI*, (*rule allI*)+)
by(*erule-tac x=P X* **in** *allE*, *auto*)

lemma *cp-const* : $cp(\lambda\text{-}.\ c)$
by (*simp add*: *cp-def*, *fast*)

lemma *cp-id* : $\quad cp(\lambda X.\ X)$
by (*simp add*: *cp-def*, *fast*)

lemmas *cp-intro*[*simp*,*intro*!] =
cp-const
cp-id
cp-defined[*THEN allI*[*THEN allI*[*THEN cpI1*], *of defined*]]
cp-valid[*THEN allI*[*THEN allI*[*THEN cpI1*], *of valid*]]
cp-not[*THEN allI*[*THEN allI*[*THEN cpI1*], *of not*]]
cp-ocl-and[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of op and*]]
cp-ocl-or[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of op or*]]
cp-ocl-implies[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of op implies*]]
cp-StrongEq[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]],
of StrongEq]]

3.3.4. Laws to Establish Definedness (Delta-Closure)

For the logical connectives, we have — beyond $?\tau \models ?P \Longrightarrow ?\tau \models \delta\ ?P$ — the following facts:

lemma *ocl-not-defargs*:
$\tau \models (not\ P) \Longrightarrow \tau \models \delta\ P$
by(*auto simp*: *not-def OclValid-def true-def invalid-def defined-def false-def bot-fun-def bot-option-def null-fun-def null-option-def*
split: *bool.split-asm HOL.split-if-asm option.split option.split-asm*)

So far, we have only one strict Boolean predicate (-family): The strict equality.

3.4. Miscellaneous: OCL's if then else endif

definition *if-ocl* :: $[('\mathfrak{A})Boolean\ ,\ ('\mathfrak{A},'\alpha{::}null)\ val,\ ('\mathfrak{A},'\alpha)\ val] \Rightarrow ('\mathfrak{A},'\alpha)\ val$
$(if\ (\text{-})\ then\ (\text{-})\ else\ (\text{-})\ endif\ [10,10,10]50)$
where $(if\ C\ then\ B_1\ else\ B_2\ endif) = (\lambda\ \tau.\ if\ (\delta\ C)\ \tau = true\ \tau$
$then\ (if\ (C\ \tau) = true\ \tau$
$then\ B_1\ \tau$
$else\ B_2\ \tau)$
$else\ invalid\ \tau)$

lemma *cp-if-ocl*:$((if\ C\ then\ B_1\ else\ B_2\ endif)\ \tau =$
$(if\ (\lambda\ \text{-}.\ C\ \tau)\ then\ (\lambda\ \text{-}.\ B_1\ \tau)\ else\ (\lambda\ \text{-}.\ B_2\ \tau)\ endif)\ \tau)$
by(*simp only*: *if-ocl-def*, *subst cp-defined*, *rule refl*)

lemmas *cp-intro'*[*simp*,*intro*!] =
cp-intro
cp-if-ocl[*THEN allI*[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI3*]]], *of if-ocl*]]

lemma *if-ocl-invalid* [*simp*]: $(if\ invalid\ then\ B_1\ else\ B_2\ endif) = invalid$

by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-null* [*simp*]: (*if null then* B_1 *else* B_2 *endif*) = *invalid*
by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-true* [*simp*]: (*if true then* B_1 *else* B_2 *endif*) = B_1
by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-true'* [*simp*]: $\tau \models P \Longrightarrow$ (*if P then* B_1 *else* B_2 *endif*)τ = B_1 τ
apply(*subst cp-if-ocl*,*auto simp*: *OclValid-def*)
by(*simp add*:*cp-if-ocl*[*symmetric*])

lemma *if-ocl-false* [*simp*]: (*if false then* B_1 *else* B_2 *endif*) = B_2
by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-false'* [*simp*]: $\tau \models$ *not* $P \Longrightarrow$ (*if P then* B_1 *else* B_2 *endif*)τ = B_2 τ
apply(*subst cp-if-ocl*)
apply(*auto simp*: *foundation14*[*symmetric*] *foundation22*)
by(*auto simp*: *cp-if-ocl*[*symmetric*])

lemma *if-ocl-idem1*[*simp*]:(*if* δ *X then A else A endif*) = *A*
by(*rule ext*, *auto simp*: *if-ocl-def*)

lemma *if-ocl-idem2*[*simp*]:(*if* υ *X then A else A endif*) = *A*
by(*rule ext*, *auto simp*: *if-ocl-def*)

end

4. Part II: Library Definitions

theory *OCL-lib*
imports *OCL-core*
begin

4.1. Basic Types: Void and Integer

4.1.1. The construction of the Void Type

type-synonym ($'\mathfrak{A}$) *Void* = ($'\mathfrak{A}$,*unit option*) *val*

This *minimal* OCL type contains only two elements: *undefined* and *null*. *Void* could initially be defined as *unit option option*, however the cardinal of this type is more than two, so it would have the cost to consider *Some None* and *Some* (*Some* ()) seemingly everywhere.

4.1.2. The construction of the Integer Type

Since *Integer* is again a basic type, we define its semantic domain as the valuations over *int option option*.

type-synonym ($'\mathfrak{A}$)*Integer* = ($'\mathfrak{A}$,*int option option*) *val*

Although the remaining part of this library reasons about integers abstractly, we provide here some shortcuts to some usual integers.

definition *ocl-zero* ::($'\mathfrak{A}$)*Integer* (**0**)
where $\mathbf{0} = (\lambda\ \text{-}\ .\ \lfloor\lfloor 0{::}int\rfloor\rfloor)$

definition *ocl-one* ::($'\mathfrak{A}$)*Integer* (**1**)
where $\mathbf{1} = (\lambda\ \text{-}\ .\ \lfloor\lfloor 1{::}int\rfloor\rfloor)$

definition *ocl-two* ::($'\mathfrak{A}$)*Integer* (**2**)
where $\mathbf{2} = (\lambda\ \text{-}\ .\ \lfloor\lfloor 2{::}int\rfloor\rfloor)$

definition *ocl-three* ::($'\mathfrak{A}$)*Integer* (**3**)
where $\mathbf{3} = (\lambda\ \text{-}\ .\ \lfloor\lfloor 3{::}int\rfloor\rfloor)$

definition *ocl-four* ::($'\mathfrak{A}$)*Integer* (**4**)
where $\mathbf{4} = (\lambda\ \text{-}\ .\ \lfloor\lfloor 4{::}int\rfloor\rfloor)$

definition *ocl-five* ::($'\mathfrak{A}$)*Integer* (**5**)
where $\mathbf{5} = (\lambda\ \text{-}\ .\ \lfloor\lfloor 5{::}int\rfloor\rfloor)$

definition *ocl-six* $::('\mathfrak{A})Integer$ $(\mathbf{6})$
where $\mathbf{6} = (\lambda\ \text{-}\ .\ \lfloor\lfloor \mathit{6}::int\rfloor\rfloor)$

definition *ocl-seven* $::('\mathfrak{A})Integer$ $(\mathbf{7})$
where $\mathbf{7} = (\lambda\ \text{-}\ .\ \lfloor\lfloor \mathit{7}::int\rfloor\rfloor)$

definition *ocl-eight* $::('\mathfrak{A})Integer$ $(\mathbf{8})$
where $\mathbf{8} = (\lambda\ \text{-}\ .\ \lfloor\lfloor \mathit{8}::int\rfloor\rfloor)$

definition *ocl-nine* $::('\mathfrak{A})Integer$ $(\mathbf{9})$
where $\mathbf{9} = (\lambda\ \text{-}\ .\ \lfloor\lfloor \mathit{9}::int\rfloor\rfloor)$

definition *ocl-ten* $::('\mathfrak{A})Integer$ $(\mathbf{10})$
where $\mathbf{10} = (\lambda\ \text{-}\ .\ \lfloor\lfloor \mathit{10}::int\rfloor\rfloor)$

4.1.3. Validity and Definedness Properties

lemma $\delta(null::('\mathfrak{A})Integer) = \mathit{false}$ **by** *simp*
lemma $\upsilon(null::('\mathfrak{A})Integer) = \mathit{true}$ **by** *simp*

lemma [*simp*,*code-unfold*]: $\delta\ (\lambda\text{-}.\ \lfloor\lfloor n\rfloor\rfloor) = \mathit{true}$
by(*simp add*:*defined-def true-def*
bot-fun-def bot-option-def null-fun-def null-option-def)

lemma [*simp*,*code-unfold*]: $\upsilon\ (\lambda\text{-}.\ \lfloor\lfloor n\rfloor\rfloor) = \mathit{true}$
by(*simp add*:*valid-def true-def*
bot-fun-def bot-option-def)

lemma [*simp*,*code-unfold*]:$\upsilon\ \mathbf{0} = \mathit{true}$ **by**(*simp add*:*ocl-zero-def*)
lemma [*simp*,*code-unfold*]:$\delta\ \mathbf{1} = \mathit{true}$ **by**(*simp add*:*ocl-one-def*)
lemma [*simp*,*code-unfold*]:$\upsilon\ \mathbf{1} = \mathit{true}$ **by**(*simp add*:*ocl-one-def*)
lemma [*simp*,*code-unfold*]:$\delta\ \mathbf{2} = \mathit{true}$ **by**(*simp add*:*ocl-two-def*)
lemma [*simp*,*code-unfold*]:$\upsilon\ \mathbf{2} = \mathit{true}$ **by**(*simp add*:*ocl-two-def*)
lemma [*simp*,*code-unfold*]: $\upsilon\ \mathbf{6} = \mathit{true}$ **by**(*simp add*:*ocl-six-def*)
lemma [*simp*,*code-unfold*]: $\upsilon\ \mathbf{8} = \mathit{true}$ **by**(*simp add*:*ocl-eight-def*)
lemma [*simp*,*code-unfold*]: $\upsilon\ \mathbf{9} = \mathit{true}$ **by**(*simp add*:*ocl-nine-def*)

4.1.4. Arithmetical Operations on Integer

Definition

Here is a common case of a built-in operation on built-in types. Note that the arguments must be both defined (non-null, non-bot).

Note that we can not follow the lexis of standard OCL for Isabelle- technical reasons; these operators are heavily overloaded in the library that a further overloading would lead to heavy technical buzz in this document...

definition *ocl-add-int* ::($'\mathfrak{A}$)*Integer* ⇒ ($'\mathfrak{A}$)*Integer* ⇒ ($'\mathfrak{A}$)*Integer* (**infix** ⊕ *40*)
where $x \oplus y \equiv \lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\delta\ y)\ \tau = true\ \tau$
$then\ \lfloor\lfloor\lceil\lceil x\ \tau\rceil\rceil + \lceil\lceil y\ \tau\rceil\rceil\rfloor\rfloor$
$else\ invalid\ \tau$

definition *ocl-less-int* ::($'\mathfrak{A}$)*Integer* ⇒ ($'\mathfrak{A}$)*Integer* ⇒ ($'\mathfrak{A}$)*Boolean* (**infix** ≺ *40*)
where $x \prec y \equiv \lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\delta\ y)\ \tau = true\ \tau$
$then\ \lfloor\lfloor\lceil\lceil x\ \tau\rceil\rceil < \lceil\lceil y\ \tau\rceil\rceil\rfloor\rfloor$
$else\ invalid\ \tau$

definition *ocl-le-int* ::($'\mathfrak{A}$)*Integer* ⇒ ($'\mathfrak{A}$)*Integer* ⇒ ($'\mathfrak{A}$)*Boolean* (**infix** ⪯ *40*)
where $x \preceq y \equiv \lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\delta\ y)\ \tau = true\ \tau$
$then\ \lfloor\lfloor\lceil\lceil x\ \tau\rceil\rceil \leq \lceil\lceil y\ \tau\rceil\rceil\rfloor\rfloor$
$else\ invalid\ \tau$

Test Statements

Here follows a list of code-examples, that explain the meanings of the above definitions by compilation to code and execution to *True*.

value $\tau_0 \models (\mathbf{9} \preceq \mathbf{10}\ )$
value $\tau_0 \models ((\ \mathbf{4} \oplus \mathbf{4}\ ) \preceq \mathbf{10}\ )$
value $\neg(\tau_0 \models ((\mathbf{4} \oplus(\ \mathbf{4} \oplus \mathbf{4}\ )) \prec \mathbf{10}\ ))$

4.2. Fundamental Predicates on Boolean and Integer: Strict Equality

4.2.1. Definition

The strict equality on basic types (actually on all types) must be exceptionally defined on *null* — otherwise the entire concept of null in the language does not make much sense. This is an important exception from the general rule that null arguments — especially if passed as "self"-argument — lead to invalid results.

consts *StrictRefEq* :: [($'\mathfrak{A}$,$'a$)*val*,($'\mathfrak{A}$,$'a$)*val*] ⇒ ($'\mathfrak{A}$)*Boolean* (**infixl** $\doteq$ *30*)

syntax
notequal :: ($'\mathfrak{A}$)*Boolean* ⇒ ($'\mathfrak{A}$)*Boolean* ⇒ ($'\mathfrak{A}$)*Boolean* (**infix** <> *40*)
translations
$a <> b == CONST\ not(\ a \doteq b)$

defs *StrictRefEq-bool*[*code-unfold*] :
$(x::('\mathfrak{A})Boolean) \doteq y \equiv \lambda\ \tau.\ if\ (\upsilon\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ (x \triangleq y)\tau$
$else\ invalid\ \tau$

defs *StrictRefEq-int*[*code-unfold*] :
$(x::('\mathfrak{A})Integer) \doteq y \equiv \lambda\ \tau.\ if\ (\upsilon\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$

then $(x \triangleq y)\ \tau$
else invalid τ

4.2.2. Logic and Algebraic Layer on Basic Types

Validity and Definedness Properties (I)

lemma *StrictRefEq-bool-defined-args-valid*:
$(\tau \models \delta((x::('\mathfrak{A})Boolean) \doteq y)) = ((\tau \models(\upsilon\ x)) \wedge (\tau \models(\upsilon\ y)))$
by(*auto simp*: *StrictRefEq-bool OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def null-fun-def bot-fun-def null-option-def bot-option-def*
split: *bool.split-asm HOL.split-if-asm option.split*)

lemma *StrictRefEq-int-defined-args-valid*:
$(\tau \models \delta((x::('\mathfrak{A})Integer) \doteq y)) = ((\tau \models(\upsilon\ x)) \wedge (\tau \models(\upsilon\ y)))$
by(*auto simp*: *StrictRefEq-int OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def null-fun-def bot-fun-def null-option-def bot-option-def*
split: *bool.split-asm HOL.split-if-asm option.split*)

Validity and Definedness Properties (II)

lemma *StrictRefEq-bool-defargs*:
$\tau \models ((x::('\mathfrak{A})Boolean) \doteq y) \Longrightarrow (\tau \models (\upsilon\ x)) \wedge (\tau \models(\upsilon\ y))$
by(*simp add*: *StrictRefEq-bool OclValid-def true-def invalid-def bot-option-def*
split: *bool.split-asm HOL.split-if-asm*)

lemma *StrictRefEq-int-defargs*:
$\tau \models ((x::('\mathfrak{A})Integer) \doteq y) \Longrightarrow (\tau \models (\upsilon\ x)) \wedge (\tau \models (\upsilon\ y))$
by(*simp add*: *StrictRefEq-int OclValid-def true-def invalid-def valid-def bot-option-def*
split: *bool.split-asm HOL.split-if-asm*)

Validity and Definedness Properties (III) Miscellaneous

lemma *StrictRefEq-bool-strict''* : $\delta\ ((x::('\mathfrak{A})Boolean) \doteq y) = (\upsilon(x)\ \mathit{and}\ \upsilon(y))$
by(*auto intro*!: *transform2-rev defined-and-I simp*:*foundation10 StrictRefEq-bool-defined-args-valid*)

lemma *StrictRefEq-int-strict''* : $\delta\ ((x::('\mathfrak{A})Integer) \doteq y) = (\upsilon(x)\ \mathit{and}\ \upsilon(y))$
by(*auto intro*!: *transform2-rev defined-and-I simp*:*foundation10 StrictRefEq-int-defined-args-valid*)

lemma *StrictRefEq-int-strict* :
assumes *A*: $\upsilon\ (x::('\mathfrak{A})Integer) = \mathit{true}$
and *B*: $\upsilon\ y = \mathit{true}$
shows $\upsilon\ (x \doteq y) = \mathit{true}$
apply(*insert A B*)
apply(*rule ext*, *simp add*: *StrongEq-def StrictRefEq-int true-def valid-def defined-def bot-fun-def bot-option-def*)
done

lemma *StrictRefEq-int-strict'* :
  assumes *A*: υ $(((x::(\mathfrak{'A})Integer)) \doteq y) = true$
  shows $\upsilon\ x = true \wedge \upsilon\ y = true$
  apply(*insert A*, *rule conjI*)
  apply(*rule ext*, *drule-tac x=xa* **in** *fun-cong*)
  prefer *2*
  apply(*rule ext*, *drule-tac x=xa* **in** *fun-cong*)
  apply(*simp-all add*: *StrongEq-def StrictRefEq-int*
                false-def true-def valid-def defined-def)
  apply(*case-tac y xa*, *auto*)
  apply(*simp-all add*: *true-def invalid-def bot-fun-def*)
  done

Reflexivity

lemma *StrictRefEq-bool-refl*[*simp*,*code-unfold*] :
$((x::(\mathfrak{'A})Boolean) \doteq x) = (if\ (\upsilon\ x)\ then\ true\ else\ invalid\ endif)$
by(*rule ext*, *simp add*: *StrictRefEq-bool if-ocl-def*)

lemma *StrictRefEq-int-refl*[*simp*,*code-unfold*] :
$((x::(\mathfrak{'A})Integer) \doteq x) = (if\ (\upsilon\ x)\ then\ true\ else\ invalid\ endif)$
by(*rule ext*, *simp add*: *StrictRefEq-int if-ocl-def*)

Execution with invalid or null as argument

lemma *StrictRefEq-bool-strict1*[*simp*] : $((x::(\mathfrak{'A})Boolean) \doteq invalid) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-bool true-def false-def*)

lemma *StrictRefEq-bool-strict2*[*simp*] : $(invalid \doteq (x::(\mathfrak{'A})Boolean)) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-bool true-def false-def*)

lemma *StrictRefEq-int-strict1*[*simp*] : $((x::(\mathfrak{'A})Integer) \doteq invalid) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-int true-def false-def*)

lemma *StrictRefEq-int-strict2*[*simp*] : $(invalid \doteq (x::(\mathfrak{'A})Integer)) = invalid$
by(*rule ext*, *simp add*: *StrictRefEq-int true-def false-def*)

lemma *integer-non-null* [*simp*]: $((\lambda\text{-}.\ \lfloor\lfloor n\rfloor\rfloor) \doteq (null::(\mathfrak{'A})Integer)) = false$
by(*rule ext*,*auto simp*: *StrictRefEq-int valid-def*
        bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

lemma *null-non-integer* [*simp*]: $((null::(\mathfrak{'A})Integer) \doteq (\lambda\text{-}.\ \lfloor\lfloor n\rfloor\rfloor)) = false$
by(*rule ext*,*auto simp*: *StrictRefEq-int valid-def*
        bot-fun-def bot-option-def null-fun-def null-option-def StrongEq-def)

lemma *zero-non-null* [*simp*]: $(\mathbf{0} \doteq null) = false$ **by**(*simp add*: *ocl-zero-def*)
lemma *null-non-zero* [*simp*]: $(null \doteq \mathbf{0}) = false$ **by**(*simp add*: *ocl-zero-def*)
lemma *one-non-null* [*simp*]: $(\mathbf{1} \doteq null) = false$ **by**(*simp add*: *ocl-one-def*)

lemma *null-non-one* [*simp*]: $(null \doteq \mathbf{1}) = false$ **by**(*simp add*: *ocl-one-def*)
lemma *two-non-null* [*simp*]: $(\mathbf{2} \doteq null) = false$ **by**(*simp add*: *ocl-two-def*)
lemma *null-non-two* [*simp*]: $(null \doteq \mathbf{2}) = false$ **by**(*simp add*: *ocl-two-def*)
lemma *six-non-null* [*simp*]: $(\mathbf{6} \doteq null) = false$ **by**(*simp add*: *ocl-six-def*)
lemma *null-non-six* [*simp*]: $(null \doteq \mathbf{6}) = false$ **by**(*simp add*: *ocl-six-def*)
lemma *eight-non-null* [*simp*]: $(\mathbf{8} \doteq null) = false$ **by**(*simp add*: *ocl-eight-def*)
lemma *null-non-eight* [*simp*]: $(null \doteq \mathbf{8}) = false$ **by**(*simp add*: *ocl-eight-def*)
lemma *nine-non-null* [*simp*]: $(\mathbf{9} \doteq null) = false$ **by**(*simp add*: *ocl-nine-def*)
lemma *null-non-nine* [*simp*]: $(null \doteq \mathbf{9}) = false$ **by**(*simp add*: *ocl-nine-def*)

Behavior vs StrongEq

lemma *StrictRefEq-bool-vs-strongEq*:
$\tau \models(\upsilon\ x) \Longrightarrow \tau \models(\upsilon\ y) \Longrightarrow (\tau \models (((x{::}('\mathfrak{A})Boolean) \doteq y) \triangleq (x \triangleq y)))$
apply(*simp add*: *StrictRefEq-bool OclValid-def*)
apply(*subst cp-StrongEq*)**back**
by *simp*

lemma *StrictRefEq-int-vs-strongEq*:
$\tau \models(\upsilon\ x) \Longrightarrow \tau \models(\upsilon\ y) \Longrightarrow (\tau \models (((x{::}('\mathfrak{A})Integer) \doteq y) \triangleq (x \triangleq y)))$
apply(*simp add*: *StrictRefEq-int OclValid-def*)
apply(*subst cp-StrongEq*)**back**
by *simp*

Context Passing

lemma *cp-StrictRefEq-bool*:
$((X{::}('\mathfrak{A})Boolean) \doteq Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau) \doteq (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
by(*auto simp*: *StrictRefEq-bool StrongEq-def defined-def valid-def cp-defined*[*symmetric*])

lemma *cp-StrictRefEq-int*:
$((X{::}('\mathfrak{A})Integer) \doteq Y)\ \tau = ((\lambda\ \text{-}.\ X\ \tau) \doteq (\lambda\ \text{-}.\ Y\ \tau))\ \tau$
by(*auto simp*: *StrictRefEq-int StrongEq-def valid-def cp-defined*[*symmetric*])

lemmas *cp-intro′*[*simp*,*intro*!] =
cp-intro′
cp-StrictRefEq-bool[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of StrictRefEq*]]
cp-StrictRefEq-int[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of StrictRefEq*]]

4.2.3. Test Statements on Basic Types.

Here follows a list of code-examples, that explain the meanings of the above definitions by compilation to code and execution to *True*.

Elementary computations on Booleans

value $\tau_0 \models \upsilon(true)$
value $\tau_0 \models \delta(false)$

value $\neg(\tau_0 \models \delta(\mathit{null}))$
value $\neg(\tau_0 \models \delta(\mathit{invalid}))$
value $\tau_0 \models \upsilon((\mathit{null}::('\mathfrak{A})\mathit{Boolean}))$
value $\neg(\tau_0 \models \upsilon(\mathit{invalid}))$
value $\tau_0 \models (\mathit{true}\ \mathit{and}\ \mathit{true})$
value $\tau_0 \models (\mathit{true}\ \mathit{and}\ \mathit{true} \triangleq \mathit{true})$
value $\tau_0 \models ((\mathit{null}\ \mathit{or}\ \mathit{null}) \triangleq \mathit{null})$
value $\tau_0 \models ((\mathit{null}\ \mathit{or}\ \mathit{null}) \doteq \mathit{null})$
value $\tau_0 \models ((\mathit{true} \triangleq \mathit{false}) \triangleq \mathit{false})$
value $\tau_0 \models ((\mathit{invalid} \triangleq \mathit{false}) \triangleq \mathit{false})$
value $\tau_0 \models ((\mathit{invalid} \doteq \mathit{false}) \triangleq \mathit{invalid})$

Elementary computations on Integer

value $\tau_0 \models \upsilon(\mathbf{4})$
value $\tau_0 \models \delta(\mathbf{4})$
value $\tau_0 \models \upsilon((\mathit{null}::('\mathfrak{A})\mathit{Integer}))$
value $\tau_0 \models (\mathit{invalid} \triangleq \mathit{invalid}\ )$
value $\tau_0 \models (\mathit{null} \triangleq \mathit{null}\ )$
value $\tau_0 \models (\mathbf{4} \triangleq \mathbf{4})$
value $\neg(\tau_0 \models (\mathbf{9} \triangleq \mathbf{10}\ ))$
value $\neg(\tau_0 \models (\mathit{invalid} \triangleq \mathbf{10}\ ))$
value $\neg(\tau_0 \models (\mathit{null} \triangleq \mathbf{10}\ ))$
value $\neg(\tau_0 \models (\mathit{invalid} \doteq (\mathit{invalid}::('\mathfrak{A})\mathit{Integer})))$
value $\tau_0 \models (\mathit{null} \doteq (\mathit{null}::('\mathfrak{A})\mathit{Integer})\ )$
value $\tau_0 \models (\mathit{null} \doteq (\mathit{null}::('\mathfrak{A})\mathit{Integer})\ )$
value $\tau_0 \models (\mathbf{4} \doteq \mathbf{4})$
value $\neg(\tau_0 \models (\mathbf{4} \doteq \mathbf{10}\ ))$

4.3. Complex Types: The Set-Collection Type (I)

4.3.1. The construction of the Set Type

no-notation *None* $(\bot)$
notation *bot* $(\bot)$

For the semantic construction of the collection types, we have two goals:

1. we want the types to be *fully abstract*, i.e. the type should not contain junk-elements that are not representable by OCL expressions, and
2. we want a possibility to nest collection types (so, we want the potential to talking about $\mathit{Set}(\mathit{Set}(\mathit{Sequences}(\mathit{Pairs}(X,Y))))$).

The former principe rules out the option to define $'\alpha$ *Set* just by $('\mathfrak{A}$, $('\alpha$ *option option*) *set*) *val*. This would allow sets to contain junk elements such as $\{\bot\}$ which we need to identify with undefinedness itself. Abandoning fully abstractness of rules would later on produce all sorts of problems when quantifying over the elements of a type. However, if we build an own type, then it must conform to our abstract interface in order to have nested types: arguments of type-constructors must conform to our abstract interface, and the result type too.

The core of an own type construction is done via a type definition which provides the raw-type $'\alpha$ *Set-0*. it is shown that this type "fits" indeed into the abstract type interface discussed in the previous section.

typedef $'\alpha$ *Set-0* $=\{X::('\alpha::null)$ *set option option*.
$X = bot \lor X = null \lor (\forall x \in \lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
by (*rule-tac* x=*bot* **in** *exI*, *simp*)

instantiation *Set-0* :: (*null*)*bot*
begin

definition *bot-Set-0-def*: $(bot::('a::null)$ *Set-0*$) \equiv$ *Abs-Set-0 None*

instance proof show $\exists x::'a$ *Set-0*. $x \neq bot$
apply(*rule-tac* x=*Abs-Set-0* $\lfloor None\rfloor$ **in** *exI*)
apply(*simp add*:*bot-Set-0-def*)
apply(*subst Abs-Set-0-inject*)
apply(*simp-all add*: *bot-Set-0-def*
null-option-def bot-option-def)
done
qed
end

instantiation *Set-0* :: (*null*)*null*
begin

definition *null-Set-0-def*: $(null::('a::null)$ *Set-0*$) \equiv$ *Abs-Set-0* $\lfloor$ *None* $\rfloor$

instance proof show $(null::('a::null)$ *Set-0*$) \neq bot$
apply(*simp add*:*null-Set-0-def bot-Set-0-def*)
apply(*subst Abs-Set-0-inject*)
apply(*simp-all add*: *bot-Set-0-def*
null-option-def bot-option-def)
done
qed
end

... and lifting this type to the format of a valuation gives us:

type-synonym $('\mathfrak{A},'\alpha)$ *Set* $= ('\mathfrak{A}, '\alpha$ *Set-0*$)$ *val*

4.3.2. Validity and Definedness Properties

Every element in a defined set is valid.

lemma *Set-inv-lemma*: $\tau \models (\delta\ X) \Longrightarrow \forall x \in \lceil\lceil$*Rep-Set-0* $(X\ \tau)\rceil\rceil.\ x \neq bot$
apply(*insert OCL-lib.Set-0.Rep-Set-0* [*of* X τ], *simp*)
apply(*auto simp*: *OclValid-def defined-def false-def true-def cp-def*
bot-fun-def bot-Set-0-def null-Set-0-def null-fun-def
split:*split-if-asm*)

apply(*erule contrapos-pp* [*of Rep-Set-0* (X τ) = *bot*])
apply(*subst Abs-Set-0-inject*[*symmetric*], *rule Rep-Set-0*, *simp*)
apply(*simp add*: *Rep-Set-0-inverse bot-Set-0-def bot-option-def*)
apply(*erule contrapos-pp* [*of Rep-Set-0* (X τ) = *null*])
apply(*subst Abs-Set-0-inject*[*symmetric*], *rule Rep-Set-0*, *simp*)
apply(*simp add*: *Rep-Set-0-inverse null-option-def*)
by (*metis bot-option-def null-Set-0-def null-option-def*)

lemma *Set-inv-lemma'* :
assumes *x-def* : $\tau \models \delta$ X
and *e-mem* : $e \in \lceil\lceil$*Rep-Set-0* (X τ)$\rceil\rceil$
shows $\tau \models \upsilon$ (λ-. e)
apply(*rule Set-inv-lemma*[*OF x-def*, *THEN ballE*[**where** x = e]])
apply (*metis foundation18'*)
by (*metis e-mem*)

lemma *abs-rep-simp'* :
assumes *S-all-def* : $\tau \models \delta$ (S :: ('$\mathfrak{A}$, '*a option option*) *Set*)
shows *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* ($S$ τ)$\rceil\rceil\rfloor\rfloor$ = S τ
proof –
have *discr-eq-false-true* : $\bigwedge\tau$. (*false* τ = *true* τ) = *False* **by** (*metis OclValid-def foundation2*)
show *?thesis*
apply(*insert S-all-def*, *simp add*: *OclValid-def defined-def*)
apply(*rule mp*[*OF Abs-Set-0-induct*[**where** $P = \lambda S$. (*if* $S = \bot$ τ $\vee$ S = *null* τ *then false* τ *else true* τ) = *true* τ $\longrightarrow$ *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* $S\rceil\rceil\rfloor\rfloor$ = S]])
apply(*simp add*: *Abs-Set-0-inverse discr-eq-false-true*)
apply(*case-tac y*) **apply**(*simp add*: *bot-fun-def bot-Set-0-def*)+
apply(*case-tac a*) **apply**(*simp add*: *null-fun-def null-Set-0-def*)+
done
qed

lemma *S-lift'* :
assumes *S-all-def* : (τ :: '$\mathfrak{A}$ *st*) $\models \delta$ S
shows $\exists S'$. (λa (-::'$\mathfrak{A}$ *st*). a) ‘ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$ = (λa (-::'$\mathfrak{A}$ *st*). $\lfloor a\rfloor$) ‘ S'
apply(*rule-tac* x = (λa. $\lceil a\rceil$) ‘ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$ **in** *exI*)
apply(*simp only*: *image-comp*[*symmetric*])
apply(*simp add*: *comp-def*)
apply(*subgoal-tac* $\forall x\in$ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$. $\lfloor\lceil x\rceil\rfloor = x$)
apply(*rule equalityI*)

apply(*rule subsetI*)
apply(*drule imageE*) **prefer** *2* **apply** *assumption*
apply(*drule-tac* x = a **in** *ballE*) **prefer** *3* **apply** *assumption*
apply(*drule-tac* f = λx τ. $\lfloor\lceil x\rceil\rfloor$ **in** *imageI*)
apply(*simp*)
apply(*simp*)

apply(*rule subsetI*)
apply(*drule imageE*) **prefer** *2* **apply** *assumption*

apply(*drule-tac* $x = xa$ **in** *ballE*) **prefer** *3* **apply** *assumption*
apply(*drule-tac* $f = \lambda x\ \tau.\ x$ **in** *imageI*)
apply(*simp*)
apply(*simp*)

apply(*rule ballI*)
apply(*drule Set-inv-lemma'*[*OF S-all-def*])
apply(*case-tac x*, *simp add*: *bot-option-def foundation18'*)
apply(*simp*)
done

lemma *invalid-set-not-defined* [*simp*,*code-unfold*]:δ(*invalid*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *false* **by** *simp*
lemma *null-set-not-defined* [*simp*,*code-unfold*]:δ(*null*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *false*
by(*simp add*: *defined-def null-fun-def*)
lemma *invalid-set-valid* [*simp*,*code-unfold*]:υ(*invalid*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *false*
by *simp*
lemma *null-set-valid* [*simp*,*code-unfold*]:υ(*null*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set*) = *true*
apply(*simp add*: *valid-def null-fun-def bot-fun-def bot-Set-0-def null-Set-0-def*)
apply(*subst Abs-Set-0-inject*,*simp-all add*: *null-option-def bot-option-def*)
done

... which means that we can have a type ($'\mathfrak{A}$,($'\mathfrak{A}$,($'\mathfrak{A}$) *Integer*) *Set*) *Set* corresponding exactly to Set(Set(Integer)) in OCL notation. Note that the parameter $\mathfrak{A}$ still refers to the object universe; making the OCL semantics entirely parametric in the object universe makes it possible to study (and prove) its properties independently from a concrete class diagram.

4.3.3. Constants on Sets

definition *mtSet*::($'\mathfrak{A}$,$'\alpha$::*null*) *Set* (*Set*{})
where *Set*{} $\equiv$ ($\lambda\ \tau.$ *Abs-Set-0* $\lfloor\lfloor\{\}::'\alpha\ set\rfloor\rfloor$)

lemma *mtSet-defined*[*simp*,*code-unfold*]:δ(*Set*{}) = *true*
apply(*rule ext*, *auto simp*: *mtSet-def defined-def null-Set-0-def*
bot-Set-0-def bot-fun-def null-fun-def)
apply(*simp-all add*: *Abs-Set-0-inject bot-option-def null-Set-0-def null-option-def*)
done

lemma *mtSet-valid*[*simp*,*code-unfold*]:υ(*Set*{}) = *true*
apply(*rule ext*,*auto simp*: *mtSet-def valid-def null-Set-0-def*
bot-Set-0-def bot-fun-def null-fun-def)
apply(*simp-all add*: *Abs-Set-0-inject bot-option-def null-Set-0-def null-option-def*)
done

Note that the collection types in OCL allow for null to be included; however, there is the null-collection into which inclusion yields invalid.

4.4. Complex Types: The Set-Collection Type (II)

This part provides a collection of operators for the Set type.

4.4.1. Computational Operations on Set

Definition

definition *OclIncluding* :: $[('\mathfrak{A},'\alpha::null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow ('\mathfrak{A},'\alpha)\ Set$
where $OclIncluding\ x\ y = (\lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil\ \cup \{y\ \tau\}\ \rfloor\rfloor$
$else\ \bot\ )$
notation *OclIncluding* (-->*including*'(-'))

syntax
-*OclFinset* :: *args* => ($'\mathfrak{A},'a::null$) *Set* (*Set*{(-)})
translations
Set{x, xs} == *CONST OclIncluding* (*Set*{xs}) x
Set{x} == *CONST OclIncluding* (*Set*{}) x

definition *OclExcluding* :: $[('\mathfrak{A},'\alpha::null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow ('\mathfrak{A},'\alpha)\ Set$
where $OclExcluding\ x\ y = (\lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil - \{y\ \tau\}\ \rfloor\rfloor$
$else\ \bot\ )$
notation *OclExcluding* (-->*excluding*'(-'))

definition *OclIncludes* :: $[('\mathfrak{A},'\alpha::null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow\ '\mathfrak{A}\ Boolean$
where $OclIncludes\ x\ y = (\lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ \lfloor\lfloor(y\ \tau) \in \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil\ \rfloor\rfloor$
$else\ \bot\ )$
notation *OclIncludes* (-->*includes*'(-') [66,65]65)

definition *OclExcludes* :: $[('\mathfrak{A},'\alpha::null)\ Set,('\mathfrak{A},'\alpha)\ val] \Rightarrow\ '\mathfrak{A}\ Boolean$
where $OclExcludes\ x\ y = (not(OclIncludes\ x\ y))$
notation *OclExcludes* (-->*excludes*'(-') [66,65]65)

The case of the size definition is somewhat special, we admit explicitly in Featherweight OCL the possibility of infinite sets. For the size definition, this requires an extra condition that assures that the cardinality of the set is actually a defined integer.

definition *OclSize* :: $('\mathfrak{A},'\alpha::null)Set \Rightarrow\ '\mathfrak{A}\ Integer$
where $OclSize\ x = (\lambda\ \tau.\ if\ (\delta\ x)\ \tau = true\ \tau \wedge finite(\lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil)$
$then\ \lfloor\lfloor\ int(card\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil)\ \rfloor\rfloor$
$else\ \bot\ )$
notation
OclSize (-->*size*'(') [66])

The following definition follows the requirement of the standard to treat null as neutral element of sets. It is a well-documented exception from the general strictness rule and the rule that the distinguished argument self should be non-null.

definition $\mathit{OclIsEmpty}$ :: $('\mathfrak{A},'\alpha{::}null)\ Set \Rightarrow {}'\mathfrak{A}\ Boolean$
where $\mathit{OclIsEmpty}\ x = ((x \doteq null)\ or\ ((\mathit{OclSize}\ x) \doteq \mathbf{0}))$
notation $\mathit{OclIsEmpty}$ $(\text{-}{-}{>}isEmpty'(')\ [66])$

definition $\mathit{OclNotEmpty}$ :: $('\mathfrak{A},'\alpha{::}null)\ Set \Rightarrow {}'\mathfrak{A}\ Boolean$
where $\mathit{OclNotEmpty}\ x = not(\mathit{OclIsEmpty}\ x)$
notation $\mathit{OclNotEmpty}$ $(\text{-}{-}{>}notEmpty'(')\ [66])$

The definition of OclForall mimics the one of *op and*: OclForall is not a strict operation.

definition $\mathit{OclForall}$ :: $[('\mathfrak{A},'\alpha{::}null)Set,('\mathfrak{A},'\alpha)val \Rightarrow ('\mathfrak{A})Boolean] \Rightarrow {}'\mathfrak{A}\ Boolean$
where $\mathit{OclForall}\ S\ P = (\lambda\ \tau.\ if\ (\delta\ S)\ \tau = true\ \tau$
$then\ if\ (\exists\, x \in \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil.\ P(\lambda\ \text{-}.\ x)\ \tau = false\ \tau)$
$then\ false\ \tau$
$else\ if\ (\exists\, x \in \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil.\ P(\lambda\ \text{-}.\ x)\ \tau = \bot\ \tau)$
$then\ \bot\ \tau$
$else\ if\ (\exists\, x \in \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil.\ P(\lambda\ \text{-}.\ x)\ \tau = null\ \tau)$
$then\ null\ \tau$
$else\ true\ \tau$
$else\ \bot)$
syntax
$\text{-}\mathit{OclForall}$ :: $[('\mathfrak{A},'\alpha{::}null)\ Set,id,('\mathfrak{A})Boolean] \Rightarrow {}'\mathfrak{A}\ Boolean$ $((\text{-}){-}{>}forall'(\text{-}|\text{-}'))$
translations
$X{-}{>}forall(x \mid P) == CONST\ \mathit{OclForall}\ X\ (\%x.\ P)$

Like OclForall, OclExists is also not strict.

definition $\mathit{OclExists}$ :: $[('\mathfrak{A},'\alpha{::}null)\ Set,('\mathfrak{A},'\alpha)val \Rightarrow ('\mathfrak{A})Boolean] \Rightarrow {}'\mathfrak{A}\ Boolean$
where $\mathit{OclExists}\ S\ P = not(\mathit{OclForall}\ S\ (\lambda\ X.\ not\ (P\ X)))$

syntax
$\text{-}\mathit{OclExist}$ :: $[('\mathfrak{A},'\alpha{::}null)\ Set,id,('\mathfrak{A})Boolean] \Rightarrow {}'\mathfrak{A}\ Boolean$ $((\text{-}){-}{>}exists'(\text{-}|\text{-}'))$
translations
$X{-}{>}exists(x \mid P) == CONST\ \mathit{OclExists}\ X\ (\%x.\ P)$

definition $\mathit{OclIterate}_{Set}$ :: $[('\mathfrak{A},'\alpha{::}null)\ Set,('\mathfrak{A},'\beta{::}null)val,$
$('\mathfrak{A},'\alpha)val \Rightarrow ('\mathfrak{A},'\beta)val \Rightarrow ('\mathfrak{A},'\beta)val] \Rightarrow ('\mathfrak{A},'\beta)val$
where $\mathit{OclIterate}_{Set}\ S\ A\ F = (\lambda\ \tau.\ if\ (\delta\ S)\ \tau = true\ \tau \wedge (\upsilon\ A)\ \tau = true\ \tau \wedge finite\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}$ $(S\ \tau)\rceil\rceil$
$then\ (\mathit{Finite\text{-}Set.fold}\ (F)\ (A)\ ((\lambda a\ \tau.\ a)\ `\ \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil))\tau$
$else\ \bot)$
syntax
$\text{-}\mathit{OclIterate}$ :: $[('\mathfrak{A},'\alpha{::}null)\ Set,\ idt,\ idt,\ '\alpha,\ '\beta] => ('\mathfrak{A},'\gamma)val$
$(\text{-}\ {-}{>}iterate'(\text{-};\text{-}{=}\text{-}\ |\ \text{-}')\ [71,100,70]50)$
translations
$X{-}{>}iterate(a;\ x = A \mid P) == CONST\ \mathit{OclIterate}_{Set}\ X\ A\ (\%a.\ (\%\ x.\ P))$

Definition (futur operators)

consts

OclUnion :: $[('\mathfrak{A},'\alpha::null)$ *Set*,$('\mathfrak{A},'\alpha)$ *Set*$] \Rightarrow ('\mathfrak{A},'\alpha)$ *Set*
OclIntersection:: $[('\mathfrak{A},'\alpha::null)$ *Set*,$('\mathfrak{A},'\alpha)$ *Set*$] \Rightarrow ('\mathfrak{A},'\alpha)$ *Set*
OclIncludesAll :: $[('\mathfrak{A},'\alpha::null)$ *Set*,$('\mathfrak{A},'\alpha)$ *Set*$] \Rightarrow '\mathfrak{A}$ *Boolean*
OclExcludesAll :: $[('\mathfrak{A},'\alpha::null)$ *Set*,$('\mathfrak{A},'\alpha)$ *Set*$] \Rightarrow '\mathfrak{A}$ *Boolean*
OclComplement :: $('\mathfrak{A},'\alpha::null)$ *Set* $\Rightarrow ('\mathfrak{A},'\alpha)$ *Set*
OclSum :: $('\mathfrak{A},'\alpha::null)$ *Set* $\Rightarrow '\mathfrak{A}$ *Integer*
OclCount :: $[('\mathfrak{A},'\alpha::null)$ *Set*,$('\mathfrak{A},'\alpha)$ *Set*$] \Rightarrow '\mathfrak{A}$ *Integer*

notation
OclCount (-->*count*'(-') [*66*,*65*]*65*)
notation
OclSum (-->*sum*'(') [*66*])
notation
OclIncludesAll (-->*includesAll*'(-') [*66*,*65*]*65*)
notation
OclExcludesAll (-->*excludesAll*'(-') [*66*,*65*]*65*)
notation
OclComplement (-->*complement*'('))
notation
OclUnion (-->*union*'(-') [*66*,*65*]*65*)
notation
OclIntersection(-->*intersection*'(-') [*71*,*70*]*70*)

4.4.2. Validity and Definedness Properties

OclIncluding

lemma *including-defined-args-valid*:
$(\tau \models \delta(X{-}{>}including(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof –
have $A : \bot \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *bot-option-def*)
have $B : \lfloor\bot\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def* *bot-option-def*)
have $C : (\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
have *D*: $(\tau \models \delta(X{-}{>}including(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclIncluding-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
have *E*: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \delta(X{-}{>}including(x)))$
apply(*subst OclIncluding-def*, *subst OclValid-def*, *subst defined-def*)

apply(*auto simp*: *OclValid-def null-Set-0-def bot-Set-0-def null-fun-def bot-fun-def*)
apply(*frule Abs-Set-0-inject*[*OF C A*, *simplified OclValid-def*, *THEN iffD1*], *simp-all add*: *bot-option-def*)
apply(*frule Abs-Set-0-inject*[*OF C B*, *simplified OclValid-def*, *THEN iffD1*], *simp-all add*: *bot-option-def*)
done
show *?thesis* **by**(*auto dest*:*D intro*:*E*)
qed

lemma *including-valid-args-valid*:
$(\tau \models \upsilon(X{-}{>}including(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof –
have *D*: $(\tau \models \upsilon(X{-}{>}including(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclIncluding-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
have *E*: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \upsilon(X{-}{>}including(x)))$
by(*simp add*: *foundation20 including-defined-args-valid*)
show *?thesis* **by**(*auto dest*:*D intro*:*E*)
qed

lemma *including-defined-args-valid'*[*simp*,*code-unfold*]:
$\delta(X{-}{>}including(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*including-defined-args-valid foundation10 defined-and-I*)

lemma *including-valid-args-valid''*[*simp*,*code-unfold*]:
$\upsilon(X{-}{>}including(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*including-valid-args-valid foundation10 defined-and-I*)

OclExcluding

lemma *excluding-defined-args-valid*:
$(\tau \models \delta(X{-}{>}excluding(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof –
have *A* : $\bot \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *bot-option-def*)
have *B* : $\lfloor\bot\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)
have *C* : $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor\lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
have *D*: $(\tau \models \delta(X{-}{>}excluding(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclExcluding-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
have *E*: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \delta(X{-}{>}excluding(x)))$

apply(*subst OclExcluding-def*, *subst OclValid-def*, *subst defined-def*)
apply(*auto simp*: *OclValid-def null-Set-0-def bot-Set-0-def null-fun-def bot-fun-def*)
apply(*frule Abs-Set-0-inject*[*OF C A*, *simplified OclValid-def*, *THEN iffD1*], *simp-all add*: *bot-option-def*)
apply(*frule Abs-Set-0-inject*[*OF C B*, *simplified OclValid-def*, *THEN iffD1*], *simp-all add*: *bot-option-def*)
done
show *?thesis* **by**(*auto dest*:D *intro*:E)
qed

lemma *excluding-valid-args-valid*:
$(\tau \models \upsilon(X{-}{>}excluding(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof –
have D: $(\tau \models \upsilon(X{-}{>}excluding(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclExcluding-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
have E: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \upsilon(X{-}{>}excluding(x)))$
by(*simp add*: *foundation20 excluding-defined-args-valid*)
show *?thesis* **by**(*auto dest*:D *intro*:E)
qed

lemma *excluding-valid-args-valid'*[*simp*,*code-unfold*]:
$\delta(X{-}{>}excluding(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*excluding-defined-args-valid foundation10 defined-and-I*)

lemma *excluding-valid-args-valid''*[*simp*,*code-unfold*]:
$\upsilon(X{-}{>}excluding(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*excluding-valid-args-valid foundation10 defined-and-I*)

OclIncludes

lemma *includes-defined-args-valid*:
$(\tau \models \delta(X{-}{>}includes(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof –
have A: $(\tau \models \delta(X{-}{>}includes(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclIncludes-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
have B: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \delta(X{-}{>}includes(x)))$
by(*auto simp*: *OclIncludes-def OclValid-def true-def false-def StrongEq-def defined-def invalid-def valid-def bot-fun-def null-fun-def bot-option-def null-option-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
show *?thesis* **by**(*auto dest*:A *intro*:B)
qed

lemma *includes-valid-args-valid*:
$(\tau \models \upsilon(X{-}{>}includes(x))) = ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
proof −
have *A*: $(\tau \models \upsilon(X{-}{>}includes(x))) \Longrightarrow ((\tau \models(\delta\ X)) \wedge (\tau \models(\upsilon\ x)))$
by(*auto simp*: *OclIncludes-def OclValid-def true-def valid-def false-def StrongEq-def defined-def invalid-def bot-fun-def null-fun-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
have *B*: $(\tau \models(\delta\ X)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow (\tau \models \upsilon(X{-}{>}includes(x)))$
by(*auto simp*: *OclIncludes-def OclValid-def true-def false-def StrongEq-def defined-def invalid-def valid-def bot-fun-def null-fun-def bot-option-def null-option-def*
split: *bool.split-asm HOL.split-if-asm option.split*)
show *?thesis* **by**(*auto dest*:*A intro*:*B*)
qed

lemma *includes-valid-args-valid'*[*simp*,*code-unfold*]:
$\delta(X{-}{>}includes(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*includes-defined-args-valid foundation10 defined-and-I*)

lemma *includes-valid-args-valid''*[*simp*,*code-unfold*]:
$\upsilon(X{-}{>}includes(x)) = ((\delta\ X)\ and\ (\upsilon\ x))$
by(*auto intro*!: *transform2-rev simp*:*includes-valid-args-valid foundation10 defined-and-I*)

4.4.3. Execution with invalid or null as argument

OclIncluding

lemma *including-strict1*[*simp*,*code-unfold*]:$(invalid{-}{>}including(x)) = invalid$
by(*simp add*: *bot-fun-def OclIncluding-def invalid-def defined-def valid-def false-def true-def*)

lemma *including-strict2*[*simp*,*code-unfold*]:$(X{-}{>}including(invalid)) = invalid$
by(*simp add*: *OclIncluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *including-strict3*[*simp*,*code-unfold*]:$(null{-}{>}including(x)) = invalid$
by(*simp add*: *OclIncluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

OclExcluding

lemma *excluding-strict1*[*simp*,*code-unfold*]:$(invalid{-}{>}excluding(x)) = invalid$
by(*simp add*: *bot-fun-def OclExcluding-def invalid-def defined-def valid-def false-def true-def*)

lemma *excluding-strict2*[*simp*,*code-unfold*]:$(X{-}{>}excluding(invalid)) = invalid$
by(*simp add*: *OclExcluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *excluding-strict3*[*simp*,*code-unfold*]:$(null{-}{>}excluding(x)) = invalid$
by(*simp add*: *OclExcluding-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

OclIncludes

lemma *includes-strict1*[*simp*,*code-unfold*]:(*invalid*$->$*includes*(x)) = *invalid*
by(*simp add*: *bot-fun-def OclIncludes-def invalid-def defined-def valid-def false-def true-def*)

lemma *includes-strict2*[*simp*,*code-unfold*]:($X$$->$*includes*(*invalid*)) = *invalid*
by(*simp add*: *OclIncludes-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

lemma *includes-strict3*[*simp*,*code-unfold*]:(*null*$->$*includes*(x)) = *invalid*
by(*simp add*: *OclIncludes-def invalid-def bot-fun-def defined-def valid-def false-def true-def*)

OclIterate

lemma $OclIterate_{Set}$*-strict1*[*simp*]:*invalid*$->$*iterate*(a; x = A | P a x) = *invalid*
by(*simp add*: *bot-fun-def invalid-def* $OclIterate_{Set}$*-def defined-def valid-def false-def true-def*)

lemma $OclIterate_{Set}$*-null1*[*simp*]:*null*$->$*iterate*(a; x = A | P a x) = *invalid*
by(*simp add*: *bot-fun-def invalid-def* $OclIterate_{Set}$*-def defined-def valid-def false-def true-def*)

lemma $OclIterate_{Set}$*-strict2*[*simp*]:$S$$->$*iterate*($a$; x = *invalid* | P a x) = *invalid*
by(*simp add*: *bot-fun-def invalid-def* $OclIterate_{Set}$*-def defined-def valid-def false-def true-def*)

An open question is this ...

lemma $S$$->$*iterate*($a$; x = *null* | P a x) = *invalid*
oops

4.4.4. Context Passing

lemma *cp-OclIncluding*:
($X$$->$*including*($x$)) τ = ((λ -. X τ)$->$*including*(λ -. x τ)) τ
by(*auto simp*: *OclIncluding-def StrongEq-def invalid-def*
cp-defined[*symmetric*] *cp-valid*[*symmetric*])

lemma *cp-OclExcluding*:
($X$$->$*excluding*($x$)) τ = ((λ -. X τ)$->$*excluding*(λ -. x τ)) τ
by(*auto simp*: *OclExcluding-def StrongEq-def invalid-def*
cp-defined[*symmetric*] *cp-valid*[*symmetric*])

lemma *cp-OclIncludes*:
($X$$->$*includes*($x$)) τ = (*OclIncludes* (λ -. X τ) (λ -. x τ) τ)
by(*auto simp*: *OclIncludes-def StrongEq-def invalid-def*
cp-defined[*symmetric*] *cp-valid*[*symmetric*])

lemma *cp-OclIncludes1*:
($X$$->$*includes*($x$)) τ = (*OclIncludes* X (λ -. x τ) τ)
by(*auto simp*: *OclIncludes-def StrongEq-def invalid-def*
cp-defined[*symmetric*] *cp-valid*[*symmetric*])

lemma *cp-OclSize*: $X$$->$*size*() τ = (λ-. X τ)$->$*size*() τ

by(*simp add*: *OclSize-def cp-defined*[*symmetric*])

lemma *cp-OclForall*:
$(X{-}{>}forall(x \mid P\ x))\ \tau = ((\lambda\ \text{-}.\ X\ \tau){-}{>}forall(x \mid P\ (\lambda\ \text{-}.\ x\ \tau)))\ \tau$
by(*simp add*: *OclForall-def cp-defined*[*symmetric*])

lemma *cp-OclIterate*$_{Set}$: $(X{-}{>}iterate(a;\ x = A \mid P\ a\ x))\ \tau =$
$((\lambda\ \text{-}.\ X\ \tau){-}{>}iterate(a;\ x = A \mid P\ a\ x))\ \tau$
by(*simp add*: *OclIterate*$_{Set}$*-def cp-defined*[*symmetric*])

lemmas *cp-intro''*[*simp*,*intro!*] =
cp-intro'
cp-OclIncluding [*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of OclIncluding*]]
cp-OclExcluding [*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of OclExcluding*]]
cp-OclIncludes [*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of OclIncludes*]]
cp-OclSize [*THEN allI*[*THEN allI*[*THEN cpI1*], *of OclSize*]]

4.5. Fundamental Predicates on Set: Strict Equality

4.5.1. Definition

After the part of foundational operations on sets, we detail here equality on sets. Strong Equality is inherited from the OCL core, but we have to consider the case of the strict equality. We decide to overload strict equality in the same way we do for other value's in OCL:

defs *StrictRefEq-set* :
$(x{::}('\mathfrak{A},'\alpha{::}null)Set) \doteq y \equiv \lambda\ \tau.\ if\ (\upsilon\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ (x \triangleq y)\tau$
$else\ invalid\ \tau$

One might object here that for the case of objects, this is an empty definition. The answer is no, we will restrain later on states and objects such that any object has its id stored inside the object (so the ref, under which an object can be referenced in the store will represented in the object itself). For such well-formed stores that satisfy this invariant (the WFF - invariant), the referential equality and the strong equality — and therefore the strict equality on sets in the sense above) coincides.

4.5.2. Logic and Algebraic Layer on Set

Reflexivity

To become operational, we derive:

lemma *StrictRefEq-set-refl*[*simp*,*code-unfold*]:
$((x{::}('\mathfrak{A},'\alpha{::}null)Set) \doteq x) = (if\ (\upsilon\ x)\ then\ true\ else\ invalid\ endif)$
by(*rule ext*, *simp add*: *StrictRefEq-set if-ocl-def*)

Symmetry

lemma *StrictRefEq-set-sym*:
$((x::('\mathfrak{A},'\alpha::null)Set) \doteq y) = (y \doteq x)$
by(*simp add*: *StrictRefEq-set*, *subst StrongEq-sym*, *rule ext*, *simp*)

Execution with invalid or null as argument

lemma *StrictRefEq-set-strict1*: $((x::('\mathfrak{A},'\alpha::null)Set) \doteq invalid) = invalid$
by(*simp add*:*StrictRefEq-set false-def true-def*)

lemma *StrictRefEq-set-strict2*: $(invalid \doteq (y::('\mathfrak{A},'\alpha::null)Set)) = invalid$
by(*simp add*:*StrictRefEq-set false-def true-def*)

lemma *StrictRefEq-set-strictEq-valid-args-valid*:
$(\tau \models \delta\ ((x::('\mathfrak{A},'\alpha::null)Set) \doteq y)) = ((\tau \models (\upsilon\ x)) \wedge (\tau \models \upsilon\ y))$
proof −
  have *A*: $\tau \models \delta\ (x \doteq y) \Longrightarrow \tau \models \upsilon\ x \wedge \tau \models \upsilon\ y$
    apply(*simp add*: *StrictRefEq-set valid-def OclValid-def defined-def*)
    apply(*simp add*: *invalid-def bot-fun-def split*: *split-if-asm*)
    done
  have *B*: $(\tau \models \upsilon\ x) \wedge (\tau \models \upsilon\ y) \Longrightarrow \tau \models \delta\ (x \doteq y)$
    apply(*simp add*: *StrictRefEq-set*, *elim conjE*)
    apply(*drule foundation13*[*THEN iffD2*],*drule foundation13*[*THEN iffD2*])
    apply(*rule cp-validity*[*THEN iffD2*])
    apply(*subst cp-defined*, *simp add*: *foundation22*)
    apply(*simp add*: *cp-defined*[*symmetric*] *cp-validity*[*symmetric*])
    done
  show *?thesis* **by**(*auto intro*!: *A B*)
qed

Behavior vs StrongEq

lemma *strictRefEq-set-vs-strongEq*:
$\tau \models \upsilon\ x \Longrightarrow \tau \models \upsilon\ y \Longrightarrow (\tau \models (((x::('\mathfrak{A},'\alpha::null)Set) \doteq y) \triangleq (x \triangleq y)))$
apply(*drule foundation13*[*THEN iffD2*],*drule foundation13*[*THEN iffD2*])
by(*simp add*:*StrictRefEq-set foundation22*)

Context Passing

lemma *cp-StrictRefEq-set*:$((X::('\mathfrak{A},'\alpha::null)Set) \doteq Y)\ \tau = ((\lambda\text{-}.\ X\ \tau) \doteq (\lambda\text{-}.\ Y\ \tau))\ \tau$
by(*simp add*:*StrictRefEq-set cp-StrongEq*[*symmetric*] *cp-valid*[*symmetric*])

4.6. Execution on Set's Operators

4.6.1. OclIncluding

lemma *including-charn0*[*simp*]:
assumes *val-x*:$\tau \models (\upsilon\ x)$
shows $\tau \models not(Set\{\}{-}{>}includes(x))$

```
using val-x
apply(auto simp: OclValid-def OclIncludes-def not-def false-def true-def)
apply(auto simp: mtSet-def OCL-lib.Set-0.Abs-Set-0-inverse)
done
```

```
lemma including-charn0'[simp,code-unfold]:
Set{}->includes(x) = (if υ x then false else invalid endif)
proof -
  have A: ⋀ τ. (Set{}->includes(invalid)) τ = (if (υ invalid) then false else invalid endif) τ
        by simp
  have B: ⋀ τ x. τ ⊨ (υ x) ⟹ (Set{}->includes(x)) τ = (if υ x then false else invalid endif)
τ
        apply(frule including-charn0, simp add: OclValid-def)
        apply(rule foundation21[THEN fun-cong, simplified StrongEq-def,simplified,
                 THEN iffD1, of - - false])
        by simp
  show ?thesis
    apply(rule ext, rename-tac τ)
    apply(case-tac τ ⊨ (υ x))
    apply(simp-all add: B foundation18)
    apply(subst cp-OclIncludes, simp add: cp-OclIncludes[symmetric] A)
  done
qed
```

```
lemma including-charn1:
assumes def-X:τ ⊨ (δ X)
assumes val-x:τ ⊨ (υ x)
shows        τ ⊨ (X->including(x)->includes(x))
proof -
 have C : ⌊⌊insert (x τ) ⌈⌈Rep-Set-0 (X τ)⌉⌉⌋⌋ ∈ {X. X = bot ∨ X = null ∨ (∀ x∈⌈⌈X⌉⌉. x
≠ bot)}
        apply(insert val-x Set-inv-lemma[OF def-X])
        apply(simp add: foundation18 invalid-def)
        done
 show ?thesis
 apply(subst OclIncludes-def, simp add: def-X[simplified OclValid-def] val-x[simplified OclValid-def]
foundation10[simplified OclValid-def] OclValid-def)
 apply(simp add: OclIncluding-def def-X[simplified OclValid-def] val-x[simplified OclValid-def]
Abs-Set-0-inverse[OF C] true-def)
 done
qed
```

```
lemma including-charn2:
assumes def-X:τ ⊨ (δ X)
and     val-x:τ ⊨ (υ x)
```

and *val-y*:$\tau \models (\upsilon\ y)$
and *neq* :$\tau \models not(x \triangleq y)$
shows $\tau \models (X{-}{>}including(x){-}{>}includes(y)) \triangleq (X{-}{>}includes(y))$
proof –
have C : $\lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*insert val-x Set-inv-lemma*[*OF def-X*])
apply(*simp add*: *foundation18 invalid-def*)
done
show *?thesis*
apply(*subst OclIncludes-def*, *simp add*: *def-X*[*simplified OclValid-def*] *val-x*[*simplified OclValid-def*] *val-y*[*simplified OclValid-def*] *foundation10*[*simplified OclValid-def*] *OclValid-def StrongEq-def*)
apply(*simp add*: *OclIncluding-def OclIncludes-def def-X*[*simplified OclValid-def*] *val-x*[*simplified OclValid-def*] *val-y*[*simplified OclValid-def*] *Abs-Set-0-inverse*[*OF C*] *true-def*)
by(*metis foundation22 foundation6 foundation9 neq*)
qed

One would like a generic theorem of the form:

```
lemma includes_execute[code_unfold]:
"(X->including(x)->includes(y)) = (if \<delta> X then if x \<doteq> y
                                                   then true
                                                   else X->includes(y)
                                                   endif
                                              else invalid endif)"
```

Unfortunately, this does not hold in general, since referential equality is an overloaded concept and has to be defined for each type individually. Consequently, it is only valid for concrete type instances for Boolean, Integer, and Sets thereof...

The computational law `includes_execute` becomes generic since it uses strict equality which in itself is generic. It is possible to prove the following generic theorem and instantiate it if a number of properties that link the polymorphic logical, Strong Equality with the concrete instance of strict quality.

lemma *includes-execute-generic*:
assumes *strict1*: $(x \doteq invalid) = invalid$
and *strict2*: $(invalid \doteq y) = invalid$
and *cp-StrictRefEq*: $\bigwedge (X::('\mathfrak{A},'a::null)val)\ Y\ \tau.\ (X \doteq Y)\ \tau = ((\lambda\text{-}.\ X\ \tau) \doteq (\lambda\text{-}.\ Y\ \tau))\ \tau$
and *StrictRefEq-vs-strongEq*: $\bigwedge (x::('\mathfrak{A},'a::null)val)\ y\ \tau.$
$\tau \models \upsilon\ x \Longrightarrow \tau \models \upsilon\ y \Longrightarrow (\tau \models ((x \doteq y) \triangleq (x \triangleq y)))$
shows
$(X{-}{>}including(x::('\mathfrak{A},'a::null)val){-}{>}includes(y)) =$
$(if\ \delta\ X\ then\ if\ x \doteq y\ then\ true\ else\ X{-}{>}includes(y)\ endif\ else\ invalid\ endif)$
proof –
have A: $\bigwedge\tau.\ \tau \models (X \triangleq invalid) \Longrightarrow$
$(X{-}{>}including(x){-}{>}includes(y))\ \tau = invalid\ \tau$
apply(*rule foundation22*[*THEN iffD1*])

by(*erule StrongEq-L-subst2-rev*,*simp*,*simp*)
have B: $\bigwedge\tau.\ \tau \models (X \triangleq null) \Longrightarrow$
$(X{-}{>}including(x){-}{>}includes(y))\ \tau = invalid\ \ \tau$
apply(*rule foundation22*[*THEN iffD1*])
by(*erule StrongEq-L-subst2-rev*,*simp*,*simp*)

note [*simp*] = *cp-StrictRefEq* [*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of StrictRefEq*]]

have C: $\bigwedge\tau.\ \tau \models (x \triangleq invalid) \Longrightarrow$
$(X{-}{>}including(x){-}{>}includes(y))\ \tau =$
$(if\ x \doteq y\ then\ true\ else\ X{-}{>}includes(y)\ endif)\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
apply(*erule StrongEq-L-subst2-rev*,*simp*,*simp*)
by (*simp add*: *strict2*)
have D:$\bigwedge\tau.\ \tau \models (y \triangleq invalid) \Longrightarrow$
$(X{-}{>}including(x){-}{>}includes(y))\ \tau =$
$(if\ x \doteq y\ then\ true\ else\ X{-}{>}includes(y)\ endif)\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
apply(*erule StrongEq-L-subst2-rev*,*simp*,*simp*)
by (*simp add*: *strict1*)
have E: $\bigwedge\tau.\ \tau \models \upsilon\ x \Longrightarrow \tau \models \upsilon\ y \Longrightarrow$
$(if\ x \doteq y\ then\ true\ else\ X{-}{>}includes(y)\ endif)\ \tau =$
$(if\ x \triangleq y\ then\ true\ else\ X{-}{>}includes(y)\ endif)\ \tau$
apply(*subst cp-if-ocl*)
apply(*subst StrictRefEq-vs-strongEq*[*THEN foundation22*[*THEN iffD1*]])
by(*simp-all add*: *cp-if-ocl*[*symmetric*])
have F: $\bigwedge\tau.\ \tau \models (x \triangleq y) \Longrightarrow$
$(X{-}{>}including(x){-}{>}includes(y))\ \tau = (X{-}{>}including(x){-}{>}includes(x))\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
by(*erule StrongEq-L-subst2-rev*,*simp*, *simp*)
show *?thesis*
apply(*rule ext*, *rename-tac* τ)
apply(*case-tac* $\neg\ (\tau \models (\delta\ X))$, *simp add*:*def-split-local*,*elim disjE A B*)
apply(*case-tac* $\neg\ (\tau \models (\upsilon\ x))$,
simp add:*foundation18 foundation22*[*symmetric*],
drule StrongEq-L-sym)
apply(*simp add*: *foundation22 C*)
apply(*case-tac* $\neg\ (\tau \models (\upsilon\ y))$,
simp add:*foundation18 foundation22*[*symmetric*],
drule StrongEq-L-sym, *simp add*: *foundation22 D*, *simp*)
apply(*subst E*,*simp-all*)
apply(*case-tac* $\tau \models not(x \triangleq y)$)
apply(*simp add*: *including-charn2*[*simplified foundation22*])
apply(*simp add*: *foundation9 F*
including-charn1[*THEN foundation13*[*THEN iffD2*],
THEN foundation22[*THEN iffD1*]]])
done
qed

schematic-lemma *includes-execute-int*[*code-unfold*]: *?X*
by(*rule includes-execute-generic*[*OF StrictRefEq-int-strict1 StrictRefEq-int-strict2*
cp-StrictRefEq-int
StrictRefEq-int-vs-strongEq], *simp-all*)

schematic-lemma *includes-execute-bool*[*code-unfold*]: *?X*
by(*rule includes-execute-generic*[*OF StrictRefEq-bool-strict1 StrictRefEq-bool-strict2*
cp-StrictRefEq-bool
StrictRefEq-bool-vs-strongEq], *simp-all*)

schematic-lemma *includes-execute-set*[*code-unfold*]: *?X*
by(*rule includes-execute-generic*[*OF StrictRefEq-set-strict1 StrictRefEq-set-strict2*
cp-StrictRefEq-set
strictRefEq-set-vs-strongEq], *simp-all*)

lemma *finite-including-exec* :
assumes *X-def* : $\tau \models \delta\ X$
and *x-val* : $\tau \models \upsilon\ x$
shows *finite* $\lceil\lceil$*Rep-Set-0* (*X*−>*including*(*x*) τ)$\rceil\rceil$ = *finite* $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$
proof −
have *C* : $\lfloor\lfloor$*insert* (*x* τ) $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*insert X-def x-val*, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
show *?thesis*
by(*insert X-def x-val*,
auto simp: *OclIncluding-def Abs-Set-0-inverse*[*OF C*]
dest: *foundation13*[*THEN iffD2*, *THEN foundation22*[*THEN iffD1*]])
qed

lemma *including-includes* :
assumes *a-val* : $\tau \models \upsilon\ a$
and *x-val* : $\tau \models \upsilon\ x$
and *S-incl* : $\tau \models$ (*S* :: ('$\mathfrak{A}$, *int option option*) *Set*)−>*includes*(*x*)
shows $\tau \models$ *S*−>*including*(*a*)−>*includes*(*x*)
proof −
have *discr-eq-bot1-true* : $\bigwedge\tau.\ (\perp\ \tau = true\ \tau) = False$ **by** (*metis OCL-core.bot-fun-def foundation1 foundation18′ valid3*)
have *discr-eq-bot2-true* : $\bigwedge\tau.\ (\perp = true\ \tau) = False$ **by** (*metis bot-fun-def discr-eq-bot1-true*)
have *discr-neq-invalid-true* : $\bigwedge\tau.\ (invalid\ \tau \neq true\ \tau) = True$ **by** (*metis discr-eq-bot2-true invalid-def*)
have *discr-eq-invalid-true* : $\bigwedge\tau.\ (invalid\ \tau = true\ \tau) = False$ **by** (*metis bot-option-def invalid-def option.simps*(*2*) *true-def*)

show *?thesis*
apply(*simp add*: *includes-execute-int*)
apply(*subgoal-tac* $\tau \models \delta$ *S*)
prefer *2*
apply(*insert S-incl*[*simplified OclIncludes-def*], *simp add*: *OclValid-def*)
apply(*metis discr-eq-bot2-true*)
apply(*simp add*: *cp-if-ocl*[*of* δ *S*] *OclValid-def if-ocl-def discr-neq-invalid-true discr-eq-invalid-true x-val*[*simplified OclValid-def*])
by (*metis OclValid-def S-incl StrictRefEq-int-strict*$''$ *a-val foundation10 foundation6 x-val*)
qed

4.6.2. OclExcluding

lemma *excluding-charn0*[*simp*]:
assumes *val-x*:$\tau \models (\upsilon\ x)$
shows $\tau \models ((Set\{\}{-}{>}excluding(x)) \triangleq Set\{\})$
proof –
have $A : \lfloor None \rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)
have $B : \lfloor\lfloor\{\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *mtSet-def*)

show *?thesis* **using** *val-x*
apply(*auto simp*: *OclValid-def OclIncludes-def not-def false-def true-def StrongEq-def OclExcluding-def mtSet-def defined-def bot-fun-def null-fun-def null-Set-0-def*)
apply(*auto simp*: *mtSet-def OCL-lib.Set-0.Abs-Set-0-inverse OCL-lib.Set-0.Abs-Set-0-inject*[*OF B A*])
done
qed

lemma *excluding-charn0-exec*[*code-unfold*]:
$(Set\{\}{-}{>}excluding(x)) = (if\ (\upsilon\ x)\ then\ Set\{\}\ else\ invalid\ endif)$
proof –
have *A*: $\bigwedge \tau.\ (Set\{\}{-}{>}excluding(invalid))\ \tau = (if\ (\upsilon\ invalid)\ then\ Set\{\}\ else\ invalid\ endif)\ \tau$
by *simp*
have *B*: $\bigwedge \tau\ x.\ \tau \models (\upsilon\ x) \Longrightarrow (Set\{\}{-}{>}excluding(x))\ \tau = (if\ (\upsilon\ x)\ then\ Set\{\}\ else\ invalid\ endif)\ \tau$
by(*simp add*: *excluding-charn0*[*THEN foundation22*[*THEN iffD1*]])
show *?thesis*
apply(*rule ext*, *rename-tac* τ)
apply(*case-tac* $\tau \models (\upsilon\ x)$)
apply(*simp add*: *B*)
apply(*simp add*: *foundation18*)
apply(*subst cp-OclExcluding*, *simp*)
apply(*simp add*: *cp-if-ocl*[*symmetric*] *cp-OclExcluding*[*symmetric*] *cp-valid*[*symmetric*] *A*)
done
qed

lemma *excluding-charn1*:
assumes *def-X*:$\tau \models (\delta\ X)$
and *val-x*:$\tau \models (\upsilon\ x)$
and *val-y*:$\tau \models (\upsilon\ y)$
and *neq* :$\tau \models not(x \triangleq y)$
shows $\tau \models ((X{-}{>}including(x)){-}{>}excluding(y)) \triangleq ((X{-}{>}excluding(y)){-}{>}including(x))$
proof −
have $A : \bot \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *bot-option-def*)
have $B : \lfloor\bot\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)
have $C : \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*insert def-X val-x*, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
have $D : \lfloor\lfloor\lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil - \{y\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*insert def-X val-x*, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
have $E : x\ \tau \neq y\ \tau$
apply(*insert neq*)
by(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def defined-def valid-def bot-Set-0-def null-fun-def null-Set-0-def StrongEq-def not-def*)

have $G1 : Abs\text{-}Set\text{-}0\ \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \neq Abs\text{-}Set\text{-}0\ None$
apply(*insert C*, *simp*)
apply(*simp add*: *def-X val-x A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)
done
have $G2 : Abs\text{-}Set\text{-}0\ \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor \neq Abs\text{-}Set\text{-}0\ \lfloor None\rfloor$
apply(*insert C*, *simp*)
apply(*simp add*: *def-X val-x A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)
done

have $G : (\delta\ (\lambda\text{-}.\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil\rfloor\rfloor))\ \tau = true\ \tau$
apply(*auto simp*: *OclValid-def false-def true-def defined-def bot-fun-def bot-Set-0-def null-fun-def null-Set-0-def G1 G2*)
done

have $H1 : Abs\text{-}Set\text{-}0\ \lfloor\lfloor\lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil - \{y\ \tau\}\rfloor\rfloor \neq Abs\text{-}Set\text{-}0\ None$
apply(*insert D*, *simp*)
apply(*simp add*: *A Abs-Set-0-inject Abs-Set-0-inverse B C OclExcluding-def OclValid-def Option.set.simps*(*2*) *Rep-Set-0-inverse bot-Set-0-def bot-option-def null-Set-0-def null-option-def option.distinct*(*1*))

done
have *H2* : *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$ − {*y* τ}$\rfloor\rfloor$ ≠ *Abs-Set-0* $\lfloor$*None*$\rfloor$
apply(*insert D*, *simp*)
apply(*simp add*: *A Abs-Set-0-inject Abs-Set-0-inverse B C OclExcluding-def OclValid-def Option.set.simps*(*2*) *Rep-Set-0-inverse bot-Set-0-def bot-option-def null-Set-0-def null-option-def option.distinct*(*1*))
done
have *H* : (δ (λ-. *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$ − {*y* τ}$\rfloor\rfloor$)) τ = *true* τ
apply(*auto simp*: *OclValid-def false-def true-def defined-def bot-fun-def bot-Set-0-def null-fun-def null-Set-0-def H1 H2*)
done

have *Z*:*insert* (*x* τ) $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$ − {*y* τ} = *insert* (*x* τ) ($\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$ − {*y* τ})
by(*auto simp*: *E*)
show *?thesis*
apply(*insert def-X*[*THEN foundation13*[*THEN iffD2*]] *val-x*[*THEN foundation13*[*THEN iffD2*]] *val-y*[*THEN foundation13*[*THEN iffD2*]])
apply(*simp add*: *foundation22 OclIncluding-def OclExcluding-def def-X*[*THEN foundation17*])
apply(*subst cp-defined*, *simp*)+
apply(*simp add*: *G H Abs-Set-0-inverse*[*OF C*] *Abs-Set-0-inverse*[*OF D*] *Z*)
done
qed

lemma *excluding-charn2*:
assumes *def-X*:$\tau \models (\delta\ X)$
and *val-x*:$\tau \models (\upsilon\ x)$
shows $\tau \models$ (((*X*−>*including*(*x*))−>*excluding*(*x*)) ≜ (*X*−>*excluding*(*x*)))
proof −
have *A* : ⊥ ∈ {*X*. *X* = *bot* ∨ *X* = *null* ∨ (∀ *x*∈$\lceil\lceil$*X*$\rceil\rceil$. *x* ≠ *bot*)} **by**(*simp add*: *bot-option-def*)
have *B* : $\lfloor$⊥$\rfloor$ ∈ {*X*. *X* = *bot* ∨ *X* = *null* ∨ (∀ *x*∈$\lceil\lceil$*X*$\rceil\rceil$. *x* ≠ *bot*)} **by**(*simp add*: *null-option-def bot-option-def*)
have *C* : $\lfloor\lfloor$*insert* (*x* τ) $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil\rfloor\rfloor$ ∈ {*X*. *X* = *bot* ∨ *X* = *null* ∨ (∀ *x*∈$\lceil\lceil$*X*$\rceil\rceil$. *x* ≠ *bot*)}
apply(*insert def-X val-x*, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
have *G1* : *Abs-Set-0* $\lfloor\lfloor$*insert* (*x* τ) $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil\rfloor\rfloor$ ≠ *Abs-Set-0 None*
apply(*insert C*, *simp*)
apply(*simp add*: *def-X val-x A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)
done
have *G2* : *Abs-Set-0* $\lfloor\lfloor$*insert* (*x* τ) $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil\rfloor\rfloor$ ≠ *Abs-Set-0* $\lfloor$*None*$\rfloor$
apply(*insert C*, *simp*)
apply(*simp add*: *def-X val-x A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)

done
show *?thesis*
  apply(*insert def-X*[*THEN foundation17*] *val-x*[*THEN foundation19*])
  apply(*auto simp*: *OclValid-def bot-fun-def OclIncluding-def OclIncludes-def false-def true-def invalid-def defined-def valid-def bot-Set-0-def null-fun-def null-Set-0-def StrongEq-def*)
  apply(*subst cp-OclExcluding*) **back**
  apply(*auto simp*:*OclExcluding-def*)
  apply(*simp add*: *Abs-Set-0-inverse*[*OF C*])
  apply(*simp-all add*: *false-def true-def defined-def valid-def null-fun-def bot-fun-def null-Set-0-def bot-Set-0-def split*: *bool.split-asm HOL.split-if-asm option.split*)
  apply(*auto simp*: *G1 G2*)
 done
qed

lemma *excluding-charn-exec*[*code-unfold*]:
assumes *strict1*: $(x \doteq \mathit{invalid}) = \mathit{invalid}$
and *strict2*: $(\mathit{invalid} \doteq y) = \mathit{invalid}$
and *StrictRefEq-valid-args-valid*: $\bigwedge (x::('\mathfrak{A},'a::null)val)\ y\ \tau.$
$(\tau \models \delta\ (x \doteq y)) = ((\tau \models (\upsilon\ x)) \wedge (\tau \models \upsilon\ y))$
and *cp-StrictRefEq*: $\bigwedge (X::('\mathfrak{A},'a::null)val)\ Y\ \tau.\ (X \doteq Y)\ \tau = ((\lambda\text{-}.\ X\ \tau) \doteq (\lambda\text{-}.\ Y\ \tau))\ \tau$
and *StrictRefEq-vs-strongEq*: $\bigwedge (x::('\mathfrak{A},'a::null)val)\ y\ \tau.$
$\tau \models \upsilon\ x \Longrightarrow \tau \models \upsilon\ y \Longrightarrow (\tau \models ((x \doteq y) \triangleq (x \triangleq y)))$
shows $(X{-}{>}including(x::('\mathfrak{A},'a::null)val){-}{>}excluding(y)) =$
$(if\ \delta\ X\ then\ if\ x \doteq y$
$then\ X{-}{>}excluding(y)$
$else\ X{-}{>}excluding(y){-}{>}including(x)$
$endif$
$else\ invalid\ endif)$
proof –

have *A1*: $\bigwedge\tau.\ \tau \models (X \triangleq invalid) \Longrightarrow$
$(X{-}{>}including(x){-}{>}includes(y))\ \tau = invalid\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
by(*erule StrongEq-L-subst2-rev*, *simp*,*simp*)

have *B1*: $\bigwedge\tau.\ \tau \models (X \triangleq null) \Longrightarrow$
$(X{-}{>}including(x){-}{>}includes(y))\ \tau = invalid\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
by(*erule StrongEq-L-subst2-rev*, *simp*,*simp*)

have *A2*: $\bigwedge\tau.\ \tau \models (X \triangleq invalid) \Longrightarrow X{-}{>}including(x){-}{>}excluding(y)\ \tau = invalid\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
by(*erule StrongEq-L-subst2-rev*, *simp*,*simp*)

have *B2*: $\bigwedge\tau.\ \tau \models (X \triangleq null) \Longrightarrow X{-}{>}including(x){-}{>}excluding(y)\ \tau = invalid\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
by(*erule StrongEq-L-subst2-rev*, *simp*,*simp*)

note [*simp*] = *cp-StrictRefEq* [*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]], *of StrictRefEq*]]

have C: $\bigwedge\tau.\ \tau \models (x \triangleq \mathit{invalid}) \Longrightarrow$
$(X{-}{>}\mathit{including}(x){-}{>}\mathit{excluding}(y))\ \tau =$
$(\mathit{if}\ x \doteq y\ \mathit{then}\ X{-}{>}\mathit{excluding}(y)\ \mathit{else}\ X{-}{>}\mathit{excluding}(y){-}{>}\mathit{including}(x)\ \mathit{endif})\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
apply(*erule StrongEq-L-subst2-rev*,*simp*,*simp*)
by(*simp add*: *strict2*)

have D: $\bigwedge\tau.\ \tau \models (y \triangleq \mathit{invalid}) \Longrightarrow$
$(X{-}{>}\mathit{including}(x){-}{>}\mathit{excluding}(y))\ \tau =$
$(\mathit{if}\ x \doteq y\ \mathit{then}\ X{-}{>}\mathit{excluding}(y)\ \mathit{else}\ X{-}{>}\mathit{excluding}(y){-}{>}\mathit{including}(x)\ \mathit{endif})\ \tau$
apply(*rule foundation22*[*THEN iffD1*])
apply(*erule StrongEq-L-subst2-rev*,*simp*,*simp*)
by (*simp add*: *strict1*)

have E: $\bigwedge\tau.\ \tau \models \upsilon\ x \Longrightarrow \tau \models \upsilon\ y \Longrightarrow$
$(\mathit{if}\ x \doteq y\ \mathit{then}\ X{-}{>}\mathit{excluding}(y)\ \mathit{else}\ X{-}{>}\mathit{excluding}(y){-}{>}\mathit{including}(x)\ \mathit{endif})\ \tau =$
$(\mathit{if}\ x \triangleq y\ \mathit{then}\ X{-}{>}\mathit{excluding}(y)\ \mathit{else}\ X{-}{>}\mathit{excluding}(y){-}{>}\mathit{including}(x)\ \mathit{endif})\ \tau$
apply(*subst cp-if-ocl*)
apply(*subst StrictRefEq-vs-strongEq*[*THEN foundation22*[*THEN iffD1*]])
by(*simp-all add*: *cp-if-ocl*[*symmetric*])

have F: $\bigwedge\tau.\ \tau \models \delta\ X \Longrightarrow \tau \models \upsilon\ x \Longrightarrow \tau \models (x \triangleq y) \Longrightarrow$
$(X{-}{>}\mathit{including}(x){-}{>}\mathit{excluding}(y)\ \tau) = (X{-}{>}\mathit{excluding}(y)\ \tau)$
apply(*drule StrongEq-L-sym*)
apply(*rule foundation22*[*THEN iffD1*])
apply(*erule StrongEq-L-subst2-rev*,*simp*)
by(*simp add*: *excluding-charn2*)

show *?thesis*
apply(*rule ext*, *rename-tac* τ)
apply(*case-tac* $\neg\ (\tau \models (\delta\ X))$, *simp add*:*def-split-local*,*elim disjE A1 B1 A2 B2*)
apply(*case-tac* $\neg\ (\tau \models (\upsilon\ x))$,
simp add:*foundation18 foundation22*[*symmetric*],
drule StrongEq-L-sym)
apply(*simp add*: *foundation22 C*)
apply(*case-tac* $\neg\ (\tau \models (\upsilon\ y))$,
simp add:*foundation18 foundation22*[*symmetric*],
drule StrongEq-L-sym, *simp add*: *foundation22 D*, *simp*)
apply(*subst E*,*simp-all*)
apply(*case-tac* $\tau \models \mathit{not}\ (x \triangleq y)$)
apply(*simp add*: *excluding-charn1*[*simplified foundation22*]
excluding-charn2[*simplified foundation22*])
apply(*simp add*: *foundation9 F*)
done
qed

schematic-lemma *excluding-charn-exec-int*[*code-unfold*]: *?X*
by(*rule excluding-charn-exec*[*OF StrictRefEq-int-strict1 StrictRefEq-int-strict2*
StrictRefEq-int-defined-args-valid
cp-StrictRefEq-int StrictRefEq-int-vs-strongEq], *simp-all*)

schematic-lemma *excluding-charn-exec-bool*[*code-unfold*]: *?X*
by(*rule excluding-charn-exec*[*OF StrictRefEq-bool-strict1 StrictRefEq-bool-strict2*
StrictRefEq-bool-defined-args-valid
cp-StrictRefEq-bool StrictRefEq-bool-vs-strongEq], *simp-all*)

schematic-lemma *excluding-charn-exec-set*[*code-unfold*]: *?X*
by(*rule excluding-charn-exec*[*OF StrictRefEq-set-strict1 StrictRefEq-set-strict2*
StrictRefEq-set-strictEq-valid-args-valid
cp-StrictRefEq-set strictRefEq-set-vs-strongEq], *simp-all*)

lemma *finite-excluding-exec* :
assumes *X-def* : $\tau \models \delta\ X$
and *x-val* : $\tau \models \upsilon\ x$
shows *finite* $\lceil\lceil Rep\text{-}Set\text{-}0\ (X{-}{>}excluding(x)\ \tau)\rceil\rceil$ = *finite* $\lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil$
proof –
have C : $\lfloor\lfloor\lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*insert X-def x-val*, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
show *?thesis*
by(*insert X-def x-val*,
auto simp: *OclExcluding-def Abs-Set-0-inverse*[*OF C*]
dest: *foundation13*[*THEN iffD2*, *THEN foundation22*[*THEN iffD1*]])
qed

4.6.3. OclSize

lemma *OclSize-infinite*:
assumes *non-finite*:$\tau \models not(\delta(S{-}{>}size()))$
shows $(\tau \models not(\delta(S))) \lor \neg$ *finite* $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil$
apply(*insert non-finite*, *simp*)
apply(*rule impI*)
apply(*simp add*: *OclSize-def OclValid-def defined-def*)
apply(*case-tac finite* $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil$,
simp-all add:*null-fun-def null-option-def bot-fun-def bot-option-def*)
done

lemma [*simp*]: $\delta\ (Set\{\}\ {-}{>}size()) = true$
proof –
have $A1$: $\lfloor\lfloor\{\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *mtSet-def*)
have $A2$: $None \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*:

bot-option-def)
have *A3* : $\lfloor None \rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x \in \lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *bot-option-def null-option-def*)
show *?thesis*
apply(*rule ext*)
apply(*simp add*: *defined-def mtSet-def OclSize-def*
bot-Set-0-def bot-fun-def
null-Set-0-def null-fun-def)
apply(*subst Abs-Set-0-inject*, *simp-all add*: *A1 A2 A3 bot-option-def null-option-def*) +
by(*simp add*: *A1 Abs-Set-0-inverse bot-fun-def bot-option-def null-fun-def null-option-def*)
qed

lemma *including-size-defined*[*simp*]: δ $((X\ {-}{>}including(x))\ {-}{>}size()) = (\delta(X{-}{>}size())$ *and* $\upsilon(x))$
proof −

have *defined-inject-true* : $\bigwedge\tau\ P.\ (\delta\ P)\ \tau \neq true\ \tau \Longrightarrow (\delta\ P)\ \tau = false\ \tau$
apply(*simp add*: *defined-def true-def false-def bot-fun-def bot-option-def*
null-fun-def null-option-def)
by (*case-tac* $P\ \tau = \bot \lor P\ \tau = null$, *simp-all add*: *true-def*)

have *valid-inject-true* : $\bigwedge\tau\ P.\ (\upsilon\ P)\ \tau \neq true\ \tau \Longrightarrow (\upsilon\ P)\ \tau = false\ \tau$
apply(*simp add*: *valid-def true-def false-def bot-fun-def bot-option-def*
null-fun-def null-option-def)
by (*case-tac* $P\ \tau = \bot$, *simp-all add*: *true-def*)

have *finite-including-exec* : $\bigwedge\tau.\ (\delta\ X\ and\ \upsilon\ x)\ \tau = true\ \tau \Longrightarrow$
$finite\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X{-}{>}including(x)\ \tau)\rceil\rceil = finite\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil$
apply(*rule finite-including-exec*)
apply(*metis OclValid-def foundation5*)+
done

have *card-including-exec* : $\bigwedge\tau.\ (\delta\ (\lambda\text{-}.\ \lfloor\lfloor int\ (card\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X{-}{>}including(x)\ \tau)\rceil\rceil)\rfloor\rfloor))\ \tau = (\delta\ (\lambda\text{-}.\ \lfloor\lfloor int\ (card\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil)\rfloor\rfloor))\ \tau$
apply(*simp add*: *defined-def bot-fun-def bot-option-def null-fun-def null-option-def*)
done

show *?thesis*

apply(*rule ext*, *rename-tac* τ)
apply(*case-tac* $(\delta\ (X{-}{>}including(x){-}{>}size()))\ \tau = true\ \tau$, *simp*)
apply(*subst cp-ocl-and*)
apply(*subst cp-defined*)
apply(*simp only*: *cp-defined*[*of* $X{-}{>}including(x){-}{>}size()$])
apply(*simp add*: *OclSize-def*)
apply(*case-tac* $((\delta\ X\ and\ \upsilon\ x)\ \tau = true\ \tau \land finite\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X{-}{>}including(x)\ \tau)\rceil\rceil)$, *simp*)
prefer *2*
apply(*simp*)

apply(*simp add*: *defined-def true-def false-def bot-fun-def bot-option-def*)
apply(*erule conjE*)
apply(*simp add*: *finite-including-exec*[*simplified OclValid-def*] *card-including-exec*
cp-ocl-and[*of* δ *X* υ *x*]
cp-ocl-and[*of true*, *THEN sym*])
apply(*subgoal-tac* (δ *X*) τ = *true* τ $\wedge$ (υ *x*) τ = *true* τ, *simp*)
apply(*rule foundation5*[*of* - δ *X* υ *x*, *simplified OclValid-def*], *simp only*: *cp-ocl-and*[*THEN sym*])

apply(*drule defined-inject-true*[*of X*−>*including*(*x*)−>*size*()], *simp*)
apply(*simp only*: *cp-ocl-and*[*of* δ (*X*−>*size*()) υ *x*])
apply(*simp add*: *cp-defined*[*of X*−>*including*(*x*)−>*size*()] *cp-defined*[*of X*−>*size*()])
apply(*simp add*: *OclSize-def card-including-exec*)
apply(*case-tac* (δ *X and* υ *x*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$,
simp add: *finite-including-exec*[*simplified OclValid-def*] *card-including-exec*)
apply(*simp only*: *cp-ocl-and*[*THEN sym*])
apply(*simp add*: *defined-def bot-fun-def*)

apply(*split split-if-asm*)
apply(*simp add*: *finite-including-exec*[*simplified OclValid-def*])
apply(*simp add*: *finite-including-exec*[*simplified OclValid-def*] *card-including-exec*)
apply(*simp only*: *cp-ocl-and*[*THEN sym*])
apply(*simp*)
apply(*rule impI*)
apply(*erule conjE*)
apply(*case-tac* (υ *x*) τ = *true* τ, *simp add*: *cp-ocl-and*[*of* δ *X* υ *x*])
apply(*drule valid-inject-true*[*of x*], *simp add*: *cp-ocl-and*[*of* - υ *x*])
done
qed

lemma *excluding-size-defined*[*simp*]: δ ((*X* −>*excluding*(*x*)) −>*size*()) = (δ(*X*−>*size*()) *and* υ(*x*))
proof −

have *defined-inject-true* : $\bigwedge\tau$ *P*. (δ *P*) τ $\neq$ *true* τ $\Longrightarrow$ (δ *P*) τ = *false* τ
apply(*simp add*: *defined-def true-def false-def bot-fun-def*
bot-option-def null-fun-def null-option-def)
by (*case-tac* *P* τ = $\perp$ $\vee$ *P* τ = *null*, *simp-all add*: *true-def*)

have *valid-inject-true* : $\bigwedge\tau$ *P*. (υ *P*) τ $\neq$ *true* τ $\Longrightarrow$ (υ *P*) τ = *false* τ
apply(*simp add*: *valid-def true-def false-def bot-fun-def bot-option-def*
null-fun-def null-option-def)
by(*case-tac P* τ = $\perp$, *simp-all add*: *true-def*)

have *finite-excluding-exec* : $\bigwedge\tau$. (δ *X and* υ *x*) τ = *true* τ $\Longrightarrow$
finite $\lceil\lceil$*Rep-Set-0* (*X*−>*excluding*(*x*) τ)$\rceil\rceil$ =
finite $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$
apply(*rule finite-excluding-exec*)

apply(*metis OclValid-def foundation5*)+
done

have *card-excluding-exec* : $\bigwedge\tau.$ (δ (λ-. $\lfloor\lfloor$*int* (*card* $\lceil\lceil$*Rep-Set-0* (*X*−>*excluding*(*x*) τ)$\rceil\rceil$)$\rfloor\rfloor$)) τ =
(δ (λ-. $\lfloor\lfloor$*int* (*card* $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$)$\rfloor\rfloor$)) τ
apply(*simp add*: *defined-def bot-fun-def bot-option-def null-fun-def null-option-def*)
done

show *?thesis*

apply(*rule ext*, *rename-tac* τ)
apply(*case-tac* (δ (*X*−>*excluding*(*x*)−>*size*())) τ = *true* τ, *simp*)
apply(*subst cp-ocl-and*)
apply(*subst cp-defined*)
apply(*simp only*: *cp-defined*[*of X*−>*excluding*(*x*)−>*size*()])
apply(*simp add*: *OclSize-def*)
apply(*case-tac* ((δ *X and* υ *x*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*X*−>*excluding*(*x*) τ)$\rceil\rceil$), *simp*)
prefer *2*
apply(*simp*)
apply(*simp add*: *defined-def true-def false-def bot-fun-def bot-option-def*)
apply(*erule conjE*)
apply(*simp add*: *finite-excluding-exec card-excluding-exec*
cp-ocl-and[*of* δ *X* υ *x*]
cp-ocl-and[*of true*, *THEN sym*])
apply(*subgoal-tac* (δ *X*) τ = *true* τ $\wedge$ (υ *x*) τ = *true* τ, *simp*)
apply(*rule foundation5*[*of* - δ *X* υ *x*, *simplified OclValid-def*], *simp only*: *cp-ocl-and*[*THEN sym*])

apply(*drule defined-inject-true*[*of X*−>*excluding*(*x*)−>*size*()], *simp*)
apply(*simp only*: *cp-ocl-and*[*of* δ (*X*−>*size*()) υ *x*])
apply(*simp add*: *cp-defined*[*of X*−>*excluding*(*x*)−>*size*()] *cp-defined*[*of X*−>*size*()])
apply(*simp add*: *OclSize-def finite-excluding-exec card-excluding-exec*)
apply(*case-tac* (δ *X and* υ *x*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*X* τ)$\rceil\rceil$,
simp add: *finite-excluding-exec card-excluding-exec*)
apply(*simp only*: *cp-ocl-and*[*THEN sym*])
apply(*simp add*: *defined-def bot-fun-def*)

apply(*split split-if-asm*)
apply(*simp add*: *finite-excluding-exec*)
apply(*simp add*: *finite-excluding-exec card-excluding-exec*)
apply(*simp only*: *cp-ocl-and*[*THEN sym*])
apply(*simp*)
apply(*rule impI*)
apply(*erule conjE*)
apply(*case-tac* (υ *x*) τ = *true* τ, *simp add*: *cp-ocl-and*[*of* δ *X* υ *x*])
apply(*drule valid-inject-true*[*of x*], *simp add*: *cp-ocl-and*[*of* - υ *x*])
done

qed

lemma *size-defined*:
assumes *X-finite*: $\bigwedge\tau.$ *finite* $\lceil\lceil$*Rep-Set-0* $(X\ \tau)\rceil\rceil$
shows δ $(X{-}{>}size()) = \delta\ X$
apply(*rule ext*, *simp add*: *cp-defined*[*of X*$-$>*size*()] *OclSize-def*)
apply(*simp add*: *defined-def bot-option-def bot-fun-def null-option-def null-fun-def X-finite*)
done

lemma [*simp*]:
assumes *X-finite*: $\bigwedge\tau.$ *finite* $\lceil\lceil$*Rep-Set-0* $(X\ \tau)\rceil\rceil$
shows δ $((X\ {-}{>}including(x))\ {-}{>}size()) = (\delta(X)\ and\ \upsilon(x))$
by(*simp add*: *size-defined*[*OF X-finite*])

4.6.4. OclForall

lemma *forall-set-null-exec*[*simp*,*code-unfold*] :
$(null{-}{>}forall(z|\ P(z))) = invalid$
by(*simp add*: *OclForall-def invalid-def false-def true-def*)

lemma *forall-set-mt-exec*[*simp*,*code-unfold*] :
$((Set\{\}){-}{>}forall(z|\ P(z))) = true$
apply(*simp add*: *OclForall-def*)
apply(*subst mtSet-def*)+
apply(*subst Abs-Set-0-inverse*, *simp-all add*: *true-def*)+
done

lemma *forall-set-including-exec*[*simp*,*code-unfold*] :
assumes *cp*: $\bigwedge\tau.\ P\ x\ \tau = P\ (\lambda\text{-}.\ x\ \tau)\ \tau$
shows $((S{-}{>}including(x)){-}{>}forall(z\ |\ P(z))) = (if\ \delta\ S\ and\ \upsilon\ x$
$then\ P\ x\ and\ (S{-}{>}forall(z\ |\ P(z)))$
$else\ invalid$
$endif)$
proof $-$

have *insert-in-Set-0* : $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor$
$\in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done

have *d-and-v-destruct-defined* : $\bigwedge\tau\ S\ x.\ \tau \models (\delta\ S\ and\ \upsilon\ x) \Longrightarrow \tau \models \delta\ S$
by (*simp add*: *foundation5*[*THEN conjunct1*])
have *d-and-v-destruct-valid* :$\bigwedge\tau\ S\ x.\ \tau \models (\delta\ S\ and\ \upsilon\ x) \Longrightarrow \tau \models \upsilon\ x$
by (*simp add*: *foundation5*[*THEN conjunct2*])

have *forall-including-invert* : $\bigwedge\tau\ f.\ (f\ x\ \tau = f\ (\lambda\ \text{-}.\ x\ \tau)\ \tau) \Longrightarrow$
$\tau \models (\delta\ S\ and\ \upsilon\ x) \Longrightarrow$
$(\forall x{\in}\lceil\lceil Rep\text{-}Set\text{-}0\ (S{-}{>}including(x)\ \tau)\rceil\rceil.\ f\ (\lambda\text{-}.\ x)\ \tau) =$

$(f\ x\ \tau \wedge (\forall\, x{\in}\lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil.\ f\ (\lambda\text{-}.\ x)\ \tau))$

apply(*simp add*: *OclIncluding-def*)
apply(*subst Abs-Set-0-inverse*)
apply(*rule insert-in-Set-0*)
apply(*rule d-and-v-destruct-defined*, *assumption*)
apply(*rule d-and-v-destruct-valid*, *assumption*)
apply(*simp add*: *d-and-v-destruct-defined d-and-v-destruct-valid*)
apply(*frule d-and-v-destruct-defined*, *drule d-and-v-destruct-valid*)
apply(*simp add*: *OclValid-def*)
done

have *exists-including-invert* : $\bigwedge\tau\ f.\ (f\ x\ \tau = f\ (\lambda\ \text{-}.\ x\ \tau)\ \tau) \Longrightarrow$
$\tau \models (\delta\ S\ \textit{and}\ \upsilon\ x) \Longrightarrow$
$(\exists\, x{\in}\lceil\lceil \textit{Rep-Set-0}\ (S{-}{>}\textit{including}(x)\ \tau)\rceil\rceil.\ f\ (\lambda\text{-}.\ x)\ \tau) =$
$(f\ x\ \tau \vee (\exists\, x{\in}\lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil.\ f\ (\lambda\text{-}.\ x)\ \tau))$
apply(*subst arg-cong*[**where** $f = \lambda x.\ \neg x$,
OF forall-including-invert[**where** $f = \lambda x\ \tau.\ \neg\ (f\ x\ \tau)$],
simplified])
by *simp-all*

have *cp-eq* : $\bigwedge\tau\ v.\ (P\ x\ \tau = v) = (P\ (\lambda\text{-}.\ x\ \tau)\ \tau = v)$ **by**(*subst cp*, *simp*)
have *cp-not-eq* : $\bigwedge\tau\ v.\ (P\ x\ \tau \neq v) = (P\ (\lambda\text{-}.\ x\ \tau)\ \tau \neq v)$ **by**(*subst cp*, *simp*)

have *foundation10'*: $\bigwedge\tau\ x\ y.\ (\tau \models x) \wedge (\tau \models y) \Longrightarrow \tau \models (x\ \textit{and}\ y)$
apply(*erule conjE*)
apply(*subst foundation10*)
apply(*rule foundation6*, *simp*)
apply(*rule foundation6*, *simp*)
by *simp*

have *contradict-Rep-Set-0*: $\bigwedge\tau\ S\ f.$
$\exists\, x{\in}\lceil\lceil \textit{Rep-Set-0}\ S\rceil\rceil.\ f\ (\lambda\text{-}.\ x)\ \tau \Longrightarrow$
$(\forall\, x{\in}\lceil\lceil \textit{Rep-Set-0}\ S\rceil\rceil.\ \neg\ (f\ (\lambda\text{-}.\ x)\ \tau)) = \textit{False}$
by(*case-tac* $(\forall\, x{\in}\lceil\lceil \textit{Rep-Set-0}\ S\rceil\rceil.\ \neg\ (f\ (\lambda\text{-}.\ x)\ \tau)) = \textit{True}$, *simp-all*)

show *?thesis*

apply(*rule ext*, *rename-tac* τ)
apply(*simp add*: *if-ocl-def*)
apply(*simp add*: *cp-defined*[*of* $\delta\ S$ *and* $\upsilon\ x$])
apply(*simp add*: *cp-defined*[*THEN sym*])
apply(*rule conjI*, *rule impI*)

apply(*subgoal-tac* $\tau \models \delta\ S$)
prefer *2*
apply(*drule foundation5*[*simplified OclValid-def*], *erule conjE*)+ **apply**(*simp add*: *OclValid-def*)

apply(*subst OclForall-def*)
apply(*simp add*: *cp-ocl-and*[*THEN sym*] *OclValid-def*

foundation10'[**where** $x = \delta\ S$ **and** $y = \upsilon\ x$, *simplified OclValid-def*])

apply(*subgoal-tac* $\tau \models (\delta\ S\ \mathit{and}\ \upsilon\ x)$)
prefer *2*
apply(*simp add*: *OclValid-def*)

apply(*case-tac* $\exists x \in \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S{-}{>}\mathit{including}(x)\ \tau)\rceil\rceil.\ P\ (\lambda\text{-}.\ x)\ \tau = \mathit{false}\ \tau$, *simp-all*)
apply(*subst contradict-Rep-Set-0*[**where** $f = \lambda\ x\ \tau.\ P\ x\ \tau = \mathit{false}\ \tau$], *simp*)+
apply(*simp add*: *exists-including-invert*[**where** $f = \lambda\ x\ \tau.\ P\ x\ \tau = \mathit{false}\ \tau$, *OF cp-eq*])

apply(*simp add*: *cp-ocl-and*[*of P x*])
apply(*erule disjE*)
apply(*simp only*: *cp-ocl-and*[*symmetric*], *simp*)

apply(*subgoal-tac OclForall S P* $\tau = \mathit{false}\ \tau$)
apply(*simp only*: *cp-ocl-and*[*symmetric*], *simp*)
apply(*simp add*: *OclForall-def*)

apply(*simp add*: *forall-including-invert*[**where** $f = \lambda\ x\ \tau.\ P\ x\ \tau \neq \mathit{false}\ \tau$, *OF cp-not-eq*],
erule conjE)

apply(*case-tac* $\exists x \in \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S{-}{>}\mathit{including}(x)\ \tau)\rceil\rceil.\ P\ (\lambda\text{-}.\ x)\ \tau = \mathit{bot}\ \tau$, *simp-all*)
apply(*subst contradict-Rep-Set-0*[**where** $f = \lambda\ x\ \tau.\ P\ x\ \tau = \mathit{bot}\ \tau$], *simp*)+
apply(*simp add*: *exists-including-invert*[**where** $f = \lambda\ x\ \tau.\ P\ x\ \tau = \mathit{bot}\ \tau$, *OF cp-eq*])

apply(*simp add*: *cp-ocl-and*[*of P x*])
apply(*erule disjE*)

apply(*subgoal-tac OclForall S P* $\tau \neq \mathit{false}\ \tau$)
apply(*simp only*: *cp-ocl-and*[*symmetric*], *simp*)
apply(*simp add*: *OclForall-def null-fun-def null-option-def bot-fun-def bot-option-def true-def false-def*)

apply(*subgoal-tac OclForall S P* $\tau = \mathit{bot}\ \tau$)
apply(*simp only*: *cp-ocl-and*[*symmetric*], *simp*)
apply(*simp add*: *OclForall-def null-fun-def null-option-def bot-fun-def bot-option-def true-def false-def*)

apply(*simp add*: *forall-including-invert*[**where** $f = \lambda\ x\ \tau.\ P\ x\ \tau \neq \mathit{bot}\ \tau$, *OF cp-not-eq*],
erule conjE)

apply(*case-tac* $\exists x\in\lceil\lceil$*Rep-Set-0* (*S*−>*including*(*x*) τ)$\rceil\rceil$. *P* (λ-. *x*) τ = *null* τ, *simp-all*)
apply(*subst contradict-Rep-Set-0*[**where** *f* = λ *x* τ. *P x* τ = *null* τ], *simp*)+
apply(*simp add*: *exists-including-invert*[**where** *f* = λ *x* τ. *P x* τ = *null* τ, *OF cp-eq*])

apply(*simp add*: *cp-ocl-and*[*of P x*])
apply(*erule disjE*)

apply(*subgoal-tac OclForall S P* $\tau \neq$ *false* τ $\wedge$ *OclForall S P* $\tau \neq$ *bot* τ)
apply(*simp only*: *cp-ocl-and*[*symmetric*], *simp*)
apply(*simp add*: *OclForall-def null-fun-def null-option-def bot-fun-def bot-option-def true-def false-def*)

apply(*subgoal-tac OclForall S P* τ = *null* τ)
apply(*simp only*: *cp-ocl-and*[*symmetric*], *simp*)
apply(*simp add*: *OclForall-def null-fun-def null-option-def bot-fun-def bot-option-def true-def false-def*)

apply(*simp add*: *forall-including-invert*[**where** *f* = λ *x* τ. *P x* $\tau \neq$ *null* τ, *OF cp-not-eq*],
erule conjE)

apply(*simp add*: *cp-ocl-and*[*of P x*] *OclForall-def*)
apply(*subgoal-tac P x* τ = *true* τ, *simp*)
apply(*metis bot-fun-def bool-split foundation18′ foundation2 valid1*)

by(*metis OclForall-def including-defined-args-valid′ invalid-def*)
qed

lemma *forall-includes* :
assumes *x-def* : $\tau \models \delta$ *x*
and *y-def* : $\tau \models \delta$ *y*
shows ($\tau \models$ *OclForall x* (*OclIncludes y*)) = ($\lceil\lceil$*Rep-Set-0* (*x* τ)$\rceil\rceil \subseteq \lceil\lceil$*Rep-Set-0* (*y* τ)$\rceil\rceil$)
proof −
have *discr-eq-false-true* : $\bigwedge\tau$. (*false* τ = *true* τ) = *False* **by** (*metis OclValid-def foundation2*)
have *discr-eq-bot1-true* : $\bigwedge\tau$. ($\bot$ τ = *true* τ) = *False* **by** (*metis defined3 defined-def discr-eq-false-true*)
have *discr-eq-bot2-true* : $\bigwedge\tau$. ($\bot$ = *true* τ) = *False* **by** (*metis bot-fun-def discr-eq-bot1-true*)
have *discr-eq-null-true* : $\bigwedge\tau$. (*null* τ = *true* τ) = *False* **by** (*metis OclValid-def foundation4*)
show *?thesis*
apply(*case-tac* $\tau \models$ *OclForall x* (*OclIncludes y*))

apply(*simp add*: *OclValid-def OclForall-def*)
apply(*split split-if-asm*, *simp-all add*: *discr-eq-false-true discr-eq-bot1-true discr-eq-null-true discr-eq-bot2-true*)+
apply(*subgoal-tac* $\forall x\in\lceil\lceil$*Rep-Set-0* (*x* τ)$\rceil\rceil$. ($\tau \models$ *y*−>*includes*((λ-. *x*))))
prefer *2*
apply(*simp add*: *OclValid-def*)
apply (*metis* (*full-types*) *bot-fun-def bool-split invalid-def null-fun-def*)

apply(*rule subsetI*, *rename-tac e*)
apply(*drule-tac* $P = \lambda x.\ \tau \models y{-}{>}includes((\lambda\text{-}.\ x))$ **and** $x = e$ **in** *ballE*) **prefer** *3* **apply** *assumption*
apply(*simp add*: *OclIncludes-def OclValid-def*)
apply (*metis discr-eq-bot2-true option.inject true-def*)
apply(*simp*)

apply(*simp add*: *OclValid-def OclForall-def x-def*[*simplified OclValid-def*])
apply(*subgoal-tac* ($\exists\, x{\in}\lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil.\ (y{-}{>}includes((\lambda\text{-}.\ x)))\ \tau = false\ \tau$
$\lor\ (y{-}{>}includes((\lambda\text{-}.\ x)))\ \tau = \bot\ \tau$
$\lor\ (y{-}{>}includes((\lambda\text{-}.\ x)))\ \tau = null\ \tau$))
prefer *2*
apply *metis*
apply(*erule bexE*, *rename-tac e*)
apply(*simp add*: *OclIncludes-def y-def*[*simplified OclValid-def*])

apply(*case-tac* $\tau \models \upsilon\ (\lambda\text{-}.\ e)$, *simp add*: *OclValid-def*)
apply(*erule disjE*)
apply(*metis* (*mono-tags*) *discr-eq-false-true set-mp true-def*)
apply(*simp add*: *bot-fun-def bot-option-def null-fun-def null-option-def*)
apply(*erule contrapos-nn*[*OF - Set-inv-lemma'*[*OF x-def*]], *simp*)
done
qed

lemma *forall-not-includes* :
assumes *x-def* : $\tau \models \delta\ x$
and *y-def* : $\tau \models \delta\ y$
shows $(OclForall\ x\ (OclIncludes\ y)\ \tau = false\ \tau) = (\neg\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil \subseteq \lceil\lceil Rep\text{-}Set\text{-}0\ (y\ \tau)\rceil\rceil)$
proof –
have *discr-eq-false-true* : $\bigwedge\tau.\ (false\ \tau = true\ \tau) = False$ **by** (*metis OclValid-def foundation2*)
have *discr-eq-null-true* : $\bigwedge\tau.\ (null\ \tau = true\ \tau) = False$ **by** (*metis OclValid-def foundation4*)
have *discr-eq-null-false* : $\bigwedge\tau.\ (null\ \tau = false\ \tau) = False$ **by** (*metis defined4 foundation1 foundation16 null-fun-def*)
have *discr-neq-false-true* : $\bigwedge\tau.\ (false\ \tau \neq true\ \tau) = True$ **by** (*metis discr-eq-false-true*)
have *discr-neq-true-false* : $\bigwedge\tau.\ (true\ \tau \neq false\ \tau) = True$ **by** (*metis discr-eq-false-true*)
have *discr-eq-bot1-true* : $\bigwedge\tau.\ (\bot\ \tau = true\ \tau) = False$ **by** (*metis defined3 defined-def discr-eq-false-true*)
have *discr-eq-bot2-true* : $\bigwedge\tau.\ (\bot = true\ \tau) = False$ **by** (*metis bot-fun-def discr-eq-bot1-true*)
have *discr-eq-bot1-false* : $\bigwedge\tau.\ (\bot\ \tau = false\ \tau) = False$ **by** (*metis OCL-core.bot-fun-def defined4 foundation1 foundation16*)
have *discr-eq-bot2-false* : $\bigwedge\tau.\ (\bot = false\ \tau) = False$ **by** (*metis foundation1 foundation18' valid4*)
show *?thesis*
apply(*subgoal-tac* $\neg\ (OclForall\ x\ (OclIncludes\ y)\ \tau = false\ \tau) = (\neg\ (\neg\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil \subseteq \lceil\lceil Rep\text{-}Set\text{-}0\ (y\ \tau)\rceil\rceil))$, *simp*)
apply(*subst forall-includes*[*symmetric*], *simp add*: *x-def*, *simp add*: *y-def*)
apply(*subst OclValid-def*)
apply(*simp add*: *OclForall-def*

discr-neq-false-true
discr-neq-true-false
discr-eq-bot1-false
discr-eq-bot2-false
discr-eq-bot1-true
discr-eq-bot2-true
discr-eq-null-false
discr-eq-null-true)
apply(*simp add*: *x-def*[*simplified OclValid-def*])
apply(*subgoal-tac* ($\forall x \in \lceil\lceil$*Rep-Set-0* ($x$ τ)$\rceil\rceil$. ((*y*−>*includes*((λ-. x))) τ = *true* τ $\vee$ (*y*−>*includes*((λ-. x))) τ = *false* τ)))
apply(*metis bot-fun-def discr-eq-bot2-true discr-eq-null-true null-fun-def*)
apply(*rule ballI*, *rename-tac e*)
apply(*simp add*: *OclIncludes-def*, *rule conjI*)
apply (*metis* (*full-types*) *false-def true-def*)

apply(*simp add*: *y-def*[*simplified OclValid-def*], *rule impI*)
apply(*drule contrapos-nn*[*OF - Set-inv-lemma'*[*OF x-def*], *simplified OclValid-def*], *blast* +)
done
qed

4.6.5. OclExists

lemma *exists-set-null-exec*[*simp*,*code-unfold*] :
(*null*−>*exists*(z | $P(z)$)) = *invalid*
by(*simp add*: *OclExists-def*)

lemma *exists-set-mt-exec*[*simp*,*code-unfold*] :
((*Set*{})−>*exists*(z | $P(z)$)) = *false*
by(*simp add*: *OclExists-def*)

lemma *not-if*[*simp*]:
not(*if P then C else E endif*) = (*if P then not C else not E endif*)

apply(*rule not-inject*, *simp*)
apply(*rule ext*)
apply(*subst cp-not*, *simp add*: *if-ocl-def*)
apply(*subst cp-not*[*symmetric*] *not-not*)+
by *simp*

lemma *exists-set-including-exec*[*simp*,*code-unfold*] :
assumes *cp*: $\bigwedge\tau$. $P\ x\ \tau = P\ (\lambda\text{-}.\ x\ \tau)\ \tau$
shows ((*S*−>*including*(x))−>*exists*(z | $P(z)$)) = (*if* δ *S and* υ *x*
then P x or (*S*−>*exists*(z | $P(z)$))
else invalid
endif)
apply(*simp add*: *OclExists-def ocl-or-def*)

apply(*rule not-inject*)

apply(*simp*)
apply(*rule forall-set-including-exec*)
apply(*rule sym*, *subst cp-not*)
apply(*simp only*: *cp*[*symmetric*] *cp-not*[*symmetric*])
done

4.6.6. OclIterate

lemma $OclIterate_{Set}$*-infinite*:
assumes *non-finite*: $\tau \models not(\delta(S{-}{>}size()))$
shows $(OclIterate_{Set}\ S\ A\ F)\ \tau = invalid\ \tau$
apply(*insert non-finite* [*THEN OclSize-infinite*])
apply(*erule disjE*)
apply(*simp-all add*: $OclIterate_{Set}$*-def invalid-def*)
apply(*erule contrapos-np*)
apply(*simp add*: *OclValid-def*)
done

lemma $OclIterate_{Set}$*-empty*[*simp*]: $((Set\{\}){-}{>}iterate(a;\ x = A \mid P\ a\ x)) = A$
proof –
have *A1* : $\lfloor\lfloor\{\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *mtSet-def*)
have *C* : $\bigwedge\ \tau.\ (\delta\ (\lambda\tau.\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor\{\}\rfloor\rfloor))\ \tau = true\ \tau$
by (*metis A1 Abs-Set-0-cases Abs-Set-0-inverse cp-defined defined-def false-def mtSet-def mtSet-defined null-fun-def null-option-def null-set-not-defined true-def*)
show *?thesis*
apply(*simp add*: $OclIterate_{Set}$*-def mtSet-def Abs-Set-0-inverse valid-def C*)
apply(*rule ext*)
apply(*case-tac* $A\ \tau = \bot\ \tau$, *simp-all*, *simp add*:*true-def false-def bot-fun-def*)
apply(*simp add*: *A1 Abs-Set-0-inverse*)
done
qed

In particular, this does hold for A = null.

lemma $OclIterate_{Set}$*-including*:
assumes *S-finite*: $\tau \models \delta(S{-}{>}size())$
and *F-valid-arg*: $(\upsilon\ A)\ \tau = (\upsilon\ (F\ a\ A))\ \tau$
and *F-commute*: *comp-fun-commute F*
and *F-cp*: $\bigwedge\ x\ y\ \tau.\ F\ x\ y\ \tau = F\ (\lambda\ \text{-}.\ x\ \tau)\ y\ \tau$
shows $((S{-}{>}including(a)){-}{>}iterate(a;\ x = A \mid F\ a\ x))\ \tau =$
$((S{-}{>}excluding(a)){-}{>}iterate(a;\ x = F\ a\ A \mid F\ a\ x))\ \tau$
proof –

have *valid-inject-true* : $\bigwedge\tau\ P.\ (\upsilon\ P)\ \tau \neq true\ \tau \Longrightarrow (\upsilon\ P)\ \tau = false\ \tau$
apply(*simp add*: *valid-def true-def false-def*
bot-fun-def bot-option-def
null-fun-def null-option-def)
by (*case-tac* $P\ \tau = \bot$, *simp-all add*: *true-def*)

have *insert-in-Set-0* : $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ a)) \Longrightarrow \lfloor\lfloor insert\ (a\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor$

$\in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done

have *insert-defined* : $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ a)) \Longrightarrow$
$(\delta\ (\lambda\text{-}.\ \mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor \mathit{insert}\ (a\ \tau)\ \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil\rfloor\rfloor))\ \tau = \mathit{true}\ \tau$
apply(*subst defined-def*)
apply(*simp add*: *bot-fun-def bot-option-def bot-Set-0-def null-Set-0-def null-option-def null-fun-def false-def true-def*)
apply(*subst Abs-Set-0-inject*)
apply(*rule insert-in-Set-0*, *simp-all add*: *bot-option-def*)

apply(*subst Abs-Set-0-inject*)
apply(*rule insert-in-Set-0*, *simp-all add*: *null-option-def bot-option-def*)
done

have *remove-finite* : *finite* $\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil \Longrightarrow$ *finite* $((\lambda a\ \tau.\ a)\ \text{`}\ (\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{a\ \tau\}))$
by(*simp*)

have *remove-in-Set-0* : $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ a)) \Longrightarrow \lfloor\lfloor\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{a\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done

have *remove-defined* : $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ a)) \Longrightarrow$
$(\delta\ (\lambda\text{-}.\ \mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{a\ \tau\}\rfloor\rfloor))\ \tau = \mathit{true}\ \tau$
apply(*subst defined-def*)
apply(*simp add*: *bot-fun-def bot-option-def bot-Set-0-def null-Set-0-def null-option-def null-fun-def false-def true-def*)
apply(*subst Abs-Set-0-inject*)
apply(*rule remove-in-Set-0*, *simp-all add*: *bot-option-def*)

apply(*subst Abs-Set-0-inject*)
apply(*rule remove-in-Set-0*, *simp-all add*: *null-option-def bot-option-def*)
done

have *abs-rep*: $\bigwedge x.\ \lfloor\lfloor x\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\} \Longrightarrow \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (\mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor x\rfloor\rfloor)\rceil\rceil = x$
by(*subst Abs-Set-0-inverse*, *simp-all*)

have *inject* : *inj* $(\lambda a\ \tau.\ a)$
by(*rule inj-fun*, *simp*)

show *?thesis*
apply(*simp only*: *cp-OclIterate*$_{Set}$[*of* S*->including*(a)] *cp-OclIterate*$_{Set}$[*of* S*->excluding*(a)])
apply(*subst OclIncluding-def*, *subst OclExcluding-def*)

apply(*case-tac* $\neg$ ((δ *S*) τ = *true* τ $\wedge$ (υ *a*) τ = *true* τ), *simp*)

apply(*subgoal-tac* $OclIterate_{Set}$ (λ-. $\bot$) *A F* τ = $OclIterate_{Set}$ (λ-. $\bot$) (*F a A*) *F* τ, *simp*)
apply(*rule conjI*)
apply(*blast*)
apply(*blast*)
apply(*auto*)

apply(*simp add*: $OclIterate_{Set}$*-def*) **apply**(*auto*)
apply(*simp add*: *defined-def bot-option-def bot-fun-def false-def true-def*)
apply(*simp add*: *defined-def bot-option-def bot-fun-def false-def true-def*)
apply(*simp add*: *defined-def bot-option-def bot-fun-def false-def true-def*)

apply(*simp add*: $OclIterate_{Set}$*-def*) **apply**(*auto*)
apply(*simp add*: *defined-def bot-option-def bot-fun-def false-def true-def*)
apply(*simp add*: *defined-def bot-option-def bot-fun-def false-def true-def*)
apply(*simp add*: *defined-def bot-option-def bot-fun-def false-def true-def*)

apply(*simp add*: $OclIterate_{Set}$*-def*)

apply(*subst abs-rep*[*OF insert-in-Set-0*[*simplified OclValid-def*], *of* τ], *simp-all*)+
apply(*subst abs-rep*[*OF remove-in-Set-0*[*simplified OclValid-def*], *of* τ], *simp-all*)+
apply(*subst insert-defined*, *simp-all add*: *OclValid-def*)+
apply(*subst remove-defined*, *simp-all add*: *OclValid-def*)+

apply(*case-tac* $\neg$ ((υ *A*) τ = *true* τ), *simp add*: *F-valid-arg*)
apply(*simp add*: *valid-inject-true F-valid-arg*)
apply(*rule impI*)

apply(*subst Finite-Set.comp-fun-commute.fold-fun-comm*[**where** *f* = *F* **and** *z* = *A* **and** *x* = *a* **and** *A* = ((λ*a* τ. *a*) ‘ ($\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ − {*a* τ})), *symmetric*, *OF F-commute*])
apply(*rule remove-finite*, *simp*)

apply(*subst image-set-diff*[*OF inject*], *simp*)
apply(*subgoal-tac Finite-Set.fold F A* (*insert* ($\lambda\tau'$. *a* τ) ((λ*a* τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)) τ
=
F ($\lambda\tau'$. *a* τ) (*Finite-Set.fold F A* ((λ*a* τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ − {$\lambda\tau'$. *a* τ})) τ)
apply(*subst F-cp*)
apply(*simp*)

apply(*subst Finite-Set.comp-fun-commute.fold-insert-remove*[*OF F-commute*])
apply(*simp*)+
done
qed

4.6.7. Strict Equality

lemma *StrictRefEq-set-exec*[*simp*,*code-unfold*] :
((x::($'\mathfrak{A}$,$'\alpha$::*null*)*Set*) $\doteq$ *y*) =

$(if\ \delta\ x\ then\ (if\ \delta\ y$
$then\ ((x{-}{>}forall(z|\ y{-}{>}includes(z))\ and\ (y{-}{>}forall(z|\ x{-}{>}includes(z)))))$
$else\ if\ \upsilon\ y$
$then\ false\ (*\ x'{-}{>}includes = null\ *)$
$else\ invalid$
$endif$
$endif)$
$else\ if\ \upsilon\ x\ (*\ null = ???\ *)$
$then\ if\ \upsilon\ y\ then\ not(\delta\ y)\ else\ invalid\ endif$
$else\ invalid$
$endif$
$endif)$

proof −

have *defined-inject-true* : $\bigwedge\tau\ P.\ \neg\ (\tau \models \delta\ P) \Longrightarrow (\delta\ P)\ \tau = false\ \tau$
by(*metis bot-fun-def defined-def foundation16 null-fun-def*)

have *valid-inject-true* : $\bigwedge\tau\ P.\ \neg\ (\tau \models \upsilon\ P) \Longrightarrow (\upsilon\ P)\ \tau = false\ \tau$
by(*metis bot-fun-def foundation18' valid-def*)

have *valid-inject-defined* : $\bigwedge\tau\ P.\ \neg\ (\tau \models \upsilon\ P) \Longrightarrow \neg\ (\tau \models \delta\ P)$
by(*metis foundation20*)

have *null-simp* : $\bigwedge\tau\ y.\ \tau \models \upsilon\ y \Longrightarrow \neg\ (\tau \models \delta\ y) \Longrightarrow y\ \tau = null\ \tau$
by (*simp add*: *foundation16 foundation18' null-fun-def*)

have *discr-eq-false-true* : $\bigwedge\tau.\ (false\ \tau = true\ \tau) = False$ **by** (*metis OclValid-def foundation2*)
have *discr-neq-true-false* : $\bigwedge\tau.\ (true\ \tau \neq false\ \tau) = True$ **by** (*metis discr-eq-false-true*)

have *strongeq-true* : $\bigwedge\ \tau\ x\ y.\ (\lfloor\lfloor x\ \tau = y\ \tau\rfloor\rfloor = true\ \tau) = (x\ \tau = y\ \tau)$
by(*simp add*: *foundation22*[*simplified OclValid-def StrongEq-def*])

have *strongeq-false* : $\bigwedge\ \tau\ x\ y.\ (\lfloor\lfloor x\ \tau = y\ \tau\rfloor\rfloor = false\ \tau) = (x\ \tau \neq y\ \tau)$
apply(*case-tac* $x\ \tau \neq y\ \tau$, *simp add*: *false-def*)
apply(*simp add*: *false-def true-def*)
done

have *rep-set-inj* : $\bigwedge\tau.\ (\delta\ x)\ \tau = true\ \tau \Longrightarrow$
$(\delta\ y)\ \tau = true\ \tau \Longrightarrow$
$x\ \tau \neq y\ \tau \Longrightarrow$
$\lceil\lceil Rep\text{-}Set\text{-}0\ (y\ \tau)\rceil\rceil \neq \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil$
apply(*simp add*: *defined-def*)
apply(*split split-if-asm*, *simp add*: *false-def true-def*)+
apply(*simp add*: *null-fun-def null-Set-0-def bot-fun-def bot-Set-0-def*)

apply(*case-tac* $x\ \tau$)
apply(*case-tac ya*, *simp-all*)
apply(*case-tac a*, *simp-all*)

apply(*case-tac* y τ)
apply(*case-tac* *yaa*, *simp-all*)
apply(*case-tac* *ab*, *simp-all*)

apply(*simp add*: *Abs-Set-0-inverse*)

apply(*blast*)
done

show *?thesis*
apply(*rule ext*, *rename-tac* τ)

apply(*simp add*: *cp-if-ocl*[*of* δ x])
apply(*case-tac* $\neg$ ($\tau \models \upsilon$ x))
apply(*subgoal-tac* $\neg$ ($\tau \models \delta$ x))
prefer *2*
apply(*metis foundation20*)
apply(*simp add*: *defined-inject-true*)
apply(*simp add*: *cp-if-ocl*[*symmetric*] *OclValid-def StrictRefEq-set*)

apply(*simp*)

apply(*case-tac* $\neg$ ($\tau \models \upsilon$ y))
apply(*subgoal-tac* $\neg$ ($\tau \models \delta$ y))
prefer *2*
apply(*metis foundation20*)
apply(*simp add*: *defined-inject-true*)
apply(*simp add*: *cp-if-ocl*[*symmetric*] *OclValid-def StrictRefEq-set*)

apply(*simp*)

apply(*simp add*: *cp-if-ocl*[*of* δ y])
apply(*simp add*: *cp-if-ocl*[*symmetric*])

apply(*simp add*: *cp-if-ocl*[*of* δ x])
apply(*case-tac* $\neg$ ($\tau \models \delta$ x))
apply(*simp add*: *defined-inject-true*)
apply(*simp add*: *cp-if-ocl*[*symmetric*])
apply(*simp add*: *cp-not*[*of* δ y])
apply(*case-tac* $\neg$ ($\tau \models \delta$ y))
apply(*simp add*: *defined-inject-true*)
apply(*simp add*: *cp-not*[*symmetric*])
apply(*metis* (*hide-lams*, *no-types*) *OclValid-def StrongEq-sym foundation22 null-fun-def null-simp strictRefEq-set-vs-strongEq true-def*)
apply(*simp add*: *OclValid-def cp-not*[*symmetric*])

apply(*simp add*: *null-simp*[*simplified OclValid-def*, *of x*] *StrictRefEq-set StrongEq-def false-def*)
apply(*simp add*: *defined-def*[*of y*])

apply(*metis discr-neq-true-false*)

apply(*simp*)
apply(*simp add*: *OclValid-def*)
apply(*simp add*: *cp-if-ocl*[*of* δ *y*])
apply(*case-tac* $\neg$ ($\tau \models \delta$ *y*))
apply(*simp add*: *defined-inject-true*)
apply(*simp add*: *cp-if-ocl*[*symmetric*])

apply(*drule null-simp*[*simplified OclValid-def*, *of y*])
apply(*simp add*: *OclValid-def*)
apply(*simp add*: *cp-StrictRefEq-set*[*of x*])
apply(*simp add*: *cp-StrictRefEq-set*[*symmetric*])
apply(*simp add*: *null-simp*[*simplified OclValid-def*, *of y*] *StrictRefEq-set StrongEq-def false-def*)
apply(*simp add*: *defined-def*[*of x*])
apply (*metis discr-neq-true-false*)

apply(*simp add*: *OclValid-def*)

apply(*simp add*: *StrictRefEq-set StrongEq-def*)

apply(*subgoal-tac* $\lfloor\lfloor x\ \tau = y\ \tau\rfloor\rfloor = true\ \tau \vee \lfloor\lfloor x\ \tau = y\ \tau\rfloor\rfloor = false\ \tau$)
prefer *2*
apply(*case-tac* $x\ \tau = y\ \tau$)
apply(*rule disjI1*, *simp add*: *true-def*)
apply(*rule disjI2*, *simp add*: *false-def*)

apply(*erule disjE*)
apply(*simp add*: *strongeq-true*)

apply(*subgoal-tac* ($\tau \models$ *OclForall x* (*OclIncludes y*)) $\wedge$ ($\tau \models$ *OclForall y* (*OclIncludes x*)))
apply(*simp add*: *cp-ocl-and*[*of OclForall x* (*OclIncludes y*)] *true-def OclValid-def*)
apply(*simp add*: *OclValid-def*)
apply(*simp add*: *forall-includes*[*simplified OclValid-def*])

apply(*simp add*: *strongeq-false*)

apply(*subgoal-tac OclForall x* (*OclIncludes y*) τ = *false* $\tau \vee$ *OclForall y* (*OclIncludes x*) τ = *false* τ)
apply(*simp add*: *cp-ocl-and*[*of OclForall x* (*OclIncludes y*)] *false-def*)
apply(*erule disjE*)
apply(*simp*)
apply(*subst cp-ocl-and*[*symmetric*])
apply(*simp only*: *ocl-and-false1*[*simplified false-def*])

apply(*simp*)
apply(*subst cp-ocl-and*[*symmetric*])
apply(*simp only*: *ocl-and-false2*[*simplified false-def*])
apply(*simp add*: *forall-not-includes*[*simplified OclValid-def*] *rep-set-inj*)
done
qed

4.7. Gogolla's Challenge on Sets

4.7.1. Introduction

$\mathit{OclIterate}_{Set}$ is defined with the function *Finite-Set.fold*. So when proving properties where the term $\mathit{OclIterate}_{Set}$ appears at some point, most lemmas defined in the library *Finite-Set* could be helpful for the proof. However, for some part of the Gogolla's Challenge proof, it is required to have this statement *Finite-Set.fold ?f ?z* (*insert ?x ?A*) = *?f ?x* (*Finite-Set.fold ?f ?z ?A*) (coming from *comp-fun-commute.fold-insert*), but *comp-fun-commute.fold-insert* requires *comp-fun-commute*, which is not trivial to prove on two OCL terms without extra hypothesis (like finiteness on sets). Thus, we overload here this *comp-fun-commute*.

definition *is-int* $x \equiv \forall\ \tau.\ \tau \models \upsilon\ x \wedge (\forall \tau 0.\ x\ \tau = x\ \tau 0)$

lemma *int-is-valid* : $\bigwedge x.\ \mathit{is\text{-}int}\ x \Longrightarrow \tau \models \upsilon\ x$
by (*metis foundation18′ is-int-def*)

definition *all-int-set* $S \equiv \mathit{finite}\ S \wedge (\forall x{\in}S.\ \mathit{is\text{-}int}\ x)$
definition *all-int* $\tau\ S \equiv (\tau \models \delta\ S) \wedge \mathit{all\text{-}int\text{-}set}\ \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil$
definition *all-defined-set* $\tau\ S \equiv \mathit{finite}\ S \wedge (\forall x{\in}S.\ (\tau \models \upsilon\ (\lambda\text{-}.\ x)))$
definition *all-defined-set′* $\tau\ S \equiv \mathit{finite}\ S$
definition *all-defined* $\tau\ S \equiv (\tau \models \delta\ S) \wedge \mathit{all\text{-}defined\text{-}set'}\ \tau\ \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil$

lemma *all-def-to-all-int* : $\bigwedge\tau.\ \mathit{all\text{-}defined}\ \tau\ S \Longrightarrow$
$\mathit{all\text{-}int\text{-}set}\ ((\lambda a\ \tau.\ a)\ `\ \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil)$
apply(*simp add*: *all-defined-def*, *erule conjE*, *frule Set-inv-lemma*)
apply(*simp add*: *all-defined-def all-defined-set′-def all-int-set-def is-int-def defined-def OclValid-def*)
by (*metis* (*no-types*) *OclValid-def foundation18′ true-def Set-inv-lemma′*)

term *all-defined* τ (*f* **0** *Set*{**0**}) = (*all-defined* τ *Set*{**0**})

lemma *int-trivial* : *is-int* $(\lambda\text{-}.\ \lfloor a\rfloor)$ **by**(*simp add*: *is-int-def OclValid-def valid-def bot-fun-def bot-option-def*)

lemma *EQ-sym* : $(x{::}(\text{-},\text{-})\ \mathit{Set}) = y \Longrightarrow \tau \models \upsilon\ x \Longrightarrow \tau \models (x \doteq y)$
apply(*simp add*: *OclValid-def*)
done

lemma *StrictRefEq-set-L-subst1* : $\mathit{cp}\ P \Longrightarrow \tau \models \upsilon\ x \Longrightarrow \tau \models \upsilon\ y \Longrightarrow \tau \models \upsilon\ P\ x \Longrightarrow \tau \models \upsilon$

$P\ y \Longrightarrow \tau \models (x::('\mathfrak{A},'\alpha::null)Set) \doteq y \Longrightarrow \tau \models (P\ x\ ::('\mathfrak{A},'\alpha::null)Set) \doteq P\ y$
apply(*simp only*: *StrictRefEq-set OclValid-def*)
apply(*split split-if-asm*)
apply(*simp add*: *StrongEq-L-subst1*[*simplified OclValid-def*])
by (*simp add*: *invalid-def bot-option-def true-def*)

lemma *abs-rep-simp* :
assumes *S-all-def* : *all-defined* τ (S :: ($'\mathfrak{A}$, $'a$ *option option*) *Set*)
shows *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil\rfloor\rfloor = S\ \tau$
by(*rule abs-rep-simp'*, *simp add*: *assms*[*simplified all-defined-def*])

lemma *cp-all-def* : *all-defined* τ f = *all-defined* τ' (λ-. f τ)
apply(*simp add*: *all-defined-def all-defined-set'-def OclValid-def*)
apply(*subst cp-defined*)
by (*metis* (*no-types*) *OclValid-def cp-defined cp-valid defined2 defined-def foundation1 foundation16 foundation17 foundation18' foundation6 foundation9 not3 ocl-and-true1 ocl-and-true2 transform1-rev valid-def*)

lemma *cp-all-def'* : ($\forall\,\tau$. *all-defined* τ f) = ($\forall\,\tau\ \tau'$. *all-defined* τ' (λ-. f τ))
apply(*rule iffI*)
apply(*rule allI*) **apply**(*erule-tac* $x = \tau$ **in** *allE*) **apply**(*rule allI*)
apply(*simp add*: *cp-all-def*[*THEN iffD1*])
apply(*subst cp-all-def*, *blast*)
done

lemma *S-lift* :
assumes *S-all-def* : *all-defined* (τ :: $'\mathfrak{A}$ *st*) S
shows $\exists\,S'$. (λa (-::$'\mathfrak{A}$ *st*). a) ‘ $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil$ = (λa (-::$'\mathfrak{A}$ *st*). $\lfloor a\rfloor$) ‘ S'
by(*rule S-lift'*, *simp add*: *assms*[*simplified all-defined-def*])

lemma *destruct-int* : *is-int* $i \Longrightarrow \exists!\ j.\ i = (\lambda$-. $j)$
proof − **fix** τ **show** *is-int* $i \Longrightarrow$ *?thesis*
apply(*rule-tac* $a = i\ \tau$ **in** *ex1I*)
apply(*rule ext*, *simp add*: *is-int-def*)
apply (*metis surj-pair*)
apply(*simp*)
done
apply-end(*simp*)
qed

4.7.2. mtSet

lemma *mtSet-all-def* : *all-defined* τ *Set*{}
proof −
have B : $\lfloor\lfloor\{\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall\,x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *mtSet-def*)
show *?thesis*
apply(*simp add*: *all-defined-def all-defined-set'-def mtSet-def Abs-Set-0-inverse B*)
by (*metis* (*no-types*) *foundation16 mtSet-def mtSet-defined transform1*)
qed

lemma *cp-mtSet* : $\bigwedge x.\ Set\{\} = (\lambda\text{-}.\ Set\{\}\ x)$
by (*metis* (*hide-lams*, *no-types*) *mtSet-def*)

4.7.3. OclIncluding

Identity

lemma *including-id'* : $all\text{-}defined\ \tau\ (S{::}\ ('\mathfrak{A},\ 'a\ option\ option)\ Set) \Longrightarrow$
$x \in \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \Longrightarrow$
$S{-}{>}including(\lambda\tau.\ x)\ \tau = S\ \tau$
proof –
have *discr-eq-invalid-true* : $\bigwedge\tau.\ (invalid\ \tau = true\ \tau) = False$ **by** (*metis bot-option-def invalid-def option.simps*(*2*) *true-def*)

have *all-defined1* : $\bigwedge r2.\ all\text{-}defined\ \tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)

show $all\text{-}defined\ \tau\ S \Longrightarrow$
$x \in \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \Longrightarrow$
?thesis
apply(*simp add*: *OclIncluding-def all-defined1*[*simplified OclValid-def*] *OclValid-def insert-absorb abs-rep-simp del*: *StrictRefEq-set-exec*)
by (*metis OCL-core.bot-fun-def all-defined-def foundation18' valid-def Set-inv-lemma'*)
qed

lemma *including-id* :
assumes *S-all-def* : $\bigwedge\tau.\ all\text{-}defined\ \tau\ (S :: ('\mathfrak{A},\ 'a\ option\ option)\ Set)$
shows $\forall\tau.\ x \in \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \Longrightarrow$
$S{-}{>}including(\lambda\tau.\ x) = S$
proof –
have *all-defined1* : $\bigwedge r2\ \tau.\ all\text{-}defined\ \tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)

have *x-val* : $\bigwedge\tau.\ (\forall\tau.\ x \in \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil) \Longrightarrow$
$\tau \models \upsilon\ (\lambda\tau.\ x)$
apply(*insert S-all-def*)
apply(*simp add*: *all-defined-def all-defined-set-def*)
by (*metis* (*no-types*) *foundation18' Set-inv-lemma'*)

show $(\forall\tau.\ x \in \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil) \Longrightarrow$
?thesis
apply(*rule ext*, *rename-tac* τ', *simp add*: *OclIncluding-def*)
apply(*subst insert-absorb*) **apply** (*metis* (*full-types*) *surj-pair*)
apply(*subst abs-rep-simp*, *simp add*: *S-all-def*, *simp*)
proof – **fix** τ' **show** $\forall a\ b.\ x \in \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ (a,\ b))\rceil\rceil \Longrightarrow ((\delta\ S)\ \tau' = true\ \tau' \longrightarrow (\upsilon\ (\lambda\tau.\ x))\ \tau' \neq true\ \tau') \longrightarrow \bot = S\ \tau'$
apply(*frule x-val*[*simplified*, **where** $\tau = \tau'$])
apply(*insert S-all-def*[**where** $\tau = \tau'$])
apply(*subst all-defined1*[*simplified OclValid-def*], *simp*)
by (*metis OclValid-def*)
qed

apply-end(*simp*)
qed

Commutativity

lemma *including-swap-* :
assumes *S-def* : $\tau \models \delta\ S$
and *i-val* : $\tau \models \upsilon\ i$
and *j-val* : $\tau \models \upsilon\ j$
shows $\tau \models ((S :: ('\mathfrak{A}, \mathit{int\ option\ option})\ \mathit{Set})\text{->}\mathit{including}(i)\text{->}\mathit{including}(j) \doteq (S\text{->}\mathit{including}(j)\text{->}\mathit{includin}$
proof –

have *ocl-and-true* : $\bigwedge a\ b.\ \tau \models a \Longrightarrow \tau \models b \Longrightarrow \tau \models a\ \mathit{and}\ b$
by (*simp add*: *foundation10 foundation6*)

have *discr-eq-false-true* : $(\mathit{false}\ \tau = \mathit{true}\ \tau) = \mathit{False}$ **by** (*metis OclValid-def foundation2*)
have *discr-eq-false-true* : $\bigwedge\tau.\ (\mathit{false}\ \tau = \mathit{true}\ \tau) = \mathit{False}$ **by** (*metis OclValid-def foundation2*)
have *discr-eq-false-bot* : $\bigwedge\tau.\ (\mathit{false}\ \tau = \mathit{bot}\ \tau) = \mathit{False}$ **by** (*metis OCL-core.bot-fun-def bot-option-def false-def option.simps*(*2*))
have *discr-eq-false-null* : $\bigwedge\tau.\ (\mathit{false}\ \tau = \mathit{null}\ \tau) = \mathit{False}$ **by** (*metis defined4 foundation1 foundation17 null-fun-def*)
have *discr-eq-invalid-true* : $\bigwedge\tau.\ (\mathit{invalid}\ \tau = \mathit{true}\ \tau) = \mathit{False}$ **by** (*metis bot-option-def invalid-def option.simps*(*2*) *true-def*)
have *discr-eq-null-false* : $\bigwedge\tau.\ (\mathit{null}\ \tau = \mathit{false}\ \tau) = \mathit{False}$ **by** (*metis defined4 foundation1 foundation16 null-fun-def*)
have *discr-eq-null-true* : $\bigwedge\tau.\ (\mathit{null}\ \tau = \mathit{true}\ \tau) = \mathit{False}$ **by** (*metis OclValid-def foundation4*)
have *discr-eq-bot1-true* : $\bigwedge\tau.\ (\bot\ \tau = \mathit{true}\ \tau) = \mathit{False}$ **by** (*metis defined3 defined-def discr-eq-false-true*)
have *discr-eq-bot2-true* : $\bigwedge\tau.\ (\bot = \mathit{true}\ \tau) = \mathit{False}$ **by** (*metis bot-fun-def discr-eq-bot1-true*)
have *discr-eq-bot1-false* : $\bigwedge\tau.\ (\bot\ \tau = \mathit{false}\ \tau) = \mathit{False}$ **by** (*metis OCL-core.bot-fun-def defined4 foundation1 foundation16*)
have *discr-eq-bot2-false* : $\bigwedge\tau.\ (\bot = \mathit{false}\ \tau) = \mathit{False}$ **by** (*metis foundation1 foundation18′ valid4*)
have *discr-neq-false-true* : $\bigwedge\tau.\ (\mathit{false}\ \tau \neq \mathit{true}\ \tau) = \mathit{True}$ **by** (*metis discr-eq-false-true*)
have *discr-neq-true-false* : $\bigwedge\tau.\ (\mathit{true}\ \tau \neq \mathit{false}\ \tau) = \mathit{True}$ **by** (*metis discr-eq-false-true*)
have *discr-neq-true-bot* : $\bigwedge\tau.\ (\mathit{true}\ \tau \neq \mathit{bot}\ \tau) = \mathit{True}$ **by** (*metis OCL-core.bot-fun-def discr-eq-bot2-true*)
have *discr-neq-true-null* : $\bigwedge\tau.\ (\mathit{true}\ \tau \neq \mathit{null}\ \tau) = \mathit{True}$ **by** (*metis discr-eq-null-true*)
have *discr-neq-invalid-true* : $\bigwedge\tau.\ (\mathit{invalid}\ \tau \neq \mathit{true}\ \tau) = \mathit{True}$ **by** (*metis discr-eq-bot2-true invalid-def*)
have *discr-neq-invalid-bot* : $\bigwedge\tau.\ (\mathit{invalid}\ \tau \neq \bot\ \tau) = \mathit{False}$ **by** (*metis bot-fun-def invalid-def*)

have *bot-in-set-0* : $\lfloor\bot\rfloor \in \{X.\ X = \mathit{bot} \lor X = \mathit{null} \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq \mathit{bot})\}$ **by**(*simp add*: *null-option-def bot-option-def*)

have *forall-includes-id* : $\bigwedge a\ b.\ \tau \models \delta\ S \Longrightarrow \tau \models (\mathit{OclForall}\ S\ (\mathit{OclIncludes}\ S))$
by(*simp add*: *forall-includes*)

have *forall-includes2* : $\bigwedge a\ b.\ \tau \models \upsilon\ a \Longrightarrow \tau \models \upsilon\ b \Longrightarrow \tau \models \delta\ S \Longrightarrow \tau \models (\mathit{OclForall}\ S\ (\mathit{OclIncludes}\ (S\text{->}\mathit{including}(a)\text{->}\mathit{including}(b))))$
proof –

```
  have consist : ⋀x. (δ S) τ = true τ ⟹ x ∈ ⌈⌈Rep-Set-0 (S τ)⌉⌉ ⟹ (υ (λ-. x)) τ = true τ
  by(simp add: Set-inv-lemma'[simplified OclValid-def])
  show ⋀a b. τ ⊨ υ a ⟹ τ ⊨ υ b ⟹ τ ⊨ δ S ⟹ ?thesis a b
  apply(simp add: OclForall-def OclValid-def discr-eq-false-true discr-eq-bot1-true discr-eq-null-true)
  apply(subgoal-tac ∀x∈⌈⌈Rep-Set-0 (S τ)⌉⌉. (S->including(a)->including(b)->includes((λ-.
x))) τ = true τ)
   apply(simp add: discr-neq-true-null discr-neq-true-bot discr-neq-true-false)
   apply(rule ballI)
   apply(rule including-includes[simplified OclValid-def], simp, rule consist, simp-all)+
   apply(frule Set-inv-lemma'[simplified OclValid-def]) apply assumption
   apply(simp add: OclIncludes-def true-def)
  done
 qed

 show τ ⊨ δ S ⟹ τ ⊨ υ i ⟹ τ ⊨ υ j ⟹ ?thesis
  apply(simp add:
   cp-if-ocl[of δ S and υ i and υ j]
   cp-if-ocl[of δ S and υ j and υ i]
   cp-not[of δ S and υ j and υ i])
  apply(subgoal-tac (δ S and υ i and υ j) = (δ S and υ j and υ i))
   prefer 2
   apply (metis ocl-and-assoc ocl-and-commute)
  apply(subgoal-tac τ ⊨ δ S and υ i and υ j)
   prefer 2
   apply (metis foundation10 foundation6)
  apply(simp add: OclValid-def)
  apply(rule ocl-and-true[simplified OclValid-def])

  apply(subst forall-set-including-exec)
  apply(simp add: cp-OclIncludes1[where x = j])
  apply(simp)
  apply(simp add:
   cp-if-ocl[of δ S and υ i and υ j]
   cp-if-ocl[of δ S and υ j and υ i]
   cp-not[of δ S and υ j and υ i])
  apply(simp add: cp-if-ocl[symmetric])
  apply(rule ocl-and-true[simplified OclValid-def])
  apply(simp add: includes-execute-int)
  apply(simp add: cp-if-ocl[of δ S and υ j] cp-if-ocl[of i ≐ j] cp-if-ocl[of δ S] cp-if-ocl[of if υ
j then true else invalid endif] cp-if-ocl[of υ j])
  apply(subgoal-tac τ ⊨ (δ S and υ j))
   prefer 2
   apply (metis OclValid-def foundation10 foundation6)
  apply(simp add: cp-if-ocl[symmetric])
  apply(simp add: if-ocl-def discr-eq-invalid-true)
  apply (metis OclValid-def StrictRefEq-int-defined-args-valid)

  apply(subst forall-set-including-exec)
  apply(simp add: cp-OclIncludes1[where x = i])
```

apply(*simp add*:
 cp-if-ocl[*of* δ *S and* υ *i*])
apply(*subgoal-tac* $\tau \models (\delta\ S\ and\ \upsilon\ i)$)
 prefer *2*
 apply (*metis OclValid-def foundation10 foundation6*)
apply(*simp add*: *cp-if-ocl*[*symmetric*])
apply(*rule ocl-and-true*[*simplified OclValid-def*])
apply(*simp add*: *includes-execute-int*)
apply(*simp add*: *cp-if-ocl*[*of* δ *S and* υ *j*] *cp-if-ocl*[*of* $i \doteq j$] *cp-if-ocl*[*of* δ *S*] *cp-if-ocl*[*of if* υ *i then true else invalid endif*] *cp-if-ocl*[*of* υ *i*])
apply(*subgoal-tac* $\tau \models (\delta\ S\ and\ \upsilon\ j)$)
 prefer *2*
 apply (*metis OclValid-def foundation10 foundation6*)
apply(*simp add*: *cp-if-ocl*[*symmetric*])

apply(*rule forall-includes2*[*simplified OclValid-def*]) **apply**(*simp*) **apply**(*simp*) **apply**(*simp*)

apply(*subst forall-set-including-exec*)
apply(*simp add*: *cp-OclIncludes1*[**where** $x = i$])
apply(*simp*)
apply(*simp add*:
 cp-if-ocl[*of* δ *S and* υ *i and* υ *j*]
 cp-if-ocl[*of* δ *S and* υ *j and* υ *i*])
apply(*simp add*: *cp-if-ocl*[*symmetric*])
apply(*rule ocl-and-true*[*simplified OclValid-def*])
apply(*simp add*: *includes-execute-int*)
apply(*simp add*: *cp-if-ocl*[*of* δ *S and* υ *i*] *cp-if-ocl*[*of* $j \doteq i$] *cp-if-ocl*[*of* δ *S*] *cp-if-ocl*[*of if* υ *i then true else invalid endif*] *cp-if-ocl*[*of* υ *i*])
apply(*subgoal-tac* $\tau \models (\delta\ S\ and\ \upsilon\ i)$)
 prefer *2*
 apply (*metis OclValid-def foundation10 foundation6*)
apply(*simp add*: *cp-if-ocl*[*symmetric*])
apply(*simp add*: *if-ocl-def discr-eq-invalid-true*)
apply (*metis OclValid-def StrictRefEq-int-defined-args-valid*)

apply(*subst forall-set-including-exec*)
apply(*simp add*: *cp-OclIncludes1*[**where** $x = j$])
apply(*simp add*:
 cp-if-ocl[*of* δ *S and* υ *j*])
apply(*subgoal-tac* $\tau \models (\delta\ S\ and\ \upsilon\ j)$)
 prefer *2*
 apply (*metis OclValid-def foundation10 foundation6*)
apply(*simp add*: *cp-if-ocl*[*symmetric*])
apply(*rule ocl-and-true*[*simplified OclValid-def*])
apply(*simp add*: *includes-execute-int*)
apply(*simp add*: *cp-if-ocl*[*of* δ *S and* υ *i*] *cp-if-ocl*[*of* $j \doteq i$] *cp-if-ocl*[*of* δ *S*] *cp-if-ocl*[*of if* υ *j then true else invalid endif*] *cp-if-ocl*[*of* υ *j*])
apply(*subgoal-tac* $\tau \models (\delta\ S\ and\ \upsilon\ i)$)
 prefer *2*

apply (*metis OclValid-def foundation10 foundation6*)
apply(*simp add*: *cp-if-ocl*[*symmetric*])

apply(*rule forall-includes2*[*simplified OclValid-def*]) **apply**(*simp*) **apply**(*simp*) **apply**(*simp*)
done
apply-end(*simp-all add*: *assms*)
qed

lemma *including-swap'* : $\tau \models \delta\ S \Longrightarrow \tau \models \upsilon\ i \Longrightarrow \tau \models \upsilon\ j \Longrightarrow ((S :: ('\mathfrak{A}, int\ option\ option)\ Set){-}{>}including(i){-}{>}including(j)\ \tau = (S{-}{>}including(j){-}{>}including(i))\ \tau)$
apply(*frule including-swap-*[**where** $i = i$ **and** $j = j$], *simp-all del*: *StrictRefEq-set-exec*)
apply(*simp add*: *StrictRefEq-set OclValid-def del*: *StrictRefEq-set-exec*)
apply(*subgoal-tac* $(\delta\ S\ and\ \upsilon\ i\ and\ \upsilon\ j)\ \tau = true\ \tau \wedge (\delta\ S\ and\ \upsilon\ j\ and\ \upsilon\ i)\ \tau = true\ \tau$)
prefer *2*
apply(*metis OclValid-def foundation3*)
apply(*simp add*: *StrongEq-def true-def*)
done

lemma *including-swap* : $\forall \tau.\ \tau \models \delta\ S \Longrightarrow \forall \tau.\ \tau \models \upsilon\ i \Longrightarrow \forall \tau.\ \tau \models \upsilon\ j \Longrightarrow ((S :: ('\mathfrak{A}, int\ option\ option)\ Set){-}{>}including(i){-}{>}including(j) = (S{-}{>}including(j){-}{>}including(i)))$
apply(*rule ext*, *rename-tac* τ)
apply(*erule-tac* $x = \tau$ **in** *allE*)+
apply(*frule including-swap-*[**where** $i = i$ **and** $j = j$], *simp-all del*: *StrictRefEq-set-exec*)
apply(*simp add*: *StrictRefEq-set OclValid-def del*: *StrictRefEq-set-exec*)
apply(*subgoal-tac* $(\delta\ S\ and\ \upsilon\ i\ and\ \upsilon\ j)\ \tau = true\ \tau \wedge (\delta\ S\ and\ \upsilon\ j\ and\ \upsilon\ i)\ \tau = true\ \tau$)
prefer *2*
apply(*metis OclValid-def foundation3*)
apply(*simp add*: *StrongEq-def true-def*)
done

Congruence

lemma *including-subst-set* : $(s{::}('\mathfrak{A},'a{::}null)Set) = t \Longrightarrow s{-}{>}including(x) = (t{-}{>}including(x))$
by(*simp*)

lemma *including-subst-set'* :
shows $\tau \models \delta\ s \Longrightarrow \tau \models \delta\ t \Longrightarrow \tau \models \upsilon\ x \Longrightarrow \tau \models ((s{::}('\mathfrak{A},'a{::}null)Set) \doteq t) \Longrightarrow \tau \models (s{-}{>}including(x) \doteq (t{-}{>}including(x)))$
proof −
have *cp*: *cp* $(\lambda s.\ (s{-}{>}including(x)))$
apply(*simp add*: *cp-def*, *subst cp-OclIncluding*)
by (*rule-tac* $x = (\lambda xab\ ab.\ ((\lambda\text{-}.\ xab){-}{>}including(\lambda\text{-}.\ x\ ab))\ ab)$ **in** *exI*, *simp*)

show $\tau \models \delta\ s \Longrightarrow \tau \models \delta\ t \Longrightarrow \tau \models \upsilon\ x \Longrightarrow \tau \models (s \doteq t) \Longrightarrow$ *?thesis*
apply(*rule-tac* $P = \lambda s.\ (s{-}{>}including(x))$ **in** *StrictRefEq-set-L-subst1*)
apply(*rule cp*)
apply(*simp add*: *foundation20*) **apply**(*simp add*: *foundation20*)
apply (*simp add*: *foundation10 foundation6*)+
done

qed

lemma *including-subst-set''* : $\tau \models \delta\ s \Longrightarrow \tau \models \delta\ t \Longrightarrow \tau \models \upsilon\ x \Longrightarrow (s::('\mathfrak{A},'a::null)Set)\ \tau = t\ \tau \Longrightarrow s{-}{>}including(x)\ \tau = (t{-}{>}including(x))\ \tau$
apply(*frule including-subst-set'*[**where** $s = s$ **and** $t = t$ **and** $x = x$], *simp-all del*: *StrictRefEq-set-exec*)
apply(*simp add*: *StrictRefEq-set OclValid-def del*: *StrictRefEq-set-exec*)
apply (*metis* (*hide-lams*, *no-types*) *OclValid-def foundation20 foundation22*)
by (*metis cp-OclIncluding*)

all defined (construction)

lemma *cons-all-def* :
assumes *S-all-def* : $\bigwedge\tau.$ *all-defined* τ *S*
assumes *x-val* : $\bigwedge\tau.\ \tau \models \upsilon\ x$
shows *all-defined* τ $S{-}{>}including(x)$
proof −

have *discr-eq-false-true* : $\bigwedge\tau.$ (*false* τ = *true* τ) = *False* **by** (*metis OclValid-def foundation2*)

have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

have *A* : $\bot \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *bot-option-def*)
have *B* : $\lfloor\bot\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)

have *C* : $\bigwedge\tau.\ \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
proof − **fix** τ **show** *?thesis* τ
apply(*insert S-all-def*[*simplified all-defined-def*, *THEN conjunct1*, *of* τ]
x-val, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
qed

have *G1* : $\bigwedge\tau.$ *Abs-Set-0* $\lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \neq$ *Abs-Set-0 None*
proof − **fix** τ **show** *?thesis* τ
apply(*insert C*, *simp*)
apply(*simp add*: *S-all-def*[*simplified all-defined-def*, *THEN conjunct1*, *of* τ] *x-val*[*of* τ] *A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)
done
qed

have *G2* : $\bigwedge\tau.$ *Abs-Set-0* $\lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \neq$ *Abs-Set-0* $\lfloor None\rfloor$
proof − **fix** τ **show** *?thesis* τ
apply(*insert C*, *simp*)
apply(*simp add*: *S-all-def*[*simplified all-defined-def*, *THEN conjunct1*, *of* τ] *x-val*[*of* τ] *A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)

done
qed

have $G : \bigwedge\tau.\ (\delta\ (\lambda\text{-}.\ \textit{Abs-Set-0}\ \lfloor\lfloor \textit{insert}\ (x\ \tau)\ \lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil\rfloor\rfloor))\ \tau = \textit{true}\ \tau$
proof $-$ **fix** τ **show** *?thesis* τ
apply(*auto simp*: *OclValid-def false-def true-def defined-def bot-fun-def bot-Set-0-def null-fun-def null-Set-0-def G1 G2*)
done
qed

have *invert-all-defined-aux* : $(\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor \textit{insert}\ (x\ \tau)\ \lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = \textit{bot} \lor X = \textit{null} \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq \textit{bot})\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done

show *?thesis*
apply(*subgoal-tac* $\tau \models \upsilon\ x$) **prefer** *2* **apply**(*simp add*: *x-val*)
apply(*simp add*: *all-defined-def OclIncluding-def OclValid-def*)
apply(*simp add*: *x-val*[*simplified OclValid-def*] *S-all-def*[*simplified all-defined-def OclValid-def*])
apply(*insert Abs-Set-0-inverse*[*OF invert-all-defined-aux*]
S-all-def[*simplified all-defined-def*, *of* τ]
x-val[*of* τ], *simp*)
apply(*simp add*: *cp-defined*[*of* $\lambda\tau.\ \textit{Abs-Set-0}\ \lfloor\lfloor \textit{insert}\ (x\ \tau)\ \lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil\rfloor\rfloor$])
apply(*simp add*: *all-defined-set′-def OclValid-def*)
apply(*simp add*: *cp-valid*[*symmetric*] *x-val*[*simplified OclValid-def*])
apply(*rule G*)
done
qed

lemma *cons-all-def′* :
assumes *S-all-def* : *all-defined* τ *S*
assumes *x-val* : $\tau \models \upsilon\ x$
shows *all-defined* τ (*S*−>*including*(*x*))
proof $-$

have *discr-eq-false-true* : $\bigwedge\tau.\ (\textit{false}\ \tau = \textit{true}\ \tau) = \textit{False}$ **by** (*metis OclValid-def foundation2*)

have *all-defined1* : $\bigwedge r2\ \tau.\ \textit{all-defined}\ \tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)

have $A : \bot \in \{X.\ X = \textit{bot} \lor X = \textit{null} \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq \textit{bot})\}$ **by**(*simp add*: *bot-option-def*)
have $B : \lfloor\bot\rfloor \in \{X.\ X = \textit{bot} \lor X = \textit{null} \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq \textit{bot})\}$ **by**(*simp add*: *null-option-def bot-option-def*)

have $C : \lfloor\lfloor \textit{insert}\ (x\ \tau)\ \lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = \textit{bot} \lor X = \textit{null} \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq \textit{bot})\}$
apply(*insert S-all-def*[*simplified all-defined-def*, *THEN conjunct1*]
x-val, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)

done

have $G1$: $Abs\text{-}Set\text{-}0\ \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \neq Abs\text{-}Set\text{-}0\ None$
apply(*insert C*, *simp*)
apply(*simp add*: *S-all-def*[*simplified all-defined-def*, *THEN conjunct1*] *x-val A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)
done

have $G2$: $Abs\text{-}Set\text{-}0\ \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \neq Abs\text{-}Set\text{-}0\ \lfloor None\rfloor$
apply(*insert C*, *simp*)
apply(*simp add*: *S-all-def*[*simplified all-defined-def*, *THEN conjunct1*] *x-val A Abs-Set-0-inject B C OclValid-def Rep-Set-0-cases Rep-Set-0-inverse bot-Set-0-def bot-option-def insert-compr insert-def not-Some-eq null-Set-0-def null-option-def*)
done

have G : $(\delta\ (\lambda\text{-}.\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor))\ \tau = true\ \tau$
apply(*auto simp*: *OclValid-def false-def true-def defined-def bot-fun-def bot-Set-0-def null-fun-def null-Set-0-def G1 G2*)
done

have *invert-all-defined-aux* : $(\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
show *?thesis*
apply(*subgoal-tac* $\tau \models \upsilon\ x$) **prefer** *2* **apply**(*simp add*: *x-val*)
apply(*simp add*: *all-defined-def OclIncluding-def OclValid-def*)
apply(*simp add*: *x-val*[*simplified OclValid-def*] *S-all-def*[*simplified all-defined-def OclValid-def*])
apply(*insert Abs-Set-0-inverse*[*OF invert-all-defined-aux*]
S-all-def[*simplified all-defined-def*]
x-val, *simp*)
apply(*simp add*: *cp-defined*[*of* $\lambda\tau.$ *if* $(\delta\ S)\ \tau = true\ \tau \land (\upsilon\ x)\ \tau = true\ \tau$ *then* $Abs\text{-}Set\text{-}0\ \lfloor\lfloor\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \cup \{x\ \tau\}\rfloor\rfloor$ *else* $\bot$])
apply(*simp add*: *all-defined-set'-def OclValid-def*)
apply(*simp add*: *cp-valid*[*symmetric*] *x-val*[*simplified OclValid-def*])
apply(*rule G*)
done
qed

all defined (inversion)

lemma *invert-all-defined* : $all\text{-}defined\ \tau\ (S{-}{>}including(x)) \Longrightarrow \tau \models \upsilon\ x \land all\text{-}defined\ \tau\ S$
proof –
have *invert-all-defined-aux* : $(\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)

done

have *finite-including-exec* : $\bigwedge\tau\ X\ x.\ \bigwedge\tau.\ \tau \models (\delta\ X\ and\ \upsilon\ x) \Longrightarrow$
$finite\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X{-}{>}including(x)\ \tau)\rceil\rceil = finite\ \lceil\lceil Rep\text{-}Set\text{-}0\ (X\ \tau)\rceil\rceil$
apply(*rule finite-including-exec*)
apply(*metis OclValid-def foundation5*)+
done

show *all-defined* τ $(S{-}{>}including(x)) \Longrightarrow$ *?thesis*
apply(*simp add*: *all-defined-def all-defined-set'-def*)
apply(*erule conjE*, *frule finite-including-exec*[*of* τ *S x*], *simp*)
by (*metis foundation5*)
qed

lemma *invert-all-defined'* : $(\forall\tau.\ all\text{-}defined\ \tau\ (S{-}{>}including(\lambda(\text{-}{::}\ '\mathfrak{A}\ st).\ x))) \Longrightarrow is\text{-}int\ (\lambda\ (\text{-}{::}\ '\mathfrak{A}\ st).\ x) \wedge (\forall\tau.\ all\text{-}defined\ \tau\ S)$
apply(*rule conjI*)
apply(*simp only*: *is-int-def*, *rule allI*)
apply(*erule-tac* $x = \tau$ **in** *allE*, *simp*)
apply(*drule invert-all-defined*, *simp*)
apply(*rule allI*)
apply(*erule-tac* $x = \tau$ **in** *allE*)
apply(*drule invert-all-defined*, *simp*)
done

Preservation of cp

lemma *including-cp-gen* : $cp\ f \Longrightarrow cp\ (\lambda r2.\ ((f\ r2){-}{>}including(x)))$
apply(*unfold cp-def*)
apply(*subst cp-OclIncluding*[*of - x*])
apply(*drule exE*) **prefer** *2* **apply** *assumption*
apply(*rule-tac* $x = \lambda\ X\text{-}\tau\ \tau.\ ((\lambda\text{-}.\ fa\ X\text{-}\tau\ \tau){-}{>}including(\lambda\text{-}.\ x\ \tau))\ \tau$ **in** *exI*, *simp*)
done

lemma *including-cp* : $cp\ (\lambda r2.\ (r2{-}{>}including(x)))$
apply(*unfold cp-def*)
apply(*subst cp-OclIncluding*[*of - x*])
apply(*rule-tac* $x = \lambda\ X\text{-}\tau\ \tau.\ ((\lambda\text{-}.\ X\text{-}\tau){-}{>}including(\lambda\text{-}.\ x\ \tau))\ \tau$ **in** *exI*, *simp*)
done

lemma *including-cp'* : *cp* (*OclIncluding S*)
apply(*unfold cp-def*)
apply(*subst cp-OclIncluding*)
apply(*rule-tac* $x = \lambda\ X\text{-}\tau\ \tau.\ ((\lambda\text{-}.\ S\ \tau){-}{>}including(\lambda\text{-}.\ X\text{-}\tau))\ \tau$ **in** *exI*, *simp*)
done

lemma *including-cp'''* : $cp\ (OclIncluding\ S{-}{>}including(i){-}{>}including(j))$
apply(*unfold cp-def*)
apply(*subst cp-OclIncluding*)

apply(*rule-tac* $x = \lambda$ *X-τ* τ. ((λ-. *S*−>*including*(*i*)−>*including*(*j*) τ)−>*including*(λ-. *X-τ*)) τ **in** *exI*, *simp*)
done

lemma *including-cp2* : *cp* (λ*r2*. (*r2*−>*including*(*x*))−>*including*(*y*))
by(*rule including-cp-gen*, *simp add*: *including-cp*)

lemma *including-cp3* : *cp* (λ*r2*. ((*r2*−>*including*(*x*))−>*including*(*y*))−>*including*(*z*))
by(*rule including-cp-gen*, *simp add*: *including-cp2*)

Preservation of global judgment

lemma *including-cp-all* :
assumes *x-int* : *is-int x*
and *S-def* : $\bigwedge\tau$. $\tau \models \delta$ *S*
and *S-incl* : *S* $\tau 1$ = *S* $\tau 2$
shows *S*−>*including*(*x*) $\tau 1$ = *S*−>*including*(*x*) $\tau 2$
proof −
have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)
show *?thesis*
apply(*unfold OclIncluding-def*)
apply(*simp add*: *S-def*[*simplified OclValid-def*] *int-is-valid*[*OF x-int*, *simplified OclValid-def*] *S-incl*)
apply(*subgoal-tac x* $\tau 1$ = *x* $\tau 2$, *simp*)
apply(*insert x-int*[*simplified is-int-def*, *THEN spec*, *of* $\tau 1$, *THEN conjunct2*, *THEN spec*], *simp*)
done
qed

Preservation of non-emptiness

lemma *including-notempty* :
assumes *S-def* : $\tau \models \delta$ *S*
and *x-val* : $\tau \models \upsilon$ *x*
and *S-notempty* : $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil \neq \{\}$
shows $\lceil\lceil$*Rep-Set-0* (*S*−>*including*(*x*) τ)$\rceil\rceil \neq \{\}$
proof −
have *insert-in-Set-0* : $\bigwedge\tau$. ($\tau \models$(δ *S*)) $\Longrightarrow$ ($\tau \models$(υ *x*)) $\Longrightarrow \lfloor\lfloor$*insert* (*x* τ) $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil\rfloor\rfloor$ $\in$ {*X*. *X* = *bot* $\vee$ *X* = *null* $\vee$ ($\forall$ *x*$\in\lceil\lceil$*X*$\rceil\rceil$. *x* $\neq$ *bot*)}
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
show *?thesis*
apply(*unfold OclIncluding-def*)
apply(*simp add*: *S-def*[*simplified OclValid-def*] *x-val*[*simplified OclValid-def*] *Abs-Set-0-inverse*[*OF insert-in-Set-0*[*OF S-def x-val*]])
done
qed

lemma *including-notempty′* :

assumes *x-val* : $\tau \models \upsilon\ x$
shows $\lceil\lceil$*Rep-Set-0* (*Set*$\{x\}$ τ)$\rceil\rceil \neq \{\}$
proof –
have *insert-in-Set-0* : $\bigwedge S\ \tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor insert\ (x\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
show *?thesis*
apply(*unfold OclIncluding-def*)
apply(*simp add*: *x-val*[*simplified OclValid-def*])
apply(*subst Abs-Set-0-inverse*)
apply(*rule insert-in-Set-0*)
apply(*simp add*: *mtSet-all-def*)
apply(*simp-all add*: *x-val*)
done
qed

4.7.4. Constant set

lemma *cp-singleton* :
assumes *x-int* : *is-int* (λ(-:: $'\mathfrak{A}$ *st*). *x*)
shows (λ-. *Set*$\{\lambda$(-:: $'\mathfrak{A}$ *st*). $x\}$ τ) = *Set*$\{\lambda$(-:: $'\mathfrak{A}$ *st*). $x\}$
apply(*rule ext*, *rename-tac* τ')
apply(*rule including-cp-all*, *simp add*: *x-int*, *simp*)
apply(*subst* (*1 2*) *cp-mtSet*, *simp*)
done

lemma *cp-doubleton* :
assumes *x-int* : *is-int* (λ(-:: $'\mathfrak{A}$ *st*). *x*)
and *a-int* : *is-int a*
shows (λ-. *Set*$\{\lambda$(-:: $'\mathfrak{A}$ *st*). *x*, $a\}$ τ) = *Set*$\{\lambda$(-:: $'\mathfrak{A}$ *st*). *x*, $a\}$
apply(*rule ext*, *rename-tac* τ')
apply(*rule including-cp-all*, *simp add*: *x-int*, *simp add*: *a-int int-is-valid*)
apply(*rule including-cp-all*, *simp add*: *a-int*, *simp*)
apply(*subst* (*1 2*) *cp-mtSet*, *simp*)
done

lemma *flatten-int'* :
assumes *a-all-def* : $\bigwedge\tau$. *all-defined* τ *Set*$\{\lambda(\tau$:: $'\mathfrak{A}$ *st*). (*a* :: *'a option option*)$\}$
and *a-int* : *is-int* ($\lambda(\tau$:: $'\mathfrak{A}$ *st*). *a*)
shows *let a* = $\lambda(\tau$:: $'\mathfrak{A}$ *st*). (*a* :: -) *in Set*$\{a,a\}$ = *Set*$\{a\}$
proof –
have $B : \lfloor\lfloor\{\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *mtSet-def*)
have $B' : \lfloor\lfloor\{a\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply(*simp*) **apply**(*rule disjI2*)+ **apply**(*insert int-is-valid*[*OF a-int*]) **by** (*metis foundation18'*)
have $C : \bigwedge\ \tau.\ (\delta\ (\lambda\tau.\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor\{\}\rfloor\rfloor))\ \tau = true\ \tau$
by (*metis B Abs-Set-0-cases Abs-Set-0-inverse cp-defined defined-def false-def mtSet-def mtSet-defined*

null-fun-def null-option-def null-set-not-defined true-def)

show *?thesis*
apply(*simp add*: *Let-def*)
apply(*rule including-id*, *simp add*: *a-all-def*)
apply(*rule allI*, *simp add*: *OclIncluding-def int-is-valid*[*OF a-int*, *simplified OclValid-def*] *mtSet-def Abs-Set-0-inverse*[*OF B*] *C Abs-Set-0-inverse*[*OF B'*])
done
qed

lemma *flatten-int* :
assumes *a-int* : *is-int* (a :: ($'\mathfrak{A}$, $'a$ *option option*) *val*)
shows $Set\{a,a\} = Set\{a\}$
proof −
have *all-def* : $\bigwedge\tau$. *all-defined* τ $Set\{a\}$
apply(*rule cons-all-def*)
apply(*simp add*: *mtSet-all-def int-is-valid*[*OF a-int*])+
done

show *?thesis*
apply(*insert a-int*, *drule destruct-int*)
apply(*drule ex1E*) **prefer** *2* **apply** *assumption*
apply(*simp*)
apply(*rule flatten-int'*[*simplified Let-def*])
apply(*insert all-def*, *simp*)
apply(*insert a-int*, *simp*)
done
qed

4.7.5. OclExcluding

Identity

lemma *excluding-id* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ (S :: ($'\mathfrak{A}$, $'a$ *option option*) *Set*)
and *x-int* : *is-int* ($\lambda(\tau$:: $'\mathfrak{A}$ *st*). x)
shows $\forall\tau.\ x \notin \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \Longrightarrow$
$S{-}{>}excluding(\lambda\tau.\ x) = S$
proof −

have *S-incl* : $\forall (x$ :: ($'\mathfrak{A}$, $'a$ *option option*) *Set*). ($\forall\tau$. *all-defined* τ x) $\longrightarrow$ ($\forall\tau$. $\lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil = \{\}$) $\longrightarrow Set\{\} = x$
apply(*rule allI*) **apply**(*rule impI*)+
apply(*rule ext*, *rename-tac* τ)
apply(*drule-tac* $x = \tau$ **in** *allE*) **prefer** *2* **apply** *assumption*
apply(*drule-tac* $x = \tau$ **in** *allE*) **prefer** *2* **apply** *assumption*
apply(*simp add*: *mtSet-def*)
by (*metis abs-rep-simp*)

have *discr-eq-invalid-true* : $\bigwedge\tau$. (*invalid* τ = *true* τ) = *False* **by** (*metis bot-option-def invalid-def*

option.simps(*2*) *true-def*)
have *discr-eq-false-true* : $\bigwedge\tau$. (*false* τ = *true* τ) = *False* **by** (*metis OclValid-def foundation2*)

have *all-defined1* : $\bigwedge r2\ \tau$. *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

show ($\forall\tau$. $x \notin \lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$) $\Longrightarrow$
?thesis
apply(*rule ext*, *rename-tac* τ', *simp add*: *OclExcluding-def S-all-def*[*simplified all-defined-def OclValid-def*] *int-is-valid*[*OF x-int*, *simplified OclValid-def*])

proof − **fix** τ' **show** $\forall$ *a b*. $x \notin \lceil\lceil$*Rep-Set-0* (S (*a*, *b*))$\rceil\rceil \Longrightarrow$ *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* ($S$ τ')$\rceil\rceil - \{x\}\rfloor\rfloor = S\ \tau'$

apply(*subst finite-induct*[**where** $P = \lambda$*set*. $x \notin$ *set* $\longrightarrow$ ($\forall$ *set'*. *all-defined* τ' *set'* $\longrightarrow$ *set* = $\lceil\lceil$*Rep-Set-0* (*set'* τ')$\rceil\rceil \longrightarrow$ *Abs-Set-0* $\lfloor\lfloor$*set* $- \{x\}\rfloor\rfloor$ = *set'* τ'), *THEN mp*, *THEN spec*, *THEN mp*])
apply(*simp add*: *S-all-def*[*simplified all-defined-def all-defined-set'-def*])
apply(*simp*)
apply(*rule allI*, *rename-tac* S') **apply**(*rule impI*)+
apply(*drule-tac* $f = \lambda x$. *Abs-Set-0* $\lfloor\lfloor x\rfloor\rfloor$ **in** *arg-cong*)
apply(*simp*)

apply(*subst abs-rep-simp*, *simp*)
apply(*simp*)
apply(*rename-tac* x' F)
apply(*rule impI*, *rule allI*, *rename-tac* S') **apply**(*rule impI*)+
proof − **fix** x' F S' **show** $\forall$ *a b*. $x \notin \lceil\lceil$*Rep-Set-0* (S (*a*, *b*))$\rceil\rceil \Longrightarrow$
finite $F \Longrightarrow$
$x' \notin F \Longrightarrow$
$x \notin F \longrightarrow$ ($\forall$ *xa*. *all-defined* τ' *xa* $\longrightarrow F = \lceil\lceil$*Rep-Set-0* (*xa* τ')$\rceil\rceil \longrightarrow$ *Abs-Set-0* $\lfloor\lfloor F - \{x\}\rfloor\rfloor$ = *xa* τ') $\Longrightarrow$
$x \notin$ *insert* x' $F \Longrightarrow$ *all-defined* τ' $S' \Longrightarrow$ *insert* x' $F = \lceil\lceil$*Rep-Set-0* (S' τ')$\rceil\rceil \Longrightarrow$ *Abs-Set-0* $\lfloor\lfloor$*insert* x' $F - \{x\}\rfloor\rfloor = S'\ \tau'$
apply(*subgoal-tac* $x \notin F$, *simp*)
apply(*rule abs-rep-simp*, *simp*)
by (*metis insertCI*)
apply-end(*simp*)+
apply-end(*metis surj-pair*)
prefer *3*
apply-end(*rule refl*)
apply-end(*simp add*: *S-all-def*, *simp*)
qed
qed
qed

all defined (construction)

lemma *cons-all-def-e* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ S

assumes *x-val* : $\bigwedge\tau.\ \tau \models \upsilon\ x$
shows *all-defined* τ *S−>excluding*(x)
proof −

have *discr-eq-false-true* : $\bigwedge\tau.\ (\mathit{false}\ \tau = \mathit{true}\ \tau) = \mathit{False}$ **by** (*metis OclValid-def foundation2*)

have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

have $A : \bot \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *bot-option-def*)
have $B : \lfloor\bot\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)

have $C : \bigwedge\tau.\ \lfloor\lfloor\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
proof − **fix** τ **show** *?thesis* τ
 apply(*insert S-all-def*[*simplified all-defined-def*, *THEN conjunct1*, *of* τ]
  x-val, *frule Set-inv-lemma*)
 apply(*simp add*: *foundation18 invalid-def*)
 done
qed

have $G1 : \bigwedge\tau.\ \mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \neq \mathit{Abs\text{-}Set\text{-}0}\ \mathit{None}$
proof − **fix** τ **show** *?thesis* τ
 apply(*insert C*[*of* τ], *simp*)
 apply(*simp add*: *Abs-Set-0-inject bot-option-def*)
done
qed

have $G2 : \bigwedge\tau.\ \mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \neq \mathit{Abs\text{-}Set\text{-}0}\ \lfloor \mathit{None}\rfloor$
proof − **fix** τ **show** *?thesis* τ
 apply(*insert C*[*of* τ], *simp*)
 apply(*simp add*: *Abs-Set-0-inject bot-option-def null-option-def*)
done
qed

have $G : \bigwedge\tau.\ (\delta\ (\lambda\text{-}.\ \mathit{Abs\text{-}Set\text{-}0}\ \lfloor\lfloor\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor))\ \tau = \mathit{true}\ \tau$
proof − **fix** τ **show** *?thesis* τ
 apply(*auto simp*: *OclValid-def false-def true-def defined-def*
  bot-fun-def bot-Set-0-def null-fun-def null-Set-0-def G1 G2)
done
qed

have *invert-all-defined-aux* : $(\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
 apply(*frule Set-inv-lemma*)
 apply(*simp add*: *foundation18 invalid-def*)
 done

show *?thesis*

apply(*subgoal-tac* $\tau \models \upsilon\ x$) **prefer** *2* **apply**(*simp add*: *x-val*)
apply(*simp add*: *all-defined-def OclExcluding-def OclValid-def*)
apply(*simp add*: *x-val*[*simplified OclValid-def*] *S-all-def*[*simplified all-defined-def OclValid-def*])
apply(*insert Abs-Set-0-inverse*[*OF invert-all-defined-aux*]
S-all-def[*simplified all-defined-def*, *of* τ]
x-val[*of* τ], *simp*)
apply(*simp add*: *cp-defined*[*of* $\lambda\tau$. *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor$])
apply(*simp add*: *all-defined-set'-def OclValid-def*)
apply(*simp add*: *cp-valid*[*symmetric*] *x-val*[*simplified OclValid-def*])
apply(*rule G*)
done
qed

Execution

lemma *excluding-unfold* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ S
and *x-val* : $\bigwedge\tau.\ \tau \models \upsilon\ x$
shows $\lceil\lceil$*Rep-Set-0* $(S{-}{>}excluding(x)\ \tau)\rceil\rceil = \lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil - \{x\ \tau\}$
proof −
have *all-defined1* : $\bigwedge r2\ \tau$. *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

have C : $\bigwedge\tau.\ \lfloor\lfloor\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
proof − **fix** τ **show** *?thesis* τ
apply(*insert S-all-def*[*simplified all-defined-def*, *THEN conjunct1*, *of* τ]
x-val, *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
qed
show *?thesis*
apply(*simp add*: *OclExcluding-def all-defined1*[*OF S-all-def*, *simplified OclValid-def*] *x-val*[*simplified OclValid-def*] *Abs-Set-0-inverse*[*OF C*])
done
qed

4.7.6. OclIncluding and OclExcluding

Identity

lemma *Ocl-insert-Diff* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ $(S :: ('\mathfrak{A}, 'a\ option\ option)\ Set)$
and *x-mem* : $\bigwedge\tau.\ x \in (\lambda a\ (\tau{::}\ '\mathfrak{A}\ st).\ a)$ ‘ $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil$
and *x-int* : *is-int* x
shows $S{-}{>}excluding(x){-}{>}including(x) = S$
proof −
have *all-defined1* : $\bigwedge r2\ \tau$. *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

have *remove-in-Set-0* : $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ x)) \Longrightarrow \lfloor\lfloor\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil - \{x\ \tau\}\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$

apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
have *remove-in-Set-0* : $\bigwedge\tau$. *?this* τ
apply(*rule remove-in-Set-0*)
by(*simp add*: *S-all-def*[*simplified all-defined-def*] *int-is-valid*[*OF x-int*])+
have *inject* : *inj* ($\lambda a\ \tau.\ a$) **by**(*rule inj-fun*, *simp*)

show *?thesis*

apply(*rule ext*, *rename-tac* τ)
apply(*subgoal-tac* $\tau \models \delta$ (*S*−>*excluding*(*x*)))
prefer *2*
apply(*simp add*: *foundation10 all-defined1*[*OF S-all-def*] *int-is-valid*[*OF x-int*])
apply(*simp add*: *OclExcluding-def OclIncluding-def all-defined1*[*OF S-all-def*, *simplified OclValid-def*] *Abs-Set-0-inverse*[*OF remove-in-Set-0*] *int-is-valid*[*OF x-int*, *simplified OclValid-def*] *OclValid-def*)
proof − **fix** τ **show** *Abs-Set-0* $\lfloor\lfloor$*insert* ($x\ \tau$) $\lceil\lceil$*Rep-Set-0* ($S\ \tau$)$\rceil\rceil\rfloor\rfloor = S\ \tau$
apply(*rule ex1E*[*OF destruct-int*[*OF x-int*]], *rename-tac* x', *simp*)
apply(*subgoal-tac* $x' \in \lceil\lceil$*Rep-Set-0* ($S\ \tau$)$\rceil\rceil$)
apply(*drule insert-Diff*[*symmetric*], *simp*)
apply(*simp add*: *abs-rep-simp*[*OF S-all-def*[**where** $\tau = \tau$]])
apply(*insert x-mem*[*of* τ], *simp*)
apply(*rule inj-image-mem-iff*[*THEN iffD1*]) **prefer** *2* **apply** *assumption*
apply(*simp add*: *inject*)
done
qed
qed

4.7.7. OclIterate

all defined (inversion)

lemma *i-invert-all-defined-not* :
assumes *A-all-def* : $\exists\tau.\ \neg$ *all-defined* τ *S*
shows $\exists\tau.\ \neg$ *all-defined* τ (*OclIterate*$_{Set}$ *S S F*)
proof −
have *A* : $\perp \in \{X.\ X = bot \lor X = null \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *bot-option-def*)
have *B* : $\lfloor\perp\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)
have *C* : $\lfloor None\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)

show *?thesis*
apply(*insert A-all-def*)
apply(*drule exE*) **prefer** *2* **apply** *assumption*
apply(*rule-tac* $x = \tau$ **in** *exI*)
proof − **fix** τ **show** $\neg$ *all-defined* τ *S* $\Longrightarrow \neg$ *all-defined* τ (*OclIterate*$_{Set}$ *S S F*)
apply(*unfold OclIterate*$_{Set}$*-def*)
apply(*case-tac* $\tau \models (\delta\ S) \land \tau \models (\upsilon\ S) \land$ *finite* $\lceil\lceil$*Rep-Set-0* ($S\ \tau$)$\rceil\rceil$, *simp add*: *OclValid-def all-defined-def*)

apply(*simp add*: *all-defined-set′-def*)
apply(*simp add*: *all-defined-def all-defined-set′-def defined-def OclValid-def false-def true-def bot-fun-def*)
done
qed
qed

lemma *i-invert-all-defined* :
assumes *A-all-def* : $\bigwedge\tau$. *all-defined* τ (*OclIterate*$_{Set}$ *S S F*)
shows *all-defined* τ *S*
by (*metis A-all-def i-invert-all-defined-not*)

lemma *i-invert-all-defined′* :
assumes *A-all-def* : $\forall\tau$. *all-defined* τ (*OclIterate*$_{Set}$ *S S F*)
shows $\forall\tau$. *all-defined* τ *S*
by (*metis A-all-def i-invert-all-defined*)

4.7.8. comp fun commute

Main

TODO add some comment on comparison with inductively constructed OCL term

locale *EQ-comp-fun-commute0-gen0-bis′′* =
fixes *f000* :: $'b \Rightarrow 'c$
fixes *is-i* :: $'\mathfrak{A}$ *st* $\Rightarrow 'c \Rightarrow$ *bool*
fixes *is-i′* :: $'\mathfrak{A}$ *st* $\Rightarrow 'c \Rightarrow$ *bool*
fixes *all-i-set* :: $'c$ *set* $\Rightarrow$ *bool*

fixes *f* :: $'c$
$\Rightarrow$ ($'\mathfrak{A}$, $'a$ *option option*) *Set*
$\Rightarrow$ ($'\mathfrak{A}$, $'a$ *option option*) *Set*
assumes *i-set* : $\bigwedge x\ A$. *all-i-set* (*insert x A*) $\Longrightarrow$ (($\forall\tau$. *is-i* τ *x*) $\wedge$ *all-i-set A*)
assumes *i-set′* : $\bigwedge x\ A$. (($\forall\tau$. *is-i* τ (*f000 x*)) $\wedge$ *all-i-set A*) $\Longrightarrow$ *all-i-set* (*insert* (*f000 x*) *A*)
assumes *i-set′′* : $\bigwedge x\ A$. (($\forall\tau$. *is-i* τ (*f000 x*)) $\wedge$ *all-i-set A*) $\Longrightarrow$ *all-i-set* ($A - \{$*f000 x*$\}$)
assumes *i-set-finite* : *all-i-set A* $\Longrightarrow$ *finite A*
assumes *i-val* : $\bigwedge x\ \tau$. *is-i* τ *x* $\Longrightarrow$ *is-i′* τ *x*
assumes *f000-inj* : $\bigwedge x\ y$. $x \neq y \Longrightarrow$ *f000 x* $\neq$ *f000 y*

assumes *cp-set* : $\bigwedge x\ S\ \tau$. $\forall\tau$. *all-defined* τ *S* $\Longrightarrow$ *f* (*f000 x*) *S* τ = *f* (*f000 x*) (λ-. *S* τ) τ
assumes *all-def*: $\bigwedge x\ y$. ($\forall\tau$. *all-defined* τ (*f* (*f000 x*) *y*)) = (($\forall\tau$. *is-i′* τ (*f000 x*)) $\wedge$ ($\forall\tau$. *all-defined* τ *y*))
assumes *commute*: $\bigwedge x\ y\ S$.
($\bigwedge\tau$. *is-i′* τ (*f000 x*)) $\Longrightarrow$
($\bigwedge\tau$. *is-i′* τ (*f000 y*)) $\Longrightarrow$
($\bigwedge\tau$. *all-defined* τ *S*) $\Longrightarrow$
f (*f000 y*) (*f* (*f000 x*) *S*) = *f* (*f000 x*) (*f* (*f000 y*) *S*)

inductive *EQG-fold-graph* :: ($'b \Rightarrow 'c$)
$\Rightarrow$ ($'c$

$\Rightarrow ('\mathfrak{A}, 'a)\ Set$
$\Rightarrow ('\mathfrak{A}, 'a)\ Set)$
$\Rightarrow ('\mathfrak{A}, 'a)\ Set$
$\Rightarrow 'c\ set$
$\Rightarrow ('\mathfrak{A}, 'a)\ Set$
$\Rightarrow bool$
for *is-i* **and** F **and** z **where**
EQG-emptyI [*intro*]: *EQG-fold-graph is-i F z* {} *z* |
EQG-insertI [*intro*]: *is-i x* $\notin A \Longrightarrow$
EQG-fold-graph is-i F z A y $\Longrightarrow$
EQG-fold-graph is-i F z (*insert* (*is-i x*) *A*) (*F* (*is-i x*) *y*)

inductive-cases *EQG-empty-fold-graphE* [*elim*!]: *EQG-fold-graph is-i f z* {} *x*
definition *foldG is-i f z A* = (*THE y. EQG-fold-graph is-i f z A y*)

lemma *eqg-fold-of-fold* :
assumes *fold-g* : *fold-graph F z* (*f000* ' *A*) *y*
shows *EQG-fold-graph f000 F z* (*f000* ' *A*) *y*
apply(*insert fold-g*)
apply(*subgoal-tac* $\bigwedge A'$. *fold-graph F z A' y* $\Longrightarrow A' \subseteq$ *f000* ' *A* $\Longrightarrow$ *EQG-fold-graph f000 F z A' y*)
apply(*simp*)
proof − **fix** A' **show** *fold-graph F z A' y* $\Longrightarrow A' \subseteq$ *f000* ' *A* $\Longrightarrow$ *EQG-fold-graph f000 F z A' y*
apply(*induction set*: *fold-graph*)
apply(*rule EQG-emptyI*)
apply(*simp*, *erule conjE*)
apply(*drule imageE*) **prefer** *2* **apply** *assumption*
apply(*simp*)
apply(*rule EQG-insertI*, *simp*, *simp*)
done
qed

lemma *fold-of-eqg-fold* :
assumes *fold-g* : *EQG-fold-graph f000 F z A y*
shows *fold-graph F z A y*
apply(*insert fold-g*)
apply(*induction set*: *EQG-fold-graph*)
apply(*rule emptyI*)
apply(*simp add*: *insertI*)
done

context *EQ-comp-fun-commute0-gen0-bis''*
begin

lemma *fold-graph-insertE-aux*:
assumes *y-defined* : $\bigwedge\tau$. *all-defined* τ *y*
assumes *a-valid* : $\forall \tau$. *is-i'* τ (*f000 a*)
shows

EQG-fold-graph f000 f z A y $\Longrightarrow$ (*f000 a*) $\in$ *A* $\Longrightarrow$ $\exists$ *y'*. *y* = *f* (*f000 a*) *y'* $\wedge$ ($\forall$ τ. *all-defined* τ *y'*) $\wedge$ *EQG-fold-graph f000 f z* (*A* − {(*f000 a*)}) *y'*
apply(*insert y-defined*)
proof (*induct set*: *EQG-fold-graph*)
case (*EQG-insertI x A y*)
assume $\bigwedge\tau$. *all-defined* τ (*f* (*f000 x*) *y*)
then show $\forall$ τ. *is-i'* τ (*f000 x*) $\Longrightarrow$ ($\bigwedge\tau$. *all-defined* τ *y*) $\Longrightarrow$ *?case*
proof (*cases x* = *a*) **assume** *x* = *a* **with** *EQG-insertI* **show** ($\bigwedge\tau$. *all-defined* τ *y*) $\Longrightarrow$ *?case* **by** (*metis Diff-insert-absorb all-def*)
next apply-end(*simp*)

assume *f000 x* $\neq$ *f000 a* $\wedge$ ($\forall$ τ. *all-defined* τ *y*)
then obtain *y'* **where** *y*: *y* = *f* (*f000 a*) *y'* **and** ($\forall$ τ. *all-defined* τ *y'*) **and** *y'*: *EQG-fold-graph f000 f z* (*A* − {(*f000 a*)}) *y'*
using *EQG-insertI* **by** (*metis OCL-core.drop.simps insert-iff*)
have ($\bigwedge\tau$. *all-defined* τ *y*) $\Longrightarrow$ ($\bigwedge\tau$. *all-defined* τ *y'*)
apply(*subgoal-tac* $\forall$ τ. *is-i'* τ (*f000 a*) $\wedge$ ($\forall$ τ. *all-defined* τ *y'*)) **apply**(*simp only*:)
apply(*subst* (*asm*) *cp-all-def*) **unfolding** *y* **apply**(*subst* (*asm*) *cp-all-def*[*symmetric*])
apply(*insert all-def*[**where** *x* = *a* **and** *y* = *y'*, *THEN iffD1*], *blast*)
done
moreover have $\forall$ τ. *is-i'* τ (*f000 x*) $\Longrightarrow$ $\forall$ τ. *is-i'* τ (*f000 a*) $\Longrightarrow$ ($\bigwedge\tau$. *all-defined* τ *y'*) $\Longrightarrow$ *f* (*f000 x*) *y* = *f* (*f000 a*) (*f* (*f000 x*) *y'*)
unfolding *y*
by(*rule commute*, *blast*+)
moreover have *EQG-fold-graph f000 f z* (*insert* (*f000 x*) *A* − {*f000 a*}) (*f* (*f000 x*) *y'*)
using *y'* **and** ‹*f000 x* $\neq$ *f000 a* $\wedge$ ($\forall$ τ. *all-defined* τ *y*)› **and** ‹*f000 x* $\notin$ *A*›
apply (*simp add*: *insert-Diff-if OCL-lib.EQG-insertI*)
done
apply-end(*subgoal-tac f000 x* $\neq$ *f000 a* $\wedge$ ($\forall$ τ. *all-defined* τ *y*) $\Longrightarrow$ $\exists$ *y'*. *f* (*f000 x*) *y* = *f* (*f000 a*) *y'* $\wedge$ ($\forall$ τ. *all-defined* τ *y'*) $\wedge$ *EQG-fold-graph f000 f z* (*insert* (*f000 x*) *A* − {(*f000 a*)}) *y'*)
ultimately show ($\forall$ τ. *is-i'* τ (*f000 x*)) $\wedge$ *f000 x* $\neq$ *f000 a* $\wedge$ ($\forall$ τ. *all-defined* τ *y*) $\Longrightarrow$ *?case* **apply**(*auto simp*: *a-valid*)
by (*metis* (*mono-tags*) ‹$\bigwedge\tau$. *all-defined* τ (*f* (*f000 x*) *y*)› *all-def*)
apply-end(*drule f000-inj*, *blast*)+
qed
apply-end *simp*

fix *x y*
show ($\bigwedge\tau$. *all-defined* τ (*f* (*f000 x*) *y*)) $\Longrightarrow$ $\forall$ τ. *is-i'* τ (*f000 x*)
apply(*rule all-def*[**where** *x* = *x* **and** *y* = *y*, *THEN iffD1*, *THEN conjunct1*], *simp*) **done**
apply-end *blast*
fix *x y* τ
show ($\bigwedge\tau$. *all-defined* τ (*f* (*f000 x*) *y*)) $\Longrightarrow$ *all-defined* τ *y*
apply(*rule all-def*[**where** *x* = *x*, *THEN iffD1*, *THEN conjunct2*, *THEN spec*], *simp*) **done**
apply-end *blast*
qed

lemma *fold-graph-insertE*:

```
  assumes v-defined : ⋀τ. all-defined τ v
      and x-valid : ∀ τ. is-i' τ (f000 x)
      and EQG-fold-graph f000 f z (insert (f000 x) A) v and (f000 x) ∉ A
   obtains y where v = f (f000 x) y and is-i' τ (f000 x) and ⋀τ. all-defined τ y and
EQG-fold-graph f000 f z A y
  apply(insert fold-graph-insertE-aux[OF v-defined x-valid ‹EQG-fold-graph f000 f z (insert
(f000 x) A) v› insertI1] x-valid ‹(f000 x) ∉ A›)
 apply(drule exE) prefer 2 apply assumption
 apply(drule Diff-insert-absorb, simp only:)
done

lemma fold-graph-determ:
 assumes x-defined : ⋀τ. all-defined τ x
     and y-defined : ⋀τ. all-defined τ y
   shows EQG-fold-graph f000 f z A x ⟹ EQG-fold-graph f000 f z A y ⟹ y = x
apply(insert x-defined y-defined)
proof (induct arbitrary: y set: EQG-fold-graph)
  case (EQG-insertI x A y v)
  from ‹⋀τ. all-defined τ (f (f000 x) y)›
  have ∀ τ. is-i' τ (f000 x) by(metis all-def)
  from ‹⋀τ. all-defined τ v› and ‹∀ τ. is-i' τ (f000 x)› and ‹EQG-fold-graph f000 f z (insert
(f000 x) A) v› and ‹(f000 x) ∉ A›
  obtain y' where v = f (f000 x) y' and ⋀τ. all-defined τ y' and EQG-fold-graph f000 f z A
y'
     by (rule fold-graph-insertE, simp)
  from EQG-insertI have ⋀τ. all-defined τ y by (metis all-def)
  from ‹⋀τ. all-defined τ y› and ‹⋀τ. all-defined τ y'› and ‹EQG-fold-graph f000 f z A y'›
have y' = y by (metis EQG-insertI.hyps(3))
  with ‹v = f (f000 x) y'› show v = f (f000 x) y by (simp)
  apply-end(rule-tac f = f in EQG-empty-fold-graphE, auto)
qed

lemma det-init2 :
  assumes z-defined : ∀ (τ :: '𝔄 st). all-defined τ z
      and A-int : all-i-set A
    shows EQG-fold-graph f000 f z A x ⟹ ∀ τ. all-defined τ x
 apply(insert z-defined A-int)
 proof (induct set: EQG-fold-graph)
  apply-end(simp)
  apply-end(subst all-def, drule i-set, auto, rule i-val, blast)
qed

lemma fold-graph-determ3 :
  assumes z-defined : ⋀τ. all-defined τ z
      and A-int : all-i-set A
    shows EQG-fold-graph f000 f z A x ⟹ EQG-fold-graph f000 f z A y ⟹ y = x
 apply(insert z-defined A-int)
 apply(rule fold-graph-determ)
 apply(rule det-init2[THEN spec]) apply(blast)+
```

apply(*rule det-init2*[*THEN spec*]) **apply**(*blast*)+
done

lemma *fold-graph-fold*:
assumes *z-int* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-i-set* (*f000* ‘ *A*)
shows *EQG-fold-graph f000 f z* (*f000* ‘ *A*) (*foldG f000 f z* (*f000* ‘ *A*))
proof −
from *A-int* **have** *finite* (*f000* ‘ *A*) **by** (*simp add*: *i-set-finite*)
then have $\exists$ *x*. *fold-graph f z* (*f000* ‘ *A*) *x* **by** (*rule finite-imp-fold-graph*)
then have $\exists$ *x*. *EQG-fold-graph f000 f z* (*f000* ‘ *A*) *x* **by** (*metis eqg-fold-of-fold*)
moreover note *fold-graph-determ3*[*OF z-int A-int*]
ultimately have $\exists!$*x*. *EQG-fold-graph f000 f z* (*f000* ‘ *A*) *x* **by**(*rule ex-ex1I*)
then have *EQG-fold-graph f000 f z* (*f000* ‘ *A*) (*The* (*EQG-fold-graph f000 f z* (*f000* ‘ *A*))) **by** (*rule theI*′)
then show *?thesis* **by**(*unfold foldG-def*)
qed

lemma *fold-equality*:
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-i-set* (*f000* ‘ *A*)
shows *EQG-fold-graph f000 f z* (*f000* ‘ *A*) *y* $\Longrightarrow$ *foldG f000 f z* (*f000* ‘ *A*) = *y*
apply(*rule fold-graph-determ3*[*OF z-defined A-int*], *simp*)
apply(*rule fold-graph-fold*[*OF z-defined A-int*])
done

lemma *fold-insert*:
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-i-set* (*f000* ‘ *A*)
and *x-int* : $\forall\tau$. *is-i* τ (*f000 x*)
and *x-nA* : *f000 x* $\notin$ *f000* ‘ *A*
shows *foldG f000 f z* (*f000* ‘ (*insert x A*)) = *f* (*f000 x*) (*foldG f000 f z* (*f000* ‘ *A*))
proof (*rule fold-equality*)
have *EQG-fold-graph f000 f z* (*f000* ‘*A*) (*foldG f000 f z* (*f000* ‘*A*)) **by** (*rule fold-graph-fold*[*OF z-defined A-int*])
with *x-nA* **show** *EQG-fold-graph f000 f z* (*f000* ‘(*insert x A*)) (*f* (*f000 x*) (*foldG f000 f z* (*f000* ‘*A*))) **apply**(*simp add*: *image-insert*) **by**(*rule EQG-insertI*, *simp*, *simp*)
apply-end (*simp add*: *z-defined*)
apply-end (*simp only*: *image-insert*)
apply-end(*rule i-set*′, *simp add*: *x-int A-int*)
qed

lemma *fold-insert*′:
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-i-set* (*f000* ‘ *A*)
and *x-int* : $\forall\tau$. *is-i* τ (*f000 x*)
and *x-nA* : *x* $\notin$ *A*
shows *Finite-Set.fold f z* (*f000* ‘ *insert x A*) = *f* (*f000 x*) (*Finite-Set.fold f z* (*f000* ‘ *A*))
proof −

have *eq-f* : $\bigwedge A$. *Finite-Set.fold f z* (*f000* ‘ *A*) = *foldG f000 f z* (*f000* ‘ *A*)
apply(*simp add*: *Finite-Set.fold-def foldG-def*)
by (*metis eqg-fold-of-fold fold-of-eqg-fold*)

have *x-nA* : *f000 x* $\notin$ *f000* ‘ *A*
apply(*simp add*: *image-iff*)
by (*metis x-nA f000-inj*)

have *foldG f000 f z* (*f000* ‘ *insert x A*) = *f* (*f000 x*) (*foldG f000 f z* (*f000* ‘ *A*))
apply(*rule fold-insert*) **apply**(*simp add*: *assms x-nA*)+
done

thus *?thesis* **by** (*subst* (*1 2*) *eq-f*, *simp*)
qed

lemma *all-int-induct* :
assumes *i-fin* : *all-i-set* (*f000* ‘ *F*)
assumes *P* {}
and *insert*: $\bigwedge x$ *F*. *all-i-set* (*f000* ‘ *F*) $\Longrightarrow \forall \tau$. *is-i* τ (*f000 x*) $\Longrightarrow x \notin F \Longrightarrow P$ (*f000* ‘ *F*) $\Longrightarrow P$ (*f000* ‘ (*insert x F*))
shows *P* (*f000* ‘ *F*)
proof −
from *i-fin* **have** *finite* (*f000* ‘ *F*) **by** (*simp add*: *i-set-finite*)
then have *finite F* **apply**(*rule finite-imageD*) **apply**(*simp add*: *inj-on-def*, *insert f000-inj*, *blast*) **done**
show *?thesis*
using ‹*finite F*› **and** *i-fin*
proof *induct*
apply-end(*simp*)
show *P* {} **by** *fact*
apply-end(*simp add*: *i-set*)
apply-end(*rule insert*[*simplified*], *simp add*: *i-set*, *simp add*: *i-set*)
apply-end(*simp*, *simp*)
qed
qed

lemma *all-defined-fold-rec* :
assumes *A-defined* : $\bigwedge \tau$. *all-defined* τ *A*
and *x-notin* : *x* $\notin$ *Fa*
shows
all-i-set (*f000* ‘ *insert x Fa*) $\Longrightarrow$
($\bigwedge \tau$. *all-defined* τ (*Finite-Set.fold f A* (*f000* ‘ *Fa*))) $\Longrightarrow$
all-defined τ (*Finite-Set.fold f A* (*f000* ‘ *insert x Fa*))
apply(*subst* (*asm*) *image-insert*)
apply(*frule i-set*[*THEN conjunct1*])
apply(*subst fold-insert'*[*OF A-defined*])
apply(*rule i-set*[*THEN conjunct2*], *simp*)
apply(*simp*)
apply(*simp add*: *x-notin*)

apply(*rule all-def* [*THEN iffD2*, *THEN spec*])
apply(*simp add*: *i-val*)
done

lemma (**in** −) *fold-empty* [*simp*]: *foldG f000 f z* {} = *z*
by (*unfold foldG-def*) *blast*

lemma *fold-def* :
assumes *z-def* : $\bigwedge\tau$. *all-defined* τ *z*
assumes *F-int* : *all-i-set* (*f000* ‘ *F*)
shows *all-defined* τ (*Finite-Set.fold f z* (*f000* ‘ *F*))
apply(*subgoal-tac* $\forall\tau$. *all-defined* τ (*Finite-Set.fold f z* (*f000* ‘ *F*)), *blast*)
proof (*induct rule*: *all-int-induct*[*OF F-int*])
apply-end(*simp add*:*z-def*)
apply-end(*rule allI*)
apply-end(*rule all-defined-fold-rec*[*OF z-def*], *simp*, *simp add*: *i-set′*, *blast*)
qed

lemma *fold-fun-comm*:
assumes *z-def* : $\bigwedge\tau$. *all-defined* τ *z*
assumes *A-int* : *all-i-set* (*f000* ‘ *A*)
and *x-val* : $\bigwedge\tau$. *is-i′* τ (*f000 x*)
shows *f* (*f000 x*) (*Finite-Set.fold f z* (*f000* ‘ *A*)) = *Finite-Set.fold f* (*f* (*f000 x*) *z*) (*f000* ‘ *A*)
proof −
have *fxz-def*: $\bigwedge\tau$. *all-defined* τ (*f* (*f000 x*) *z*)
by(*rule all-def* [*THEN iffD2*, *THEN spec*], *simp add*: *z-def x-val*)

show *?thesis*
proof (*induct rule*: *all-int-induct*[*OF A-int*])
apply-end(*simp*)
apply-end(*rename-tac x′ F*)
apply-end(*subst fold-insert′*[*OF z-def*], *simp*, *simp*, *simp*)
apply-end(*subst fold-insert′*[*OF fxz-def*], *simp*, *simp*, *simp*)
apply-end(*subst commute*[*symmetric*])
apply-end(*simp add*: *x-val*)
apply-end(*rule i-val*, *blast*)
apply-end(*subst fold-def* [*OF z-def*], *simp-all*)
qed
qed

lemma *fold-rec*:
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-i-set* (*f000* ‘ *A*)
and *x-int* : $\forall\tau$. *is-i* τ (*f000 x*)
and $x \in A$
shows *Finite-Set.fold f z* (*f000* ‘ *A*) = *f* (*f000 x*) (*Finite-Set.fold f z* (*f000* ‘ (*A* − {*x*})))
proof −
have *f-inj* : *inj f000* **by** (*simp add*: *inj-on-def*, *insert f000-inj*, *blast*)

from *A-int* **have** *A-int*: *all-i-set* (*f000* ‘ ($A - \{x\}$)) **apply**(*subst image-set-diff*[*OF f-inj*]) **apply**(*simp*, *rule i-set''*, *simp add*: *x-int*) **done**
have *A*: *f000* ‘ A = *insert* (*f000* x) (*f000* ‘ ($A - \{x\}$)) **using** ‹$x \in A$› **by** *blast*
then have *Finite-Set.fold f z* (*f000* ‘ A) = *Finite-Set.fold f z* (*insert* (*f000* x) (*f000* ‘ ($A - \{x\}$))) **by** *simp*
also have ... = *f* (*f000* x) (*Finite-Set.fold f z* (*f000* ‘ ($A - \{x\}$))) **by**(*simp only*: *image-insert*[*symmetric*], *rule fold-insert'*[*OF z-defined A-int x-int*], *simp*)
finally show *?thesis* **.**
qed

lemma *fold-insert-remove*:
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-i-set* (*f000* ‘ A)
and *x-int* : $\forall\tau$. *is-i* τ (*f000* x)
shows *Finite-Set.fold f z* (*f000* ‘ *insert* x A) = *f* (*f000* x) (*Finite-Set.fold f z* (*f000* ‘ ($A - \{x\}$)))
proof −
from *A-int* **have** *finite* (*f000* ‘ A) **by** (*simp add*: *i-set-finite*)
then have *finite* (*f000* ‘ *insert* x A) **by** *auto*
moreover have $x \in$ *insert* x A **by** *auto*
moreover from *A-int* **have** *A-int*: *all-i-set* (*f000* ‘ *insert* x A) **by** (*simp*, *subst i-set'*, *simp-all add*: *x-int*)
ultimately have *Finite-Set.fold f z* (*f000* ‘ *insert* x A) = *f* (*f000* x) (*Finite-Set.fold f z* (*f000* ‘ (*insert* x $A - \{x\}$)))
by (*subst fold-rec*[*OF z-defined A-int x-int*], *simp-all*)
then show *?thesis* **by** *simp*
qed

lemma *finite-fold-insert* :
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-i-set* (*f000* ‘ A)
and *x-int* : $\forall\tau$. *is-i* τ (*f000* x)
and $x \notin A$
shows *finite* ⌈⌈*Rep-Set-0* (*Finite-Set.fold f z* (*f000* ‘ *insert* x A) τ)⌉⌉ = *finite* ⌈⌈*Rep-Set-0* (*f* (*f000* x) (*Finite-Set.fold f z* (*f000* ‘ A)) τ)⌉⌉
apply(*subst fold-insert'*, *simp-all add*: *assms*)
done
end

locale *EQ-comp-fun-commute0-gen0-bis'* = *EQ-comp-fun-commute0-gen0-bis''* +
assumes *cp-gen* : $\bigwedge x$ S $\tau 1$ $\tau 2$. $\forall\tau$. *is-i* τ (*f000* x) $\Longrightarrow$ ($\bigwedge\tau$. *all-defined* τ S) $\Longrightarrow$ S $\tau 1$ = S $\tau 2$ $\Longrightarrow$ *f* (*f000* x) S $\tau 1$ = *f* (*f000* x) S $\tau 2$
assumes *notempty* : $\bigwedge x$ S τ. $\forall\tau$. *all-defined* τ S $\Longrightarrow$ $\forall\tau$. *is-i* τ (*f000* x) $\Longrightarrow$ ⌈⌈*Rep-Set-0* (S τ)⌉⌉ $\neq \{\}$ $\Longrightarrow$ ⌈⌈*Rep-Set-0* (*f* (*f000* x) S τ)⌉⌉ $\neq \{\}$

context *EQ-comp-fun-commute0-gen0-bis'*
begin
lemma *downgrade-up* : *EQ-comp-fun-commute0-gen0-bis'' f000 is-i is-i' all-i-set f* **by** *default*
lemma *downgrade* : *EQ-comp-fun-commute0-gen0-bis' f000 is-i is-i' all-i-set f* **by** *default*

end

lemma *fold-cong′′′* :
assumes *f-comm* : *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set f*
and *g-comm* : *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set g*
and *a-def* : *all-i-set* (*f000* ‘ *A*)
and *s-def* : $\bigwedge\tau$. *all-defined* τ *s*
and *t-def* : $\bigwedge\tau$. *all-defined* τ *t*
and *cong* : ($\bigwedge x$ *s*. $\forall\tau$. *is-i* τ (*f000 x*) $\Longrightarrow$ *P s* τ $\Longrightarrow$ *f* (*f000 x*) *s* τ = *g* (*f000 x*) *s* τ)
and *ab* : *A* = *B*
and *st* : *s* τ = *t* τ'
and *P0* : *P s* τ
and *Prec* : $\bigwedge x$ *F*.
all-i-set (*f000* ‘ *F*) $\Longrightarrow$
$\forall\tau$. *is-i* τ (*f000 x*) $\Longrightarrow$
$x \notin F$ $\Longrightarrow$
P (*Finite-Set.fold f s* (*f000* ‘ *F*)) τ $\Longrightarrow$ *P* (*Finite-Set.fold f s* (*f000* ‘ *insert x F*)) τ
shows *Finite-Set.fold f s* (*f000* ‘ *A*) τ = *Finite-Set.fold g t* (*f000* ‘ *B*) τ'
proof −
interpret *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set f* **by** (*rule f-comm*)
note *g-comm-down* = *g-comm*[*THEN EQ-comp-fun-commute0-gen0-bis′.downgrade-up*]
note *g-fold-insert′* = *EQ-comp-fun-commute0-gen0-bis′′.fold-insert′*[*OF g-comm-down*]
note *g-cp-set* = *EQ-comp-fun-commute0-gen0-bis′′.cp-set*[*OF g-comm-down*]
note *g-fold-def* = *EQ-comp-fun-commute0-gen0-bis′′.fold-def*[*OF g-comm-down*]
note *g-cp-gen* = *EQ-comp-fun-commute0-gen0-bis′.cp-gen*[*OF g-comm*]
have *Finite-Set.fold f s* (*f000* ‘ *A*) τ = *Finite-Set.fold g t* (*f000* ‘ *A*) τ'
apply(*rule all-int-induct*[*OF a-def*], *simp add*: *st*)
apply(*subst fold-insert′*, *simp add*: *s-def*, *simp*, *simp*, *simp*)
apply(*subst g-fold-insert′*, *simp add*: *t-def*, *simp*, *simp*, *simp*)
apply(*subst g-cp-set*, *rule allI*, *rule g-fold-def*, *simp add*: *t-def*, *simp*)
apply(*drule sym*, *simp*)
apply(*subst g-cp-gen*[*of - - -* τ], *simp*, *subst cp-all-def*[**where** τ' = τ], *subst cp-all-def*[*symmetric*], *rule fold-def*, *simp add*: *s-def*, *simp*, *simp*)
apply(*subst g-cp-set*[*symmetric*], *rule allI*, *rule fold-def*, *simp add*: *s-def*, *simp*)
apply(*rule cong*, *simp*)

apply(*rule all-int-induct*, *simp*, *simp add*: *P0*, *simp add*: *st*[*symmetric*] *P0*)
apply(*rule Prec*[*simplified*], *simp-all*)
done
thus *?thesis* **by** (*simp add*: *st*[*symmetric*] *ab*[*symmetric*])
qed

lemma *fold-cong′′* :
assumes *f-comm* : *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set f*
and *g-comm* : *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set g*
and *a-def* : *all-i-set* (*f000* ‘ *A*)
and *s-def* : $\bigwedge\tau$. *all-defined* τ *s*
and *cong* : ($\bigwedge x$ *s*. $\forall\tau$. *is-i* τ (*f000 x*) $\Longrightarrow$ *P s* τ $\Longrightarrow$ *f* (*f000 x*) *s* τ = *g* (*f000 x*) *s* τ)
and *ab* : *A* = *B*

and *st* : $s = t$
and *stau* : $s\ \tau = s\ \tau'$
and *P0* : $P\ s\ \tau$
and *Prec* : $\bigwedge x\ F.$
all-i-set (*f000* ‘ F) $\Longrightarrow$
$\forall \tau.$ *is-i* τ (*f000* x) $\Longrightarrow$
$x \notin F \Longrightarrow$
P (*Finite-Set.fold f s* (*f000* ‘ F)) $\tau \Longrightarrow P$ (*Finite-Set.fold f s* (*f000* ‘ *insert* x F)) τ
shows *Finite-Set.fold f s* (*f000* ‘ A) τ = *Finite-Set.fold g t* (*f000* ‘ B) τ'
proof −
interpret *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set f* **by** (*rule f-comm*)
note *g-comm-down* = *g-comm*[*THEN EQ-comp-fun-commute0-gen0-bis′.downgrade-up*]
note *g-fold-insert′* = *EQ-comp-fun-commute0-gen0-bis′′.fold-insert′*[*OF g-comm-down*]
note *g-cp-set* = *EQ-comp-fun-commute0-gen0-bis′′.cp-set*[*OF g-comm-down*]
note *g-fold-def* = *EQ-comp-fun-commute0-gen0-bis′′.fold-def*[*OF g-comm-down*]
note *g-cp-gen* = *EQ-comp-fun-commute0-gen0-bis′.cp-gen*[*OF g-comm*]
have *Finite-Set.fold g s* (*f000* ‘ A) τ' = *Finite-Set.fold f s* (*f000* ‘ A) τ
apply(*rule all-int-induct*[*OF a-def*], *simp add*: *stau*)
apply(*subst fold-insert′*, *simp add*: *s-def*, *simp*, *simp*, *simp*)
apply(*subst g-fold-insert′*, *simp add*: *s-def*, *simp*, *simp*, *simp*)
apply(*subst g-cp-set*, *rule allI*, *rule g-fold-def*, *simp add*: *s-def*, *simp*)
apply(*simp*, *subst g-cp-set*[*symmetric*], *rule allI*, *subst cp-all-def*[**where** $\tau' = \tau$], *subst cp-all-def*[*symmetric*], *rule fold-def*, *simp add*: *s-def*, *simp*)
apply(*subst g-cp-gen*[*of - - -* τ], *simp*, *subst cp-all-def*[**where** $\tau' = \tau$], *subst cp-all-def*[*symmetric*], *rule fold-def*, *simp add*: *s-def*, *simp*, *simp*)
apply(*subst g-cp-set*[*symmetric*], *rule allI*, *subst cp-all-def*[**where** $\tau' = \tau$], *subst cp-all-def*[*symmetric*], *rule fold-def*, *simp add*: *s-def*, *simp*)
apply(*rule cong*[*symmetric*], *simp*)

apply(*rule all-int-induct*, *simp*, *simp add*: *P0*, *simp add*: *st*[*symmetric*] *P0*)
apply(*rule Prec*[*simplified*], *simp-all*)
done
thus *?thesis* **by** (*simp add*: *st*[*symmetric*] *ab*[*symmetric*])
qed

lemma *fold-cong′* :
assumes *f-comm* : *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set f*
and *g-comm* : *EQ-comp-fun-commute0-gen0-bis′ f000 is-i is-i′ all-i-set g*
and *a-def* : *all-i-set* (*f000* ‘ A)
and *s-def* : $\bigwedge\tau.$ *all-defined* τ s
and *cong* : ($\bigwedge x\ s.\ \forall \tau.$ *is-i* τ (*f000* x) $\Longrightarrow P\ s\ \tau \Longrightarrow f$ (*f000* x) $s\ \tau = g$ (*f000* x) $s\ \tau$)
and *ab* : $A = B$
and *st* : $s = t$
and *P0* : $P\ s\ \tau$
and *Prec* : $\bigwedge x\ F.$
all-i-set (*f000* ‘ F) $\Longrightarrow$
$\forall \tau.$ *is-i* τ (*f000* x) $\Longrightarrow$
$x \notin F \Longrightarrow$
P (*Finite-Set.fold f s* (*f000* ‘ F)) $\tau \Longrightarrow P$ (*Finite-Set.fold f s* (*f000* ‘ *insert* x F)) τ

shows *Finite-Set.fold f s (f000 ‘ A)* τ = *Finite-Set.fold g t (f000 ‘ B)* τ
by(*rule fold-cong''[OF f-comm g-comm a-def s-def cong ab st]*, *simp*, *simp*, *simp*, *rule P0*, *rule Prec*, *blast+*)

lemma *fold-cong* :
assumes *f-comm* : *EQ-comp-fun-commute0-gen0-bis' f000 is-i is-i' all-i-set f*
and *g-comm* : *EQ-comp-fun-commute0-gen0-bis' f000 is-i is-i' all-i-set g*
and *a-def* : *all-i-set (f000 ‘ A)*
and *s-def* : $\bigwedge\tau$. *all-defined* τ *s*
and *cong* : ($\bigwedge x\ s.\ \forall\tau.$ *is-i* τ *(f000 x)* $\Longrightarrow$ *P s* $\Longrightarrow$ *f (f000 x) s = g (f000 x) s*)
and *ab* : *A = B*
and *st* : *s = t*
and *P0* : *P s*
and *Prec* : $\bigwedge x\ F$.
all-i-set (f000 ‘ F) $\Longrightarrow$
$\forall\tau$. *is-i* τ *(f000 x)* $\Longrightarrow$
$x \notin F$ $\Longrightarrow$
P (Finite-Set.fold f s (f000 ‘ F)) $\Longrightarrow$ *P (Finite-Set.fold f s (f000 ‘ insert x F))*
shows *Finite-Set.fold f s (f000 ‘ A) = Finite-Set.fold g t (f000 ‘ B)*
apply(*rule ext*, *rule fold-cong'[OF f-comm g-comm a-def s-def]*)
apply(*subst cong*, *simp*, *simp*, *simp*, *rule ab*, *rule st*, *rule P0*)
apply(*rule Prec*, *simp-all*)
done

Sublocale

locale *EQ-comp-fun-commute* =
fixes *f* :: *('𝔄, 'a option option) val*
$\Rightarrow$ *('𝔄, 'a option option) Set*
$\Rightarrow$ *('𝔄, 'a option option) Set*
assumes *cp-x* : $\bigwedge x\ S\ \tau.\ f\ x\ S\ \tau = f\ (\lambda\text{-}.\ x\ \tau)\ S\ \tau$
assumes *cp-set* : $\bigwedge x\ S\ \tau.\ f\ x\ S\ \tau = f\ x\ (\lambda\text{-}.\ S\ \tau)\ \tau$
assumes *cp-gen* : $\bigwedge x\ S\ \tau 1\ \tau 2.$ *is-int x* $\Longrightarrow$ ($\bigwedge\tau$. *all-defined* τ *S*) $\Longrightarrow S\ \tau 1 = S\ \tau 2 \Longrightarrow f\ x\ S\ \tau 1 = f\ x\ S\ \tau 2$
assumes *notempty* : $\bigwedge x\ S\ \tau.$ ($\bigwedge\tau$. *all-defined* τ *S*) $\Longrightarrow \tau \models \upsilon\ x \Longrightarrow \lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow \lceil\lceil$*Rep-Set-0* $(f\ x\ S\ \tau)\rceil\rceil \neq \{\}$
assumes *all-def*: $\bigwedge x\ y\ \tau.$ *all-defined* τ $(f\ x\ y) = (\tau \models \upsilon\ x \wedge$ *all-defined* $\tau\ y)$
assumes *commute*: $\bigwedge x\ y\ S\ \tau.$
$\tau \models \upsilon\ x \Longrightarrow$
$\tau \models \upsilon\ y \Longrightarrow$
all-defined τ *S* $\Longrightarrow$
$f\ y\ (f\ x\ S)\ \tau = f\ x\ (f\ y\ S)\ \tau$

sublocale *EQ-comp-fun-commute* < *EQ-comp-fun-commute0-gen0-bis'* $\lambda x.\ x$ λ-. *is-int* $\lambda\tau\ x.\ \tau \models \upsilon\ x$ *all-int-set*
apply(*default*)
apply(*simp add*: *all-int-set-def*) **apply**(*simp add*: *all-int-set-def*) **apply**(*simp add*: *all-int-set-def is-int-def*)
apply(*simp add*: *all-int-set-def*)

apply(*simp add*: *int-is-valid*, *simp*)
apply(*rule cp-set*)
apply(*rule iffI*)
apply(*rule conjI*) **apply**(*rule allI*) **apply**(*drule-tac* $x = \tau$ **in** *allE*) **prefer** *2* **apply** *assumption* **apply**(*rule all-def*[*THEN iffD1*, *THEN conjunct1*], *blast*)
apply(*rule allI*) **apply**(*drule allE*) **prefer** *2* **apply** *assumption* **apply**(*rule all-def*[*THEN iffD1*, *THEN conjunct2*], *blast*)
apply(*erule conjE*) **apply**(*rule allI*) **apply**(*rule all-def*[*THEN iffD2*], *blast*)
apply(*rule ext*, *rename-tac* τ)
apply(*rule commute*) **apply**(*blast*)+
apply(*rule cp-gen*, *simp*, *blast*, *simp*)
apply(*rule notempty*, *blast*, *simp add*: *int-is-valid*, *simp*)
done

locale *EQ-comp-fun-commute0-gen0* =
fixes *f000* :: $'b \Rightarrow ('\mathfrak{A}, 'a\ option\ option)\ val$
fixes *all-def-set* :: $'\mathfrak{A}\ st \Rightarrow 'b\ set \Rightarrow bool$
fixes *f* :: $'b$
$\Rightarrow ('\mathfrak{A}, 'a\ option\ option)\ Set$
$\Rightarrow ('\mathfrak{A}, 'a\ option\ option)\ Set$
assumes *def-set* : $\bigwedge x\ A.\ (\forall \tau.\ all\text{-}def\text{-}set\ \tau\ (insert\ x\ A)) = (is\text{-}int\ (f000\ x) \wedge (\forall \tau.\ all\text{-}def\text{-}set\ \tau\ A))$
assumes *def-set'* : $\bigwedge x\ A.\ (is\text{-}int\ (f000\ x) \wedge (\forall \tau.\ all\text{-}def\text{-}set\ \tau\ A)) \Longrightarrow \forall \tau.\ all\text{-}def\text{-}set\ \tau\ (A - \{x\})$
assumes *def-set-finite* : $\forall \tau.\ all\text{-}def\text{-}set\ \tau\ A \Longrightarrow finite\ A$
assumes *all-i-set-to-def* : $all\text{-}int\text{-}set\ (f000\ `\ F) \Longrightarrow \forall \tau.\ all\text{-}def\text{-}set\ \tau\ F$

assumes *f000-inj* : $\bigwedge x\ y.\ x \neq y \Longrightarrow f000\ x \neq f000\ y$

assumes *cp-gen'* : $\bigwedge x\ S\ \tau 1\ \tau 2.\ is\text{-}int\ (f000\ x) \Longrightarrow \forall \tau.\ all\text{-}defined\ \tau\ S \Longrightarrow S\ \tau 1 = S\ \tau 2 \Longrightarrow f\ x\ S\ \tau 1 = f\ x\ S\ \tau 2$
assumes *notempty'* : $\bigwedge x\ S\ \tau.\ \forall \tau.\ all\text{-}defined\ \tau\ S \Longrightarrow is\text{-}int\ (f000\ x) \Longrightarrow \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow \lceil\lceil Rep\text{-}Set\text{-}0\ (f\ x\ S\ \tau)\rceil\rceil \neq \{\}$

assumes *cp-set* : $\bigwedge x\ S\ \tau.\ \forall \tau.\ all\text{-}defined\ \tau\ S \Longrightarrow f\ x\ S\ \tau = f\ x\ (\lambda\text{-}.\ S\ \tau)\ \tau$
assumes *all-def*: $\bigwedge x\ y.\ (\forall \tau.\ all\text{-}defined\ \tau\ (f\ x\ y)) = (is\text{-}int\ (f000\ x) \wedge (\forall \tau.\ all\text{-}defined\ \tau\ y))$
assumes *commute*: $\bigwedge x\ y\ S.$
$is\text{-}int\ (f000\ x) \Longrightarrow$
$is\text{-}int\ (f000\ y) \Longrightarrow$
$(\bigwedge \tau.\ all\text{-}defined\ \tau\ S) \Longrightarrow$
$f\ y\ (f\ x\ S) = f\ x\ (f\ y\ S)$

sublocale *EQ-comp-fun-commute0-gen0* < *EQ-comp-fun-commute0-gen0-bis'* $\lambda x.\ x\ \lambda\text{-}\ x.\ is\text{-}int\ (f000\ x)\ \lambda\text{-}\ x.\ is\text{-}int\ (f000\ x)\ \lambda x.\ \forall \tau.\ all\text{-}def\text{-}set\ \tau\ x$
apply *default*
apply(*drule def-set*[*THEN iffD1*], *blast*)
apply(*simp add*: *def-set*[*THEN iffD2*])
apply(*simp add*: *def-set'*)
apply(*simp add*: *def-set-finite*)

apply(*simp*)
apply(*simp*)
apply(*rule cp-set*, *simp*)
apply(*insert all-def*, *blast*)
apply(*rule commute*, *blast*+)
apply(*rule cp-gen'*, *blast*+)
apply(*rule notempty'*, *blast*+)
done

locale *EQ-comp-fun-commute0* =
fixes *f* :: *'a option option*
$\Rightarrow$ (*'$\mathfrak{A}$*, *'a option option*) *Set*
$\Rightarrow$ (*'$\mathfrak{A}$*, *'a option option*) *Set*
assumes *cp-set* : $\bigwedge x\ S\ \tau.\ \forall \tau.$ *all-defined* τ $S \Longrightarrow f\ x\ S\ \tau = f\ x\ (\lambda\text{-}.\ S\ \tau)\ \tau$
assumes *cp-gen'* : $\bigwedge x\ S\ \tau 1\ \tau 2.$ *is-int* (λ(-::*'$\mathfrak{A}$ st*). x) $\Longrightarrow \forall \tau.$ *all-defined* τ $S \Longrightarrow S\ \tau 1 = S\ \tau 2 \Longrightarrow f\ x\ S\ \tau 1 = f\ x\ S\ \tau 2$
assumes *notempty'* : $\bigwedge x\ S\ \tau.\ \forall \tau.$ *all-defined* τ $S \Longrightarrow$ *is-int* (λ(-::*'$\mathfrak{A}$ st*). x) $\Longrightarrow \lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow \lceil\lceil$*Rep-Set-0* $(f\ x\ S\ \tau)\rceil\rceil \neq \{\}$
assumes *all-def*: $\bigwedge x\ y.$ ($\forall \tau.$ *all-defined* τ $(f\ x\ y)$) = (*is-int* (λ(-::*'$\mathfrak{A}$ st*). x) $\wedge$ ($\forall \tau.$ *all-defined* τ y))
assumes *commute*: $\bigwedge x\ y\ S.$
is-int (λ(-::*'$\mathfrak{A}$ st*). x) $\Longrightarrow$
is-int (λ(-::*'$\mathfrak{A}$ st*). y) $\Longrightarrow$
($\bigwedge \tau.$ *all-defined* τ S) $\Longrightarrow$
$f\ y\ (f\ x\ S) = f\ x\ (f\ y\ S)$

sublocale *EQ-comp-fun-commute0* < *EQ-comp-fun-commute0-gen0* λx (-::*'$\mathfrak{A}$ st*). x *all-defined-set*
apply *default*
apply(*rule iffI*)
apply(*simp add*: *all-defined-set-def is-int-def*)
apply(*simp add*: *all-defined-set-def is-int-def*)
apply(*simp add*: *all-defined-set-def is-int-def*)
apply(*simp add*: *all-defined-set-def*)
apply(*simp add*: *all-int-set-def all-defined-set-def int-is-valid*)
apply(*rule finite-imageD*, *blast*, *metis inj-onI*)
apply *metis*
apply(*rule cp-gen'*, *simp*, *simp*, *simp*)
apply(*rule notempty'*, *simp*, *simp*, *simp*)
apply(*rule cp-set*, *simp*)
apply(*rule all-def*)
apply(*rule commute*, *simp*, *simp*, *blast*)
done

locale *EQ-comp-fun-commute000* =
fixes *f* :: (*'$\mathfrak{A}$*, *'a option option*) *val*
$\Rightarrow$ (*'$\mathfrak{A}$*, *'a option option*) *Set*
$\Rightarrow$ (*'$\mathfrak{A}$*, *'a option option*) *Set*
assumes *cp-set* : $\bigwedge x\ S\ \tau.\ \forall \tau.$ *all-defined* τ $S \Longrightarrow f$ (λ(-::*'$\mathfrak{A}$ st*). x) $S\ \tau = f$ (λ(-::*'$\mathfrak{A}$ st*). x) (λ-. $S\ \tau$) τ

assumes *all-def*: $\bigwedge x\ y.\ (\forall \tau.\ \textit{all-defined}\ \tau\ (f\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ x)\ y)) = (\textit{is-int}\ (\lambda(\text{-} :: '\mathfrak{A}\ st).\ x) \wedge (\forall \tau.\ \textit{all-defined}\ \tau\ y))$
assumes *commute*: $\bigwedge x\ y\ S.$
$\textit{is-int}\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ x) \Longrightarrow$
$\textit{is-int}\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ y) \Longrightarrow$
$(\bigwedge \tau.\ \textit{all-defined}\ \tau\ S) \Longrightarrow$
$f\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ y)\ (f\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ x)\ S) = f\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ x)\ (f\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ y)\ S)$

sublocale *EQ-comp-fun-commute000* < *EQ-comp-fun-commute0-gen0-bis''* $\lambda x\ (\text{-}::'\mathfrak{A}\ st).\ x\ \lambda\text{-}.$ *is-int* λ-. *is-int all-int-set*
apply *default*
apply(*simp add*: *all-int-set-def is-int-def*)
apply(*simp add*: *all-int-set-def is-int-def*)
apply(*simp add*: *all-int-set-def*)
apply(*simp add*: *all-int-set-def*)
apply(*simp*)
apply(*metis*)
apply(*rule cp-set*, *simp*)
apply(*insert all-def*, *blast*)
apply(*rule commute*, *simp*, *simp*, *blast*)
done

lemma *c0-of-c* :
assumes *f-comm* : *EQ-comp-fun-commute f*
shows *EQ-comp-fun-commute0* $(\lambda x.\ f\ (\lambda\text{-}.\ x))$
proof − **interpret** *EQ-comp-fun-commute f* **by** (*rule f-comm*) **show** *?thesis*
apply *default*
apply(*rule cp-set*)
apply(*subst cp-gen*, *simp*, *blast*, *simp*, *simp*)
apply(*rule notempty*, *blast*, *simp add*: *int-is-valid*, *simp*)
apply (*metis* (*mono-tags*) *all-def is-int-def*)

apply(*rule ext*, *rename-tac* τ)
apply(*subst commute*)
apply (*metis is-int-def*)+
done
qed

lemma *c000-of-c0* :
assumes *f-comm* : *EQ-comp-fun-commute0* $(\lambda x.\ f\ (\lambda\text{-}.\ x))$
shows *EQ-comp-fun-commute000 f*
proof − **interpret** *EQ-comp-fun-commute0* $\lambda x.\ f\ (\lambda\text{-}.\ x)$ **by** (*rule f-comm*) **show** *?thesis*
apply *default*
apply(*rule cp-set*, *simp*)
apply(*rule all-def*)
apply(*rule commute*)
apply(*blast*)+
done

qed

locale *EQ-comp-fun-commute0'* =
  fixes f :: *'a option*
  $\Rightarrow$ (*'𝔄*, *'a option option*) *Set*
  $\Rightarrow$ (*'𝔄*, *'a option option*) *Set*
  assumes *cp-set* : $\bigwedge x\ S\ \tau.\ \forall \tau.\ \textit{all-defined}\ \tau\ S \Longrightarrow f\ x\ S\ \tau = f\ x\ (\lambda\text{-}.\ S\ \tau)\ \tau$
  assumes *cp-gen'* : $\bigwedge x\ S\ \tau 1\ \tau 2.\ \textit{is-int}\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ \lfloor x \rfloor) \Longrightarrow \forall \tau.\ \textit{all-defined}\ \tau\ S \Longrightarrow S\ \tau 1 = S\ \tau 2 \Longrightarrow f\ x\ S\ \tau 1 = f\ x\ S\ \tau 2$
  assumes *notempty'* : $\bigwedge x\ S\ \tau.\ \forall \tau.\ \textit{all-defined}\ \tau\ S \Longrightarrow \textit{is-int}\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ \lfloor x \rfloor) \Longrightarrow \lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow \lceil\lceil \textit{Rep-Set-0}\ (f\ x\ S\ \tau)\rceil\rceil \neq \{\}$
  assumes *all-def*: $\bigwedge x\ y.\ (\forall \tau.\ \textit{all-defined}\ \tau\ (f\ x\ y)) = (\textit{is-int}\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ \lfloor x \rfloor) \land (\forall \tau.\ \textit{all-defined}\ \tau\ y))$
  assumes *commute*: $\bigwedge x\ y\ S.$
  $\textit{is-int}\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ \lfloor x \rfloor) \Longrightarrow$
  $\textit{is-int}\ (\lambda(\text{-}::'\mathfrak{A}\ st).\ \lfloor y \rfloor) \Longrightarrow$
  $(\bigwedge \tau.\ \textit{all-defined}\ \tau\ S) \Longrightarrow$
  $f\ y\ (f\ x\ S) = f\ x\ (f\ y\ S)$

sublocale *EQ-comp-fun-commute0'* < *EQ-comp-fun-commute0-gen0* $\lambda x\ (\text{-}::'\mathfrak{A}\ st).\ \lfloor x \rfloor$ *all-defined-set'*
 apply *default*
 apply(*rule iffI*)
 apply(*simp add*: *all-defined-set'-def is-int-def*, *metis bot-option-def foundation18' option.distinct*(*1*))
 apply(*simp add*: *all-defined-set'-def is-int-def*)
 apply(*simp add*: *all-defined-set'-def is-int-def*)
 apply(*simp add*: *all-defined-set'-def*)
 apply(*simp add*: *all-int-set-def all-defined-set'-def int-is-valid*)
 apply(*rule finite-imageD*, *blast*, *metis* (*full-types*) *UNIV-I inj-Some inj-fun subsetI subset-inj-on*)
 apply (*metis option.inject*)
 apply(*rule cp-gen'*, *simp*, *simp*, *simp*)
 apply(*rule notempty'*, *simp*, *simp*, *simp*)
 apply(*rule cp-set*, *simp*)
 apply(*rule all-def*)
 apply(*rule commute*, *simp*, *simp*, *blast*)
done

locale *EQ-comp-fun-commute000'* =
  fixes f :: (*'𝔄*, *'a option option*) *val*
  $\Rightarrow$ (*'𝔄*, *'a option option*) *Set*
  $\Rightarrow$ (*'𝔄*, *'a option option*) *Set*
  assumes *cp-set* : $\bigwedge x\ S\ \tau.\ \forall \tau.\ \textit{all-defined}\ \tau\ S \Longrightarrow f\ (\lambda\text{-}.\ \lfloor x \rfloor)\ S\ \tau = f\ (\lambda\text{-}.\ \lfloor x \rfloor)\ (\lambda\text{-}.\ S\ \tau)\ \tau$
  assumes *all-def*: $\bigwedge x\ y\ (\tau :: '\mathfrak{A}\ st).\ (\forall (\tau :: '\mathfrak{A}\ st).\ \textit{all-defined}\ \tau\ (f\ (\lambda(\text{-} :: '\mathfrak{A}\ st).\ \lfloor x \rfloor)\ y)) = (\tau \models \upsilon\ (\lambda(\text{-} :: '\mathfrak{A}\ st).\ \lfloor x \rfloor) \land (\forall (\tau :: '\mathfrak{A}\ st).\ \textit{all-defined}\ \tau\ y))$
  assumes *commute*: $\bigwedge x\ y\ S\ (\tau :: '\mathfrak{A}\ st).$
  $\tau \models \upsilon\ (\lambda\text{-}.\ \lfloor x \rfloor) \Longrightarrow$
  $\tau \models \upsilon\ (\lambda\text{-}.\ \lfloor y \rfloor) \Longrightarrow$
  $(\bigwedge \tau.\ \textit{all-defined}\ \tau\ S) \Longrightarrow$
  $f\ (\lambda\text{-}.\ \lfloor y \rfloor)\ (f\ (\lambda\text{-}.\ \lfloor x \rfloor)\ S) = f\ (\lambda\text{-}.\ \lfloor x \rfloor)\ (f\ (\lambda\text{-}.\ \lfloor y \rfloor)\ S)$

sublocale *EQ-comp-fun-commute000'* < *EQ-comp-fun-commute0-gen0-bis''* λx (-::'$\mathfrak{A}$ *st*). $\lfloor x \rfloor$ $\lambda\tau$ x. $\tau \models \upsilon$ x $\lambda\tau$ x. $\tau \models \upsilon$ x *all-int-set*
 apply *default*
 apply(*simp add*: *all-int-set-def is-int-def*)
 apply(*simp add*: *all-int-set-def is-int-def*)
 apply(*simp add*: *all-int-set-def*)
 apply(*simp add*: *all-int-set-def*)
 apply(*simp*)
 apply (*metis option.inject*)
 apply(*rule cp-set*, *simp*)
 apply(*rule iffI*)
 apply(*rule conjI*, *rule allI*)
 apply(*rule all-def*[*THEN iffD1*, *THEN conjunct1*], *blast*)
 apply(*rule all-def*[*THEN iffD1*, *THEN conjunct2*], *blast*)
 apply(*rule all-def*[*THEN iffD2*], *blast*)
 apply(*rule commute*, *blast*+)
done

lemma *c0'-of-c0* :
 assumes *EQ-comp-fun-commute0* (λx. f (λ-. x))
  shows *EQ-comp-fun-commute0'* (λx. f (λ-. $\lfloor x \rfloor$))
proof −
 interpret *EQ-comp-fun-commute0* λx. f (λ-. x) **by** (*rule assms*) **show** *?thesis*
 apply *default*
 apply(*rule cp-set*, *simp*)
 apply(*rule cp-gen'*, *simp*, *simp*, *simp*)
 apply(*rule notempty'*, *simp*, *simp*, *simp*)
 apply(*rule all-def*)
 apply(*rule commute*) **apply**(*blast*)+
 done
qed

lemma *c000'-of-c0'* :
 assumes *f-comm* : *EQ-comp-fun-commute0'* (λx. f (λ-. $\lfloor x \rfloor$))
  shows *EQ-comp-fun-commute000' f*
proof − **interpret** *EQ-comp-fun-commute0'* λx. f (λ-. $\lfloor x \rfloor$) **by** (*rule f-comm*) **show** *?thesis*
 apply *default*
 apply(*rule cp-set*, *simp*)
 apply(*subst all-def*, *simp only*: *is-int-def valid-def OclValid-def bot-fun-def false-def true-def*, *blast*)
 apply(*rule commute*)
 apply(*simp add*: *int-trivial*)+
 done
qed

context *EQ-comp-fun-commute*
begin
 lemmas *F-cp* = *cp-x*
 lemmas *F-cp-set* = *cp-set*

lemmas *fold-fun-comm* = *fold-fun-comm*[*simplified*]
lemmas *fold-insert-remove* = *fold-insert-remove*[*simplified*]
lemmas *fold-insert* = *fold-insert'*[*simplified*]
lemmas *all-int-induct* = *all-int-induct*[*simplified*]
lemmas *all-defined-fold-rec* = *all-defined-fold-rec*[*simplified image-ident*]
lemmas *downgrade* = *downgrade*
end

context *EQ-comp-fun-commute000*
begin
lemma *fold-insert'*:
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-int-set* ((λa (τ :: $'\mathfrak{A}$ *st*). *a*) ‘ *A*)
and *x-int* : *is-int* (λ(- :: $'\mathfrak{A}$ *st*). *x*)
and *x-nA* : $x \notin A$
shows *Finite-Set.fold f z* ((λa (τ :: $'\mathfrak{A}$ *st*). *a*) ‘ (*insert x A*)) = *f* (λ(- :: $'\mathfrak{A}$ *st*). *x*) (*Finite-Set.fold f z* ((λa (τ :: $'\mathfrak{A}$ *st*). *a*) ‘ *A*))
apply(*rule fold-insert'*, *simp-all add*: *assms*)
done

lemmas *all-defined-fold-rec* = *all-defined-fold-rec*[*simplified*]
lemmas *fold-def* = *fold-def*
end

context *EQ-comp-fun-commute000'*
begin
lemma *fold-insert'*:
assumes *z-defined* : $\bigwedge\tau$. *all-defined* τ *z*
and *A-int* : *all-int-set* ((λa (τ :: $'\mathfrak{A}$ *st*). $\lfloor a \rfloor$) ‘ *A*)
and *x-int* : *is-int* (λ(- :: $'\mathfrak{A}$ *st*). $\lfloor x \rfloor$)
and *x-nA* : $x \notin A$
shows *Finite-Set.fold f z* ((λa (τ :: $'\mathfrak{A}$ *st*). $\lfloor a \rfloor$) ‘ (*insert x A*)) = *f* (λ(- :: $'\mathfrak{A}$ *st*). $\lfloor x \rfloor$) (*Finite-Set.fold f z* ((λa (τ :: $'\mathfrak{A}$ *st*). $\lfloor a \rfloor$) ‘ *A*))
apply(*rule fold-insert'*, *simp-all only*: *assms*)
apply(*insert x-int*[*simplified is-int-def*], *auto*)
done

lemmas *fold-def* = *fold-def*
end

context *EQ-comp-fun-commute0-gen0*
begin
lemma *fold-insert*:
assumes *z-defined* : $\forall$(τ :: $'\mathfrak{A}$ *st*). *all-defined* τ *z*
and *A-int* : $\forall$(τ :: $'\mathfrak{A}$ *st*). *all-def-set* τ *A*
and *x-int* : *is-int* (*f000 x*)
and $x \notin A$
shows *Finite-Set.fold f z* (*insert x A*) = *f x* (*Finite-Set.fold f z A*)
by(*rule fold-insert'*[*simplified*], *simp-all add*: *assms*)

lemmas *downgrade* = *downgrade*
end

context *EQ-comp-fun-commute0*
begin
lemmas *fold-insert* = *fold-insert*
lemmas *all-defined-fold-rec* = *all-defined-fold-rec*[*simplified image-ident*]
end

context *EQ-comp-fun-commute0'*
begin
lemmas *fold-insert* = *fold-insert*
lemmas *all-defined-fold-rec* = *all-defined-fold-rec*[*simplified image-ident*]
end

Misc

lemma *img-fold* :
assumes *g-comm* : *EQ-comp-fun-commute0-gen0 f000 all-def-set* (λx. *G* (*f000 x*))
and *a-def* : $\forall \tau$. *all-defined* τ *A*
and *fini* : *all-int-set* (*f000* ‘ *Fa*)
and *g-fold-insert* : $\bigwedge x\ F.\ x \notin F \Longrightarrow$ *is-int* (*f000 x*) $\Longrightarrow$ *all-int-set* (*f000* ‘ *F*) $\Longrightarrow$ *Finite-Set.fold G A* (*insert* (*f000 x*) (*f000* ‘ *F*)) = *G* (*f000 x*) (*Finite-Set.fold G A* (*f000* ‘ *F*))
shows *Finite-Set.fold* (*G* :: ('$\mathfrak{A}$, -) *val*
⇒ ('$\mathfrak{A}$, -) *Set*
⇒ ('$\mathfrak{A}$, -) *Set*) *A* (*f000* ‘ *Fa*) =
Finite-Set.fold (λx. *G* (*f000 x*)) *A Fa*
proof −
have *invert-all-int-set* : $\bigwedge x\ S$. *all-int-set* (*insert x S*) $\Longrightarrow$
all-int-set S
by(*simp add*: *all-int-set-def*)
have *invert-int* : $\bigwedge x\ S$. *all-int-set* (*insert x S*) $\Longrightarrow$
is-int x
by(*simp add*: *all-int-set-def*)

interpret *EQ-comp-fun-commute0-gen0 f000 all-def-set* λx. *G* (*f000 x*) **by** (*rule g-comm*)
show *?thesis*
apply(*rule finite-induct*[**where** *P* = λ*set*. *let set'* = *f000* ‘ *set in*
all-int-set set' $\longrightarrow$
Finite-Set.fold G A set' = *Finite-Set.fold* (λx. *G* (*f000 x*)) *A set*
and *F* = *Fa*, *simplified Let-def*, *THEN mp*])
apply(*insert fini*[*simplified all-int-set-def*, *THEN conjunct1*], *rule finite-imageD*, *assumption*)
apply (*metis f000-inj inj-onI*)
apply(*simp*)
apply(*rule impI*)+

apply(*subgoal-tac all-int-set* (*f000* ‘ *F*), *simp*)

apply(*subst EQ-comp-fun-commute0-gen0*.*fold-insert*[*OF g-comm*])
apply(*simp add*: *a-def*)
apply(*simp add*: *all-i-set-to-def*)
apply(*simp add*: *invert-int*)
apply(*simp*)
apply(*drule sym*, *simp only*:)
apply(*subst g-fold-insert*, *simp*, *simp add*: *invert-int*, *simp*)
apply(*simp*)

apply(*rule invert-all-int-set*, *simp*)
apply(*simp add*: *fini*)
done
qed

context *EQ-comp-fun-commute0-gen0* **begin lemma** *downgrade'* : *EQ-comp-fun-commute0-gen0 f000 all-def-set f* **by** *default* **end**
context *EQ-comp-fun-commute0* **begin lemmas** *downgrade'* = *downgrade'* **end**
context *EQ-comp-fun-commute0'* **begin lemmas** *downgrade'* = *downgrade'* **end**

4.7.9. comp fun commute OclIncluding

Preservation of comp fun commute (main)

lemma *including-commute-gen-var* :
assumes *f-comm* : *EQ-comp-fun-commute F*
and *f-out* : $\bigwedge x\ y\ S\ \tau.\ \tau \models \delta\ S \Longrightarrow \tau \models \upsilon\ x \Longrightarrow \tau \models \upsilon\ y \Longrightarrow F\ x\ (S{-}{>}including(y))\ \tau = (F\ x\ S){-}{>}including(y)\ \tau$
and *a-int* : *is-int a*
shows *EQ-comp-fun-commute* ($\lambda j\ r2.\ ((F\ j\ r2){-}{>}including(a))$)
proof –
interpret *EQ-comp-fun-commute F* **by** (*rule f-comm*)

have *f-cp* : $\bigwedge x\ y\ \tau.\ F\ x\ y\ \tau = F\ (\lambda\text{-}.\ x\ \tau)\ (\lambda\text{-}.\ y\ \tau)\ \tau$
by (*metis F-cp F-cp-set*)

have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* $\tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)

show *?thesis*
apply(*simp only*: *EQ-comp-fun-commute-def*)
apply(*rule conjI*)+
apply(*rule allI*)+

proof – **fix** $x\ S\ \tau$ **show** $(F\ x\ S){-}{>}including(a)\ \tau = (F\ (\lambda\text{-}.\ x\ \tau)\ S){-}{>}including(a)\ \tau$
by(*subst* (*1 2*) *cp-OclIncluding*, *subst F-cp*, *simp*)

apply-end(*rule conjI*)+ **apply-end**(*rule allI*)+

fix $x\ S\ \tau$ **show** $(F\ x\ S){-}{>}including(a)\ \tau = (F\ x\ (\lambda\text{-}.\ S\ \tau)){-}{>}including(a)\ \tau$

by(*subst* (*1 2*) *cp-OclIncluding*, *subst F-cp-set*, *simp*)

apply-end(*rule allI*)+ **apply-end**(*rule impI*)+

fix *x* **fix** *S* **fix** $\tau 1$ $\tau 2$
show *is-int x* $\Longrightarrow \forall \tau$. *all-defined* τ *S* $\Longrightarrow$ *S* $\tau 1$ = *S* $\tau 2$ $\Longrightarrow$ ((*F x S*)->*including*(*a*)) $\tau 1$ = ((*F x S*)->*including*(*a*)) $\tau 2$
apply(*subgoal-tac x* $\tau 1$ = *x* $\tau 2$) **prefer** *2* **apply** (*simp add*: *is-int-def*) **apply**(*metis surj-pair*)
apply(*subgoal-tac* $\bigwedge\tau$. *all-defined* τ (*F x S*)) **prefer** *2* **apply**(*rule all-def*[*THEN iffD2*], *simp only*: *int-is-valid*, *blast*)
apply(*subst including-cp-all*[*of - -* $\tau 1$ $\tau 2$]) **apply**(*simp add*: *a-int*) **apply**(*rule all-defined1*, *blast*)
apply(*rule cp-gen*, *simp*, *blast*, *simp*)
apply(*simp*)
done
apply-end(*simp*) **apply-end**(*simp*) **apply-end**(*simp*) **apply-end**(*rule conjI*)
apply-end(*rule allI*)+ **apply-end**(*rule impI*)+

apply-end(*rule including-notempty*)
apply-end(*rule all-defined1*)
apply-end(*simp add*: *all-def*, *metis surj-pair*, *simp*)
apply-end(*simp add*: *int-is-valid*[*OF a-int*])
apply-end(*rule notempty*, *blast*, *simp*, *simp*)

apply-end(*rule conjI*) **apply-end**(*rule allI*)+
apply-end(*rule iffI*)
apply-end(*drule invert-all-defined*, *simp add*: *all-def*)
apply-end(*rule cons-all-def'*, *simp add*: *all-def*)
apply-end(*simp add*: *int-is-valid*[*OF a-int*])

apply-end(*rule allI*)+ **apply-end**(*rule impI*)+

fix *x y S* τ **show** $\tau \models \upsilon$ *x* $\Longrightarrow \tau \models \upsilon$ *y* $\Longrightarrow$ *all-defined* τ *S* $\Longrightarrow$
(*F y* ((*F x S*)->*including*(*a*)))->*including*(*a*) τ =
(*F x* ((*F y S*)->*including*(*a*)))->*including*(*a*) τ
apply(*rule including-subst-set''*)
apply(*rule all-defined1*)
apply(*simp add*: *all-def*, *rule cons-all-def'*, *simp add*: *all-def*)
apply(*simp add*: *int-is-valid*[*OF a-int*])
apply(*rule all-defined1*)
apply(*simp add*: *all-def*, *rule cons-all-def'*, *simp add*: *all-def*)
apply(*simp add*: *int-is-valid*[*OF a-int*])+
apply(*subst f-out*)
apply(*rule all-defined1*, *simp add*: *all-def*, *simp*)
apply(*simp add*: *int-is-valid*[*OF a-int*])
apply(*subst cp-OclIncluding*)
apply(*subst commute*, *simp-all add*: *cp-OclIncluding*[*symmetric*] *f-out*[*symmetric*])
apply(*subst f-out*[*symmetric*])
apply(*rule all-defined1*, *simp add*: *all-def*, *simp*)

apply(*simp add*: *int-is-valid*[*OF a-int*])
apply(*simp*)
done
apply-end(*simp*)+
qed
qed

Preservation of comp fun commute (instance)

lemma *including-commute* : *EQ-comp-fun-commute* (λj (*r2* :: ($'\mathfrak{A}$, *int option option*) *Set*). (*r2*−>*including*(*j*)))
proof −
have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow$ $\tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)
show *?thesis*
apply(*simp only*: *EQ-comp-fun-commute-def including-cp including-cp'*)
apply(*rule conjI*, *rule conjI*) **apply**(*subst* (*1 2*) *cp-OclIncluding*, *simp*) **apply**(*rule conjI*)
apply(*subst* (*1 2*) *cp-OclIncluding*, *simp*) **apply**(*rule allI*)+
apply(*rule impI*)+
apply(*rule including-cp-all*) **apply**(*simp*) **apply**(*rule all-defined1*, *blast*) **apply**(*simp*)
apply(*rule conjI*) **apply**(*rule allI*)+
apply(*rule impI*)+ **apply**(*rule including-notempty*) **apply**(*rule all-defined1*, *blast*) **apply**(*simp*)
apply(*simp*)
apply(*rule conjI*) **apply**(*rule allI*)+
apply(*rule iff*[*THEN mp*, *THEN mp*], *rule impI*)
apply(*rule invert-all-defined*, *simp*)
apply(*rule impI*, *rule cons-all-def'*) **apply**(*simp*) **apply**(*simp*)
apply(*rule allI*)+ **apply**(*rule impI*)+
apply(*rule including-swap'*, *simp-all add*: *all-defined-def*)
done
qed

lemma *including-commute2* :
assumes *i-int* : *is-int i*
shows *EQ-comp-fun-commute* (λx (*acc* :: ($'\mathfrak{A}$, *int option option*) *Set*). ((*acc*−>*including*(*x*))−>*including*(*i*)))
apply(*rule including-commute-gen-var*)
apply(*rule including-commute*)
apply(*rule including-swap'*, *simp-all add*: *i-int*)
done

lemma *including-commute3* :
assumes *i-int* : *is-int i*
shows *EQ-comp-fun-commute* (λx (*acc* :: ($'\mathfrak{A}$, *int option option*) *Set*). *acc*−>*including*(*i*)−>*including*(*x*))
proof −
have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow$ $\tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)
have *i-val* : $\bigwedge\tau$. $\tau \models \upsilon$ *i* **by** (*simp add*: *int-is-valid*[*OF i-int*])
show *?thesis*
apply(*simp only*: *EQ-comp-fun-commute-def including-cp2 including-cp'*)
apply(*rule conjI*, *rule conjI*) **apply**(*subst* (*1 2*) *cp-OclIncluding*, *simp*) **apply**(*rule conjI*)
apply(*subst* (*1 2*) *cp-OclIncluding*, *subst* (*1 3*) *cp-OclIncluding*, *simp*) **apply**(*rule allI*)+

apply(*rule impI*)+
apply(*rule including-cp-all*) **apply**(*simp*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 foundation10 foundation6 i-val including-defined-args-valid'*)
apply(*rule including-cp-all*) **apply**(*simp add*: *i-int*) **apply**(*rule all-defined1*, *blast*) **apply**(*simp*)
apply(*rule conjI*) **apply**(*rule allI*)+

apply(*rule impI*)+
apply(*rule including-notempty*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 foundation10 foundation6 i-val including-defined-args-valid'*) **apply**(*simp*)
apply(*rule including-notempty*) **apply**(*rule all-defined1*, *blast*) **apply**(*simp add*: *i-val*) **apply**(*simp*)
apply(*rule conjI*) **apply**(*rule allI*)+

apply(*rule iff* [*THEN mp*, *THEN mp*], *rule impI*)
apply(*drule invert-all-defined*, *drule conjE*) **prefer** *2* **apply** *assumption*
apply(*drule invert-all-defined*, *drule conjE*) **prefer** *2* **apply** *assumption*
apply(*simp*)

apply(*rule impI*, *rule cons-all-def'*, *rule cons-all-def'*) **apply**(*simp*) **apply**(*simp add*: *i-val*) **apply**(*simp*)
apply(*rule allI*)+ **apply**(*rule impI*)+
apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def' i-val*)
apply(*simp add*: *i-val*)
apply(*simp*)
apply(*rule sym*)
apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def' i-val*)
apply(*simp add*: *i-val*)
apply(*simp*)

apply(*rule including-subst-set''*)
apply(*rule all-defined1*)
apply(*rule cons-all-def'*)+ **apply**(*simp-all add*: *i-val*)
apply(*insert i-val*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 foundation10 foundation6*)
apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def'*)
apply(*simp*)+
done
qed

lemma *including-commute4* :
assumes *i-int* : *is-int i*
and *j-int* : *is-int j*
shows *EQ-comp-fun-commute* (λx (*acc* :: ('$\mathfrak{A}$, *int option option*) *Set*). *acc*−>*including*(*i*)−>*including*(*x*)−>
proof −
have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)
have *i-val* : $\bigwedge \tau.\ \tau \models \upsilon\ i$ **by** (*simp add*: *int-is-valid*[*OF i-int*])
have *j-val* : $\bigwedge \tau.\ \tau \models \upsilon\ j$ **by** (*simp add*: *int-is-valid*[*OF j-int*])

show *?thesis*
apply(*rule including-commute-gen-var*)
apply(*rule including-commute3*)
apply(*simp-all add*: *i-int j-int*)
apply(*subgoal-tac* $S{-}{>}including(y){-}{>}including(i){-}{>}including(x)\ \tau = S{-}{>}including(i){-}{>}including(y){-}{>}including(x)\ \tau$)
prefer *2*
apply(*rule including-subst-set''*)
apply (*metis* (*hide-lams*, *no-types*) *foundation10 foundation6 i-val including-defined-args-valid'*)+
apply(*rule including-swap'*, *simp-all add*: *i-val*)
apply(*rule including-swap'*)
apply (*metis* (*hide-lams*, *no-types*) *foundation10 foundation6 i-val including-defined-args-valid'*)+
done
qed

lemma *including-commute5* :
assumes *i-int* : *is-int i*
and *j-int* : *is-int j*
shows *EQ-comp-fun-commute* (λx (acc :: ($'\mathfrak{A}$, *int option option*) *Set*). $acc{-}{>}including(x){-}{>}including(j){-}{>}including(i)$)
proof −
have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* $\tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)
have *i-val* : $\bigwedge\tau.\ \tau \models \upsilon\ i$ **by** (*simp add*: *int-is-valid*[*OF i-int*])
have *j-val* : $\bigwedge\tau.\ \tau \models \upsilon\ j$ **by** (*simp add*: *int-is-valid*[*OF j-int*])
show *?thesis*
apply(*rule including-commute-gen-var*)+
apply(*simp add*: *including-commute*)
apply(*rule including-swap'*, *simp-all add*: *i-int j-int*)
apply(*subgoal-tac* $S{-}{>}including(y){-}{>}including(x){-}{>}including(j)\ \tau = S{-}{>}including(x){-}{>}including(y){-}{>}including(j)\ \tau$)
prefer *2*
apply(*rule including-subst-set''*)
apply (*metis* (*hide-lams*, *no-types*) *foundation10 foundation6 j-val including-defined-args-valid'*)+
apply(*rule including-swap'*, *simp-all*)
apply(*rule including-swap'*)
apply (*metis* (*hide-lams*, *no-types*) *foundation10 foundation6 j-val including-defined-args-valid'*)+
done
qed

lemma *including-commute6* :
assumes *i-int* : *is-int i*
and *j-int* : *is-int j*
shows *EQ-comp-fun-commute* (λx (acc :: ($'\mathfrak{A}$, *int option option*) *Set*). $acc{-}{>}including(i){-}{>}including(j){-}{>}including(x)$)
proof −
have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* $\tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)
have *i-val* : $\bigwedge\tau.\ \tau \models \upsilon\ i$ **by** (*simp add*: *int-is-valid*[*OF i-int*])
have *j-val* : $\bigwedge\tau.\ \tau \models \upsilon\ j$ **by** (*simp add*: *int-is-valid*[*OF j-int*])
show *?thesis*
apply(*simp only*: *EQ-comp-fun-commute-def including-cp3 including-cp'''*)
apply(*rule conjI*, *rule conjI*) **apply**(*subst* (*1 2*) *cp-OclIncluding*, *simp*)

apply(*rule conjI*) **apply**(*subst* (*1 2*) *cp-OclIncluding*, *subst* (*1 3*) *cp-OclIncluding*, *subst* (*1 4*) *cp-OclIncluding*, *simp*) **apply**(*rule allI*)+
apply(*rule impI*)+
apply(*rule including-cp-all*) **apply**(*simp*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def i-val j-val*)
apply(*rule including-cp-all*) **apply**(*simp*) **apply**(*simp add*: *j-int*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def i-val*)
apply(*rule including-cp-all*) **apply**(*simp*) **apply**(*simp add*: *i-int*) **apply**(*rule all-defined1*, *blast*) **apply**(*simp*)
apply(*rule conjI*) **apply**(*rule allI*)+

apply(*rule impI*)+
apply(*rule including-notempty*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def i-val j-val*) **apply**(*simp*)
apply(*rule including-notempty*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def i-val*) **apply**(*simp add*: *j-val*)
apply(*rule including-notempty*) **apply**(*rule all-defined1*, *blast*) **apply**(*simp add*: *i-val*) **apply**(*simp*)
apply(*rule conjI*) **apply**(*rule allI*)+

apply(*rule iff*[*THEN mp*, *THEN mp*], *rule impI*)
apply(*drule invert-all-defined*, *drule conjE*) **prefer** *2* **apply** *assumption*
apply(*drule invert-all-defined*, *drule conjE*) **prefer** *2* **apply** *assumption*
apply(*drule invert-all-defined*, *drule conjE*) **prefer** *2* **apply** *assumption*
apply(*simp*)

apply(*rule impI*, *rule cons-all-def'*, *rule cons-all-def'*, *rule cons-all-def'*) **apply**(*simp*) **apply**(*simp add*: *i-val*) **apply**(*simp add*: *j-val*) **apply**(*simp*)
apply(*rule allI*)+ **apply**(*rule impI*)+

apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def' i-val j-val*)
apply(*simp add*: *j-val*)
apply(*simp*)
apply(*rule sym*)
apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def' i-val j-val*)
apply(*simp add*: *j-val*)
apply(*simp*)

apply(*rule including-subst-set''*)
apply(*rule all-defined1*)
apply(*rule cons-all-def'*)+ **apply**(*simp-all add*: *i-val j-val*)
apply(*insert i-val j-val*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 foundation10 foundation6*)

apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def' i-val*)
apply(*simp add*: *i-val*)

apply(*simp*)
apply(*rule sym*)
apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def' i-val*)
apply(*simp add*: *i-val*)
apply(*simp*)

apply(*rule including-subst-set''*)
apply(*rule all-defined1*)
apply(*rule cons-all-def'*)+ **apply**(*simp-all add*: *i-val j-val*)
apply(*insert i-val j-val*) **apply** (*metis* (*hide-lams*, *no-types*) *all-defined1 foundation10 foundation6*)

apply(*subst including-swap'*)
apply(*metis* (*hide-lams*, *no-types*) *all-defined1 cons-all-def'*)
apply(*simp*)+
done
qed

4.7.10. comp fun commute OclIterate

Congruence

lemma *iterate-subst-set-rec* :
assumes *A-defined* : $\forall\,\tau.$ *all-defined* τ *A*
and *F-commute* : *EQ-comp-fun-commute F*
shows *let Fa'* = ($\lambda a\ \tau.\ a$) ‘ *Fa*
; $x' = \lambda\tau.\ x$ *in*
$x \notin$ *Fa* $\longrightarrow$
all-int-set (*insert x' Fa'*) $\longrightarrow$
($\forall\,\tau.$ *all-defined* τ (*Finite-Set.fold F A Fa'*)) $\longrightarrow$
($\forall\,\tau.$ *all-defined* τ (*Finite-Set.fold F A* (*insert x' Fa'*)))
apply(*simp only*: *Let-def*) **apply**(*rule impI*)+ **apply**(*rule allI*)+
apply(*rule EQ-comp-fun-commute000.all-defined-fold-rec*[*OF F-commute*[*THEN c0-of-c*, *THEN c000-of-c0*]], *simp add*: *A-defined*, *simp*, *simp*, *blast*)
done

lemma *iterate-subst-set-rec0* :
assumes *F-commute* : *EQ-comp-fun-commute0* ($\lambda x.$ (*F*:: ('$\mathfrak{A}$, -) *val*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. x))
shows
finite Fa $\Longrightarrow$
$x \notin$ *Fa* $\Longrightarrow$
($\bigwedge\tau.$ *all-defined* τ *A*) $\Longrightarrow$
all-int-set ((λa (τ:: '$\mathfrak{A}$ *st*). a) ‘ *insert x Fa*) $\Longrightarrow$
$\forall\,\tau.$ *all-defined* τ (*Finite-Set.fold* ($\lambda x.$ *F* (λ-. x)) *A Fa*) $\Longrightarrow$
$\forall\,\tau.$ *all-defined* τ (*Finite-Set.fold* ($\lambda x.$ *F* (λ-. x)) *A* (*insert x Fa*))
apply(*rule allI*, *rule EQ-comp-fun-commute0.all-defined-fold-rec*[*OF F-commute*])
apply(*simp*, *simp*, *simp add*: *all-int-set-def all-defined-set-def is-int-def*, *blast*)

done

lemma *iterate-subst-set-rec0′* :
 assumes *F-commute* : *EQ-comp-fun-commute0′* (λx. (*F*:: ('$\mathfrak{A}$, -) *val*
 $\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
 $\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. $\lfloor x \rfloor$))
 shows
 finite Fa $\Longrightarrow$
 $x \notin$ *Fa* $\Longrightarrow$
 ($\bigwedge\tau$. *all-defined* τ *A*) $\Longrightarrow$
 all-int-set ((λa (τ:: '$\mathfrak{A}$ *st*). $\lfloor a \rfloor$) ‘ *insert x Fa*) $\Longrightarrow$
 $\forall\tau$. *all-defined* τ (*Finite-Set.fold* (λx. *F* (λ-. $\lfloor x \rfloor$)) *A Fa*) $\Longrightarrow$
 $\forall\tau$. *all-defined* τ (*Finite-Set.fold* (λx. *F* (λ-. $\lfloor x \rfloor$)) *A* (*insert x Fa*))
 apply(*rule allI*, *rule EQ-comp-fun-commute0′.all-defined-fold-rec*[*OF F-commute*])
 apply(*simp*, *simp*, *simp add*: *all-int-set-def all-defined-set′-def is-int-def*, *blast*)
done

lemma *iterate-subst-set-gen* :
 assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
 and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
 and *F-commute* : *EQ-comp-fun-commute F*
 and *G-commute* : *EQ-comp-fun-commute G*
 and *fold-eq* : $\bigwedge$*x acc*. *is-int x* $\Longrightarrow$ ($\forall\tau$. *all-defined* τ *acc*) $\Longrightarrow$ *P acc* $\Longrightarrow$ *F x acc* = *G x acc*
 and *P0* : *P A*
 and *Prec* : $\bigwedge$*x Fa*. *all-int-set Fa* $\Longrightarrow$
 is-int x $\Longrightarrow$ $x \notin$ *Fa* $\Longrightarrow$ $\forall\tau$. *all-defined* τ (*Finite-Set.fold F A Fa*) $\Longrightarrow$ *P* (*Finite-Set.fold F A Fa*) $\Longrightarrow$ *P* (*F x* (*Finite-Set.fold F A Fa*))
 shows (*S*->*iterate*(*x*;*acc*=*A*|*F x acc*)) = (*S*->*iterate*(*x*;*acc*=*A*|*G x acc*))
proof −

 have *S-all-int* : $\bigwedge\tau$. *all-int-set* ((λa τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)
 by(*rule all-def-to-all-int*, *simp add*: *assms*)

 have *A-defined* : $\forall\tau$. $\tau \models \delta$ *A*
 by(*simp add*: *A-all-def*[*simplified all-defined-def*])

 interpret *EQ-comp-fun-commute F* **by** (*rule F-commute*)
 show *?thesis*
 apply(*simp only*: *OclIterate*$_{Set}$-*def*, *rule ext*)
 proof −
 fix τ
 show (*if* (δ *S*) τ = *true* τ $\wedge$ (υ *A*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ *then Finite-Set.fold F A* ((λa τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$) τ *else* $\bot$) =
 (*if* (δ *S*) τ = *true* τ $\wedge$ (υ *A*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ *then Finite-Set.fold G A* ((λa τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$) τ *else* $\bot$)
 apply(*simp add*: *S-all-def*[*simplified all-defined-def all-defined-set-def OclValid-def*]
 A-all-def[*simplified all-defined-def OclValid-def*]
 foundation20[*OF A-defined*[*THEN spec*, *of* τ], *simplified OclValid-def*]
 del: *StrictRefEq-set-exec*)

apply(*subgoal-tac Finite-Set.fold F A* $((\lambda a\ \tau.\ a)$ ‘ $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil)$ = *Finite-Set.fold G* *A* $((\lambda a\ \tau.\ a)$ ‘ $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil)$, *simp*)

apply(*rule fold-cong*[**where** $P = \lambda s.\ \forall\tau.$ *all-defined* τ $s \wedge P\ s$, *OF downgrade EQ-comp-fun-commute.downgrade*[*OF G-commute*], *simplified image-ident*])
apply(*simp only*: *S-all-int*)
apply(*simp only*: *A-all-def*)
apply(*rule fold-eq*, *simp add*: *int-is-valid*, *simp*, *simp*)
apply(*simp*, *simp*, *simp add*: *A-all-def*)
apply(*simp add*: *P0*)
apply(*rule allI*)
apply(*subst EQ-comp-fun-commute.all-defined-fold-rec*[*OF F-commute*], *simp add*: *A-all-def*, *simp*, *simp add*: *all-int-set-def*, *blast*)
apply(*subst fold-insert*, *simp add*: *A-all-def*, *simp*, *simp*, *simp*)
apply(*simp add*: *Prec*)
done
qed
qed

lemma *iterate-subst-set* :
assumes *S-all-def* : $\bigwedge\tau.$ *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau.$ *all-defined* τ *A*
and *F-commute* : *EQ-comp-fun-commute F*
and *G-commute* : *EQ-comp-fun-commute G*
and *fold-eq* : $\bigwedge x$ *acc*. $(\forall\tau.\ (\tau \models \upsilon\ x)) \Longrightarrow (\forall\tau.$ *all-defined* τ *acc*$) \Longrightarrow F\ x\ acc = G\ x\ acc$
shows (*S*−>*iterate*(*x*;*acc*=*A*|*F x acc*)) = (*S*−>*iterate*(*x*;*acc*=*A*|*G x acc*))
by(*rule iterate-subst-set-gen*[*OF S-all-def A-all-def F-commute G-commute fold-eq*], (*simp add*: *int-is-valid*)+)

lemma *iterate-subst-set′* :
assumes *S-all-def* : $\bigwedge\tau.$ *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau.$ *all-defined* τ *A*
and *A-include* : $\bigwedge\tau 1\ \tau 2.\ A\ \tau 1 = A\ \tau 2$
and *F-commute* : *EQ-comp-fun-commute F*
and *G-commute* : *EQ-comp-fun-commute G*
and *fold-eq* : $\bigwedge x$ *acc*. *is-int* $x \Longrightarrow (\forall\tau.$ *all-defined* τ *acc*$) \Longrightarrow \forall\tau\ \tau'.\ acc\ \tau = acc\ \tau' \Longrightarrow F\ x\ acc = G\ x\ acc$
shows (*S*−>*iterate*(*x*;*acc*=*A*|*F x acc*)) = (*S*−>*iterate*(*x*;*acc*=*A*|*G x acc*))
proof −
interpret *EQ-comp-fun-commute F* **by** (*rule F-commute*)
show *?thesis*
apply(*rule iterate-subst-set-gen*[**where** $P = \lambda acc.\ \forall\tau\ \tau'.\ acc\ \tau = acc\ \tau'$, *OF S-all-def A-all-def F-commute G-commute fold-eq*], *blast*+)
apply(*simp add*: *A-include*)
apply(*rule allI*)+
apply(*rule cp-gen*, *simp*, *blast*, *blast*)
done
qed

lemma *iterate-subst-set''* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
and *A-notempty* : $\bigwedge\tau$. $\lceil\lceil$*Rep-Set-0* (*A* τ)$\rceil\rceil \neq \{\}$
and *F-commute* : *EQ-comp-fun-commute F*
and *G-commute* : *EQ-comp-fun-commute G*
and *fold-eq* : $\bigwedge$*x acc*. *is-int x* $\Longrightarrow$ ($\forall\tau$. *all-defined* τ *acc*) $\Longrightarrow$ ($\bigwedge\tau$. $\lceil\lceil$*Rep-Set-0* (*acc* τ)$\rceil\rceil$ $\neq \{\}$) $\Longrightarrow$ *F x acc* = *G x acc*
shows (*S*−>*iterate*(*x*;*acc*=*A*|*F x acc*)) = (*S*−>*iterate*(*x*;*acc*=*A*|*G x acc*))
proof −
interpret *EQ-comp-fun-commute F* **by** (*rule F-commute*)
show *?thesis*
apply(*rule iterate-subst-set-gen*[**where** *P* = λ*acc*. ($\forall\tau$. $\lceil\lceil$*Rep-Set-0* (*acc* τ)$\rceil\rceil \neq \{\}$), *OF S-all-def A-all-def F-commute G-commute fold-eq*], *blast*, *blast*, *blast*)
apply(*simp add*: *A-notempty*)
apply(*rule allI*)+
apply(*rule notempty*, *blast*, *simp add*: *int-is-valid*, *blast*)
done
qed

lemma *iterate-subst-set-gen0* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
and *F-commute* : *EQ-comp-fun-commute0-gen0 f000 all-def-set* (λ*x*. *F* (*f000 x*))
and *G-commute* : *EQ-comp-fun-commute0-gen0 f000 all-def-set* (λ*x*. (*G* :: ($'\mathfrak{A}$, -) *val*
$\Rightarrow$ ($'\mathfrak{A}$, -) *Set*
$\Rightarrow$ ($'\mathfrak{A}$, -) *Set*) (*f000 x*))
and *fold-eq* : $\bigwedge$*x acc*. *is-int* (*f000 x*) $\Longrightarrow$ ($\forall\tau$. *all-defined* τ *acc*) $\Longrightarrow$ *P acc* τ $\Longrightarrow$ *F* (*f000 x*) *acc* τ = *G* (*f000 x*) *acc* τ
and *P0* : *P A* τ
and *Prec* : $\bigwedge$*x Fa*. $\forall$(τ::$'\mathfrak{A}$ *st*). *all-def-set* τ *Fa* $\Longrightarrow$
is-int (*f000 x*) $\Longrightarrow$
x $\notin$ *Fa* $\Longrightarrow$
$\forall\tau$. *all-defined* τ (*Finite-Set.fold* (λ*x*. *F* (*f000 x*)) *A Fa*) $\Longrightarrow$
P (*Finite-Set.fold* (λ*x*. *F* (*f000 x*)) *A Fa*) τ $\Longrightarrow$
P (*F* (*f000 x*) (*Finite-Set.fold* (λ*x*. *F* (*f000 x*)) *A Fa*)) τ
and *f-fold-insert* : $\bigwedge$*x S*. *x* $\notin$ *S* $\Longrightarrow$ *is-int* (*f000 x*) $\Longrightarrow$ *all-int-set* (*f000* ‘ *S*) $\Longrightarrow$ *Finite-Set.fold F A* (*insert* (*f000 x*) (*f000* ‘ *S*)) = *F* (*f000 x*) (*Finite-Set.fold F A* (*f000* ‘ *S*))
and *g-fold-insert* : $\bigwedge$*x S*. *x* $\notin$ *S* $\Longrightarrow$ *is-int* (*f000 x*) $\Longrightarrow$ *all-int-set* (*f000* ‘ *S*) $\Longrightarrow$ *Finite-Set.fold G A* (*insert* (*f000 x*) (*f000* ‘ *S*)) = *G* (*f000 x*) (*Finite-Set.fold G A* (*f000* ‘ *S*))
and *S-lift* : *all-defined* τ *S* $\Longrightarrow$ $\exists$ *S'*. (λ*a* τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ = *f000* ‘ *S'*
shows (*S*−>*iterate*(*x*;*acc*=*A*|*F x acc*)) τ = (*S*−>*iterate*(*x*;*acc*=*A*|*G x acc*)) τ
proof −
have *S-all-int* : $\bigwedge\tau$. *all-int-set* ((λ*a* τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)
by(*rule all-def-to-all-int*, *simp add*: *assms*)

have *S-all-def'* : $\bigwedge\tau$ τ'. *all-defined-set'* τ' $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$

apply(*insert S-all-def*)
apply(*subst* (*asm*) *cp-all-def*, *simp add*: *all-defined-def all-defined-set'-def*, *blast*)
done

have *A-defined* : $\forall\tau.\ \tau \models \delta\ A$
by(*simp add*: *A-all-def*[*simplified all-defined-def*])

interpret *EQ-comp-fun-commute0-gen0 f000 all-def-set* $\lambda x.$ *F* (*f000 x*) **by** (*rule F-commute*)
show *?thesis*
apply(*simp only*: *OclIterate*$_{Set}$*-def*)
proof −
show (*if* (δ *S*) τ = *true* τ $\wedge$ (υ *A*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ *then Finite-Set.fold F A* (($\lambda a\ \tau.\ a$) ' $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$) τ *else* $\perp$) =
(*if* (δ *S*) τ = *true* τ $\wedge$ (υ *A*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ *then Finite-Set.fold G A* (($\lambda a\ \tau.\ a$) ' $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$) τ *else* $\perp$)
apply(*simp add*: *S-all-def*[*simplified all-defined-def all-defined-set'-def OclValid-def*]
A-all-def[*simplified all-defined-def OclValid-def*]
foundation20[*OF A-defined*[*THEN spec*, *of* τ], *simplified OclValid-def*]
del: *StrictRefEq-set-exec*)
apply(*rule S-lift*[*OF S-all-def*, *THEN exE*], *simp*)
apply(*subst img-fold*[*OF F-commute*], *simp add*: *A-all-def*, *drule sym*, *simp add*: *S-all-int*, *rule f-fold-insert*, *simp-all*) **apply**(*subst img-fold*[*OF G-commute*], *simp add*: *A-all-def*, *drule sym*, *simp add*: *S-all-int*, *rule g-fold-insert*, *simp-all*)
apply(*rule fold-cong'*[**where** $P = \lambda s\ \tau.\ (\forall\tau.$ *all-defined* τ $s) \wedge P\ s\ \tau$, *OF downgrade EQ-comp-fun-commute0-gen0.downgra G-commute*], *simplified image-ident*])
apply(*rule all-i-set-to-def*)
apply(*drule sym*, *simp add*: *S-all-int*, *simp add*: *A-all-def*)
apply(*rule fold-eq*, *simp add*: *int-is-valid*, *blast*, *simp*)
apply(*simp*, *simp*, *simp add*: *A-all-def*, *rule P0*)
apply(*rule conjI*)+
apply(*subst all-defined-fold-rec*[*simplified*], *simp add*: *A-all-def*, *simp*) **apply**(*subst def-set*[*THEN iffD2*, *THEN spec*], *simp*) **apply**(*simp*, *blast*, *simp*)
apply(*subst fold-insert*, *simp add*: *A-all-def*, *simp*, *simp*, *simp*)
apply(*rule Prec*, *simp*+)
done
qed
qed

lemma *iterate-subst-set0-gen* :
assumes *S-all-def* : $\bigwedge\tau.$ *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau.$ *all-defined* τ *A*
and *F-commute* : *EQ-comp-fun-commute0* ($\lambda x.$ *F* (λ-. *x*))
and *G-commute* : *EQ-comp-fun-commute0* ($\lambda x.$ (*G* :: ('$\mathfrak{A}$, -) *val*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. *x*))
and *fold-eq* : $\bigwedge$*x acc*. *is-int* (λ(-::'$\mathfrak{A}$ *st*). *x*) $\Longrightarrow$ ($\forall\tau.$ *all-defined* τ *acc*) $\Longrightarrow$ *P acc* τ $\Longrightarrow$ *F* (λ-. *x*) *acc* τ = *G* (λ-. *x*) *acc* τ
and *P0* : *P A* τ
and *Prec* : $\bigwedge$*x Fa*. $\forall$(τ::'$\mathfrak{A}$ *st*). *all-defined-set* τ *Fa* $\Longrightarrow$

is-int (λ(-::'$\mathfrak{A}$ *st*). *x*) $\Longrightarrow$
$x \notin$ *Fa* $\Longrightarrow$
$\forall\tau$. *all-defined* τ (*Finite-Set.fold* (λx. *F* (λ-. *x*)) *A Fa*) $\Longrightarrow$
P (*Finite-Set.fold* (λx. *F* (λ-. *x*)) *A Fa*) τ $\Longrightarrow$
P (*F* (λ-. *x*) (*Finite-Set.fold* (λx. *F* (λ-. *x*)) *A Fa*)) τ
shows (*S*$-$>*iterate*(*x*;*acc*=*A*|*F x acc*)) τ = (*S*$-$>*iterate*(*x*;*acc*=*A*|*G x acc*)) τ
apply(*rule iterate-subst-set-gen0*[*OF S-all-def A-all-def F-commute*[*THEN EQ-comp-fun-commute0.downgrad*
G-commute[*THEN EQ-comp-fun-commute0.downgrade'*]])
apply(*rule fold-eq*, *simp*, *simp*, *simp*)
apply(*rule P0*, *rule Prec*, *blast*+)
apply(*subst EQ-comp-fun-commute000.fold-insert'*[*OF F-commute*[*THEN c000-of-c0*[**where** *f* = *F*]], *simplified*], *simp add*: *A-all-def*, *blast*+)
apply(*subst EQ-comp-fun-commute000.fold-insert'*[*OF G-commute*[*THEN c000-of-c0*[**where** *f* = *G*]], *simplified*], *simp add*: *A-all-def*, *blast*+)
done

lemma *iterate-subst-set0* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
and *F-commute* : *EQ-comp-fun-commute0* (λx. *F* (λ-. *x*))
and *G-commute* : *EQ-comp-fun-commute0* (λx. (*G* :: ('$\mathfrak{A}$, -) *val*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. *x*))
and *fold-eq* : $\bigwedge$*x acc*. ($\forall\tau$. ($\tau \models \upsilon$ (λ(-:: '$\mathfrak{A}$ *st*). *x*))) $\Longrightarrow$ ($\forall\tau$. *all-defined* τ *acc*) $\Longrightarrow$ *F* (λ-. *x*) *acc* = *G* (λ-. *x*) *acc*
shows (*S*$-$>*iterate*(*x*;*acc*=*A*|*F x acc*)) = (*S*$-$>*iterate*(*x*;*acc*=*A*|*G x acc*))
apply(*rule ext*, *rule iterate-subst-set0-gen*, *simp-all add*: *assms*)
apply(*subst fold-eq*, *simp-all add*: *int-is-valid*)
done

lemma *iterate-subst-set'0* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
and *A-include* : $\bigwedge\tau 1$ $\tau 2$. *A* $\tau 1$ = *A* $\tau 2$
and *F-commute* : *EQ-comp-fun-commute0* (λx. *F* (λ-. *x*))
and *G-commute* : *EQ-comp-fun-commute0* (λx. (*G* :: ('$\mathfrak{A}$, -) *val*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. *x*))
and *fold-eq* : $\bigwedge$*x acc* τ. *is-int* (λ(-::'$\mathfrak{A}$ *st*). *x*) $\Longrightarrow$ ($\forall\tau$. *all-defined* τ *acc*) $\Longrightarrow$ $\forall\tau$ τ'. *acc* τ = *acc* τ' $\Longrightarrow$ *F* (λ-. *x*) *acc* = *G* (λ-. *x*) *acc*
shows (*S*$-$>*iterate*(*x*;*acc*=*A*|*F x acc*)) = (*S*$-$>*iterate*(*x*;*acc*=*A*|*G x acc*))
proof $-$
interpret *EQ-comp-fun-commute0* λx. *F* (λ-. *x*) **by** (*rule F-commute*)
show *?thesis*
apply(*rule ext*, *rule iterate-subst-set0-gen*[**where** *P* = λ*acc* -. $\forall\tau$ τ'. *acc* τ = *acc* τ', *OF S-all-def A-all-def F-commute G-commute*])
apply(*subst fold-eq*, *simp*+, *simp add*: *A-include*)
apply(*rule allI*)+
apply(*rule cp-gen'*, *simp*, *blast*, *blast*)

done
qed

lemma *iterate-subst-set''0* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
and *F-commute* : *EQ-comp-fun-commute0* (λx. *F* (λ-. *x*))
and *G-commute* : *EQ-comp-fun-commute0* (λx. (*G* :: ('$\mathfrak{A}$, -) *val*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. *x*))
and *fold-eq* : $\bigwedge$*x acc*. *is-int* (λ(-::'$\mathfrak{A}$ *st*). *x*) $\Longrightarrow$ ($\forall\tau$. *all-defined* τ *acc*) $\Longrightarrow$ $\lceil\lceil$*Rep-Set-0* (*acc* τ)$\rceil\rceil$ $\neq$ {} $\Longrightarrow$ *F* (λ-. *x*) *acc* τ = *G* (λ-. *x*) *acc* τ
shows $\lceil\lceil$*Rep-Set-0* (*A* τ)$\rceil\rceil$ $\neq$ {} $\Longrightarrow$ (*S*$-$>*iterate*(*x*;*acc*=*A*|*F x acc*)) τ = (*S*$-$>*iterate*(*x*;*acc*=*A*|*G x acc*)) τ
proof –
interpret *EQ-comp-fun-commute0* λx. *F* (λ-. *x*) **by** (*rule F-commute*)
show $\lceil\lceil$*Rep-Set-0* (*A* τ)$\rceil\rceil$ $\neq$ {} $\Longrightarrow$ *?thesis*
apply(*rule iterate-subst-set0-gen*[**where** *P* = λ*acc* τ. $\lceil\lceil$*Rep-Set-0* (*acc* τ)$\rceil\rceil$ $\neq$ {}, *OF S-all-def A-all-def F-commute G-commute*])
apply(*subst fold-eq*, *simp*+)
apply(*rule notempty'*, *simp*+)
done
qed

lemma *iterate-subst-set---* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
and *A-include* : $\bigwedge\tau 1$ $\tau 2$. *A* $\tau 1$ = *A* $\tau 2$
and *F-commute* : *EQ-comp-fun-commute0'* (λx. *F* (λ-. $\lfloor x\rfloor$))
and *G-commute* : *EQ-comp-fun-commute0'* (λx. (*G* :: ('$\mathfrak{A}$, -) *val*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. $\lfloor x\rfloor$))
and *fold-eq* : $\bigwedge$*x acc*. *is-int* (λ(-::'$\mathfrak{A}$ *st*). $\lfloor x\rfloor$) $\Longrightarrow$ ($\forall\tau$. *all-defined* τ *acc*) $\Longrightarrow$ $\forall\tau$ τ'. *acc* τ = *acc* τ' $\Longrightarrow$ $\lceil\lceil$*Rep-Set-0* (*acc* τ)$\rceil\rceil$ $\neq$ {} $\Longrightarrow$ *F* (λ-. $\lfloor x\rfloor$) *acc* τ = *G* (λ-. $\lfloor x\rfloor$) *acc* τ
shows $\lceil\lceil$*Rep-Set-0* (*A* τ)$\rceil\rceil$ $\neq$ {} $\Longrightarrow$ (*S*$-$>*iterate*(*x*;*acc*=*A*|*F x acc*)) τ = (*S*$-$>*iterate*(*x*;*acc*=*A*|*G x acc*)) τ
proof –
interpret *EQ-comp-fun-commute0'* λx. *F* (λ-. $\lfloor x\rfloor$) **by** (*rule F-commute*)
show $\lceil\lceil$*Rep-Set-0* (*A* τ)$\rceil\rceil$ $\neq$ {} $\Longrightarrow$ *?thesis*
apply(*rule iterate-subst-set-gen0*[**where** *P* = λ*acc* τ. ($\forall\tau$ τ'. *acc* τ = *acc* τ') $\wedge$ $\lceil\lceil$*Rep-Set-0* (*acc* τ)$\rceil\rceil$ $\neq$ {}, *OF S-all-def A-all-def F-commute*[*THEN EQ-comp-fun-commute0'.downgrade'*] *G-commute*[*THEN EQ-comp-fun-commute0'.downgrade'*]])
apply(*rule fold-eq*, *blast*+, *simp add*: *A-include*)
apply(*rule conjI*)+
apply(*rule allI*)+
apply(*rule cp-gen'*, *blast*+)
apply(*rule notempty'*, *blast*+)
apply(*subst EQ-comp-fun-commute000'.fold-insert'*[*OF F-commute*[*THEN c000'-of-c0'*[**where** *f* = *F*]], *simplified*], *simp add*: *A-all-def*, *blast*+)

apply(*subst EQ-comp-fun-commute000'.fold-insert'*[*OF G-commute*[*THEN c000'-of-c0'*[**where** $f = G$]], *simplified*], *simp add*: *A-all-def*, *blast+*)
apply(*rule S-lift*, *simp*)
done
qed

Context passing

lemma *cp-OclIterate*$_{Set}$*1-gen*:
assumes *f-comm* : *EQ-comp-fun-commute0-gen0 f000 all-def-set* ($\lambda x.\ f\ (f000\ x)$)
and *A-all-def* : $\bigwedge\tau.$ *all-defined* τ A
and *f-fold-insert* : $\bigwedge x\ S\ A.$ ($\bigwedge\tau.$ *all-defined* τ A) $\Longrightarrow x \notin S \Longrightarrow$ *is-int* ($f000\ x$) $\Longrightarrow$ *all-int-set* ($f000$ ‘ S) $\Longrightarrow$ *Finite-Set.fold f A* (*insert* ($f000\ x$) ($f000$ ‘ S)) $= f$ ($f000\ x$) (*Finite-Set.fold f A* ($f000$ ‘ S))
and *S-lift* : *all-defined* τ $X \Longrightarrow \exists S'.$ ($\lambda a\ \tau.\ a$) ‘ $\lceil\lceil$*Rep-Set-0* ($X\ \tau$)$\rceil\rceil = f000$ ‘ S'
shows (X−>*iterate*(a; $x = A$ | $f\ a\ x$)) τ =
((λ -. $X\ \tau$)−>*iterate*(a; x = (λ-. $A\ \tau$) | $f\ a\ x$)) τ
proof −
have B : $\lfloor\bot\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ **by**(*simp add*: *null-option-def bot-option-def*)
have *A-all-def'* : $\bigwedge\tau\ \tau'.$ *all-defined* τ ($\lambda a.\ A\ \tau'$) **by**(*subst cp-all-def*[*symmetric*], *simp add*: *A-all-def*)

interpret *EQ-comp-fun-commute0-gen0 f000 all-def-set* $\lambda x.\ f\ (f000\ x)$ **by** (*rule f-comm*)
show *?thesis*
apply(*subst cp-OclIterate*$_{Set}$[*symmetric*])
apply(*simp add*: *OclIterate*$_{Set}$*-def cp-valid*[*symmetric*])
apply(*case-tac* $\neg((\delta\ X)\ \tau = true\ \tau \wedge (\upsilon\ A)\ \tau = true\ \tau \wedge$ *finite* $\lceil\lceil$*Rep-Set-0* ($X\ \tau$)$\rceil\rceil$), *blast*)
apply(*simp*)
apply(*erule conjE*)+
apply(*frule Set-inv-lemma*[*simplified OclValid-def*])
proof −
assume ($\delta\ X$) $\tau = true\ \tau$
finite $\lceil\lceil$*Rep-Set-0* ($X\ \tau$)$\rceil\rceil$
$\forall x\in\lceil\lceil$*Rep-Set-0* ($X\ \tau$)$\rceil\rceil.\ x \neq \bot$
then have *X-def* : *all-defined* τ X **by** (*metis* (*lifting*, *no-types*) *OclValid-def all-defined-def all-defined-set'-def foundation18'*)
show *Finite-Set.fold f A* (($\lambda a\ \tau.\ a$) ‘ $\lceil\lceil$*Rep-Set-0* ($X\ \tau$)$\rceil\rceil$) τ = *Finite-Set.fold f* (λ-. $A\ \tau$) (($\lambda a\ \tau.\ a$) ‘ $\lceil\lceil$*Rep-Set-0* ($X\ \tau$)$\rceil\rceil$) τ
apply(*rule S-lift*[*OF X-def*, *THEN exE*], *simp*)
apply(*subst* (*1 2*) *img-fold*[*OF f-comm*], *simp add*: *A-all-def'*, *drule sym*, *simp add*: *all-def-to-all-int*[*OF X-def*])
apply(*rule f-fold-insert*, *simp-all add*: *A-all-def' A-all-def*)+
apply(*rule fold-cong'''*[**where** $P = \lambda$- -. *True*, *OF downgrade downgrade*, *simplified image-ident*])
apply(*rule all-i-set-to-def*)
apply(*drule sym*, *simp add*: *all-def-to-all-int*[*OF X-def*], *simp add*: *A-all-def*) **apply**(*subst cp-all-def*[*symmetric*], *simp add*: *A-all-def*)
apply(*blast+*)
done

qed
qed

lemma *cp-OclIterate$_{Set}$1*:
assumes *f-comm* : *EQ-comp-fun-commute0'* ($\lambda x.$ *f* (λ-. $\lfloor x \rfloor$))
and *A-all-def* : $\bigwedge\tau.$ *all-defined* τ *A*
shows (*X*−>*iterate*(*a*; *x* = *A* | *f a x*)) τ =
((λ -. *X* τ)−>*iterate*(*a*; *x* = (λ-. *A* τ) | *f a x*)) τ
proof −
interpret *EQ-comp-fun-commute0'* $\lambda x.$ *f* (λ-. $\lfloor x \rfloor$) **by** (*rule f-comm*)
show *?thesis*
apply(*rule cp-OclIterate$_{Set}$1-gen*[*OF downgrade' A-all-def*])
apply(*subst EQ-comp-fun-commute000'.fold-insert'*[*OF f-comm*[*THEN c000'-of-c0'*[**where** *f* = *f*]], *simplified*], *simp-all*)
apply(*rule S-lift*, *simp*)
done
qed

all defined (construction)

lemma *i-cons-all-def* :
assumes *F-commute* : *EQ-comp-fun-commute0* ($\lambda x.$ (*F* :: ('$\mathfrak{A}$, -) *val*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*
$\Rightarrow$ ('$\mathfrak{A}$, -) *Set*) (λ-. *x*))
and *A-all-def* : $\bigwedge\tau.$ *all-defined* τ *S*
shows *all-defined* τ (*OclIterate$_{Set}$ S S F*)
proof −
have *A-all-def'* : $\forall\tau.$ *all-int-set* ((λa (τ:: '$\mathfrak{A}$ *st*). *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)
apply(*rule allI*, *rule all-def-to-all-int*, *simp add*: *A-all-def*)
done

show *?thesis*
apply(*unfold OclIterate$_{Set}$-def*)
apply(*simp add*: *A-all-def*[*simplified all-defined-def OclValid-def*]
A-all-def[*simplified all-defined-def all-defined-set'-def*]
A-all-def[*simplified all-defined-def*, *THEN conjunct1*, *THEN foundation20*, *simplified OclValid-def*]
)
apply(*subgoal-tac* $\forall\tau'.$ *all-defined* τ' (*Finite-Set.fold F S* ((λa $\tau.$ *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)), *metis* (*lifting*, *no-types*) *foundation16 all-defined-def*)
apply(*rule allI*, *rule EQ-comp-fun-commute000.fold-def*[*OF F-commute*[*THEN c000-of-c0*]], *simp add*: *A-all-def*, *simp add*: *A-all-def'*)
done
qed

lemma *i-cons-all-def''* :
assumes *F-commute* : *EQ-comp-fun-commute0'* ($\lambda x.$ *F* (λ-. $\lfloor x \rfloor$))
and *S-all-def* : $\bigwedge\tau.$ *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau.$ *all-defined* τ *A*

shows *all-defined* τ (*OclIterate*$_{Set}$ *S A F*)
proof –
have *A-all-def'* : $\forall\tau$. *all-int-set* ((λa (τ:: $'\mathfrak{A}$ *st*). *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)
apply(*rule allI*, *rule all-def-to-all-int*, *simp add*: *S-all-def*)
done

show *?thesis*
apply(*unfold OclIterate*$_{Set}$*-def*)
apply(*simp add*: *S-all-def*[*simplified all-defined-def OclValid-def*]
S-all-def[*simplified all-defined-def all-defined-set'-def*]
A-all-def[*simplified all-defined-def*, *THEN conjunct1*, *THEN foundation20*, *simplified OclValid-def*]
)
apply(*subgoal-tac* $\forall\tau'$. *all-defined* τ' (*Finite-Set.fold F A* ((λa τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)), *metis* (*lifting*, *no-types*) *foundation16 all-defined-def*)
apply(*rule S-lift*[*THEN exE*, *OF S-all-def*[*of* τ]], *simp only*:)
apply(*rule allI*, *rule EQ-comp-fun-commute000'.fold-def*[*OF F-commute*[*THEN c000'-of-c0'*]], *simp add*: *A-all-def*, *drule sym*, *simp add*: *A-all-def'*)
done
qed

lemma *i-cons-all-def''cp* :
assumes *F-commute* : *EQ-comp-fun-commute0'* (λx. *F* (λ-. $\lfloor x\rfloor$))
and *S-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
shows *all-defined* τ ($\lambda\tau$. *OclIterate*$_{Set}$ (λ-. *S* τ) (λ-. *A* τ) *F* τ)
apply(*subst cp-OclIterate*$_{Set}$*1*[*symmetric*, *OF F-commute A-all-def*])
apply(*rule i-cons-all-def''*[*OF F-commute S-all-def A-all-def*])
done

lemma *i-cons-all-def'* :
assumes *F-commute* : *EQ-comp-fun-commute0'* (λx. *F* (λ-. $\lfloor x\rfloor$))
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *S*
shows *all-defined* τ (*OclIterate*$_{Set}$ *S S F*)
by(*rule i-cons-all-def''*, *simp-all add*: *assms*)

Preservation of global jugdment

lemma *iterate-cp-all-gen* :
assumes *F-commute* : *EQ-comp-fun-commute0-gen0 f000 all-def-set* (λx. *F* (*f000 x*))
and *A-all-def* : $\forall\tau$. *all-defined* τ *S*
and *S-cp* : *S* ($\tau 1$ :: $'\mathfrak{A}$ *st*) = *S* $\tau 2$
and *f-fold-insert* : $\bigwedge$*x A S*. $x \notin S \Longrightarrow$ ($\bigwedge\tau$. *all-defined* τ *A*) $\Longrightarrow$ *is-int* (*f000 x*) $\Longrightarrow$ *all-int-set* (*f000* ‘ *S*) $\Longrightarrow$ *Finite-Set.fold F A* (*insert* (*f000 x*) (*f000* ‘ *S*)) = *F* (*f000 x*) (*Finite-Set.fold F A* (*f000* ‘ *S*))
and *S-lift* : *all-defined* $\tau 2$ *S* $\Longrightarrow \exists S'$. ($\lambda a$ τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* $\tau 2$)$\rceil\rceil$ = *f000* ‘ *S'*
shows *OclIterate*$_{Set}$ *S S F* $\tau 1$ = *OclIterate*$_{Set}$ *S S F* $\tau 2$
proof –
have *A-all-def'* : $\forall\tau$. *all-int-set* ((λa (τ:: $'\mathfrak{A}$ *st*). *a*) ‘ $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$)

apply(*rule allI*, *rule all-def-to-all-int*, *simp add*: *A-all-def*)
done

interpret *EQ-comp-fun-commute0-gen0 f000 all-def-set* $\lambda x.$ *F* (*f000 x*) **by** (*rule F-commute*)
show *?thesis*
apply(*unfold OclIterate*$_{Set}$*-def*)
apply(*simp add*: *A-all-def*[*THEN spec*, *simplified all-defined-def OclValid-def*]
A-all-def[*THEN spec*, *simplified all-defined-def all-defined-set'-def*]
A-all-def[*THEN spec*, *simplified all-defined-def*, *THEN conjunct1*, *THEN foundation20*, *simplified OclValid-def*]
S-cp)
apply(*rule S-lift*[*OF A-all-def*[*THEN spec*], *THEN exE*], *simp*)
apply(*subst* (*1 2*) *img-fold*[*OF F-commute*], *simp add*: *A-all-def*, *drule sym*, *simp add*: *A-all-def'*, *rule f-fold-insert*, *simp-all add*: *A-all-def*)
apply(*subst* (*1 2*) *image-ident*[*symmetric*])
apply(*rule fold-cong''*[**where** $P = \lambda$- -. *True*, *OF F-commute*[*THEN EQ-comp-fun-commute0-gen0.downgrade*] *F-commute*[*THEN EQ-comp-fun-commute0-gen0.downgrade*]])
apply(*rule all-i-set-to-def*)
apply(*drule sym*, *simp add*: *A-all-def'*, *simp add*: *A-all-def*)
apply(*simp-all add*: *S-cp*)
done
qed

lemma *iterate-cp-all* :
assumes *F-commute* : *EQ-comp-fun-commute0* ($\lambda x.$ *F* (λ-. *x*))
and *A-all-def* : $\forall \tau.$ *all-defined* τ *S*
and *S-cp* : *S* ($\tau 1$ :: $'\mathfrak{A}$ *st*) = *S* $\tau 2$
shows *OclIterate*$_{Set}$ *S S F* $\tau 1$ = *OclIterate*$_{Set}$ *S S F* $\tau 2$
apply(*rule iterate-cp-all-gen*[*OF F-commute*[*THEN EQ-comp-fun-commute0.downgrade'*] *A-all-def S-cp*])
apply(*subst EQ-comp-fun-commute000.fold-insert'*[*OF F-commute*[*THEN c000-of-c0*[**where** *f* = *F*]], *simplified*], *blast+*)
done

lemma *iterate-cp-all'* :
assumes *F-commute* : *EQ-comp-fun-commute0'* ($\lambda x.$ *F* (λ-. $\lfloor x \rfloor$))
and *A-all-def* : $\forall \tau.$ *all-defined* τ *S*
and *S-cp* : *S* ($\tau 1$ :: $'\mathfrak{A}$ *st*) = *S* $\tau 2$
shows *OclIterate*$_{Set}$ *S S F* $\tau 1$ = *OclIterate*$_{Set}$ *S S F* $\tau 2$
apply(*rule iterate-cp-all-gen*[*OF F-commute*[*THEN EQ-comp-fun-commute0'.downgrade'*] *A-all-def S-cp*])
apply(*subst EQ-comp-fun-commute000'.fold-insert'*[*OF F-commute*[*THEN c000'-of-c0'*[**where** *f* = *F*]], *simplified*], *blast+*)
apply(*rule S-lift*, *simp*)
done

Preservation of non-emptiness

lemma *iterate-notempty-gen* :

assumes *F-commute* : *EQ-comp-fun-commute0-gen0 f000 all-def-set* ($\lambda x.$ (F:: ($'\mathfrak{A}$, $'a$ *option option*) *val*
$\Rightarrow$ ($'\mathfrak{A}$, -) *Set*
$\Rightarrow$ ($'\mathfrak{A}$, -) *Set*) (*f000 x*))
and *A-all-def* : $\forall\tau$. *all-defined* τ *S*
and *S-notempty* : $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil \neq \{\}$
and *f-fold-insert* : $\bigwedge x$ *A S*. $x \notin S \Longrightarrow (\bigwedge\tau.$ *all-defined* τ *A*) $\Longrightarrow$ *is-int* (*f000 x*) $\Longrightarrow$ *all-int-set* (*f000* ‘ *S*) $\Longrightarrow$ *Finite-Set.fold F A* (*insert* (*f000 x*) (*f000* ‘ *S*)) = *F* (*f000 x*) (*Finite-Set.fold F A* (*f000* ‘ *S*))
and *S-lift* : *all-defined* τ *S* $\Longrightarrow \exists S'$. ($\lambda a$ τ. *a*) ‘ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$ = *f000* ‘ S'
shows $\lceil\lceil$*Rep-Set-0* ($OclIterate_{Set}$ *S S F* τ)$\rceil\rceil \neq \{\}$
proof −
have *A-all-def'* : $\forall\tau$. *all-int-set* ((λa (τ:: $'\mathfrak{A}$ *st*). *a*) ‘ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$)
apply(*rule allI*, *rule all-def-to-all-int*, *simp add*: *A-all-def*)
done

interpret *EQ-comp-fun-commute0-gen0 f000 all-def-set* λx. *F* (*f000 x*) **by** (*rule F-commute*)
show *?thesis*
apply(*unfold* $OclIterate_{Set}$*-def*)
apply(*simp add*: *A-all-def*[*THEN spec*, *simplified all-defined-def OclValid-def*]
A-all-def[*THEN spec*, *simplified all-defined-def all-defined-set'-def*]
A-all-def[*THEN spec*, *simplified all-defined-def*, *THEN conjunct1*, *THEN foundation20*, *simplified OclValid-def*]
)
apply(*insert S-notempty*)
apply(*rule S-lift*[*OF A-all-def*[*THEN spec*], *THEN exE*], *simp*)
apply(*subst img-fold*[*OF F-commute*], *simp add*: *A-all-def*, *drule sym*, *simp add*: *A-all-def'*, *rule f-fold-insert*, *simp-all add*: *A-all-def*)
apply(*subst* (*2*) *image-ident*[*symmetric*])
apply(*rule all-int-induct*)
apply(*rule all-i-set-to-def*)
apply(*drule sym*, *simp add*: *A-all-def'*)
apply(*simp*)
apply(*simp*)
apply(*subst fold-insert*[*OF A-all-def*], *metis surj-pair*, *simp*, *simp*)
apply(*rule notempty*, *rule allI*, *rule fold-def*[*simplified*], *simp add*: *A-all-def*, *blast+*)
done
qed

lemma *iterate-notempty* :
assumes *F-commute* : *EQ-comp-fun-commute0* (λx. (F:: ($'\mathfrak{A}$, -) *val*
$\Rightarrow$ ($'\mathfrak{A}$, -) *Set*
$\Rightarrow$ ($'\mathfrak{A}$, -) *Set*) (λ-. *x*))
and *A-all-def* : $\forall\tau$. *all-defined* τ *S*
and *S-notempty* : $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil \neq \{\}$
shows $\lceil\lceil$*Rep-Set-0* ($OclIterate_{Set}$ *S S F* τ)$\rceil\rceil \neq \{\}$
apply(*rule iterate-notempty-gen*[*OF F-commute*[*THEN EQ-comp-fun-commute0.downgrade'*] *A-all-def S-notempty*])
apply(*subst EQ-comp-fun-commute000.fold-insert'*[*OF F-commute*[*THEN c000-of-c0*[**where** *f*

$= F$]], *simplified*], *blast*+)
done

lemma *iterate-notempty'* :
assumes *F-commute* : *EQ-comp-fun-commute0'* (λx. F (λ-. $\lfloor x \rfloor$))
and *A-all-def* : $\forall \tau$. *all-defined* τ S
and *S-notempty* : $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil \neq \{\}$
shows $\lceil\lceil$*Rep-Set-0* (*OclIterate*$_{Set}$ S S F τ)$\rceil\rceil \neq \{\}$
apply(*rule iterate-notempty-gen*[*OF F-commute*[*THEN EQ-comp-fun-commute0'.downgrade'*] *A-all-def S-notempty*])
apply(*subst EQ-comp-fun-commute000'.fold-insert'*[*OF F-commute*[*THEN c000'-of-c0'*[**where** $f = F$]], *simplified*], *blast*+)
apply(*rule S-lift*, *simp*)
done

Preservation of comp fun commute (main)

lemma *iterate-commute'* :
assumes *f-comm* : $\bigwedge a$. *EQ-comp-fun-commute0'* (λx. F a (λ-. $\lfloor x \rfloor$))

assumes *f-notempty* : $\bigwedge S\ x\ y\ \tau$. *is-int* (λ(-::$'\mathfrak{A}$ *st*). $\lfloor x \rfloor$) $\Longrightarrow$
is-int (λ(-::$'\mathfrak{A}$ *st*). $\lfloor y \rfloor$) $\Longrightarrow$
($\forall$ (τ::$'\mathfrak{A}$ *st*). *all-defined* τ S) $\Longrightarrow$
$\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil \neq \{\} \Longrightarrow$
OclIterate$_{Set}$ (*OclIterate*$_{Set}$ S S (F x)) (*OclIterate*$_{Set}$ S S (F x)) (F y) τ =
OclIterate$_{Set}$ (*OclIterate*$_{Set}$ S S (F y)) (*OclIterate*$_{Set}$ S S (F y)) (F x) τ

shows *EQ-comp-fun-commute0'* (λx S. S −>*iterate*(j;S=S | F x j S))
proof − **interpret** *EQ-comp-fun-commute0'* λx. F a (λ-. $\lfloor x \rfloor$) **by** (*rule f-comm*)
apply-end(*simp only*: *EQ-comp-fun-commute0'-def*)
apply-end(*rule conjI*)+ **apply-end**(*rule allI*)+ **apply-end**(*rule impI*)+
apply-end(*subst cp-OclIterate*$_{Set}$*1*[*OF f-comm*], *blast*, *simp*)
apply-end(*rule allI*)+ **apply-end**(*rule impI*)+
apply-end(*subst iterate-cp-all'*, *simp add*: *f-comm*, *simp*, *simp*, *simp*)

apply-end(*rule conjI*)+ **apply-end**(*rule allI*)+ **apply-end**(*rule impI*)+

show $\bigwedge x\ S\ \tau$.
$\forall \tau$. *all-defined* τ S $\Longrightarrow$
is-int (λ-. $\lfloor x \rfloor$) $\Longrightarrow$ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil \neq \{\} \Longrightarrow$ $\lceil\lceil$*Rep-Set-0* (*OclIterate*$_{Set}$ S S (F x) τ)$\rceil\rceil \neq \{\}$
by(*rule iterate-notempty'*[*OF f-comm*], *simp-all*)

apply-end(*simp*) **apply-end**(*simp*) **apply-end**(*simp*)
apply-end(*rule conjI*)+ **apply-end**(*rule allI*)+
fix x y τ
show ($\forall \tau$. *all-defined* τ (*OclIterate*$_{Set}$ y y (F x))) = (*is-int* (λ(-:: $'\mathfrak{A}$ *st*). $\lfloor x \rfloor$) $\wedge$ ($\forall \tau$. *all-defined* τ y))
apply(*rule iffI*, *rule conjI*) **apply**(*simp add*: *is-int-def OclValid-def valid-def bot-fun-def bot-option-def*)

apply(*rule i-invert-all-defined'*[**where** $F = F\ x$], *simp*)
apply(*rule allI*, *rule i-cons-all-def'*[**where** $F = F\ x$, *OF f-comm*], *blast*)
done

apply-end(*rule allI*)+ **apply-end**(*rule impI*)+
apply-end(*rule ext*, *rename-tac* τ)
fix S **and** x **and** y **and** τ
show *is-int* $(\lambda(\text{-}::'\mathfrak{A}\ st).\ \lfloor x \rfloor) \Longrightarrow$
is-int $(\lambda(\text{-}::'\mathfrak{A}\ st).\ \lfloor y \rfloor) \Longrightarrow$
$(\forall (\tau::'\mathfrak{A}\ st).\ \textit{all-defined}\ \tau\ S) \Longrightarrow$
$\textit{OclIterate}_{Set}\ (\textit{OclIterate}_{Set}\ S\ S\ (F\ x))\ (\textit{OclIterate}_{Set}\ S\ S\ (F\ x))\ (F\ y)\ \tau =$
$\textit{OclIterate}_{Set}\ (\textit{OclIterate}_{Set}\ S\ S\ (F\ y))\ (\textit{OclIterate}_{Set}\ S\ S\ (F\ y))\ (F\ x)\ \tau$
apply(*case-tac* $\lceil\lceil \textit{Rep-Set-0}\ (S\ \tau)\rceil\rceil = \{\}$)
apply(*subgoal-tac* $S\ \tau = \textit{Set}\{\}\ \tau$)
prefer *2*
apply(*drule-tac* $f = \lambda s.\ \textit{Abs-Set-0}\ \lfloor\lfloor s \rfloor\rfloor$ **in** *arg-cong*)
apply(*subgoal-tac* $S\ \tau = \textit{Abs-Set-0}\ \lfloor\lfloor\{\}\rfloor\rfloor$)
prefer *2*
apply(*metis abs-rep-simp*)
apply(*simp add*: *mtSet-def*)

apply(*subst* (*1 2*) $\textit{cp-OclIterate}_{Set}\textit{1}$[*OF f-comm*]) **apply**(*rule i-cons-all-def'*[*OF f-comm*], *blast*)+
apply(*subst* (*1 2 3 4 5 6*) $\textit{cp-OclIterate}_{Set}\textit{1}$[*OF f-comm*])
apply(*subst cp-all-def*[*symmetric*]) **apply**(*rule i-cons-all-def'*[*OF f-comm*], *blast*) **apply**(*blast*)
apply(*subst cp-all-def*[*symmetric*]) **apply**(*rule i-cons-all-def'*[*OF f-comm*], *blast*)
apply(*simp*)
apply(*subst* (*1 2 3 4 5 6*) $\textit{cp-OclIterate}_{Set}\textit{1}$[*OF f-comm*, *symmetric*])
apply(*subst* (*1 2*) *cp-mtSet*[*symmetric*])
apply(*rule i-cons-all-def'*[*OF f-comm*]) **apply**(*simp add*: *mtSet-all-def*)+
apply(*subst* (*1 2*) *cp-mtSet*[*symmetric*])
apply(*rule i-cons-all-def'*[*OF f-comm*]) **apply**(*simp add*: *mtSet-all-def*)+

apply(*subst* (*1 2*) $\textit{cp-OclIterate}_{Set}\textit{1}$[*OF f-comm*])
apply(*rule i-cons-all-def'*[*OF f-comm*], *metis surj-pair*)
apply(*rule i-cons-all-def'*[*OF f-comm*], *metis surj-pair*)
apply(*subst* (*1 2 3 4 5 6*) $\textit{cp-OclIterate}_{Set}\textit{1}$[*OF f-comm*])
apply(*subst cp-all-def*[*symmetric*]) **apply**(*rule i-cons-all-def'*[*OF f-comm*]) **apply**(*metis surj-pair*)+
apply(*subst cp-all-def*[*symmetric*]) **apply**(*rule i-cons-all-def'*[*OF f-comm*]) **apply**(*metis surj-pair*)+
apply(*subst* (*1 2 3 4 5 6*) $\textit{cp-OclIterate}_{Set}\textit{1}$[*OF f-comm*, *symmetric*])
apply(*rule i-cons-all-def''cp*[*OF f-comm*]) **apply**(*metis surj-pair*) **apply**(*metis surj-pair*) **apply**(*metis surj-pair*)
apply(*rule i-cons-all-def''cp*[*OF f-comm*]) **apply**(*metis surj-pair*) **apply**(*metis surj-pair*)

apply(*rule f-notempty*, *simp-all*)

done

qed

4.7.11. comp fun commute OclIterate and OclIncluding

Identity

lemma *i-including-id'* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ (S :: ('$\mathfrak{A}$, *int option option*) *Set*)
shows (*Finite-Set.fold* (λj *r2*. *r2*−>*including*(j)) S ((λa τ. a) ‘ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$)) τ = S τ
proof −
have *invert-set-0* : $\bigwedge x$ F. $\lfloor\lfloor$*insert* x $F\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$ $\Longrightarrow \lfloor\lfloor F\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
by(*auto simp*: *bot-option-def null-option-def*)

have *invert-all-def-set* : $\bigwedge x$ F τ. *all-defined-set* τ (*insert* x F) $\Longrightarrow$ *all-defined-set* τ F
apply(*simp add*: *all-defined-set-def*)
done

have *all-def-to-all-int-* : $\bigwedge$*set* τ. *all-defined-set* τ *set* $\Longrightarrow$ *all-int-set* ((λa τ. a) ‘ *set*)
apply(*simp add*: *all-defined-set-def all-int-set-def is-int-def*)
by (*metis foundation18'*)

have *invert-int* : $\bigwedge x$ S. *all-int-set* (*insert* x S) $\Longrightarrow$
is-int x
by(*simp add*: *all-int-set-def*)

have *inject* : *inj* (λa τ. a)
by(*rule inj-fun*, *simp*)

have *image-cong*: $\bigwedge x$ *Fa* f. *inj* f $\Longrightarrow$ $x \notin$ *Fa* $\Longrightarrow$ f $x \notin f$ ‘ *Fa*
apply(*simp add*: *image-def*)
apply(*rule ballI*)
apply(*case-tac* x = *xa*, *simp*)
apply(*simp add*: *inj-on-def*)
apply(*blast*)
done
show *Finite-Set.fold* (λj *r2*. *r2*−>*including*(j)) S ((λa τ. a) ‘ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$) τ = S τ
apply(*subst finite-induct*[**where** P = λ*set*. *all-defined-set* τ *set* $\land$ $\lfloor\lfloor$*set*$\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x\in\lceil\lceil X\rceil\rceil.\ x \neq bot)\} \longrightarrow$
($\forall$(s :: ('$\mathfrak{A}$, -) *Set*). ($\forall\tau$. *all-defined* τ s) $\longrightarrow$
($\forall\tau$. *all-defined* τ (*Finite-Set.fold* (λj *r2*. (*r2*−>*including*(j))) s ((λa τ. a) ‘ *set*)))) $\land$
($\forall s$. ($\forall\tau$. *all-defined* τ s) $\land$ (*set* $\subseteq \lceil\lceil$*Rep-Set-0* (s τ)$\rceil\rceil$) $\longrightarrow$
(*Finite-Set.fold* (λj *r2*. (*r2*−>*including*(j))) s ((λa τ. a) ‘ *set*)) τ = s τ)
and F = $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$])
apply(*simp add*: *S-all-def*[*simplified all-defined-def all-defined-set'-def*])
apply(*simp*)
defer

apply(*insert S-all-def*[*simplified all-defined-def*, *THEN conjunct1*, *of* τ], *frule Set-inv-lemma*)
apply(*simp add*: *foundation18 all-defined-set-def invalid-def S-all-def*[*simplified all-defined-def all-defined-set′-def*])
apply (*metis assms order-refl*)
apply(*simp*)

apply(*rule impI*) **apply**(*erule conjE*)+
apply(*drule invert-set-0*, *simp del*: *StrictRefEq-set-exec*)
apply(*frule invert-all-def-set*, *simp del*: *StrictRefEq-set-exec*)
apply(*erule conjE*)+

apply(*rule conjI*)
apply(*rule allI*, *rename-tac SSS*, *rule impI*, *rule allI*, *rule allI*)
apply(*rule iterate-subst-set-rec*[*simplified Let-def*, *THEN mp*, *THEN mp*, *THEN mp*, *THEN spec*, *OF - including-commute*], *simp*)
apply(*simp*)
apply(*simp add*: *all-int-set-def all-defined-set-def is-int-def*) **apply** (*metis* (*mono-tags*) *foundation18′*)
apply(*simp*)

apply(*rule allI*, *rename-tac SS*, *rule impI*)
apply(*drule all-def-to-all-int-*)+
apply(*subst EQ-comp-fun-commute.fold-insert*[**where** $f = (\lambda j\ r2.\ (r2{-}{>}including(j)))$, *OF including-commute*])
apply(*metis PairE*)
apply(*simp*)+
apply(*rule invert-int*, *simp*)

apply(*rule image-cong*)
apply(*rule inject*)
apply(*simp*)

apply(*simp*)
apply(*subst including-id′*)
apply(*metis prod.exhaust*)
apply(*auto*)
done
qed

lemma *iterate-including-id* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ (S :: ($'\mathfrak{A}$, *int option option*) *Set*)
shows (S −>*iterate*(j;$r2$=S | $r2$−>*including*(j))) = S
apply(*simp add*: *OclIterate*$_{Set}$*-def OclValid-def del*: *StrictRefEq-set-exec*, *rule ext*)
apply(*subgoal-tac* $(\delta\ S)\ \tau = true\ \tau \wedge (\upsilon\ S)\ \tau = true\ \tau \wedge finite\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil$, *simp del*: *StrictRefEq-set-exec*)
prefer *2*
proof −

fix τ
show $(\delta\ S)\ \tau = true\ \tau \wedge (\upsilon\ S)\ \tau = true\ \tau \wedge finite\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil$
apply(*simp add*: *S-all-def*[*of* τ, *simplified all-defined-def OclValid-def all-defined-set'-def*]
foundation20[*simplified OclValid-def*])
done
apply-end(*subst i-including-id'*, *simp-all add*: *S-all-def*)
qed

lemma *i-including-id00* :
assumes *S-all-int* : $\bigwedge\tau.\ all\text{-}int\text{-}set\ ((\lambda a\ (\tau{::}\ '\mathfrak{A}\ st).\ a)\ `\ \lceil\lceil Rep\text{-}Set\text{-}0\ ((S :: ('\mathfrak{A},\ int\ option\ option)\ Set)\ \tau)\rceil\rceil)$
shows $\bigwedge\tau.\ \forall S'.\ (\forall\tau.\ all\text{-}defined\ \tau\ S') \longrightarrow (let\ img = image\ (\lambda a\ (\tau{::}\ '\mathfrak{A}\ st).\ a)\ ;\ set' = img\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\ ;\ f = (\lambda x.\ x)\ in$
$(\forall\tau.\ f\ `\ set' = img\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S'\ \tau)\rceil\rceil) \longrightarrow$
$(Finite\text{-}Set.fold\ (\lambda j\ r2.\ r2{-}{>}including(f\ j))\ Set\{\}\ set') = S')$
proof –
have *S-incl* : $\forall (x :: ('\mathfrak{A},\ 'a\ option\ option)\ Set).\ (\forall\tau.\ all\text{-}defined\ \tau\ x) \longrightarrow (\forall\tau.\ \lceil\lceil Rep\text{-}Set\text{-}0\ (x\ \tau)\rceil\rceil = \{\}) \longrightarrow Set\{\} = x$
apply(*rule allI*) **apply**(*rule impI*)+
apply(*rule ext*, *rename-tac* τ)
apply(*drule-tac* $x = \tau$ **in** *allE*) **prefer** *2* **apply** *assumption*
apply(*drule-tac* $x = \tau$ **in** *allE*) **prefer** *2* **apply** *assumption*
apply(*simp add*: *mtSet-def*)
by (*metis abs-rep-simp*)

have *invert-set-0* : $\bigwedge x\ F.\ \lfloor\lfloor insert\ x\ F\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\} \Longrightarrow \lfloor\lfloor F\rfloor\rfloor \in \{X.\ X = bot \vee X = null \vee (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
by(*auto simp*: *bot-option-def null-option-def*)

have *invert-all-def-set* : $\bigwedge x\ F\ \tau.\ all\text{-}defined\text{-}set\ \tau\ (insert\ x\ F) \Longrightarrow all\text{-}defined\text{-}set\ \tau\ F$
apply(*simp add*: *all-defined-set-def*)
done

have *all-def-to-all-int-* : $\bigwedge set\ \tau.\ all\text{-}defined\text{-}set\ \tau\ set \Longrightarrow all\text{-}int\text{-}set\ ((\lambda a\ \tau.\ a)\ `\ set)$
apply(*simp add*: *all-defined-set-def all-int-set-def is-int-def*)
by (*metis foundation18'*)

have *invert-int* : $\bigwedge x\ S.\ all\text{-}int\text{-}set\ (insert\ x\ S) \Longrightarrow$
is-int x
by(*simp add*: *all-int-set-def*)

have *inject* : *inj* $(\lambda a\ \tau.\ a)$
by(*rule inj-fun*, *simp*)

have *image-cong*: $\bigwedge x\ Fa\ f.\ inj\ f \Longrightarrow x \notin Fa \Longrightarrow f\ x \notin f\ `\ Fa$
apply(*simp add*: *image-def*)
apply(*rule ballI*)
apply(*case-tac* $x = xa$, *simp*)
apply(*simp add*: *inj-on-def*)

apply(*blast*)
done

have *rec* : $\bigwedge x$ (F :: $'\mathfrak{A}$ *Integer set*). *all-int-set* $F \Longrightarrow$
is-int $x \Longrightarrow$
$x \notin F \Longrightarrow$
$\forall x.$ ($\forall \tau.$ *all-defined* τ x) $\longrightarrow$
(*let img* = *op* ‘ (λa τ. a); *set'* = F; f = λx. x
in ($\forall \tau$. f ‘ *set'* = *img* $\lceil\lceil$*Rep-Set-0* (x τ)$\rceil\rceil$) $\longrightarrow$ *Finite-Set.fold* (λj *r2*. *r2*−>*including*(f j)) *Set*{} *set'* = x) $\Longrightarrow$
$\forall$ *xa*. ($\forall \tau$. *all-defined* τ *xa*) $\longrightarrow$
(*let img* = *op* ‘ (λa τ. a); *set'* = *insert* x F; f = λx. x
in ($\forall \tau$. f ‘ *set'* = *img* $\lceil\lceil$*Rep-Set-0* (*xa* τ)$\rceil\rceil$) $\longrightarrow$ *Finite-Set.fold* (λj *r2*. *r2*−>*including*(f j)) *Set*{} *set'* = *xa*)
apply(*simp only*: *Let-def image-ident*)

proof − **fix** τ **fix** x **fix** F :: $'\mathfrak{A}$ *Integer set*
show *all-int-set* $F \Longrightarrow$
is-int $x \Longrightarrow$
$x \notin F \Longrightarrow$
$\forall x$. ($\forall \tau$. *all-defined* τ x) $\longrightarrow$ ($\forall \tau$. F = (λa τ. a) ‘ $\lceil\lceil$*Rep-Set-0* (x τ)$\rceil\rceil$) $\longrightarrow$ *Finite-Set.fold* (λj *r2*. *r2*−>*including*(j)) *Set*{} F = $x \Longrightarrow$
$\forall$ *xa*. ($\forall \tau$. *all-defined* τ *xa*) $\longrightarrow$ ($\forall \tau$. *insert* x F = (λa τ. a) ‘ $\lceil\lceil$*Rep-Set-0* (*xa* τ)$\rceil\rceil$) $\longrightarrow$ *Finite-Set.fold* (λj *r2*. *r2*−>*including*(j)) *Set*{} (*insert* x F) = *xa*
apply(*rule allI*, *rename-tac S*) **apply**(*rule impI*)+
apply(*subst sym*[*of insert x F*], *blast*)
apply(*drule-tac* x = *S*−>*excluding*(x) **in** *allE*) **prefer** *2* **apply** *assumption*
apply(*subgoal-tac* $\bigwedge\tau$. (λa τ. a) ‘ $\lceil\lceil$*Rep-Set-0* (*S*−>*excluding*(x) τ)$\rceil\rceil$ = ((λa τ. a) ‘ $\lceil\lceil$*Rep-Set-0* (S τ)$\rceil\rceil$) − {x}, *simp only*:)
apply(*subgoal-tac* ($\forall \tau$. *all-defined* τ *S*−>*excluding*(x)))
prefer *2*
apply(*rule allI*)
apply(*rule cons-all-def-e*, *metis*)
apply(*rule int-is-valid*, *simp*)
apply(*simp*)
apply(*subst EQ-comp-fun-commute.fold-insert*[*OF including-commute*]) **prefer** *5*
apply(*drule arg-cong*[**where** f = λS. (*S*−>*including*(x))], *simp*)
apply(*rule Ocl-insert-Diff*)
apply(*metis surj-pair*)
apply(*subst sym*[*of insert x F*], *metis surj-pair*)
apply(*simp*)+
apply(*subst mtSet-all-def*)
apply(*simp*)+
apply(*subst excluding-unfold*)
apply(*metis surj-pair*)
apply(*rule int-is-valid*, *simp*)
apply(*subst image-set-diff*, *simp add*: *inject*)
apply(*simp*)
apply(*drule destruct-int*)

apply(*frule-tac P* = λj. *x* = (λ-. *j*) **in** *ex1E*) **prefer** *2* **apply** *assumption*
apply(*blast*)
done
qed

fix τ
show $\forall S'$. ($\forall \tau$. *all-defined* τ S') $\longrightarrow$ (*let img* = *image* (λa (τ:: '$\mathfrak{A}$ *st*). *a*); *set'* = *img* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$; *f* = (λx. *x*) *in*
($\forall \tau$. *f* ' *set'* = *img* $\lceil\lceil$*Rep-Set-0* (S' τ)$\rceil\rceil$) $\longrightarrow$
(*Finite-Set.fold* (λj *r2*. *r2*−>*including*(*f j*)) *Set*{} *set'*) = S')
apply(*rule allI*)
proof − **fix** S' :: ('$\mathfrak{A}$, -) *Set* **show** ($\forall \tau$. *all-defined* τ S') $\longrightarrow$ (*let img* = *op* ' (λa τ. *a*); *set'* = *img* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$; *f* = λx. *x*
in ($\forall \tau$. *f* ' *set'* = *img* $\lceil\lceil$*Rep-Set-0* (S' τ)$\rceil\rceil$) $\longrightarrow$ *Finite-Set.fold* (λj *r2*. *r2*−>*including*(*f j*)) *Set*{} *set'* = S')
apply(*simp add*: *Let-def*, *rule impI*)
apply(*subgoal-tac* (*let img* = *op* ' (λa τ. *a*); *set'* = (λa τ. *a*) ' $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$; *f* = λx. *x*
in ($\forall \tau$. *f* ' *set'* = *img* $\lceil\lceil$*Rep-Set-0* (S' τ)$\rceil\rceil$) $\longrightarrow$ *Finite-Set.fold* (λj *r2*. *r2*−>*including*(*f j*)) *Set*{} *set'* = S')) **prefer** *2*

apply(*subst EQ-comp-fun-commute.all-int-induct*[**where** *P* = λ*set*.
$\forall S'$. ($\forall \tau$. *all-defined* τ S') $\longrightarrow$ (*let img* = *image* (λa (τ:: '$\mathfrak{A}$ *st*). *a*)
; *set'* = *set* ; *f* = (λx. *x*) *in*
($\forall \tau$. *f* ' *set'* = *img* $\lceil\lceil$*Rep-Set-0* (S' τ)$\rceil\rceil$) $\longrightarrow$
(*Finite-Set.fold* (λj *r2*. *r2*−>*including*(*f j*)) *Set*{} *set'*) = S')
and *F* = (λa (τ:: '$\mathfrak{A}$ *st*). *a*) ' $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$, *OF including-commute*, *THEN spec*, *of* S'])
apply(*simp add*: *S-all-int*)
apply(*simp add*: *S-incl*)
apply(*rule rec*)
apply(*simp*) **apply**(*simp*) **apply**(*simp*) **apply**(*simp*)
apply (*metis pair-collapse*)
apply(*blast*)

apply(*simp add*: *Let-def*)

done
qed
qed

lemma *iterate-including-id00* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ (*S* :: ('$\mathfrak{A}$, *int option option*) *Set*)
and *S-incl* : $\bigwedge\tau$ τ'. *S* τ = *S* τ'
shows (*S*−>*iterate*(*j*;*r2*=*Set*{} | *r2*−>*including*(*j*))) = *S*
apply(*simp add*: *OclIterate*$_{Set}$*-def OclValid-def del*: *StrictRefEq-set-exec*, *rule ext*)
apply(*subgoal-tac* (δ *S*) τ = *true* τ $\wedge$ (υ *S*) τ = *true* τ $\wedge$ *finite* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$, *simp del*: *StrictRefEq-set-exec*)
prefer *2*

proof −
have *S-all-int* : $\bigwedge\tau$. *all-int-set* (($\lambda a\ \tau.\ a$) ‘ ⌈⌈*Rep-Set-0* ($S\ \tau$)⌉⌉)
by(*rule all-def-to-all-int*, *simp add*: *assms*)

fix τ
show ($\delta\ S$) τ = *true* τ ∧ ($\upsilon\ S$) τ = *true* τ ∧ *finite* ⌈⌈*Rep-Set-0* ($S\ \tau$)⌉⌉
apply(*simp add*: *S-all-def*[*of* τ, *simplified all-defined-def OclValid-def all-defined-set′-def*] *foundation20*[*simplified OclValid-def*])
done
fix τ **show** ($\delta\ S$) τ = *true* τ ∧ ($\upsilon\ S$) τ = *true* τ ∧ *finite* ⌈⌈*Rep-Set-0* ($S\ \tau$)⌉⌉ ⟹ *Finite-Set.fold* ($\lambda j\ r2.\ r2$−>*including*(j)) *Set*{} (($\lambda a\ \tau.\ a$) ‘ ⌈⌈*Rep-Set-0* ($S\ \tau$)⌉⌉) τ = $S\ \tau$
apply(*subst i-including-id00*[*simplified Let-def image-ident*, **where** $S = S$ **and** $\tau = \tau$])
prefer *4*
apply(*rule refl*)
apply(*simp add*: *S-all-int S-all-def*)+
by (*metis S-incl*)
qed

all defined (construction)

lemma *preserved-defined* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ (S :: ('$\mathfrak{A}$, *int option option*) *Set*)
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ A
shows *let* S' = ($\lambda a\ \tau.\ a$) ‘ ⌈⌈*Rep-Set-0* ($S\ \tau$)⌉⌉ *in*
∀τ. *all-defined* τ (*Finite-Set.fold* (λx *acc*. (*acc*−>*including*(x))) A S')
proof −
have *invert-all-int-set* : $\bigwedge x\ S$. *all-int-set* (*insert* x S) ⟹
all-int-set S
by(*simp add*: *all-int-set-def*)
show *?thesis*
apply(*subst Let-def*)
apply(*rule finite-induct*[**where** P = λ*set*.
let set′ = ($\lambda a\ \tau.\ a$) ‘ *set in*
all-int-set set′ ⟶
(∀τ'. *all-defined* τ' (*Finite-Set.fold* (λx *acc*. (*acc*−>*including*(x))) A *set′*))
and F = ⌈⌈*Rep-Set-0* ($S\ \tau$)⌉⌉, *simplified Let-def*, *THEN mp*])
apply(*simp add*: *S-all-def*[**where** $\tau = \tau$, *simplified all-defined-def all-defined-set′-def*])
apply(*simp add*: *A-all-def*)
apply(*rule impI*, *simp only*: *image-insert*, *rule iterate-subst-set-rec*[*simplified Let-def*, *THEN mp*, *THEN mp*, *THEN mp*])
apply(*simp add*: *A-all-def*)
apply(*simp add*: *including-commute*)
apply(*simp*)
apply(*simp*)
apply(*drule invert-all-int-set*, *simp*)

apply(*rule all-def-to-all-int*[*OF S-all-def*])
done

qed

Preservation of comp fun commute (main)

lemma *iterate-including-commute* :
assumes *f-comm* : *EQ-comp-fun-commute0* ($\lambda x.$ *F* (λ-. *x*))
and *f-empty* : $\bigwedge x\ y.$
is-int (λ(-:: $'\mathfrak{A}$ *st*). *x*) $\Longrightarrow$
is-int (λ(-:: $'\mathfrak{A}$ *st*). *y*) $\Longrightarrow$
OclIterate$_{Set}$ *Set*{λ(-:: $'\mathfrak{A}$ *st*). *x*} *Set*{λ(-:: $'\mathfrak{A}$ *st*). *x*} *F*−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *y*) =
OclIterate$_{Set}$ *Set*{λ(-:: $'\mathfrak{A}$ *st*). *y*} *Set*{λ(-:: $'\mathfrak{A}$ *st*). *y*} *F*−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *x*)
and *com* : $\bigwedge S\ x\ y\ \tau.$
is-int (λ(-:: $'\mathfrak{A}$ *st*). *x*) $\Longrightarrow$
is-int (λ(-:: $'\mathfrak{A}$ *st*). *y*) $\Longrightarrow$
$\forall$ (τ :: $'\mathfrak{A}$ *st*). *all-defined* τ *S* $\Longrightarrow$
$\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil \neq$ {} $\Longrightarrow$
(*OclIterate*$_{Set}$ ((*OclIterate*$_{Set}$ *S S F*)−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *x*)) ((*OclIterate*$_{Set}$ *S S F*)−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *x*)) *F*)−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *y*) τ =
(*OclIterate*$_{Set}$ ((*OclIterate*$_{Set}$ *S S F*)−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *y*)) ((*OclIterate*$_{Set}$ *S S F*)−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *y*)) *F*)−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *x*) τ
shows *EQ-comp-fun-commute0* (λ*x r1*. *r1* −>*iterate*(*j*;*r2*=*r1* | *F j r2*)−>*including*(λ(-:: $'\mathfrak{A}$ *st*). *x*))
proof −
have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow$ $\tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

show *?thesis*
apply(*simp only*: *EQ-comp-fun-commute0-def*)
apply(*rule conjI*)+ **apply**(*rule allI*)+ **apply**(*rule impI*)+
apply(*subst* (*1 2*) *cp-OclIncluding*, *subst cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0'-of-c0*]], *blast*, *simp*)
apply(*rule allI*)+ **apply**(*rule impI*)+
apply(*rule including-cp-all*, *simp*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*, *blast*)
apply(*rule iterate-cp-all*, *simp add*: *f-comm*, *simp*, *simp*)
apply(*rule conjI*)+ **apply**(*rule allI*)+ **apply**(*rule impI*)+
apply(*rule including-notempty*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*, *blast*, *simp add*: *int-is-valid*)
apply(*rule iterate-notempty*, *simp add*: *f-comm*, *simp*, *simp*)
apply(*rule conjI*)+ **apply**(*rule allI*)+
apply(*rule iffI*)
apply(*drule invert-all-defined'*, *erule conjE*, *rule conjI*, *simp*)
apply(*rule i-invert-all-defined'*[**where** *F* = *F*], *simp*)
apply(*rule allI*, *rule cons-all-def*, *rule i-cons-all-def*[*OF f-comm*], *blast*, *simp add*: *int-is-valid*)
apply(*rule allI*)+ **apply**(*rule impI*)+

apply(*rule ext*, *rename-tac* τ)
apply(*case-tac* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ = {})

apply(*subgoal-tac S* τ = *Set*{} τ)
prefer *2*
apply(*drule-tac f* = λs. *Abs-Set-0* $\lfloor\lfloor s\rfloor\rfloor$ **in** *arg-cong*)
apply(*subgoal-tac S* τ = *Abs-Set-0* $\lfloor\lfloor\{\}\rfloor\rfloor$)
prefer *2*
apply(*metis abs-rep-simp*)
apply(*simp add*: *mtSet-def*)

apply(*subst* (*1 2*) *cp-OclIncluding*)
apply(*subst* (*1 2*) *cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0'-of-c0*]])
apply(*rule cons-all-def'*) **apply**(*rule i-cons-all-def'*[**where** *F* = *F*, *OF f-comm*[*THEN c0'-of-c0*]], *blast*)+ **apply**(*simp add*: *int-is-valid*)
apply(*rule cons-all-def'*) **apply**(*rule i-cons-all-def'*[**where** *F* = *F*, *OF f-comm*[*THEN c0'-of-c0*]], *blast*)+ **apply**(*simp add*: *int-is-valid*)
apply(*subst* (*1 2 3 4 5 6*) *cp-OclIncluding*)
apply(*subst* (*1 2 4 5*) *cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0'-of-c0*]], *blast*)
apply(*simp*)
apply(*subst* (*1 2 4 5*) *cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0'-of-c0*], *symmetric*], *simp add*: *mtSet-all-def*)
apply(*simp*)
apply(*subst* (*1 2 4 5*) *cp-OclIncluding*[*symmetric*])
apply(*subst* (*1 2 3 4*) *cp-singleton*, *simp*, *simp*)
apply(*subst* (*1 2*) *cp-OclIncluding*[*symmetric*])
apply(*subst f-empty*, *simp-all*)

apply(*rule com*, *simp-all*)
done
qed

lemma *iterate-including-commute-var* :
assumes *f-comm* : *EQ-comp-fun-commute0* (λx. (*F* :: '$\mathfrak{A}$ *Integer*
⇒ ('$\mathfrak{A}$, -) *Set*
⇒ ('$\mathfrak{A}$, -) *Set*) (λ-. *x*))
and *f-empty* : $\bigwedge x\ y$.
is-int (λ(-:: '$\mathfrak{A}$ *st*). *x*) $\Longrightarrow$
is-int (λ(-:: '$\mathfrak{A}$ *st*). *y*) $\Longrightarrow$
OclIterate$_{Set}$ *Set*{λ(-:: '$\mathfrak{A}$ *st*). *x*, *a*} *Set*{λ(-:: '$\mathfrak{A}$ *st*). *x*, *a*} *F*−>*including*(λ(-:: '$\mathfrak{A}$ *st*). *y*) =
OclIterate$_{Set}$ *Set*{λ(-:: '$\mathfrak{A}$ *st*). *y*, *a*} *Set*{λ(-:: '$\mathfrak{A}$ *st*). *y*, *a*} *F*−>*including*(λ(-:: '$\mathfrak{A}$ *st*). *x*)
and *com* : $\bigwedge S\ x\ y\ \tau$.
is-int (λ(-:: '$\mathfrak{A}$ *st*). *x*) $\Longrightarrow$
is-int (λ(-:: '$\mathfrak{A}$ *st*). *y*) $\Longrightarrow$
$\forall$(τ :: '$\mathfrak{A}$ *st*). *all-defined* τ *S* $\Longrightarrow$
$\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ ≠ {} $\Longrightarrow$
(*OclIterate*$_{Set}$ (((*OclIterate*$_{Set}$ *S S F*)−>*including*(*a*))−>*including*(λ(-:: '$\mathfrak{A}$ *st*). *x*)) (((*OclIterate*$_{Set}$ *S S F*)−>*including*(*a*))−>*including*(λ(-:: '$\mathfrak{A}$ *st*). *x*)) *F*)−>*including*(λ(-:: '$\mathfrak{A}$ *st*). *y*) τ =
(*OclIterate*$_{Set}$ (((*OclIterate*$_{Set}$ *S S F*)−>*including*(*a*))−>*including*(λ(-:: '$\mathfrak{A}$ *st*).

y)) $(((OclIterate_{Set}\ S\ S\ F){-}{>}including(a)){-}{>}including(\lambda(\text{-}{::}\ '\mathfrak{A}\ st).\ y))\ F){-}{>}including(\lambda(\text{-}{::}\ '\mathfrak{A}\ st).\ x)\ \tau$
and *a-int* : *is-int a*
shows *EQ-comp-fun-commute0* $(\lambda x\, r1.\ (((r1\ {-}{>}iterate(j;r2{=}r1\mid F\,j\,r2)){-}{>}including(a)){-}{>}including(\lambda(\text{-}{::}\ '\mathfrak{A}\ st).\ x)))$
proof −
have *all-defined1* : $\bigwedge r2\ \tau.\ all\text{-}defined\ \tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)
show *?thesis*
apply(*simp only*: *EQ-comp-fun-commute0-def*)
apply(*rule conjI*)+ **apply**(*rule allI*)+ **apply**(*rule impI*)+
apply(*subst (1 2) cp-OclIncluding*, *subst (1 2 3 4) cp-OclIncluding*, *subst cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0'-of-c0*]], *blast*, *simp*)
apply(*rule allI*)+ **apply**(*rule impI*)+
apply(*rule including-cp-all*, *simp*, *rule all-defined1*, *rule cons-all-def*, *rule i-cons-all-def*, *simp add*: *f-comm*, *blast*, *simp add*: *a-int int-is-valid*)
apply(*rule including-cp-all*, *simp add*: *a-int*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*, *blast*, *simp add*: *a-int int-is-valid*)
apply(*rule iterate-cp-all*, *simp add*: *f-comm*, *simp*, *simp*)
apply(*rule conjI*)+ **apply**(*rule allI*)+ **apply**(*rule impI*)+
apply(*rule including-notempty*, *rule all-defined1*, *rule cons-all-def*, *rule i-cons-all-def*, *simp add*: *f-comm*, *blast*, *simp add*: *a-int int-is-valid*, *simp add*: *int-is-valid*)
apply(*rule including-notempty*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*, *blast*, *simp add*: *a-int int-is-valid*)
apply(*rule iterate-notempty*, *simp add*: *f-comm*, *simp*, *simp*)
apply(*rule conjI*)+ **apply**(*rule allI*)+
apply(*rule iffI*)
apply(*drule invert-all-defined'*, *erule conjE*, *rule conjI*, *simp*)
apply(*rule destruct-int*[*OF a-int*, *THEN ex1-implies-ex*, *THEN exE*], *rename-tac a'*, *simp only*:)
apply(*drule invert-all-defined'*, *erule conjE*)
apply(*rule i-invert-all-defined'*[**where** *F* = *F*], *simp*)
apply(*rule allI*, *rule cons-all-def*, *rule cons-all-def*, *rule i-cons-all-def*[*OF f-comm*], *blast*)
apply(*simp add*: *int-is-valid a-int*)+
apply((*rule allI*)+, (*rule impI*)+)+

apply(*rule ext*, *rename-tac* τ)
apply(*case-tac* $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil = \{\}$)
apply(*subgoal-tac* $S\ \tau = Set\{\}\ \tau$)
prefer *2*
apply(*drule-tac* $f = \lambda s.\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor s\rfloor\rfloor$ **in** *arg-cong*)
apply(*subgoal-tac* $S\ \tau = Abs\text{-}Set\text{-}0\ \lfloor\lfloor\{\}\rfloor\rfloor$)
prefer *2*
apply (*metis abs-rep-simp prod.exhaust*)
apply(*simp add*: *mtSet-def*)

apply(*subst (1 2) cp-OclIncluding*)
apply(*subst (1 2 3 4) cp-OclIncluding*)
apply(*subst (1 2) cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0'-of-c0*]])

apply(*rule cons-all-def′*)+ **apply**(*rule i-cons-all-def′*[**where** *F* = *F*, *OF f-comm*[*THEN c0′-of-c0*]], *metis surj-pair*) **apply**(*simp add*: *a-int int-is-valid*)+
apply(*rule cons-all-def′*)+ **apply**(*rule i-cons-all-def′*[**where** *F* = *F*, *OF f-comm*[*THEN c0′-of-c0*]], *metis surj-pair*) **apply**(*simp add*: *a-int int-is-valid*)+
apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8 9 10 11 12*) *cp-OclIncluding*)

apply(*subst* (*1 2 4 5*) *cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0′-of-c0*]], *metis surj-pair*)
apply(*simp*)
apply(*subst* (*1 2 4 5*) *cp-OclIterate*$_{Set}$*1*[*OF f-comm*[*THEN c0′-of-c0*], *symmetric*], *simp add*: *mtSet-all-def*)
apply(*simp*)
apply(*subst* (*1 2 3 4 7 8 9 10*) *cp-OclIncluding*[*symmetric*])
apply(*subst* (*1 2 3 4*) *cp-doubleton*, *simp*, *simp add*: *a-int*, *simp*)
apply(*subst* (*1 2 3 4*) *cp-OclIncluding*[*symmetric*])

apply(*subst* (*3 6*) *including-swap*)
apply(*rule allI*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule mtSet-all-def*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*)
apply(*rule allI*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule mtSet-all-def*) **apply**(*simp add*: *int-is-valid a-int*)+
apply(*rule including-subst-set′′*)
apply(*rule all-defined1*, *rule cons-all-def*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule mtSet-all-def*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*)
apply(*rule all-defined1*, *rule cons-all-def*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule mtSet-all-def*) **apply**(*simp add*: *int-is-valid a-int*)+

apply(*subst f-empty*, *simp-all*)

apply(*subst* (*3 6*) *including-swap*)
apply(*rule allI*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule i-cons-all-def*, *simp add*: *f-comm*, *metis surj-pair*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*)
apply(*rule allI*, *rule all-defined1*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule i-cons-all-def*, *simp add*: *f-comm*, *metis surj-pair*) **apply**(*simp add*: *int-is-valid a-int*)+
apply(*rule including-subst-set′′*)
apply(*rule all-defined1*, *rule cons-all-def*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule i-cons-all-def*, *simp add*: *f-comm*, *metis surj-pair*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*) **apply**(*simp add*: *int-is-valid a-int*)
apply(*rule all-defined1*, *rule cons-all-def*, *rule i-cons-all-def*, *simp add*: *f-comm*) **apply**(*rule cons-all-def*)+ **apply**(*rule i-cons-all-def*, *simp add*: *f-comm*, *metis surj-pair*) **apply**(*simp add*: *int-is-valid a-int*)+

apply(*rule com*, *simp-all*)
done
qed

Execution (OclIterate, OclIncluding to OclExcluding)

lemma $EQ\text{-}OclIterate_{Set}\text{-}including$:
assumes $S\text{-}all\text{-}int$: $\bigwedge(\tau{::}'\mathfrak{A}\ st).\ all\text{-}int\text{-}set\ ((\lambda\ a\ (\tau{::}\ '\mathfrak{A}\ st).\ a)\ `\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil)$
assumes $S\text{-}all\text{-}def$: $\bigwedge\tau.\ all\text{-}defined\ \tau\ S$
and $A\text{-}all\text{-}def$: $\bigwedge\tau.\ all\text{-}defined\ \tau\ A$
and $F\text{-}commute$: $EQ\text{-}comp\text{-}fun\text{-}commute\ F$
and $a\text{-}int$: $is\text{-}int\ a$
shows $((S{-}{>}including(a)){-}{>}iterate(a;\ x =\quad A \mid F\ a\ x)) =$
$((S{-}{>}excluding(a)){-}{>}iterate(a;\ x = F\ a\ A \mid F\ a\ x))$
proof −

have $all\text{-}defined1$: $\bigwedge r2\ \tau.\ all\text{-}defined\ \tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**($simp\ add$: $all\text{-}defined\text{-}def$)

have $F\text{-}cp$: $\bigwedge\ x\ y\ \tau.\ F\ x\ y\ \tau = F\ (\lambda\ \text{-}.\ x\ \tau)\ y\ \tau$
proof − **interpret** $EQ\text{-}comp\text{-}fun\text{-}commute\ F$ **by** ($rule\ F\text{-}commute$) **fix** $x\ y\ \tau$ **show** $F\ x\ y\ \tau = F\ (\lambda\ \text{-}.\ x\ \tau)\ y\ \tau$
by($rule\ F\text{-}cp$)
qed

have $F\text{-}val$: $\bigwedge\tau.\ \tau \models \upsilon\ (F\ a\ A)$
proof − **interpret** $EQ\text{-}comp\text{-}fun\text{-}commute\ F$ **by** ($rule\ F\text{-}commute$) **fix** τ **show** $\tau \models \upsilon\ (F\ a\ A)$
apply($insert$
$all\text{-}def$
$int\text{-}is\text{-}valid[OF\ a\text{-}int]$
$A\text{-}all\text{-}def$, $simp\ add$: $all\text{-}defined1\ foundation20$)
done
qed

have $insert\text{-}in\text{-}Set\text{-}0$: $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ a)) \Longrightarrow \lfloor\lfloor insert\ (a\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor \in \{X.\ X = bot \lor X = null \lor (\forall x{\in}\lceil\lceil X\rceil\rceil.\ x \neq bot)\}$
apply($frule\ Set\text{-}inv\text{-}lemma$)
apply($simp\ add$: $foundation18\ invalid\text{-}def$)
done
have $insert\text{-}in\text{-}Set\text{-}0$: $\bigwedge\tau.\ ?this\ \tau$
apply($rule\ insert\text{-}in\text{-}Set\text{-}0$)
by($simp\ add$: $S\text{-}all\text{-}def[simplified\ all\text{-}defined\text{-}def]\ int\text{-}is\text{-}valid[OF\ a\text{-}int]$)+

have $insert\text{-}defined$: $\bigwedge\tau.\ (\tau \models(\delta\ S)) \Longrightarrow (\tau \models(\upsilon\ a)) \Longrightarrow$
$(\delta\ (\lambda\text{-}.\ Abs\text{-}Set\text{-}0\ \lfloor\lfloor insert\ (a\ \tau)\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil\rfloor\rfloor))\ \tau = true\ \tau$
apply($subst\ defined\text{-}def$)
apply($simp\ add$: $bot\text{-}fun\text{-}def\ bot\text{-}option\text{-}def\ bot\text{-}Set\text{-}0\text{-}def\ null\text{-}Set\text{-}0\text{-}def\ null\text{-}option\text{-}def\ null\text{-}fun\text{-}def\ false\text{-}def\ true\text{-}def$)
apply($subst\ Abs\text{-}Set\text{-}0\text{-}inject$)
apply($rule\ insert\text{-}in\text{-}Set\text{-}0$, $simp\text{-}all\ add$: $bot\text{-}option\text{-}def$)

apply($subst\ Abs\text{-}Set\text{-}0\text{-}inject$)
apply($rule\ insert\text{-}in\text{-}Set\text{-}0$, $simp\text{-}all\ add$: $null\text{-}option\text{-}def\ bot\text{-}option\text{-}def$)
done

have *insert-defined* : $\bigwedge\tau.$ *?this* τ
apply(*rule insert-defined*)
by(*simp add*: *S-all-def*[*simplified all-defined-def*] *int-is-valid*[*OF a-int*])+

have *remove-finite* : $\bigwedge\tau.$ *finite* $\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil \Longrightarrow$ *finite* ((λa (τ:: $'\mathfrak{A}$ *st*). *a*) ‘ ($\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ $-$ {*a* τ}))
by(*simp*)

have *inject* : *inj* (λa τ. *a*)
by(*rule inj-fun*, *simp*)

have *remove-all-int* : $\bigwedge\tau.$ *all-int-set* ((λa (τ:: $'\mathfrak{A}$ *st*). *a*) ‘ ($\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ $-$ {*a* τ}))
proof $-$ **fix** τ **show** *?thesis* τ
apply(*insert S-all-int*[*of* τ], *simp add*: *all-int-set-def*, *rule remove-finite*)
apply(*erule conjE*, *drule finite-imageD*)
apply (*metis inj-onI*, *simp*)
done
qed

have *remove-in-Set-0* : $\bigwedge\tau.$ (τ $\models$(δ *S*)) $\Longrightarrow$ (τ $\models$(υ *a*)) $\Longrightarrow$ $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ $-$ {*a* τ}$\rfloor\rfloor$ $\in$ {*X*. *X* = *bot* $\vee$ *X* = *null* $\vee$ ($\forall x\in\lceil\lceil X\rceil\rceil$. $x \neq$ *bot*)}
apply(*frule Set-inv-lemma*)
apply(*simp add*: *foundation18 invalid-def*)
done
have *remove-in-Set-0* : $\bigwedge\tau.$ *?this* τ
apply(*rule remove-in-Set-0*)
by(*simp add*: *S-all-def*[*simplified all-defined-def*] *int-is-valid*[*OF a-int*])+

have *remove-defined* : $\bigwedge\tau.$ (τ $\models$(δ *S*)) $\Longrightarrow$ (τ $\models$(υ *a*)) $\Longrightarrow$
(δ (λ-. *Abs-Set-0* $\lfloor\lfloor\lceil\lceil$*Rep-Set-0* (*S* τ)$\rceil\rceil$ $-$ {*a* τ}$\rfloor\rfloor$)) τ = *true* τ
apply(*subst defined-def*)
apply(*simp add*: *bot-fun-def bot-option-def bot-Set-0-def null-Set-0-def null-option-def null-fun-def false-def true-def*)
apply(*subst Abs-Set-0-inject*)
apply(*rule remove-in-Set-0*, *simp-all add*: *bot-option-def*)

apply(*subst Abs-Set-0-inject*)
apply(*rule remove-in-Set-0*, *simp-all add*: *null-option-def bot-option-def*)
done
have *remove-defined* : $\bigwedge\tau.$ *?this* τ
apply(*rule remove-defined*)
by(*simp add*: *S-all-def*[*simplified all-defined-def*] *int-is-valid*[*OF a-int*])+

show *?thesis*
apply(*rule ext*, *rename-tac* τ)
proof $-$ **fix** τ **show** *OclIterate*$_{Set}$ *S*−>*including*(*a*) *A F* τ = *OclIterate*$_{Set}$ *S*−>*excluding*(*a*) (*F a A*) *F* τ
apply(*simp only*: *cp-OclIterate*$_{Set}$[*of S*−>*including*(*a*)] *cp-OclIterate*$_{Set}$[*of S*−>*excluding*(*a*)])
apply(*subst OclIncluding-def*, *subst OclExcluding-def*)

apply(*simp add*: *S-all-def*[*simplified all-defined-def OclValid-def*] *int-is-valid*[*OF a-int*, *simplified OclValid-def*])

apply(*simp add*: $OclIterate_{Set}$*-def*)
apply(*simp add*: *Abs-Set-0-inverse*[*OF insert-in-Set-0*] *Abs-Set-0-inverse*[*OF remove-in-Set-0*]
foundation20[*OF all-defined1*[*OF A-all-def*], *simplified OclValid-def*]
S-all-def[*simplified all-defined-def all-defined-set-def*]
insert-defined
remove-defined
F-val[*of* τ, *simplified OclValid-def*])

apply(*subst EQ-comp-fun-commute.fold-fun-comm*[**where** $f = F$ **and** $z = A$ **and** $x = a$ **and** $A = ((\lambda a\ \tau.\ a)$ ` $(\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil - \{a\ \tau\}))$, *symmetric*, *OF F-commute A-all-def* - *int-is-valid*[*OF a-int*]])
apply(*simp add*: *remove-all-int*)

apply(*subst image-set-diff*[*OF inject*], *simp*)
apply(*subgoal-tac Finite-Set.fold F A* (*insert* $(\lambda\tau'.\ a\ \tau)$ $((\lambda a\ \tau.\ a)$ ` $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil))\ \tau$ =
F $(\lambda\tau'.\ a\ \tau)$ (*Finite-Set.fold F A* $((\lambda a\ \tau.\ a)$ ` $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil - \{\lambda\tau'.\ a\ \tau\}))\ \tau)$
apply(*subst F-cp*)
apply(*simp*)

apply(*subst EQ-comp-fun-commute.fold-insert-remove*[*OF F-commute A-all-def S-all-int*])
apply (*metis* (*mono-tags*) *a-int foundation18′ is-int-def*)
apply(*simp*)
done
qed
qed

Execution OclIncluding out of OclIterate (theorem)

lemma *including-out1* :
assumes *S-all-def* : $\bigwedge\tau$. *all-defined* τ $(S :: ('\mathfrak{A}, \textit{int option option})\ Set)$
and *A-all-def* : $\bigwedge\tau$. *all-defined* τ A
and *i-int* : *is-int* i
shows $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow$
$((S :: ('\mathfrak{A}, \text{-})\ Set)$−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x*)−>*including*(*i*))) τ = (*S*−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x*))−>*including*(*i*)) τ
proof −

have *i-valid* : $\forall\tau.\ \tau \models \upsilon\ i$
by (*metis i-int int-is-valid*)

have *all-defined1* : $\bigwedge r2\ \tau$. *all-defined* $\tau\ r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)

have *S-finite* : $\bigwedge\tau$. *finite* $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil$
by(*simp add*: *S-all-def*[*simplified all-defined-def all-defined-set′-def*])

have *all-def-to-all-int-* : $\bigwedge$*set* τ. *all-defined-set* τ *set* $\Longrightarrow$ *all-int-set* ((λ*a* τ. *a*) ' *set*)
apply(*simp add*: *all-defined-set-def all-int-set-def is-int-def*)
by (*metis foundation18'*)

have *invert-all-def-set* : $\bigwedge$*x F* τ. *all-defined-set* τ (*insert x F*) $\Longrightarrow$ *all-defined-set* τ *F*
apply(*simp add*: *all-defined-set-def*)
done

have *invert-int* : $\bigwedge$*x S*. *all-int-set* (*insert x S*) $\Longrightarrow$
is-int x
by(*simp add*: *all-int-set-def*)

have *inject* : *inj* (λ*a* τ. *a*)
by(*rule inj-fun*, *simp*)

have *image-cong*: $\bigwedge$*x Fa f*. *inj f* $\Longrightarrow$ *x* $\notin$ *Fa* $\Longrightarrow$ *f x* $\notin$ *f* ' *Fa*
apply(*simp add*: *image-def*)
apply(*rule ballI*)
apply(*case-tac x* = *xa*, *simp*)
apply(*simp add*: *inj-on-def*)
apply(*blast*)
done

have *discr-eq-false-true* : $\bigwedge\tau$. (*false* τ = *true* τ) = *False* **by** (*metis OclValid-def foundation2*)

have *invert-all-defined-fold* : $\bigwedge$ *F x a b*. *let F'* = (λ*a* τ. *a*) ' *F in x* $\notin$ *F* $\longrightarrow$ *all-int-set* (*insert* ($\lambda\tau$. *x*) *F'*) $\longrightarrow$ *all-defined* (*a*, *b*) (*Finite-Set.fold* (λ*x acc*. *acc*−>*including*(*x*)) *A* (*insert* ($\lambda\tau$. *x*) *F'*)) $\longrightarrow$
all-defined (*a*, *b*) (*Finite-Set.fold* (λ*x acc*. *acc*−>*including*(*x*)) *A F'*)
proof − **fix** *F x a b* **show** *?thesis F x a b*
apply(*simp add*: *Let-def*) **apply**(*rule impI*)+
apply(*insert arg-cong*[**where** *f* = λ*x*. *all-defined* (*a*, *b*) *x*, *OF EQ-comp-fun-commute.fold-insert*[*OF including-commute*, **where** *x*= ($\lambda\tau$. *x*) **and** *A* = (λ*a* τ. *a*) ' *F* **and** *z* = *A*]]
allI[**where** *P* = λ*x*. *all-defined x A*, *OF A-all-def*])
apply(*simp*)
apply(*subgoal-tac all-int-set* ((λ*a* τ. *a*) ' *F*))
prefer *2*
apply(*simp add*: *all-int-set-def*, *auto*)
apply(*drule invert-int*, *simp*)
apply(*subgoal-tac* (λ(τ:: '$\mathfrak{A}$ *st*). *x*) $\notin$ (λ*a* (τ:: '$\mathfrak{A}$ *st*). *a*) ' *F*)
prefer *2*
apply(*rule image-cong*)
apply(*rule inject*)
apply(*simp*)

apply(*simp*)
apply(*rule invert-all-defined*[*THEN conjunct2*, *of* - - $\lambda\tau.\ x$], *simp*)
done
qed

have *i-out* : $\bigwedge i'\ x\ F.\ i = (\lambda\text{-}.\ i') \Longrightarrow$ *is-int* $(\lambda(\tau{::}\ '\mathfrak{A}\ st).\ x) \Longrightarrow \forall\, a\ b.$ *all-defined* $(a,\ b)$ (*Finite-Set.fold* (λx *acc. acc*−>*including*(x)) A (($\lambda a\ \tau.\ a$) ‘ F)) $\Longrightarrow$
(((*Finite-Set.fold* (λx *acc.* (*acc*−>*including*(x))) A
(($\lambda a\ \tau.\ a$) ‘ F))−>*including*($\lambda\tau.\ x$))−>*including*(i))−>*including*(i) =
(((*Finite-Set.fold* (λj *r2.* (*r2*−>*including*(j))) A (($\lambda a\ \tau.\ a$) ‘ F))−>*including*($\lambda\tau.\ x$))−>*including*(i))
proof − **fix** $i'\ x\ F$ **show** $i = (\lambda\text{-}.\ i') \Longrightarrow$ *is-int* $(\lambda(\tau{::}\ '\mathfrak{A}\ st).\ x) \Longrightarrow \forall\, a\ b.$ *all-defined* $(a,\ b)$ (*Finite-Set.fold* (λx *acc. acc*−>*including*(x)) A (($\lambda a\ \tau.\ a$) ‘ F)) $\Longrightarrow$ *?thesis* $i'\ x\ F$
apply(*simp*)
apply(*subst including-id*[**where** S = ((*Finite-Set.fold* (λj *r2.* (*r2*−>*including*(j))) A (($\lambda a\ \tau.$ a) ‘ F))−>*including*($\lambda\tau.\ x$))−>*including*($\lambda\text{-}.\ i'$)])
apply(*rule cons-all-def*)+
apply(*case-tac* τ'', *simp*)
apply (*metis* (*no-types*) *foundation18′ is-int-def*)
apply(*insert i-int*, *simp add*: *is-int-def*)
apply (*metis* (*no-types*) *foundation18′*)
apply(*rule allI*)
proof − **fix** τ **show** *is-int* $i \Longrightarrow i = (\lambda\text{-}.\ i') \Longrightarrow$ *is-int* $(\lambda(\tau{::}\ '\mathfrak{A}\ st).\ x) \Longrightarrow \forall\, a\ b.$ *all-defined* $(a,\ b)$ (*Finite-Set.fold* (λx *acc. acc*−>*including*(x)) A (($\lambda a\ \tau.\ a$) ‘ F)) $\Longrightarrow$
$i' \in \lceil\lceil$*Rep-Set-0* ((*Finite-Set.fold* (λj *r2.* (*r2*−>*including*(j))) A (($\lambda a\ \tau.\ a$) ‘ F))−>*including*($\lambda\tau.\ x$)−>*including*($\lambda\text{-}.\ i'$) τ)$\rceil\rceil$
apply(*insert including-charn1*[**where** X = (*Finite-Set.fold* (λj *r2.* (*r2*−>*including*(j))) A (($\lambda a\ \tau.\ a$) ‘ F))−>*including*($\lambda\tau.\ x$) **and** $x = \lambda\text{-}.\ i'$ **and** $\tau = \tau$])
apply(*subgoal-tac* $\tau \models \delta$ *Finite-Set.fold* (λj *r2. r2*−>*including*(j)) A (($\lambda a\ \tau.\ a$) ‘ F)−>*including*($\lambda\tau.$ x))
prefer *2*
apply(*rule all-defined1*, *rule cons-all-def*, *metis surj-pair*)
apply(*simp add*: *int-is-valid*)
apply(*subgoal-tac* $\tau \models \upsilon\ (\lambda\text{-}.\ i')$)
prefer *2*
apply(*drule int-is-valid*[**where** $\tau = \tau$], *simp add*: *foundation20*)
apply(*simp*)

apply(*simp add*: *OclIncludes-def OclValid-def*)
apply(*subgoal-tac* (δ *Finite-Set.fold* (λj *r2. r2*−>*including*(j)) A (($\lambda a\ \tau.\ a$) ‘ F) *and* υ ($\lambda\tau.$ x) *and* υ ($\lambda\text{-}.\ i'$)) τ = *true* τ)
apply (*metis option.inject true-def*)
by (*metis OclValid-def foundation10 foundation6*)
qed *simp-all*
qed

have *i-out1* : $\lceil\lceil$*Rep-Set-0* ($S\ \tau$)$\rceil\rceil \neq \{\} \Longrightarrow$
Finite-Set.fold (λx *acc.* (*acc*−>*including*(x))−>*including*(i)) A (($\lambda a\ \tau.\ a$) ‘ $\lceil\lceil$*Rep-Set-0* ($S\ \tau$)$\rceil\rceil$) =

$(Finite\text{-}Set.fold\ (\lambda x\ acc.\ acc{-}{>}including(x))\ A\ ((\lambda a\ \tau.\ a)\ `\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil)){-}{>}including(i)$

proof − **fix** τ **show** $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow$
$Finite\text{-}Set.fold\ (\lambda x\ acc.\ (acc{-}{>}including(x)){-}{>}including(i))\ A\ ((\lambda a\ \tau.\ a)\ `\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil) =$
$(Finite\text{-}Set.fold\ (\lambda x\ acc.\ acc{-}{>}including(x))\ A\ ((\lambda a\ \tau.\ a)\ `\ \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil)){-}{>}including(i)$

apply(*subst finite-induct*[**where** $P = \lambda set.\ let\ set' = (\lambda a\ \tau.\ a)\ `\ set$
$;\ fold\text{-}set = Finite\text{-}Set.fold\ (\lambda x\ acc.\ (acc{-}{>}including(x)))\ A\ set'\ in$
$(\forall \tau.\ all\text{-}defined\ \tau\ fold\text{-}set) \wedge$
$set' \neq \{\} \longrightarrow$
$all\text{-}int\text{-}set\ set' \longrightarrow$
$(Finite\text{-}Set.fold\ (\lambda x\ acc.\ (acc{-}{>}including(x)){-}{>}including(i))\ A\ set') =$
$(fold\text{-}set{-}{>}including(i))$
and $F = \lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil$, *simplified Let-def*])

apply(*simp add*: *S-finite*)
apply(*simp*)
defer

apply(*subst preserved-defined*[**where** $\tau = \tau$, *simplified Let-def*])
apply(*simp add*: *S-all-def*)
apply(*simp add*: *A-all-def*)
apply(*simp*)

apply(*rule all-def-to-all-int*, *simp add*: *S-all-def*)
apply(*simp add*: *cp-OclIncluding*[*of - i*])

apply(*rule impI*)+ **apply**(*erule conjE*)+
apply(*simp*)
apply(*subst EQ-comp-fun-commute.fold-insert*[*OF including-commute*])
apply(*simp add*: *A-all-def*)
apply(*simp add*: *all-int-set-def*)
apply(*simp add*: *invert-int*)

apply(*rule image-cong*)
apply(*rule inject*)
apply(*simp*)

apply(*subst EQ-comp-fun-commute.fold-insert*[*OF including-commute2*])
apply(*simp add*: *i-int*)
apply(*simp add*: *A-all-def*)
apply(*simp add*: *all-int-set-def*)
apply(*simp add*: *invert-int*)

apply(*rule image-cong*)
apply(*rule inject*)
apply(*simp*)

apply(*subgoal-tac* (∀ *a b*. *all-defined* (*a*, *b*) (*Finite-Set.fold* (λ*x acc*. *acc*−>*including*(*x*)) *A* ((λ*a τ*. *a*) ‘ *F*))))
prefer *2*
apply(*rule allI*) **apply**(*erule-tac x* = *a* **in** *allE*)
apply(*rule allI*) **apply**(*erule-tac x* = *b* **in** *allE*)
apply(*simp add*: *invert-all-defined-fold*[*simplified Let-def* , *THEN mp*, *THEN mp*, *THEN mp*])

apply(*simp*)

apply(*case-tac F* = {}, *simp*)
apply(*simp add*: *all-int-set-def*)
apply(*subst including-swap*)
apply(*rule allI*, *rule all-defined1*) **apply** (*metis PairE*)
apply(*rule allI*)
apply(*simp add*: *i-valid foundation20*)
apply(*simp add*: *is-int-def*)
apply(*insert destruct-int*[*OF i-int*])
apply(*frule ex1E*) **prefer** *2* **apply** *assumption*
apply(*rename-tac i*′)

proof −
fix *x F i*′
show *i* = (λ-. *i*′) =⇒
is-int (λ(τ:: ′𝔄 *st*). *x*) =⇒
∀ *a b*. *all-defined* (*a*, *b*) (*Finite-Set.fold* (λ*x acc*. *acc*−>*including*(*x*)) *A* ((λ*a τ*. *a*) ‘ *F*))
=⇒
(((*Finite-Set.fold* (λ*x acc*. (*acc*−>*including*(*x*))) *A* ((λ*a τ*. *a*) ‘ *F*))−>*including*(λτ. *x*))−>*including*(*i*))−>*including*(*i*)
=
((*Finite-Set.fold* (λ*j r2*. (*r2*−>*including*(*j*))) *A* ((λ*a τ*. *a*) ‘ *F*))−>*including*(λτ. *x*))−>*including*(*i*)
apply(*rule i-out*[**where** *i*′ = *i*′ **and** *x* = *x* **and** *F* = *F*], *simp-all*)
done
apply-end *assumption*
apply-end(*blast*)+
qed
qed *simp*

show ⌈⌈*Rep-Set-0* (*S τ*)⌉⌉ ≠ {} =⇒ *?thesis*
apply(*simp add*: *OclIterate*$_{Set}$*-def*)
apply(*simp add*: *S-all-def* [*simplified all-defined-def all-defined-set*′*-def*] *all-defined1* [*OF S-all-def* , *simplified OclValid-def*] *all-defined1* [*OF A-all-def* , *THEN foundation20*, *simplified OclValid-def*])
apply(*drule i-out1*)
apply(*simp add*: *cp-OclIncluding*[*of* - *i*])
done
qed

lemma *including-out2* :
assumes *S-all-def* : ⋀τ. *all-defined τ* (*S* :: (′𝔄, *int option option*) *Set*)

and *A-all-def* : $\bigwedge\tau$. *all-defined* τ *A*
and *i-int* : *is-int i*
and *x0-int* : *is-int x0*
shows $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow$ (*S*−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x0*)−>*including*(*x*)−>*inclu*
τ = (*S*−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x0*)−>*including*(*x*))−>*including*(*i*)) τ
proof −
have *x0-val* : $\bigwedge\tau$. $\tau \models \upsilon$ *x0* **apply**(*insert x0-int*[*simplified is-int-def*]) **by** (*metis foundation18′*)
have *i-val* : $\bigwedge\tau$. $\tau \models \upsilon$ *i* **apply**(*insert i-int*[*simplified is-int-def*]) **by** (*metis foundation18′*)

have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

have *init-out1* : (*S*−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x0*)−>*including*(*x*)−>*including*(*i*)))
= (*S*−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x*)−>*including*(*x0*)−>*including*(*i*)))
apply(*rule iterate-subst-set*[*OF S-all-def A-all-def including-commute4 including-commute5*])
apply(*simp add*: *x0-int i-int*)+
apply(*rule including-subst-set*)
apply(*rule including-swap*)
apply(*simp add*: *all-defined-def x0-val*)+
done

have *init-out2* : $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow$ (*S*−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x0*)−>*including*(*x*))−
τ = (*S*−>*iterate*(*x*;*acc*=*A* | *acc*−>*including*(*x*))−>*including*(*x0*)−>*including*(*i*)) τ
apply(*rule including-subst-set′′*) **prefer** *4*
apply(*simp add*: *including-out1*[*OF S-all-def A-all-def x0-int*, *symmetric*])
apply(*subst iterate-subst-set*[*OF S-all-def A-all-def including-commute3 including-commute2*])
apply(*simp add*: *x0-int*)+ **apply**(*rule x0-int*)
apply(*rule including-swap*)
apply(*simp add*: *all-defined-def x0-val*)+

apply(*rule all-defined1*)
apply(*rule i-cons-all-def′′*) **apply**(*rule including-commute3*[*THEN c0-of-c*, *THEN c0′-of-c0*],
simp add: *x0-int*, *simp add*: *S-all-def*, *simp add*: *A-all-def*)
apply(*rule all-defined1*)
apply(*rule cons-all-def*)
apply(*rule i-cons-all-def′′*) **apply**(*rule including-commute*[*THEN c0-of-c*, *THEN c0′-of-c0*],
simp add: *x0-int*, *simp add*: *S-all-def*, *simp add*: *A-all-def*, *simp add*: *int-is-valid*[*OF x0-int*])
apply(*simp add*: *int-is-valid*[*OF i-int*])
done

have *i-valid* : $\forall\tau$. $\tau \models \upsilon$ *i*
by (*metis i-int int-is-valid*)

have *S-finite* : $\bigwedge\tau$. *finite* $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil$
by(*simp add*: *S-all-def*[*simplified all-defined-def all-defined-set′-def*])

have *all-def-to-all-int-* : $\bigwedge$*set* τ. *all-defined-set* τ *set* $\Longrightarrow$ *all-int-set* (($\lambda a\ \tau.\ a$) ‘ *set*)
apply(*simp add*: *all-defined-set-def all-int-set-def is-int-def*)
by (*metis foundation18′*)

have *invert-all-def-set* : $\bigwedge x\ F\ \tau$. *all-defined-set* τ (*insert* x F) $\Longrightarrow$ *all-defined-set* τ F
apply(*simp add*: *all-defined-set-def*)
done

have *invert-int* : $\bigwedge x\ S$. *all-int-set* (*insert* x S) $\Longrightarrow$
is-int x
by(*simp add*: *all-int-set-def*)

have *inject* : *inj* ($\lambda a\ \tau.\ a$)
by(*rule inj-fun*, *simp*)

have *image-cong*: $\bigwedge x\ Fa\ f$. *inj* $f \Longrightarrow x \notin Fa \Longrightarrow f\ x \notin f$ ‘ Fa
apply(*simp add*: *image-def*)
apply(*rule ballI*)
apply(*case-tac* $x = xa$, *simp*)
apply(*simp add*: *inj-on-def*)
apply(*blast*)
done

have *discr-eq-false-true* : $\bigwedge\tau$. (*false* τ = *true* τ) = *False* **by** (*metis OclValid-def foundation2*)

have *invert-all-defined-fold* : $\bigwedge$ $F\ x\ a\ b$. *let* F' = ($\lambda a\ \tau.\ a$) ‘ F *in* $x \notin F \longrightarrow$ *all-int-set* (*insert* ($\lambda\tau.\ x$) F') $\longrightarrow$ *all-defined* (a, b) (*Finite-Set.fold* (λx *acc*. *acc*−>*including*(x)) A (*insert* ($\lambda\tau$. x) F')) $\longrightarrow$
all-defined (a, b) (*Finite-Set.fold* (λx *acc*. *acc*−>*including*(x)) A F')
proof − **fix** $F\ x\ a\ b$ **show** *?thesis* $F\ x\ a\ b$
apply(*simp add*: *Let-def*) **apply**(*rule impI*)+
apply(*insert arg-cong*[**where** $f = \lambda x$. *all-defined* (a, b) x, *OF EQ-comp-fun-commute.fold-insert*[*OF including-commute*, **where** x= ($\lambda\tau$. x) **and** A = ($\lambda a\ \tau.\ a$) ‘ F **and** z = A]]
allI[**where** $P = \lambda x$. *all-defined* x A, *OF A-all-def*])
apply(*simp*)
apply(*subgoal-tac all-int-set* (($\lambda a\ \tau.\ a$) ‘ F))
prefer *2*
apply(*simp add*: *all-int-set-def*, *auto*)
apply(*drule invert-int*, *simp*)
apply(*subgoal-tac* ($\lambda(\tau$:: $'\mathfrak{A}$ *st*). x) $\notin$ (λa (τ:: $'\mathfrak{A}$ *st*). a) ‘ F)
prefer *2*
apply(*rule image-cong*)
apply(*rule inject*)
apply(*simp*)

apply(*simp*)
apply(*rule invert-all-defined*[*THEN conjunct2*, *of - -* $\lambda\tau$. x], *simp*)
done
qed

have *i-out* : $\bigwedge i\ i'\ x\ F.$ *is-int* $i \Longrightarrow i = (\lambda\text{-}.\ i') \Longrightarrow$ *is-int* $(\lambda(\tau{::}\ '\mathfrak{A}\ st).\ x) \Longrightarrow \forall\, a\ b.$ *all-defined* $(a,\ b)$ (*Finite-Set.fold* (λx *acc. acc*$-{>}$*including*(x)) A $((\lambda a\ \tau.\ a)$ ‘ $F)) \Longrightarrow$
(((*Finite-Set.fold* (λx *acc.* (*acc*$-{>}$*including*(x)))) A
$((\lambda a\ \tau.\ a)$ ‘ $F))-{>}$*including*$(\lambda\tau.\ x))-{>}$*including*$(i))-{>}$*including*(i) =
(((*Finite-Set.fold* (λj *r2*. (*r2*$-{>}$*including*(j)))) A $((\lambda a\ \tau.\ a)$ ‘ $F))-{>}$*including*$(\lambda\tau.$ $x))-{>}$*including*(i))
proof − **fix** $i\ i'\ x\ F$ **show** *is-int* $i \Longrightarrow i = (\lambda\text{-}.\ i') \Longrightarrow$ *is-int* $(\lambda(\tau{::}\ '\mathfrak{A}\ st).\ x) \Longrightarrow \forall\, a\ b.$ *all-defined* $(a,\ b)$ (*Finite-Set.fold* (λx *acc. acc*$-{>}$*including*(x)) A $((\lambda a\ \tau.\ a)$ ‘ $F)) \Longrightarrow$ *?thesis* i $i'\ x\ F$
apply(*simp*)
apply(*subst including-id*[**where** S = ((*Finite-Set.fold* (λj *r2*. (*r2*$-{>}$*including*(j)))) A $((\lambda a\ \tau.$ $a)$ ‘ $F))-{>}$*including*$(\lambda\tau.\ x))-{>}$*including*$(\lambda\text{-}.\ i')$])
apply(*rule cons-all-def*)+
apply(*case-tac* τ'', *simp*)
apply (*metis* (*no-types*) *foundation18′ is-int-def*)
apply(*simp add*: *is-int-def*)
apply (*metis* (*no-types*) *foundation18′*)
apply(*rule allI*)
proof − **fix** τ **show** *is-int* $i \Longrightarrow i = (\lambda\text{-}.\ i') \Longrightarrow$ *is-int* $(\lambda(\tau{::}\ '\mathfrak{A}\ st).\ x) \Longrightarrow \forall\, a\ b.$ *all-defined* $(a,\ b)$ (*Finite-Set.fold* (λx *acc. acc*$-{>}$*including*(x)) A $((\lambda a\ \tau.\ a)$ ‘ $F)) \Longrightarrow$
$i' \in \lceil\lceil$*Rep-Set-0* ((*Finite-Set.fold* (λj *r2*. (*r2*$-{>}$*including*(j)))) A $((\lambda a\ \tau.\ a)$ ‘ $F))-{>}$*including*$(\lambda\tau.\ x)-{>}$*including*$(\lambda\text{-}.\ i')\ \tau)\rceil\rceil$
apply(*insert including-charn1*[**where** X = (*Finite-Set.fold* (λj *r2*. (*r2*$-{>}$*including*(j)))) A $((\lambda a\ \tau.\ a)$ ‘ $F))-{>}$*including*$(\lambda\tau.\ x)$ **and** $x = \lambda\text{-}.\ i'$ **and** $\tau = \tau$])
apply(*subgoal-tac* $\tau \models \delta$ *Finite-Set.fold* (λj *r2*. *r2*$-{>}$*including*(j)) A $((\lambda a\ \tau.\ a)$ ‘ $F)-{>}$*including*$(\lambda\tau.$ $x)$)
prefer *2*
apply(*rule all-defined1*, *rule cons-all-def*, *metis surj-pair*)
apply(*simp add*: *int-is-valid*)
apply(*subgoal-tac* $\tau \models \upsilon\ (\lambda\text{-}.\ i')$)
prefer *2*
apply(*drule int-is-valid*[**where** $\tau = \tau$], *simp add*: *foundation20*)
apply(*simp*)

apply(*simp add*: *OclIncludes-def OclValid-def*)
apply(*subgoal-tac* (δ *Finite-Set.fold* (λj *r2*. *r2*$-{>}$*including*(j)) A $((\lambda a\ \tau.\ a)$ ‘ $F)$ *and* $\upsilon\ (\lambda\tau.$ $x)$ *and* $\upsilon\ (\lambda\text{-}.\ i'))\ \tau = true\ \tau$)
apply (*metis option.inject true-def*)
by (*metis OclValid-def foundation10 foundation6*)
qed *simp-all*
qed

have *destruct3* : $\bigwedge A\ B\ C\ \tau.\ (\tau \models A) \wedge (\tau \models B) \wedge (\tau \models C) \Longrightarrow (\tau \models (A$ *and* B *and* $C))$
by (*metis foundation10 foundation6*)

have *i-out1* : $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow$
Finite-Set.fold (λx *acc.* (*acc*$-{>}$*including*(x))$-{>}$*including*$(x0)-{>}$*including*(i)) A $((\lambda a\ \tau.$ $a)$ ‘ $\lceil\lceil$*Rep-Set-0* $(S\ \tau)\rceil\rceil)$ =

(*Finite-Set.fold* (λ*x acc. acc*−>*including*(*x*)) *A* ((λ*a* τ. *a*) ‘ ⌈⌈*Rep-Set-0* (*S* τ)⌉⌉))−>*including*(*x0*)−>*including*(*i*)
proof − **fix** τ **show** ⌈⌈*Rep-Set-0* (*S* τ)⌉⌉ $\neq$ {} $\Longrightarrow$
Finite-Set.fold (λ*x acc.* (*acc*−>*including*(*x*))−>*including*(*x0*)−>*including*(*i*)) *A* ((λ*a* τ.
a) ‘ ⌈⌈*Rep-Set-0* (*S* τ)⌉⌉) =
(*Finite-Set.fold* (λ*x acc. acc*−>*including*(*x*)) *A* ((λ*a* τ. *a*) ‘ ⌈⌈*Rep-Set-0* (*S* τ)⌉⌉))−>*including*(*x0*)−>*including*(*i*)
apply(*subst finite-induct*[**where** *P* = λ*set. let set′* = (λ*a* τ. *a*) ‘ *set*
; *fold-set* = *Finite-Set.fold* (λ*x acc.* (*acc*−>*including*(*x*)))
A set′ in
($\forall$ τ. *all-defined* τ *fold-set*) $\wedge$
set′ $\neq$ {} $\longrightarrow$
all-int-set set′ $\longrightarrow$
(*Finite-Set.fold* (λ*x acc.* (*acc*−>*including*(*x*))−>*including*(*x0*)−>*including*(*i*))
A set′) =
(*fold-set*−>*including*(*x0*)−>*including*(*i*))
and *F* = ⌈⌈*Rep-Set-0* (*S* τ)⌉⌉, *simplified Let-def*])
apply(*simp add*: *S-finite*)
apply(*simp*)
defer

apply(*subst preserved-defined*[**where** τ = τ, *simplified Let-def*])
apply(*simp add*: *S-all-def*)
apply(*simp add*: *A-all-def*)
apply(*simp*)

apply(*rule all-def-to-all-int*, *simp add*: *S-all-def*)
apply(*simp add*: *cp-OclIncluding*[*of* - *i*])

apply(*rule impI*)+ **apply**(*erule conjE*)+
apply(*simp*)
apply(*subst EQ-comp-fun-commute.fold-insert*[*OF including-commute*])
apply(*simp add*: *A-all-def*)
apply(*simp add*: *all-int-set-def*)
apply(*simp add*: *invert-int*)

apply(*rule image-cong*)
apply(*rule inject*)
apply(*simp*)

apply(*subst EQ-comp-fun-commute.fold-insert*[*OF including-commute5*])
apply(*simp add*: *i-int*)
apply(*simp add*: *x0-int*)
apply(*simp add*: *A-all-def*)
apply(*simp add*: *all-int-set-def*)
apply(*simp add*: *invert-int*)

apply(*rule image-cong*)
apply(*rule inject*)
apply(*simp*)

apply(*subgoal-tac* ($\forall$ *a b*. *all-defined* (*a*, *b*) (*Finite-Set.fold* (λ*x acc*. *acc*−>*including*(*x*)) *A* ((λ*a* τ. *a*) ' *F*))))
prefer *2*
apply(*rule allI*) **apply**(*erule-tac x* = *a* **in** *allE*)
apply(*rule allI*) **apply**(*erule-tac x* = *b* **in** *allE*)
apply(*simp add*: *invert-all-defined-fold*[*simplified Let-def*, *THEN mp*, *THEN mp*, *THEN mp*])

apply(*simp*)

apply(*case-tac F* = {}, *simp*)
apply(*simp add*: *all-int-set-def*)

apply(*subgoal-tac* ((((*Finite-Set.fold* (λ*x acc*. (*acc*−>*including*(*x*))) *A* ((λ*a* τ. *a*) ' *F*)−>*including*(*x0*))−>*in*
x))−>*including*(*x0*))−>*including*(*i*) =
((((((*Finite-Set.fold* (λ*x acc*. (*acc*−>*including*(*x*))) *A* ((λ*a* τ. *a*) '
F)−>*including*($\lambda\tau$. *x*))−>*including*(*x0*))−>*including*(*x0*))−>*including*(*i*))−>*including*(*i*)))
prefer *2*
apply(*rule including-subst-set*)
apply(*rule sym*)
apply(*subst including-swap*[**where** *i* = *x0* **and** *j* = *i*]) **prefer** *4*
apply(*rule including-subst-set*)
apply(*subst including-swap*[**where** *j* = *x0*]) **prefer** *4*
apply(*rule including-swap*) **prefer** *4*

apply(*rule allI*, *rule all-defined1*) **apply** (*metis PairE*)
apply(*rule allI*, *rule all-defined1*) **apply**(*rule cons-all-def*) **apply** (*metis PairE*)
apply(*simp-all add*: *i-valid x0-val int-is-valid*)
apply(*rule allI*, *rule allI*, *rule destruct3*)
apply(*rule conjI*, *rule all-defined1*) **apply**(*simp*)
apply(*simp add*: *int-is-valid x0-val*)

apply(*insert destruct-int*[*OF i-int*])
apply(*frule-tac P* = λ*j*. *i* = (λ-. *j*) **in** *ex1E*) **prefer** *2* **apply** *assumption*
apply(*rename-tac i'*)

apply(*insert destruct-int*[*OF x0-int*])
apply(*frule-tac P* = λ*j*. *x0* = (λ-. *j*) **in** *ex1E*) **prefer** *2* **apply** *assumption*
apply(*rename-tac x0'*)

proof −
fix *x F i' x0'*
show *i* = (λ-. *i'*) $\Longrightarrow$
x0 = (λ-. *x0'*) $\Longrightarrow$
is-int (λ(τ:: $'\mathfrak{A}$ *st*). *x*) $\Longrightarrow$
$\forall$ *a b*. *all-defined* (*a*, *b*) (*Finite-Set.fold* (λ*x acc*. *acc*−>*including*(*x*)) *A* ((λ*a* τ. *a*) ' *F*))
$\Longrightarrow$

(((((*Finite-Set.fold* (λx *acc*. (*acc*−>*including*(x))) A ((λa τ. a) ‘ F))−>*including*($\lambda\tau$. x))−>*including*(*x0*))−>*including*(*x0*))−>*including*(i))−>*including*(i) =
(((*Finite-Set.fold* (λj *r2*. (*r2*−>*including*(j))) A ((λa τ. a) ‘ F))−>*including*($\lambda\tau$. x))−>*including*(*x0*))−>*including*(i)
apply(*subst i-out*[**where** i' = *x0'* **and** x = x **and** F = F, *OF x0-int*])
apply(*simp*) **apply**(*simp*) **apply**(*simp*)
apply(*subst including-swap*[**where** i = *x0* **and** j = i]) **prefer** *4*
apply(*subst including-swap*[**where** i = *x0* **and** j = i]) **prefer** *4*
apply(*subst including-swap*[**where** i = *x0* **and** j = i]) **prefer** *4*
apply(*rule including-subst-set*)
apply(*rule i-out*[**where** i' = i' **and** x = x **and** F = F, *OF i-int*], *simp*)
apply(*simp*) **apply**(*simp*)

apply(*rule allI*, *rule all-defined1*) **apply**(*rule cons-all-def*) **apply** (*metis PairE*)
apply (*simp add*: *int-is-valid*)
apply(*simp add*: *i-valid x0-val*)+
apply(*insert x0-val*, *simp*)
apply(*insert i-valid*, *simp*)

apply(*rule allI*, *rule all-defined1*) **apply**(*rule cons-all-def*)+ **apply** (*metis PairE*)
apply (*simp add*: *int-is-valid*)
apply(*simp add*: *i-valid x0-val*)+
by (*metis prod.exhaust*)
apply-end *assumption*
apply-end *assumption*
apply-end(*blast*)
apply-end(*blast*)
qed
qed *simp*

show ⌈⌈*Rep-Set-0* (S τ)⌉⌉ ≠ {} ⟹ *?thesis*
apply(*simp only*: *init-out1*, *subst init-out2*, *simp*)
apply(*simp add*: *OclIterate*$_{Set}$*-def*)
apply(*simp add*: *S-all-def*[*simplified all-defined-def all-defined-set'-def*] *all-defined1*[*OF S-all-def*, *simplified OclValid-def*] *all-defined1*[*OF A-all-def*, *THEN foundation20*, *simplified OclValid-def*])
apply(*simp add*: *i-out1*)
apply(*simp add*: *cp-OclIncluding*[*of - i*] *cp-OclIncluding*[*of - x0*])
done
qed

lemma *including-out0* :
assumes *S-all-def* : ⋀τ. *all-defined* τ (S :: ('𝔄, *int option option*) *Set*)
and *S-include* : ⋀τ τ'. S τ = S τ'
and *S-notempty* : ⋀τ. ⌈⌈*Rep-Set-0* (S τ)⌉⌉ ≠ {}
and *a-int* : *is-int a*
shows (S−>*iterate*(x;*acc*=*Set*{a} | *acc*−>*including*(x))) = (S−>*including*(a))

apply(*rule ex1E*[*OF destruct-int*[*OF a-int*]], *rename-tac* a', *simp*)

apply(*case-tac* $\forall \tau.\ a' \in \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil$)
proof −
have *S-all-int* : $\bigwedge\tau.$ *all-int-set* $((\lambda a\ \tau.\ a)$ ‘ $\lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil)$
by(*rule all-def-to-all-int*, *simp add*: *assms*)

have *a-all-def* : $\bigwedge\tau.$ *all-defined* τ *Set*$\{a\}$
by(*rule cons-all-def*, *rule mtSet-all-def*, *simp add*: *int-is-valid*[*OF a-int*])

have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* τ $r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)

have *Sa-include* : $\bigwedge a'\ \tau\ \tau'.\ (\lambda\text{-}.\ a') = a \Longrightarrow S{-}{>}\mathit{including}(a)\ \tau = S{-}{>}\mathit{including}(a)\ \tau'$
apply(*simp add*: *cp-OclIncluding*[*of - a*])
apply(*drule sym*[*of - a*], *simp add*: *cp-OclIncluding*[*symmetric*])
proof − **fix** $a'\ \tau\ \tau'$ **show** $a = (\lambda\text{-}.\ a') \Longrightarrow \lambda\text{-}.\ S\ \tau{-}{>}\mathit{including}(\lambda\text{-}.\ a')\ \tau = \lambda\text{-}.\ S\ \tau'{-}{>}\mathit{including}(\lambda\text{-}.\ a')\ \tau'$
apply(*simp add*: *OclIncluding-def*)
apply(*drule sym*[*of a*])
apply(*simp add*: *cp-defined*[*symmetric*])
apply(*simp add*: *all-defined1*[*OF S-all-def*, *simplified OclValid-def*] *int-is-valid*[*OF a-int*, *simplified OclValid-def*])
apply(*subst S-include*[*of* τ τ'], *simp*)
done
qed

have *gen-all* : $\bigwedge a.\ \exists\ \tau.\ a \notin \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil \Longrightarrow \forall\ \tau.\ a \notin \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil$
apply(*rule allI*)
apply(*drule exE*) **prefer** *2* **apply** *assumption*
by(*subst S-include*, *simp*)

fix a' **show** $a = (\lambda\text{-}.\ a') \Longrightarrow \forall\tau.\ a' \in \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ \tau)\rceil\rceil \Longrightarrow (S\ {-}{>}\mathit{iterate}(x;acc{=}\mathit{Set}\{\lambda\text{-}.\ a'\}\ |\ acc{-}{>}\mathit{including}(x))) = S{-}{>}\mathit{including}(\lambda\text{-}.\ a')$
apply(*subst including-id*[*OF S-all-def*, *symmetric*], *simp*)
apply(*drule sym*[*of a*], *simp*)
apply(*subst EQ-OclIterate*$_{Set}$*-including*[**where** $a = a$ **and** $A = \mathit{Set}\{a\}$ **and** $F = \lambda a\ A.\ (A{-}{>}\mathit{including}(a))$, *simplified flatten-int*[*OF a-int*], *OF S-all-int S-all-def a-all-def including-commute a-int*])
apply(*subst EQ-OclIterate*$_{Set}$*-including*[**where** $a = a$ **and** $A = \mathit{Set}\{\}$ **and** $F = \lambda a\ A.\ (A{-}{>}\mathit{including}(a))$, *symmetric*, *OF S-all-int S-all-def mtSet-all-def including-commute a-int*])
apply(*rule iterate-including-id00*)
apply(*rule cons-all-def*, *simp-all add*: *S-all-def int-is-valid*[*OF a-int*])
apply(*simp add*: *Sa-include*)
done
apply-end *simp-all*

fix a'
show $a = (\lambda\text{-}.\ a') \Longrightarrow$
$\forall y.\ (\lambda\text{-}.\ a') = (\lambda\text{-}.\ y) \longrightarrow y = a' \Longrightarrow \exists\, a\ b.\ a' \notin \lceil\lceil \mathit{Rep\text{-}Set\text{-}0}\ (S\ (a,\ b))\rceil\rceil \Longrightarrow (S\ {-}{>}\mathit{iterate}(x;acc{=}\mathit{Set}\{\lambda\text{-}.\ a'\}\ |\ acc{-}{>}\mathit{including}(x))) = S{-}{>}\mathit{including}(\lambda\text{-}.\ a')$
apply(*drule gen-all*[*simplified*])

apply(*subst excluding-id*[*OF S-all-def*, *symmetric*])
prefer *2* **apply** (*simp*)
apply(*drule sym*[*of a*], *simp add*: *a-int*)
apply(*drule sym*[*of a*], *simp*)
apply(*subst EQ-OclIterate*$_{Set}$*-including*[**where** $a = a$ **and** $A = Set\{\}$ **and** $F = \lambda a\ A.$ $(A{-}{>}including(a))$, *symmetric*, *OF S-all-int S-all-def mtSet-all-def including-commute a-int*])
apply(*rule iterate-including-id00*)
apply(*rule cons-all-def*, *simp-all add*: *S-all-def int-is-valid*[*OF a-int*])
apply(*simp add*: *Sa-include*)
done
apply-end *simp-all*
qed

Execution OclIncluding out of OclIterate (corollary)

lemma *iterate-including-id-out* :
assumes *S-def* : $\bigwedge\tau.$ *all-defined* τ (S:: ($'\mathfrak{A}$, *int option option*) *Set*)
and *a-int* : *is-int a*
shows $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow (S\ {-}{>}iterate(j;r2{=}S \mid r2{-}{>}including(a){-}{>}including(j)))$ $\tau = S{-}{>}including(a)\ \tau$
proof –
have *all-defined1* : $\bigwedge r2\ \tau.$ *all-defined* τ $r2 \Longrightarrow \tau \models \delta\ r2$ **by**(*simp add*: *all-defined-def*)
show $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow$ *?thesis*
apply(*subst iterate-subst-set0*[**where** $G = \lambda j\ r2.\ r2{-}{>}including(j){-}{>}including(a)$])
apply(*simp add*: *S-def*)
apply(*rule including-commute3*[*THEN c0-of-c*], *simp add*: *a-int*)
apply(*rule including-commute2*[*THEN c0-of-c*], *simp add*: *a-int*)
apply(*rule including-swap*)
apply (*metis* (*hide-lams*, *no-types*) *all-defined1*)
apply(*simp add*: *a-int int-is-valid*)+
apply(*subst including-out1*) **apply**(*simp add*: *S-def a-int*)+
apply(*subst iterate-including-id*, *simp add*: *S-def*, *simp*)
done
qed

lemma *iterate-including-id-out'* :
assumes *S-def* : $\bigwedge\tau.$ *all-defined* τ (S:: ($'\mathfrak{A}$, *int option option*) *Set*)
and *a-int* : *is-int a*
shows $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow (S\ {-}{>}iterate(j;r2{=}S \mid r2{-}{>}including(j){-}{>}including(a)))$ $\tau = S{-}{>}including(a)\ \tau$
apply(*subst including-out1*) **apply**(*simp add*: *S-def a-int*)+
apply(*subst iterate-including-id*, *simp add*: *S-def*, *simp*)
done

lemma *iterate-including-id-out''''* :
assumes *S-def* : $\bigwedge\tau.$ *all-defined* τ (S:: ($'\mathfrak{A}$, *int option option*) *Set*)
and *a-int* : *is-int a*
and *b-int* : *is-int b*
shows $\lceil\lceil Rep\text{-}Set\text{-}0\ (S\ \tau)\rceil\rceil \neq \{\} \Longrightarrow (S\ {-}{>}iterate(j;r2{=}S \mid r2{-}{>}including(a){-}{>}including(j){-}{>}including(b)))$

τ = S−>*including*(a)−>*including*(b) τ
proof −
have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow$ $\tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)
show ⌈⌈*Rep-Set-0* (S τ)⌉⌉ $\neq$ {} $\Longrightarrow$ *?thesis*
apply(*subst including-out2*) **apply**(*simp add*: *S-def a-int b-int*)+
apply(*rule including-subst-set''*)
apply(*rule all-defined1*, *rule i-cons-all-def*, *rule including-commute3*[*THEN c0-of-c*], *simp add*: *a-int*, *simp add*: *S-def*)
apply(*rule all-defined1*, *rule cons-all-def*, *simp add*: *S-def*, *simp add*: *int-is-valid*[*OF a-int*], *simp add*: *int-is-valid*[*OF b-int*])

apply(*rule iterate-including-id-out*) **apply**(*simp add*: *S-def a-int*)+
done
qed

lemma *iterate-including-id-out'''* :
assumes *S-def* : $\bigwedge\tau$. *all-defined* τ (S:: ('$\mathfrak{A}$, *int option option*) *Set*)
and *a-int* : *is-int a*
and *b-int* : *is-int b*
shows ⌈⌈*Rep-Set-0* (S τ)⌉⌉ $\neq$ {} $\Longrightarrow$ (S −>*iterate*(j;*r2*=S | *r2*−>*including*(a)−>*including*(b)−>*including*
τ = S−>*including*(a)−>*including*(b) τ
proof −
have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow$ $\tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)
show ⌈⌈*Rep-Set-0* (S τ)⌉⌉ $\neq$ {} $\Longrightarrow$ *?thesis*
apply(*subst iterate-subst-set0*[**where** G = λj *r2*. *r2*−>*including*(a)−>*including*(j)−>*including*(b)])
apply(*simp add*: *S-def*)
apply(*rule including-commute6*[*THEN c0-of-c*], *simp add*: *a-int*, *simp add*: *b-int*)
apply(*rule including-commute4*[*THEN c0-of-c*], *simp add*: *a-int*, *simp add*: *b-int*)
apply(*rule including-swap*)
apply(*rule allI*, *rule all-defined1*, *rule cons-all-def'*, *blast*, *simp add*: *int-is-valid*[*OF a-int*], *simp add*: *int-is-valid*[*OF b-int*], *simp*)
apply(*rule iterate-including-id-out''''*) **apply**(*simp add*: *S-def a-int b-int*)+
done
qed

4.7.12. Conclusion

lemma *GogollasChallenge-on-sets*:
$\tau \models$ (*Set*{ **6**,**8** } −>*iterate*(i;*r1*=*Set*{**9**}|
r1−>*iterate*(j;*r2*=*r1*|
r2−>*including*(**0**)−>*including*(i)−>*including*(j)))) $\doteq$ *Set*{**0**, **6**, **8**, **9**}
proof −
have *all-defined-68* : $\bigwedge\tau$. *all-defined* τ *Set*{**6**, **8**}
apply(*rule cons-all-def*)+
apply(*simp add*: *all-defined-def all-defined-set'-def mtSet-def Abs-Set-0-inverse mtSet-defined*[*simplified mtSet-def*])
apply(*simp*)+
done

have *all-defined-9* : $\bigwedge\tau$. *all-defined* τ *Set*{**9**}
apply(*rule cons-all-def*)+
apply(*simp add*: *all-defined-def all-defined-set′-def mtSet-def Abs-Set-0-inverse mtSet-defined*[*simplified mtSet-def*])
apply(*simp*)+
done

have *all-defined1* : $\bigwedge$*r2* τ. *all-defined* τ *r2* $\Longrightarrow \tau \models \delta$ *r2* **by**(*simp add*: *all-defined-def*)

have *zero-int* : *is-int* **0** **by** (*metis StrictRefEq-int-strict′ foundation1 is-int-def null-non-zero ocl-zero-def valid4*)
have *six-int* : *is-int* **6** **by** (*metis StrictRefEq-int-strict′ foundation1 is-int-def null-non-six ocl-six-def valid4*)
have *eight-int* : *is-int* **8** **by** (*metis StrictRefEq-int-strict′ foundation1 is-int-def null-non-eight ocl-eight-def valid4*)
have *nine-int* : *is-int* **9** **by** (*metis StrictRefEq-int-strict′ foundation1 is-int-def null-non-nine ocl-nine-def valid4*)

have *commute8*: *EQ-comp-fun-commute* (λ*x acc*. *acc*−>*including*(**0**)−>*including*(*x*)) **apply**(*rule including-commute3*) **by** (*simp add*: *zero-int*)
have *commute7*: *EQ-comp-fun-commute* (λ*x acc*. *acc*−>*including*(*x*)−>*including*(**0**)) **apply**(*rule including-commute2*) **by** (*simp add*: *zero-int*)
have *commute4*: $\bigwedge$*x acc*. *is-int x* $\Longrightarrow$ *EQ-comp-fun-commute* (λ*xa acc*. *acc*−>*including*(**0**)−>*including*(*xa*)−>*including*(*x*
apply(*rule including-commute4*) **by**(*simp add*: *zero-int*, *blast*)
have *commute3*: $\bigwedge$*x acc*. *is-int x* $\Longrightarrow$ *EQ-comp-fun-commute* (λ*xa acc*. *acc*−>*including*(**0**)−>*including*(*x*)−>*including*(*xa*
apply(*rule including-commute6*) **by**(*simp add*: *zero-int*, *blast*)

have *swap1* : $\bigwedge$(*S*:: ('$\mathfrak{A}$, -) *Set*) *y x*.
is-int x $\Longrightarrow$
is-int y $\Longrightarrow$
$\forall\tau$. *all-defined* τ *S* $\Longrightarrow$
((((*S*−>*including*(**0**))−>*including*(*x*))−>*including*(**0**))−>*including*(*y*)) =
((((*S*−>*including*(**0**))−>*including*(*y*))−>*including*(**0**))−>*including*(*x*))
apply(*subst* (*2 5*) *including-swap*)
apply(*rule allI*, *rule all-defined1*, *rule cons-all-def*, *blast*)
apply(*simp*, *simp add*: *int-is-valid*)+
apply(*rule including-swap*)
apply(*rule allI*, *rule all-defined1*)
apply(*rule cons-all-def*)+ **apply**(*blast*)
apply(*simp*, *simp add*: *int-is-valid*)+
done

have *commute5*: *EQ-comp-fun-commute0* (λ*x r1*. *r1* −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(**0**)−>*including*(*j*))−>*including*(
'$\mathfrak{A}$ *st*). *x*))
apply(*rule iterate-including-commute*, *rule commute8*[*THEN c0-of-c*])
apply(*rule ext*, *rename-tac* τ)
apply(*subst* (*1 2*) *cp-OclIncluding*)
apply(*subst iterate-including-id-out*)
apply (*metis cons-all-def′ is-int-def mtSet-all-def*)

apply(*simp add*: *zero-int*)
apply (*metis including-notempty′ is-int-def*)
apply(*rule sym*, *subst cp-OclIncluding*)
apply(*subst iterate-including-id-out*)
apply (*metis cons-all-def′ is-int-def mtSet-all-def*)
apply(*simp add*: *zero-int*)
apply (*metis including-notempty′ is-int-def*)

apply(*subst including-swap*)
apply (*metis* (*hide-lams*, *no-types*) *foundation1 mtSet-defined*)
apply(*simp add*: *int-is-valid*)
apply(*simp*)
apply(*rule sym*)
apply(*subst including-swap*)
apply (*metis* (*hide-lams*, *no-types*) *foundation1 mtSet-defined*)
apply(*simp add*: *int-is-valid*)
apply(*simp*)
apply(*subst* (*1 2*) *cp-OclIncluding*[*symmetric*])
apply(*rule including-swap′*)
apply(*simp add*: *int-is-valid*) **apply**(*simp add*: *int-is-valid*) **apply**(*simp add*: *int-is-valid*)

apply(*subst* (*1 2*) *cp-OclIncluding*)
apply(*subst* (*1 2*) *cp-OclIterate*$_{Set}$*1*[*OF including-commute3*[*THEN c0-of-c*, *THEN c0′-of-c0*]], *simp add*: *zero-int*)
apply(*rule cons-all-def′*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute3*[*THEN c0-of-c*], *simp add*: *zero-int*, *blast*, *simp add*: *int-is-valid*)
apply(*rule cons-all-def′*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute3*[*THEN c0-of-c*], *simp add*: *zero-int*, *blast*, *simp add*: *int-is-valid*)
apply(*subst* (*1 2 3 4 5 6*) *cp-OclIncluding*)

apply(*subst* (*1 2 3 4 5*) *iterate-including-id-out*)

apply(*metis surj-pair*, *simp add*: *zero-int*, *simp*)
apply(*subst cp-OclIncluding*[*symmetric*], *rule cp-all-def*[*THEN iffD1*])
apply(*rule cons-all-def′*, *rule i-cons-all-def*, *rule commute8*[*THEN c0-of-c*], *metis surj-pair*, *simp add*: *int-is-valid*, *simp add*: *zero-int*)

apply(*rule including-notempty*)
apply(*rule all-defined1*, *rule cp-all-def*[*THEN iffD1*], *rule i-cons-all-def*, *rule commute8*[*THEN c0-of-c*], *metis surj-pair*, *simp add*: *int-is-valid*, *simp add*: *zero-int*)
apply(*rule iterate-notempty*, *rule commute8*[*THEN c0-of-c*], *metis surj-pair*, *simp add*: *int-is-valid*, *simp add*: *zero-int*)
apply(*subst cp-OclIncluding*[*symmetric*], *rule cp-all-def*[*THEN iffD1*]) **apply**(*rule cons-all-def*)+ **apply**(*metis surj-pair*, *simp add*: *zero-int*, *simp add*: *int-is-valid*)
apply(*rule including-notempty*, *rule all-defined1*, *rule cp-all-def*[*THEN iffD1*]) **apply**(*rule cons-all-def*)+ **apply**(*metis surj-pair*, *simp add*: *zero-int*, *simp add*: *int-is-valid*)
apply(*rule including-notempty*, *rule all-defined1*) **apply**(*metis surj-pair*, *simp add*: *zero-int*, *simp add*: *int-is-valid*)

apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*[*symmetric*])
apply(*subst swap1*, *simp-all*)
done

have *commute6*: *EQ-comp-fun-commute0* (λx *r1*. *r1* −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(*j*)−>*including*(**0**))−>*including*(
'$\mathfrak{A}$ *st*). *x*))
apply(*rule iterate-including-commute*, *rule commute7*[*THEN c0-of-c*])
apply(*rule ext*, *rename-tac* τ)
apply(*subst* (*1 2*) *cp-OclIncluding*)
apply(*subst iterate-including-id-out'*)
apply (*metis cons-all-def' is-int-def mtSet-all-def*)
apply(*simp add*: *zero-int*)
apply (*metis including-notempty' is-int-def*)
apply(*rule sym*, *subst cp-OclIncluding*)
apply(*subst iterate-including-id-out'*)
apply (*metis cons-all-def' is-int-def mtSet-all-def*)
apply(*simp add*: *zero-int*)
apply (*metis including-notempty' is-int-def*)

apply(*subst including-swap*)
apply (*metis* (*hide-lams*, *no-types*) *foundation1 mtSet-defined*)
apply(*simp add*: *int-is-valid*)
apply(*simp*)
apply(*rule sym*)
apply(*subst including-swap*)
apply (*metis* (*hide-lams*, *no-types*) *foundation1 mtSet-defined*)
apply(*simp add*: *int-is-valid*)
apply(*simp*)
apply(*subst* (*1 2*) *cp-OclIncluding*[*symmetric*])
apply(*rule including-swap'*)
apply(*simp add*: *int-is-valid*) **apply**(*simp add*: *int-is-valid*) **apply**(*simp add*: *int-is-valid*)

apply(*subst* (*1 2*) *cp-OclIncluding*)
apply(*subst* (*1 2*) *cp-OclIterate*$_{Set}$*1*[*OF including-commute2*[*THEN c0-of-c*, *THEN c0'-of-c0*]], *simp add*: *zero-int*)
apply(*rule cons-all-def'*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute2*[*THEN c0-of-c*], *simp add*: *zero-int*, *blast*, *simp add*: *int-is-valid*)
apply(*rule cons-all-def'*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute2*[*THEN c0-of-c*], *simp add*: *zero-int*, *blast*, *simp add*: *int-is-valid*)
apply(*subst* (*1 2 3 4 5 6*) *cp-OclIncluding*)

apply(*subst* (*1 2 3 4 5*) *iterate-including-id-out'*)

apply(*metis surj-pair*, *simp add*: *zero-int*, *simp*)
apply(*subst cp-OclIncluding*[*symmetric*], *rule cp-all-def*[*THEN iffD1*])
apply(*rule cons-all-def'*, *rule i-cons-all-def*, *rule commute7*[*THEN c0-of-c*], *metis surj-pair*, *simp add*: *int-is-valid*, *simp add*: *zero-int*)

apply(*rule including-notempty*)
apply(*rule all-defined1*, *rule cp-all-def*[*THEN iffD1*], *rule i-cons-all-def*, *rule commute7*[*THEN c0-of-c*], *metis surj-pair*, *simp add*: *int-is-valid*, *simp add*: *zero-int*)
apply(*rule iterate-notempty*, *rule commute7*[*THEN c0-of-c*], *metis surj-pair*, *simp add*: *int-is-valid*, *simp add*: *zero-int*)
apply(*subst cp-OclIncluding*[*symmetric*], *rule cp-all-def*[*THEN iffD1*]) **apply**(*rule cons-all-def*)+
apply(*metis surj-pair*, *simp add*: *zero-int*, *simp add*: *int-is-valid*)
apply(*rule including-notempty*, *rule all-defined1*, *rule cp-all-def*[*THEN iffD1*]) **apply**(*rule cons-all-def*)+ **apply**(*metis surj-pair*, *simp add*: *zero-int*, *simp add*: *int-is-valid*)
apply(*rule including-notempty*, *rule all-defined1*) **apply**(*metis surj-pair*, *simp add*: *zero-int*, *simp add*: *int-is-valid*)

apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*[*symmetric*])
apply(*subst swap1*, *simp-all*)
done

have *commute9*: *EQ-comp-fun-commute0* (λ*x r1*. *r1* −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(*j*))−>*including*(**0**) *x*))
apply(*rule iterate-including-commute-var*, *rule including-commute*[*THEN c0-of-c*])
apply(*rule ext*, *rename-tac* τ)
apply(*subst* (*1 2*) *cp-OclIncluding*)
apply(*subst* (*1 2*) *iterate-including-id*)
apply (*metis StrictRefEq-int-strict′ cons-all-def′ foundation1 is-int-def mtSet-all-def null-non-zero valid4*)
apply (*metis StrictRefEq-int-strict′ cons-all-def′ foundation1 is-int-def mtSet-all-def null-non-zero valid4*)

apply(*subst* (*1 2*) *cp-OclIncluding*[*symmetric*])
apply(*rule including-swap′*)
apply (*metis* (*hide-lams*, *no-types*) *all-defined1 including-defined-args-valid int-is-valid mtSet-all-def zero-int*)
apply(*simp add*: *int-is-valid*) **apply**(*simp add*: *int-is-valid*)

apply(*subst* (*1 2*) *cp-OclIncluding*)
apply(*subst* (*1 2*) *cp-OclIterate*$_{Set}$*1*, *rule including-commute*[*THEN c0-of-c*, *THEN c0′-of-c0*])
apply(*rule cons-all-def′*)+ **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute*[*THEN c0-of-c*], *blast*, *simp*, *simp add*: *int-is-valid*)
apply(*rule cons-all-def′*)+ **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute*[*THEN c0-of-c*], *blast*, *simp*, *simp add*: *int-is-valid*)
apply(*subst* (*1 2 3 4 5 6*) *cp-OclIncluding*)

apply(*subst* (*1 2 3 4 5 6*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8 9 10*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5*) *iterate-including-id*)

apply(*metis surj-pair*)

apply(*subst (1 2) cp-OclIncluding*[*symmetric*], *rule cp-all-def*[*THEN iffD1*])
apply(*rule cons-all-def'*, *rule cons-all-def'*, *rule i-cons-all-def*, *rule including-commute*[*THEN c0-of-c*], *metis surj-pair*) **apply**(*simp add*: *int-is-valid*)+
apply(*subst (1 2) cp-OclIncluding*[*symmetric*], *rule cp-all-def*[*THEN iffD1*])
apply(*rule cons-all-def'*, *rule cons-all-def'*, *metis surj-pair*) **apply**(*simp add*: *int-is-valid*)+
apply(*metis surj-pair*)

apply(*subst (1 2 3 4 5 6) cp-OclIncluding*)
apply(*subst (1 2 3 4 5 6) cp-OclIncluding*[*symmetric*])
apply(*rule including-swap'*) **apply**(*rule all-defined1*, *rule cons-all-def*, *metis surj-pair*) **apply**(*simp add*: *int-is-valid zero-int*)+
done

have *commute1*: *EQ-comp-fun-commute0'* (λx *r1*. *r1* −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(**0**)−>*including*(λ(-:: '$\mathfrak{A}$ *st*). $\lfloor x \rfloor$)−>*including*(*j*)))
apply(*rule iterate-commute'*)
apply(*rule including-commute6*[*THEN c0-of-c*, *THEN c0'-of-c0*], *simp add*: *zero-int*, *simp add*: *int-trivial*)
apply(*subst (1 2) cp-OclIterate*$_{Set}$*1*)
apply(*rule including-commute6*[*THEN c0-of-c*, *THEN c0'-of-c0*], *simp add*: *zero-int*, *simp*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute6*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*, *blast*)
apply(*rule including-commute6*[*THEN c0-of-c*, *THEN c0'-of-c0*], *simp add*: *zero-int*, *simp*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute6*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*, *blast*)
apply(*subst (1 2 3 4 5) iterate-including-id-out''*)
apply(*simp-all add*: *zero-int*)
apply(*metis surj-pair*)
apply(*subst cp-all-def*[*symmetric*])
apply(*rule i-cons-all-def*)
apply(*rule including-commute6*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*)
apply(*metis surj-pair*)
apply(*rule iterate-notempty*)
apply(*rule including-commute6*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*)
apply(*metis surj-pair*)
apply(*simp*)
apply(*subst cp-all-def*[*symmetric*])
apply(*rule cons-all-def'*)+
apply(*metis surj-pair*)
apply(*simp add*: *int-is-valid*)+
apply(*rule including-notempty*)
apply(*rule all-defined1*)
apply(*rule cons-all-def'*)+
apply(*metis surj-pair*)
apply(*simp add*: *int-is-valid*)+
apply(*rule including-notempty*)
apply(*rule all-defined1*)
apply(*metis surj-pair*)
apply(*simp add*: *int-is-valid*)+

apply(*subst* (*1 2 3 4*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*[*symmetric*])
apply(*subst swap1*, *simp-all*)
done

have *commute2*: *EQ-comp-fun-commute0'* (λx *r1*. *r1* −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(**0**)−>*including*(*j*) '𝔄 *st*). ⌊*x*⌋)))
apply(*rule iterate-commute'*)
apply(*rule including-commute4*[*THEN c0-of-c*, *THEN c0'-of-c0*], *simp add*: *zero-int*, *simp add*: *int-trivial*)
apply(*subst* (*1 2*) *cp-OclIterate*$_{Set}$*1*)
apply(*rule including-commute4*[*THEN c0-of-c*, *THEN c0'-of-c0*], *simp add*: *zero-int*, *simp*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute4*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*, *blast*)
apply(*rule including-commute4*[*THEN c0-of-c*, *THEN c0'-of-c0*], *simp add*: *zero-int*, *simp*) **apply**(*rule i-cons-all-def*) **apply**(*rule including-commute4*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*, *blast*)
apply(*subst* (*1 2 3 4 5*) *iterate-including-id-out''''*)
apply(*simp-all add*: *zero-int*)
apply(*metis surj-pair*)
apply(*subst cp-all-def*[*symmetric*])
apply(*rule i-cons-all-def*)
apply(*rule including-commute4*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*)
apply(*metis surj-pair*)
apply(*rule iterate-notempty*)
apply(*rule including-commute4*[*THEN c0-of-c*], *simp add*: *zero-int*, *simp*)
apply(*metis surj-pair*)
apply(*simp*)
apply(*subst cp-all-def*[*symmetric*])
apply(*rule cons-all-def'*)+
apply(*metis surj-pair*)
apply(*simp add*: *int-is-valid*)+
apply(*rule including-notempty*)
apply(*rule all-defined1*)
apply(*rule cons-all-def'*)+
apply(*metis surj-pair*)
apply(*simp add*: *int-is-valid*)+
apply(*rule including-notempty*)
apply(*rule all-defined1*)
apply(*metis surj-pair*)
apply(*simp add*: *int-is-valid*)+
apply(*subst* (*1 2 3 4*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*)
apply(*subst* (*1 2 3 4 5 6 7 8*) *cp-OclIncluding*[*symmetric*])
apply(*subst swap1*, *simp-all*)
done

have *set68-notempty* : $\bigwedge(\tau$:: '𝔄 *st*). ⌈⌈*Rep-Set-0* (*Set*{**6**, **8**} τ)⌉⌉ ≠ {}

apply(*rule including-notempty*)
apply(*simp add*: *mtSet-all-def*)
apply(*simp add*: *int-is-valid*)
apply(*rule including-notempty'*)
by(*simp add*: *int-is-valid*)
have *set9-notempty* : $\bigwedge(\tau:: {}'\mathfrak{A}\ st).\ \lceil\lceil Rep\text{-}Set\text{-}0\ (Set\{\mathbf{9}\}\ \tau)\rceil\rceil \neq \{\}$
apply(*rule including-notempty'*)
by(*simp add*: *int-is-valid*)
have *set68-cp* : $\bigwedge(\tau:: {}'\mathfrak{A}\ st)\ (\tau':: {}'\mathfrak{A}\ st).\ Set\{\mathbf{6},\ \mathbf{8}\}\ \tau = Set\{\mathbf{6},\ \mathbf{8}\}\ \tau'$
apply(*rule including-cp-all*) **apply**(*simp add*: *six-int*) **apply**(*simp add*: *mtSet-all-def*)
apply(*rule including-cp-all*) **apply**(*simp add*: *eight-int*) **apply**(*simp add*: *mtSet-all-def*)
by (*simp add*: *mtSet-def*)
have *set9-cp* : $\bigwedge(\tau 1:: {}'\mathfrak{A}\ st)\ (\tau 2:: {}'\mathfrak{A}\ st).\ Set\{\mathbf{9}\}\ \tau 1 = Set\{\mathbf{9}\}\ \tau 2$
apply(*rule including-cp-all*) **apply**(*simp add*: *nine-int*) **apply**(*simp add*: *mtSet-all-def*)
by (*simp add*: *mtSet-def*)

note *iterate-subst-set---* = *iterate-subst-set---*[*OF all-defined-68 all-defined-9 set9-cp - - - set9-notempty*]
note *iterate-subst-set''0* = *iterate-subst-set''0*[*OF all-defined-68 all-defined-9 - - - set9-notempty*]
note *iterate-subst-set'0* = *iterate-subst-set'0*[*OF all-defined-68 all-defined-9 set9-cp*]

have *GogollasChallenge-on-sets*:
(*Set*{ **6**,**8** } −>*iterate*(*i*;*r1*=*Set*{**9**}|
r1−>*iterate*(*j*;*r2*=*r1*|
r2−>*including*(**0**)−>*including*(*i*)−>*including*(*j*)))) τ = *Set*{**0**, **6**,
8, **9**} τ

apply(*subst iterate-subst-set---*[**where** $G = \lambda i\ r1.\ r1$ −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(**0**)−>*including*(*j*)−>*including*(
apply(*simp add*: *commute1*, *simp add*: *commute2*)
apply(*subst iterate-subst-set*[**where** $G = \lambda j\ r2.\ r2$−>*including*(**0**)−>*including*(*j*)−>*including*(λ-.
$\lfloor x\rfloor$)]) **apply**(*blast*)+
apply(*simp add*: *commute3*, *simp add*: *commute4*)
apply(*rule including-swap*)
apply (*metis* (*hide-lams*, *mono-tags*) *StrictRefEq-int-strict' all-defined-def including-defined-args-valid'*
null-non-zero ocl-and-true1 transform1-rev valid4)
apply(*simp add*: *int-is-valid*)+

apply(*subst iterate-subst-set---*[**where** $G = \lambda i\ r1.\ r1$ −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(**0**)−>*including*(*j*))−>*including*
apply(*simp add*: *commute2*, *simp add*: *commute5*[*THEN c0'-of-c0*])
apply(*rule including-out2*)
apply(*blast*) **apply**(*blast*) **apply**(*blast*) **apply**(*simp add*: *zero-int*) **apply**(*simp*)

apply(*subst iterate-subst-set---*[**where** $G = \lambda i\ r1.\ r1$ −>*iterate*(*j*;*r2*=*r1* | *r2*−>*including*(*j*)−>*including*(**0**))−>*including*
apply(*simp add*: *commute5*[*THEN c0'-of-c0*], *simp add*: *commute6*[*THEN c0'-of-c0*])
apply(*rule including-subst-set''*)
apply(*rule all-defined1*, *rule i-cons-all-def*, *rule including-commute3*[*THEN c0-of-c*], *simp*
add: *zero-int*, *blast*)
apply(*rule all-defined1*, *rule i-cons-all-def*, *rule including-commute2*[*THEN c0-of-c*], *simp*
add: *zero-int*, *blast*)
apply(*simp add*: *int-is-valid*)

apply(*subst iterate-subst-set*[**where** $G = \lambda j\, r2.\ r2{-}{>}including(j){-}{>}including(\mathbf{0})$]) **apply**(*blast*)+
apply(*simp add*: *commute8*, *simp add*: *commute7*)
apply(*rule including-swap*)
apply(*simp add*: *all-defined1*) **apply**(*simp*) **apply**(*simp only*: *foundation20*, *simp*) **apply**(*simp*)

apply(*subst iterate-subst-set''0*[**where** $G = \lambda i\, r1.\ r1\ {-}{>}iterate(j;r2{=}r1 \mid r2{-}{>}including(j)){-}{>}including(\mathbf{0}$
apply(*simp add*: *commute6*, *simp add*: *commute9*)
apply(*rule including-subst-set''*)
apply(*rule all-defined1*) **apply**(*rule i-cons-all-def*, *rule including-commute2*[*THEN c0-of-c*], *simp add*: *zero-int*, *blast*)
apply(*rule all-defined1*) **apply**(*rule cons-all-def*, *rule i-cons-all-def*, *rule including-commute*[*THEN c0-of-c*], *blast*, *simp*, *simp add*: *int-is-valid*)
apply(*rule including-out1*)
apply(*blast*) **apply**(*blast*) **apply**(*simp add*: *zero-int*) **apply**(*simp*)

apply(*subst iterate-subst-set'0*[**where** $G = \lambda i\ r1.\ r1{-}{>}including(\mathbf{0}){-}{>}including(i)$])
apply(*simp add*: *commute9*, *simp add*: *commute8*[*THEN c0-of-c*])
apply(*rule including-subst-set*)+
apply(*rule iterate-including-id*) **apply**(*blast*)+

apply(*subst iterate-subst-set*[**where** $G = \lambda i\ r1.\ r1{-}{>}including(i){-}{>}including(\mathbf{0})$])
apply(*simp add*: *all-defined-68*, *simp add*: *all-defined-9*, *simp add*: *commute8*, *simp add*: *commute7*)
apply(*rule including-swap*)
apply(*simp add*: *all-defined1*) **apply**(*simp*) **apply**(*simp only*: *foundation20*, *simp*)

apply(*subst including-out1*[*OF all-defined-68 all-defined-9 zero-int set68-notempty*])

apply(*rule including-subst-set''*)
apply(*rule all-defined1*, *rule i-cons-all-def''*, *rule including-commute*[*THEN c0-of-c*, *THEN c0'-of-c0*], *simp add*: *all-defined-68*, *simp add*: *all-defined-9*)
apply (*metis* (*hide-lams*, *no-types*) *all-defined1 all-defined-68 all-defined-9 including-defined-args-valid*)
apply(*simp*)

apply(*subst including-out0*[*OF all-defined-68 set68-cp set68-notempty nine-int*])

apply(*subst including-swap*[**where** $i = \mathbf{6}$])
apply(*simp*)+

apply(*subst including-swap*)
apply(*simp*)+
done

have *valid-1* : $\tau \models \upsilon\ (Set\{\ \mathbf{6},\mathbf{8}\ \}\ {-}{>}iterate(i;r1{=}Set\{\mathbf{9}\}|$
$r1{-}{>}iterate(j;r2{=}r1|$
$r2{-}{>}including(\mathbf{0}){-}{>}including(i){-}{>}including(j))))$
by(*rule foundation20*, *rule all-defined1*, *rule i-cons-all-def''*, *rule commute1*, *rule all-defined-68*, *rule all-defined-9*)

have *valid-2* : $\tau \models \upsilon$ *Set*{**0**, **6**, **8**, **9**}
apply(*rule foundation20*, *rule all-defined1*) **apply**(*rule cons-all-def*)+
apply(*simp-all add*: *mtSet-all-def*)
done

show *?thesis*
apply(*simp only*: *StrictRefEq-set OclValid-def StrongEq-def valid-1*[*simplified OclValid-def*] *valid-2*[*simplified OclValid-def*])
apply(*simp add*: *GogollasChallenge-on-sets true-def*)
done
qed

4.8. Test Statements

lemma *syntax-test*: *Set*{**2**,**1**} = (*Set*{}−>*including*(**1**)−>*including*(**2**))
by (*rule refl*)

lemma *set-test1*: $\tau \models$ (*Set*{**2**,*null*}−>*includes*(*null*))
by(*simp add*: *includes-execute-int*)

lemma *set-test2*: $\neg(\tau \models$ (*Set*{**2**,**1**}−>*includes*(*null*)))
by(*simp add*: *includes-execute-int*)

Here is an example of a nested collection. Note that we have to use the abstract null (since we did not (yet) define a concrete constant *null* for the non-existing Sets) :

lemma *semantic-test2*:
assumes *H*:(*Set*{**2**} $\doteq$ *null*) = (*false*::($'\mathfrak{A}$)*Boolean*)
shows (τ::($'\mathfrak{A}$)*st*) $\models$ (*Set*{*Set*{**2**},*null*}−>*includes*(*null*))
by(*simp add*: *includes-execute-set H*)

lemma *semantic-test3*: $\tau \models$ (*Set*{*null*,**2**}−>*includes*(*null*))
by(*simp-all add*: *including-charn1 including-defined-args-valid*)

lemma *set-test4* : $\tau \models$ (*Set*{**2**,*null*,**2**} $\doteq$ *Set*{*null*,**2**})
proof −

have *cp-1*: $\bigwedge x\ \tau$. (*if null* $\doteq$ *x then true else if* **2** $\doteq$ *x then true else if* υ *x then false else invalid endif endif endif*) τ =
(*if null* $\doteq$ (λ-. *x* τ) *then true else if* **2** $\doteq$ (λ-. *x* τ) *then true else if* υ (λ-. *x* τ) *then false else invalid endif endif endif*) τ
apply(*subgoal-tac* (*null* $\doteq$ *x*) τ = (*null* $\doteq$ (λ-. *x* τ)) τ $\wedge$ (**2** $\doteq$ *x*) τ = (**2** $\doteq$ (λ-. *x* τ)) τ $\wedge$ (υ *x*) τ = (υ (λ-. *x* τ)) τ)
apply(*subst cp-if-ocl*[*of null* $\doteq$ *x*])

apply(*subst cp-if-ocl*[*of* $\mathbf{2} \doteq x$])
apply(*subst cp-if-ocl*[*of* υ *x*])
apply(*simp*)

apply(*subst if-ocl-def*)
apply(*rule sym*, *subst if-ocl-def*)

apply(*simp only*: *cp-if-ocl*[*symmetric*])
apply(*subgoal-tac* $(\delta\ (null \doteq (\lambda\text{-}.\ x\ \tau)))\ \tau = (\delta\ (\lambda\text{-}.\ (null \doteq (\lambda\text{-}.\ x\ \tau))\ \tau))\ \tau$)
apply(*simp only*:)
apply(*rule cp-defined*)

apply(*subst cp-StrictRefEq-int*[*of null x*])
apply(*simp add*: *null-fun-def*)

apply(*subst cp-StrictRefEq-int*[*of* **2**])
apply(*simp add*: *ocl-two-def*)

apply(*rule cp-valid*)
done

have *cp-2*: ($\bigwedge x\ \tau$. (*if* $\mathbf{2} \doteq x$ *then true else if null* $\doteq x$ *then true else if* $\mathbf{2} \doteq x$ *then true else if* υ *x then false else invalid endif endif endif endif*) τ =
(*if* $\mathbf{2} \doteq (\lambda\text{-}.\ x\ \tau)$ *then true else if null* $\doteq (\lambda\text{-}.\ x\ \tau)$ *then true else*
if $\mathbf{2} \doteq (\lambda\text{-}.\ x\ \tau)$ *then true else if* $\upsilon\ (\lambda\text{-}.\ x\ \tau)$ *then false else invalid endif endif endif endif*) τ)
apply(*subgoal-tac* $(null \doteq x)\ \tau = (null \doteq (\lambda\text{-}.\ x\ \tau))\ \tau \wedge (\mathbf{2} \doteq x)\ \tau = (\mathbf{2} \doteq (\lambda\text{-}.\ x\ \tau))\ \tau \wedge (\upsilon\ x)\ \tau = (\upsilon\ (\lambda\text{-}.\ x\ \tau))\ \tau$)
apply(*subst cp-if-ocl*[*of* $\mathbf{2} \doteq x$])
apply(*subst cp-if-ocl*[*of null* $\doteq x$])
apply(*subst cp-if-ocl*[*of* $\mathbf{2} \doteq x$])
apply(*subst cp-if-ocl*[*of* υ *x*])
apply(*simp*)

apply(*subst if-ocl-def*)
apply(*rule sym*, *subst if-ocl-def*)

apply(*simp only*: *cp-if-ocl*[*symmetric*])
apply(*subgoal-tac* $(\delta\ (\mathbf{2} \doteq (\lambda\text{-}.\ x\ \tau)))\ \tau = (\delta\ (\lambda\text{-}.\ (\mathbf{2} \doteq (\lambda\text{-}.\ x\ \tau))\ \tau))\ \tau$)
apply(*simp only*:)
apply(*rule cp-defined*)

apply(*subst cp-StrictRefEq-int*[*of null x*])
apply(*simp add*: *null-fun-def*)

apply(*subst cp-StrictRefEq-int*[*of* **2**])
apply(*simp add*: *ocl-two-def*)

apply(*rule cp-valid*)

done

show *?thesis*
apply(*simp add*: *includes-execute-int*)
apply(*simp add*: *forall-set-including-exec*[**where** $P = \lambda z.$ *if null* $\doteq$ *z then true else if* $\mathbf{2} \doteq z$ *then true else if* υ *z then false else invalid endif endif endif*,
OF cp-1])
apply(*simp add*: *forall-set-including-exec*[**where** $P = \lambda z.$ *if* $\mathbf{2} \doteq z$ *then true else if null* $\doteq$ *z then true else if* $\mathbf{2} \doteq z$ *then true else if* υ *z then false else invalid endif endif endif endif*,
OF cp-2])
done
qed

lemma *short-cut'*[*simp*]: $(\mathbf{8} \doteq \mathbf{6}) = \textit{false}$
apply(*rule ext*)
apply(*simp add*: *StrictRefEq-int StrongEq-def ocl-eight-def ocl-six-def*
true-def false-def invalid-def bot-option-def)
done

Elementary computations on Sets.

value $\neg\ (\tau_0 \models \upsilon(\textit{invalid}::('\mathfrak{A},'\alpha::\textit{null})\ \textit{Set}))$
value $\tau_0 \models \upsilon(\textit{null}::('\mathfrak{A},'\alpha::\textit{null})\ \textit{Set})$
value $\neg\ (\tau_0 \models \delta(\textit{null}::('\mathfrak{A},'\alpha::\textit{null})\ \textit{Set}))$
value $\tau_0 \models \upsilon(\textit{Set}\{\})$
value $\tau_0 \models \upsilon(\textit{Set}\{\textit{Set}\{\mathbf{2}\},\textit{null}\})$
value $\tau_0 \models \delta(\textit{Set}\{\textit{Set}\{\mathbf{2}\},\textit{null}\})$
value $\tau_0 \models (\textit{Set}\{\mathbf{2},\mathbf{1}\}{-}{>}\textit{includes}(\mathbf{1}))$
value $\neg\ (\tau_0 \models (\textit{Set}\{\mathbf{2}\}{-}{>}\textit{includes}(\mathbf{1})))$
value $\neg\ (\tau_0 \models (\textit{Set}\{\mathbf{2},\mathbf{1}\}{-}{>}\textit{includes}(\textit{null})))$
value $\tau_0 \models (\textit{Set}\{\mathbf{2},\textit{null}\}{-}{>}\textit{includes}(\textit{null}))$

end

5. Part III: State Operations and Objects

theory *OCL-state*
imports *OCL-lib*
begin

5.0.1. Recall: The generic structure of States

Next we will introduce the foundational concept of an object id (oid), which is just some infinite set.

type-synonym $oid = nat$

States are pair of a partial map from oid's to elements of an object universe $'\mathfrak{A}$ — the heap — and a map to relations of objects. The relations were encoded as lists of pairs in order to leave the possibility to have Bags, OrderedSets or Sequences as association ends.

Recall:

record ('\<AA>)state =
  heap :: "oid ⇀ '\<AA> "
  assocs :: "oid ⇀ (oid × oid) list "

type-synonym $('\mathfrak{A})st = '\mathfrak{A}\ state \times '\mathfrak{A}\ state$

Now we refine our state-interface. In certain contexts, we will require that the elements of the object universe have a particular structure; more precisely, we will require that there is a function that reconstructs the oid of an object in the state (we will settle the question how to define this function later).

class *object* = **fixes** $oid\text{-}of :: 'a \Rightarrow oid$

Thus, if needed, we can constrain the object universe to objects by adding the following type class constraint:

typ $'\mathfrak{A} :: object$

5.0.2. Referential Object Equality in States

Generic referential equality - to be used for instantiations with concrete object types ...

definition $gen\text{-}ref\text{-}eq :: ('\mathfrak{A},'a{::}\{object,null\})val \Rightarrow ('\mathfrak{A},'a)val \Rightarrow ('\mathfrak{A})Boolean$
where $gen\text{-}ref\text{-}eq\ x\ y$
$\equiv \lambda\ \tau.\ if\ (\upsilon\ x)\ \tau = true\ \tau \wedge (\upsilon\ y)\ \tau = true\ \tau$
$then\ if\ x\ \tau = null \vee y\ \tau = null$

$\quad\quad\quad then\ \lfloor\lfloor x\ \tau = null \wedge y\ \tau = null\rfloor\rfloor$
$\quad\quad\quad else\ \lfloor\lfloor(oid\text{-}of\ (x\ \tau)) = (oid\text{-}of\ (y\ \tau))\ \rfloor\rfloor$
$\quad\quad else\ invalid\ \tau$

lemma *gen-ref-eq-object-strict1*[*simp*] :
$(gen\text{-}ref\text{-}eq\ x\ invalid) = invalid$
by(*rule ext*, *simp add*: *gen-ref-eq-def true-def false-def*)

lemma *gen-ref-eq-object-strict2*[*simp*] :
$(gen\text{-}ref\text{-}eq\ invalid\ x) = invalid$
by(*rule ext*, *simp add*: *gen-ref-eq-def true-def false-def*)

lemma *cp-gen-ref-eq-object*:
$(gen\text{-}ref\text{-}eq\ x\ y\ \tau) = (gen\text{-}ref\text{-}eq\ (\lambda\text{-}.\ x\ \tau)\ (\lambda\text{-}.\ y\ \tau))\ \tau$
by(*auto simp*: *gen-ref-eq-def cp-valid*[*symmetric*])

lemmas *cp-intro''*[*simp*,*intro*!] =
  cp-intro''
  cp-gen-ref-eq-object[*THEN allI*[*THEN allI*[*THEN allI*[*THEN cpI2*]],
    of gen-ref-eq]]

Finally, we derive the usual laws on definedness for (generic) object equality:

lemma *gen-ref-eq-defargs*:
$\tau \models (gen\text{-}ref\text{-}eq\ x\ (y{::}('\mathfrak{A},'a{::}\{null,object\})val)) \Longrightarrow (\tau \models (\upsilon\ x)) \wedge (\tau \models (\upsilon\ y))$
by(*simp add*: *gen-ref-eq-def OclValid-def true-def invalid-def bot-option-def*
  split: *bool.split-asm HOL.split-if-asm*)

5.0.3. Further requirements on States

A key-concept for linking strict referential equality to logical equality: in well-formed states (i.e. those states where the self (oid-of) field contains the pointer to which the object is associated to in the state), referential equality coincides with logical equality.

definition $WFF :: ('\mathfrak{A}{::}object)st \Rightarrow bool$
where $WFF\ \tau = ((\forall\ x \in ran(heap(fst\ \tau)).\ \lceil heap(fst\ \tau)\ (oid\text{-}of\ x)\rceil = x) \wedge$
$\quad\quad (\forall\ x \in ran(heap(snd\ \tau)).\ \lceil heap(snd\ \tau)\ (oid\text{-}of\ x)\rceil = x))$

This is a generic definition of referential equality: Equality on objects in a state is reduced to equality on the references to these objects. As in HOL-OCL, we will store the reference of an object inside the object in a (ghost) field. By establishing certain invariants ("consistent state"), it can be assured that there is a "one-to-one-correspondance" of objects to their references — and therefore the definition below behaves as we expect.

Generic Referential Equality enjoys the usual properties: (quasi) reflexivity, symmetry, transitivity, substitutivity for defined values. For type-technical reasons, for each concrete object type, the equality $\doteq$ is defined by generic referential equality.

theorem *strictEqGen-vs-strongEq*:

$WFF\ \tau \Longrightarrow \tau \models (\upsilon\ x) \Longrightarrow \tau \models (\upsilon\ y) \Longrightarrow$
$(x\ \tau \in ran\ (heap(fst\ \tau)) \wedge y\ \tau \in ran\ (heap(fst\ \tau))) \wedge$
$(x\ \tau \in ran\ (heap(snd\ \tau)) \wedge y\ \tau \in ran\ (heap(snd\ \tau))) \Longrightarrow$ (* *x and y must be object representations that exist in either the pre or post state* *)
$(\tau \models (gen\text{-}ref\text{-}eq\ x\ y)) = (\tau \models (x \triangleq y))$
apply(*auto simp*: *gen-ref-eq-def OclValid-def WFF-def StrongEq-def true-def Ball-def*)
apply(*erule-tac x=x τ* **in** *allE′*, *simp-all*)
done

So, if two object descriptions live in the same state (both pre or post), the referential equality on objects implies in a WFF state the logical equality. Uffz.

5.1. Miscillaneous: Initial States (for Testing and Code Generation)

definition τ_0 :: $('\mathfrak{A})st$
where $\tau_0 \equiv ((\!|heap=Map.empty,\ assocs= Map.empty|\!), (\!|heap=Map.empty,\ assocs= Map.empty|\!))$

5.1.1. Generic Operations on States

In order to denote OCL-types occuring in OCL expressions syntactically — as, for example, as "argument" of allInstances — we use the inverses of the injection functions into the object universes; we show that this is sufficient "characterization".

definition *allinstances* :: $('\mathfrak{A} \Rightarrow '\alpha) \Rightarrow ('\mathfrak{A}{::}object,'\alpha\ option\ option)\ Set$
(- *.oclAllInstances′(′)*)
where $((H).oclAllInstances())\ \tau =$
$Abs\text{-}Set\text{-}0\ \lfloor\lfloor(Some\ o\ Some\ o\ H)\ `\ (ran(heap(snd\ \tau)) \cap \{x.\ \exists\ y.\ y=H\ x\})\ \rfloor\rfloor$

definition *allinstancesATpre* :: $('\mathfrak{A} \Rightarrow '\alpha) \Rightarrow ('\mathfrak{A}{::}object,'\alpha\ option\ option)\ Set$
(- *.oclAllInstances@pre′(′)*)
where $((H).oclAllInstances@pre())\ \tau =$
$Abs\text{-}Set\text{-}0\ \lfloor\lfloor(Some\ o\ Some\ o\ H)\ `\ (ran(heap(fst\ \tau)) \cap \{x.\ \exists\ y.\ y=H\ x\})\ \rfloor\rfloor$

lemma $\tau_0 \models H\ .oclAllInstances() \triangleq Set\{\}$
by(*simp add*: *StrongEq-def allinstances-def OclValid-def* τ_0*-def mtSet-def*)

lemma $\tau_0 \models H\ .oclAllInstances@pre() \triangleq Set\{\}$
by(*simp add*: *StrongEq-def allinstancesATpre-def OclValid-def* τ_0*-def mtSet-def*)

theorem *state-update-vs-allInstances*:
assumes $oid \notin dom\ \sigma'$
and $cp\ P$
shows $((\sigma,\ (\!|heap=\sigma'(oid \mapsto Object),\ assocs=A|\!)) \models (P(Type\ .oclAllInstances()))) =$
$((\sigma,\ (\!|heap=\sigma',\ assocs=A|\!)) \models (P((Type\ .oclAllInstances())\text{−>}including(\lambda\ \text{-}.\ Some(Some((the\text{-}inv\ Type)\ Object))))))$

sorry

theorem *state-update-vs-allInstancesATpre*:
assumes $oid \notin dom\ \sigma$
and $cp\ P$
shows $((\llparenthesis heap{=}\sigma(oid{\mapsto}Object),\ assocs{=}A\rrparenthesis,\ \sigma') \models (P(Type\ .oclAllInstances@pre()))) = ((\llparenthesis heap{=}\sigma,\ assocs{=}A\rrparenthesis,\ \sigma') \models (P((Type\ .oclAllInstances@pre()){-}{>}including(\lambda\ \text{-}.\ Some(Some((the\text{-}inv\ Type)\ Object))))))$
sorry

definition $oclisnew$:: $('\mathfrak{A},\ '\alpha{::}\{null,object\})val \Rightarrow ('\mathfrak{A})Boolean \quad ((\text{-}).oclIsNew'('))$
where $X\ .oclIsNew() \equiv (\lambda\tau\ .\ if\ (\delta\ X)\ \tau = true\ \tau$
$then\ \lfloor\lfloor oid\text{-}of\ (X\ \tau) \notin dom(heap(fst\ \tau)) \wedge oid\text{-}of\ (X\ \tau) \in dom(heap(snd\ \tau))\rfloor\rfloor$
$else\ invalid\ \tau)$

The following predicate — which is not part of the OCL standard descriptions — provides a simple, but powerful means to describe framing conditions. For any formal approach, be it animation of OCL contracts, test-case generation or die-hard theorem proving, the specification of the part of a system transistion that DOES NOT CHANGE is of premordial importance. The following operator establishes the equality between old and new objects in the state (provided that they exist in both states), with the exception of those objects

definition $oclismodified$ $::('\mathfrak{A}{::}object,'\alpha{::}\{null,object\})Set \Rightarrow '\mathfrak{A}\ Boolean$
$(\text{-}{-}{>}oclIsModifiedOnly'('))$
where $X{-}{>}oclIsModifiedOnly() \equiv (\lambda(\sigma,\sigma').\ \ let\ \ X' = (oid\text{-}of\ `\ \lceil\lceil Rep\text{-}Set\text{-}0(X(\sigma,\sigma'))\rceil\rceil);$
$S = ((dom\ (heap\ \sigma) \cap dom\ (heap\ \sigma')) - X')$
$in\ if\ (\delta\ X)\ (\sigma,\sigma') = true\ (\sigma,\sigma')$
$then\ \lfloor\lfloor\forall\ x \in S.\ (heap\ \sigma)\ x = (heap\ \sigma')\ x\rfloor\rfloor$
$else\ invalid\ (\sigma,\sigma'))$

definition $atSelf$ $::\ ('\mathfrak{A}{::}object,'\alpha{::}\{null,object\})val \Rightarrow$
$('\mathfrak{A} \Rightarrow '\alpha) \Rightarrow$
$('\mathfrak{A}{::}object,'\alpha{::}\{null,object\})val\ ((\text{-})@pre(\text{-}))$
where $x\ @pre\ H = (\lambda\tau\ .\ if\ (\delta\ x)\ \tau = true\ \tau$
$then\ if\ oid\text{-}of\ (x\ \tau) \in dom(heap(fst\ \tau)) \wedge oid\text{-}of\ (x\ \tau) \in dom(heap\ (snd\ \tau))$
$then\ \ H\ \lceil(heap(fst\ \tau))(oid\text{-}of\ (x\ \tau))\rceil$
$else\ invalid\ \tau$
$else\ invalid\ \tau)$

theorem *framing*:
assumes $modifiesclause{:}\tau \models (X{-}{>}excluding(x)){-}{>}oclIsModifiedOnly()$
and $represented\text{-}x{:}\ \tau \models \delta(x\ @pre\ H)$
and $H\text{-}is\text{-}typerepr{:}\ inj\ H$
shows $\tau \models (x\ \triangleq\ (x\ @pre\ H))$

sorry

end

theory *OCL-tools*
imports *OCL-core*
begin

end

theory *OCL-main*
imports *OCL-lib OCL-state OCL-tools*
begin

end

Part III.

Conclusion

6. Conclusion

6.1. Lessons Learned

While our paper and pencil arguments, given in [6], turned out to be essentially correct, there had also been a lesson to be learned: If the logic is not defined as a Kleene-Logic, having a structure similar to a complete partial order (CPO), reasoning becomes complicated: several important algebraic laws break down which makes reasoning in OCL inherent messy and a semantically clean compilation of OCL formulae to a two-valued presentation, that is amenable to animators like KodKod [23] or SMT-solvers like Z3 [14] completely impractical. Concretely, if the expression `not(null)` is defined `invalid` (as is the case in the present standard [21]), than standard involution does not hold, i. e., `not(not(A))` = `A` does not hold universally. Similarly, if `null and null` is `invalid`, then not even idempotence `X and X` = `X` holds. We strongly argue in favor of a lattice-like organization, where `null` represents "more information" than `invalid` and the logical operators are monotone with respect to this semantical "information ordering."

Featherweight OCL makes these two deviations from the standard, builds all logical operators on Kleene-`not` and Kleene-`and`, and shows that the entire construction of our paper "Extending OCL with Null-References" [6] is then correct, and the DNF-normaliation as well as δ-closure laws (necessary for a transition into a two-valued presentation of OCL specifications ready for interpretation in SMT solvers (see [5] for details) are valid in Featherweight OCL.

6.2. Conclusion and Future Work

Featherweight OCL concentrates on formalizing the semantics of a core subset of OCL in general and in particular on formalizing the consequences of a four-valued logic (i. e., OCL versions that support, besides the truth values `true` and `false` also the two exception values `invalid` and `null`).

In the following, we outline the necessary steps for turning Featherweight OCL into a fully fledged tool for OCL, e. g., similar to HOL-OCL as well as for supporting test case generation similar to HOL-TestGen [10]. There are essentially five extensions necessary:

- extension of the library to support all OCL data types, e. g., `Sequence(T)`, `OrderedSet(T)`. This formalization of the OCL standard library can be used for checking the consistency of the formal semantics (known as "Annex A") with the informal and semi-formal requirements in the normative part of the OCL standard.
- development of a compiler that compiles a textual or CASE tool representation

(e. g., using XMI or the textual syntax of the USE tool [22]) of class models. Such compiler could also generate the necessary casts when converting standard OCL to Featherweight OCL as well as providing "normalizations" such as converting multiplicities of class attributes to into OCL class invariants.

- a setup for translating Featherweight OCL into a two-valued representation as described in [5]. As, in real-world scenarios, large parts of UML/OCL specifications are defined (e. g., from the default multiplicity `1` of an attributes `x`, we can directly infer that for all valid states `x` is neither `invalid` nor `null`), such a translation enables an efficient test case generation approach.
- a setup in Featherweight OCL of the Nitpick animator [3]. It remains to be shown that the standard, Kodkod [23] based animator in Isabelle can give a similar quality of animation as the OCLexec Tool [16]
- a code-generator setup for Featherweight OCL for Isabelle's code generator. For example, the Isabelle code generator supports the generation of F#, which would allow to use OCL specifications for testing arbitrary .net-based applications.

The first two extensions are sufficient to provide a formal proof environment for OCL 2.3 similar to HOL-OCL while the remaining extensions are geared towards increasing the degree of proof automation and usability as well as providing a tool-supported test methodology for UML/OCL.

Our work shows that developing a machine-checked formal semantics of recent OCL standards still reveals significant inconsistencies—even though this type of research is not new. In fact, we started our work already with the 1.x series of OCL. The reasons for this ongoing consistency problems of OCL standard are manifold. For example, the consequences of adding an additional exception value to OCL 2.2 are widespread across the whole language and many of them are also quite subtle. Here, a machine-checked formal semantics is of great value, as one is forced to formalize all details and subtleties. Moreover, the standardization process of the OMG, in which standards (e. g., the UML infrastructure and the OCL standard) that need to be aligned closely are developed quite independently, are prone to ad-hoc changes that attempt to align these standards. And, even worse, updating a standard document by voting on the acceptance (or rejection) of isolated text changes does not help either. Here, a tool for the editor of the standard that helps to check the consistency of the whole standard after each and every modifications can be of great value as well.

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