Appendix C
The Dirac Delta Function

The Dirac delta function (also called the *unit impulse function*) is a mathematical abstraction which is often used to describe (i.e. approximate) some physical phenomenon. The main reason it is used has to do with some very convenient mathematical properties which will be described below. In optics, an idealized point source of light can be described using the delta function. Of course, real points of light will have finite width, but if the point is narrow enough, approximating it with a delta function can be very useful.

C.1 Definition

The Dirac delta function is in fact not a function at all, but a distribution (a generalized function, such as a probability distribution) that is also a measure (i.e. it assigns a value to a function) – terms that come from probability and set theory. However, for our purposes it will suffice to consider it a special function with infinite height, zero width and an area of 1. It can be considered the derivative of the Heaviside step function.

To help think about the Dirac delta function, consider a rectangle with one side along the x-axis centered about $x = x_o$ such that the area of the rectangle is 1 (this is equivalent to a uniform probability distribution). Obviously there are many such rectangles, as shown in Figure C.1. We can construct a Dirac delta function by starting with a square of height and width of 1. If we halve the width and double the height, the area will remain constant. We can repeat this process as many times as we wish. As the width goes to zero, the height will become infinite but the area will remain 1. Any unit area rectangle, centered at x_o, can be expressed as

Fundamental Principles of Optical Lithography: The Science of Microfabrication. Chris Mack.
© 2007 John Wiley & Sons, Ltd. ISBN: 978-0-470-01893-4

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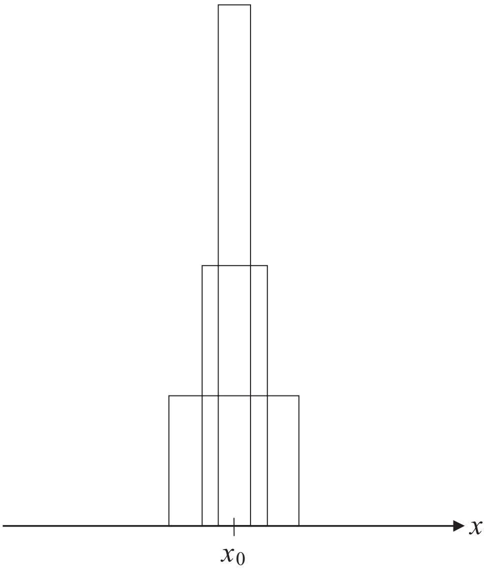

Figure C.1 *Geometrical construction of the Dirac delta function*

$$\delta_\varepsilon(x-x_o)=\begin{Bmatrix} 0, & x<x_o-\dfrac{\varepsilon}{2} \\ \dfrac{1}{\varepsilon}, & x_o-\dfrac{\varepsilon}{2}<x<x_o+\dfrac{\varepsilon}{2} \\ 0, & x>x_o+\dfrac{\varepsilon}{2} \end{Bmatrix}=\frac{1}{\varepsilon}rect\left[\frac{x-x_o}{\varepsilon}\right] \tag{C.1}$$

where *rect* is the common rectangle function. The Dirac delta function, located at $x = x_o$, can be defined as the limiting case as ε goes to zero.

$$\delta(x-x_o)=\lim_{\varepsilon\to 0}\delta_\varepsilon(x-x_o) \tag{C.2}$$

Although a rectangle is used here, in general the Dirac delta function is any pulse in the limit of zero width and unit area. Thus, the Dirac delta function can be defined by two properties:

$$\delta(x)=0 \quad \text{when} \quad x\neq 0 \tag{C.3}$$

$$\int_{-\infty}^{\infty}\delta(x)\mathrm{d}x=1 \tag{C.4}$$

Any function which has these two properties is the Dirac delta function. A consequence of Equations (C.3) and (C.4) is that $\delta(0) = \infty$.

The function $\delta_\varepsilon(x)$ is called a 'nascent' delta function, becoming a true delta function in the limit as ε goes to zero. There are many nascent delta functions, for example, the

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Gaussian pulse (a normal probability distribution, letting the standard deviation go to zero).

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathrm{e}^{-\pi x^2/\varepsilon^2} \tag{C.5}$$

Extending this form to two dimensions,

$$\delta(x,y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathrm{e}^{-\pi(x^2+y^2)/\varepsilon^2} = \delta(x)\delta(y) \tag{C.6}$$

Generalizations to more dimensions are straightforward. Other nascent delta functions include the Airy disk function, the sinc function (see section C.2.4), and the Bessel function of order $1/\varepsilon$. In general, any probability density function with a scale parameter ε is a nascent delta function as ε goes to zero.

C.2 Properties and Theorems

The following sections will state some important identities and properties of the Dirac delta function, providing proofs for some of them.

C.2.1 Sifting Property

For any function $f(x)$ continuous at x_o,

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_o)\mathrm{d}x = f(x_o) \tag{C.7}$$

It is the sifting property of the Dirac delta function that gives it the sense of a measure – it measures the value of $f(x)$ at the point x_o.

Proof

Since the delta function is zero everywhere except at $x = x_o$, the range of the integration can be changed to some infinitesimally small range ε around x_o.

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_o)\mathrm{d}x = \int_{x_o-\varepsilon}^{x_o+\varepsilon} f(x)\delta(x - x_o)\mathrm{d}x \tag{C.8}$$

Over this very small range of x, the function $f(x)$ can be thought to be constant and can be taken out of the integral.

$$\int_{x_o-\varepsilon}^{x_o+\varepsilon} f(x)\delta(x - x_o)\mathrm{d}x = f(x_o)\int_{x_o-\varepsilon}^{x_o+\varepsilon} \delta(x - x_o)\mathrm{d}x \tag{C.9}$$

From the definition of the Dirac delta function, the integral on the right-hand side will equal 1, thus proving the theorem. In fact, Equation (C.7) can be used as an alternate

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definition of the Dirac delta function. Any function $\delta(x-x_o)$ which satisfies the sifting property is the Dirac delta function.

C.2.2 Scaling Property

$$\delta(ax) = \frac{\delta(x)}{|a|} \tag{C.10}$$

C.2.3 Convolution Property

Convolution of a function f with a delta function at x_o is equivalent to shifting f by x_o.

$$f(x) * \delta(x - x_o) = f(x - x_o) \tag{C.11}$$

C.2.4 Identity 1

Another nascent delta function is the sinc function as the width of the sinc goes to zero:

$$\lim_{\varepsilon \to 0} \frac{\sin(x/\varepsilon)}{\pi x} = \lim_{a \to \infty} \frac{\sin ax}{\pi x} = \delta(x) \tag{C.12}$$

Proof

To prove identity 1, it is sufficient to show that this expression for the Dirac delta function satisfies sifting property:

$$\lim_{a \to \infty} \int_{-\infty}^{\infty} f(x) \frac{\sin ax}{\pi x} dx = f(0) \tag{C.13}$$

Breaking the integral into three sections, the outer two of which avoid the problem of dividing by zero at $x = 0$,

$$\int_{-\infty}^{\infty} f(x) \frac{\sin ax}{\pi x} dx = \int_{-\infty}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\infty} \tag{C.14}$$

The first and last integral on the right-hand side are zero by the Riemann–Lebesgue lemma (an important theorem of the Fourier integral that will not be discussed here). The center integral can be evaluated by taking ε to be very small (but not zero). Over this very small range, $f(x)$ will be about constant:

$$\int_{-\varepsilon}^{\varepsilon} f(x) \frac{\sin ax}{\pi x} dx = f(0) \int_{-\varepsilon}^{\varepsilon} \frac{\sin ax}{\pi x} dx \tag{C.15}$$

Taking the limit as a goes to infinity,

$$\lim_{a \to \infty} \int_{-\varepsilon}^{\varepsilon} \frac{\sin ax}{\pi x} dx = \lim_{a \to \infty} \int_{-a\varepsilon}^{a\varepsilon} \frac{\sin x'}{\pi x'} dx' = \int_{-\infty}^{\infty} \frac{\sin x'}{\pi x'} dx' = 1 \tag{C.16}$$

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Thus,

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} f(x)\frac{\sin ax}{\pi x}\mathrm{d}x = f(0) \tag{C.17}$$

C.2.5 Identity 2

$$\int_{-\infty}^{\infty}\cos(2\pi\nu x)\mathrm{d}x = \delta(\nu) \tag{C.18}$$

Proof

The proof simply performs the integration and then applies identity 1.

$$\int_{-\infty}^{\infty}\cos(2\pi\nu x)\mathrm{d}x = \lim_{a\to\infty}\int_{-a}^{a}\cos(2\pi\nu x)\mathrm{d}x = \lim_{a\to\infty}\frac{\sin(2\pi\nu a)}{\pi\nu} = \delta(\nu) \tag{C.19}$$

C.2.6 Identity 3 – $\mathcal{F}\{1\}$

The Fourier transform of one is the delta function:

$$\int_{-\infty}^{\infty}\mathrm{e}^{-i2\pi\nu x}\mathrm{d}x = \delta(\nu) \tag{C.20}$$

Proof

Changing the exponential into a sine and cosine,

$$\int_{-\infty}^{\infty}\mathrm{e}^{-i2\pi\nu x}\mathrm{d}x = \int_{-\infty}^{\infty}\cos(2\pi\nu x)\mathrm{d}x - i\int_{-\infty}^{\infty}\sin(2\pi\nu x)\mathrm{d}x \tag{C.21}$$

Since the sine is an odd function, the sine integral will vanish. Applying identity 2 to the cosine integral completes the proof.

C.2.7 Identity 4 – the Dirac Comb

The following identity is useful in the derivation of the diffraction pattern for a periodic line/space mask pattern with pitch p.

$$p\sum_{n=-\infty}^{\infty}\mathrm{e}^{-i2\pi\nu np} = \sum_{m=-\infty}^{\infty}\delta\left(\nu-\frac{m}{p}\right) \tag{C.22}$$

The function on the right-hand side of Equation (C.22) is called a *Dirac comb* of period p. This identity can be proved by recognizing that the Dirac comb is a periodic function

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that can be easily represented by a Fourier series. Direct calculation of the Fourier coefficients of the complex Fourier series produces Equation (C.22).

C.2.8 Relationship to the Heaviside Step Function

The Heaviside step function is defined as

$$u(x) = \begin{cases} 0, & x < 0 \\ 1 & x \geq 1 \end{cases} \tag{C.23}$$

The step function is related to the Dirac delta function by

$$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x} u(x) \quad \text{and} \quad u(x) = \int_{-\infty}^{x} \delta(t)\mathrm{d}t \tag{C.24}$$

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