More remarkable sinc integrals and sums

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Abstract

We use Poisson summation formula to calculate integrals of products of sinc functions (cf. [4]) and related integrals as in [5] and [3]. We also generalize the one in [5] and introduce other remarkable integrals.

Finally we give a sum version of Siegel-type lower bound. (cf. [2], Theorem 3) Mathematical Subject Classification: 42A38, 42B10, 42A16, 33B10, 26D15 Keywords: sinc integrals and sum, Poisson summation formula.

Introduction.

In 2001 David and Jonathan Borwein in [4] via Fourier transform theory proved that

$$\int_0^\infty \prod_{k=0}^n \operatorname{sinc}(\frac{t}{2k+1}) \, dt = \frac{\pi}{2}, \quad n = 0, 1, 2, 3, 4, 5, 6 \tag{1}$$

but less than $\frac{\pi}{2}$ if n > 6. Quite recently the integral came up again in [5] where the Fourier transformation is clarified by a graphic approach. H. Schmid also proves that

$$\int_0^\infty 2\cos(t) \prod_{k=0}^n \operatorname{sinc}(\frac{t}{2k+1}) dt = \frac{\pi}{2}, \quad n = 0, 1, 2, \dots, 55$$
 (2)

but less than $\frac{\pi}{2}$ if n > 55. We will use the Poisson summation formula to prove both (1) and (2). The formula gives us also a generalization of (2) and other curious sinc integrals. E.g. we find that

$$\int_0^\infty (2\cos(t) + 2\cos(3t)) \prod_{k=0}^n \operatorname{sinc}(\frac{t}{2k+1}) dt = \frac{\pi}{2}, \quad n = 0, 1, 2, \dots, 3090.$$
 (3)

For n > 3090 the value of the integral will be $< \frac{\pi}{2}$.

The other day Bäsel in [3] remarked that (2) via an elementary formula can be deduced from (1).

In [2] Baillie, D. Borwein and J. M. Borwein prove that (Theorem 3)

$$\int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k x) \, dx \ge \int_0^\infty \operatorname{sinc}^{n+1}(a_0 x) \, dx$$

where

$$a_0 > a_k > 0$$
 for $k = 1, 2, \dots, n$.

They also write: "Perhaps a somewhat analogous version of Theorem 3 holds for sums?" We will give an answer.

Theoretical tools.

For appropriate (see [6]) functions the Poisson summation formula may be stated as

$$\sum_{k=-\infty}^{\infty} \hat{f}(\omega - k\Omega) = T \sum_{m=-\infty}^{\infty} f(mT)e^{-im\omega T}$$
(4)

where $T\Omega = 2\pi$ and $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$ (Fourier transform).

If we choose T=1 and $\omega=0$ and if we also assume f to be an even function then we can write (4) as

$$\sum_{m=-\infty}^{\infty} f(m) = \hat{f}(0) + 2\sum_{k=1}^{\infty} \hat{f}(2k\pi) = \int_{-\infty}^{\infty} f(t) dt + 2\sum_{k=1}^{\infty} \hat{f}(2k\pi).$$
 (5)

If we instead choose $\omega = \pi$ we can write (4) as

$$\sum_{m=-\infty}^{\infty} f(m)e^{-im\pi} = \hat{f}(\pi) + \hat{f}(-\pi) + 2\sum_{k=1}^{\infty} \hat{f}(\pi + 2k\pi)$$
$$= \int_{-\infty}^{\infty} 2\cos(\pi t)f(t) dt + 2\sum_{k=1}^{\infty} \hat{f}(\pi + 2k\pi).$$
(6)

We sum this up as

Lemma 1. Let f(t) be even and sufficiently summable and integrable. Then

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} f(t) dt \Leftrightarrow \sum_{k=1}^{\infty} \hat{f}(2k\pi) = 0$$
 (7)

and

$$\sum_{m=-\infty}^{\infty} f(m)e^{-im\pi} = \int_{-\infty}^{\infty} 2\cos(\pi t)f(t) dt \Leftrightarrow \sum_{k=1}^{\infty} \hat{f}(\pi + 2k\pi) = 0.$$
 (8)

From this we build

Theorem 1. Assume that f_0, f_1, \ldots, f_n are even functions and assume that their Fourier transforms have their supports in [-1, 1]. Then

$$\sum_{m=-\infty}^{\infty} f_0(a_0 m) f_1(a_1 m) \cdot \ldots \cdot f_n(a_n m) = \int_{-\infty}^{\infty} f_0(a_0 t) f_1(a_1 t) \cdot \ldots \cdot f_n(a_n t) dt \qquad (9)$$

if a_0, a_1, \ldots, a_n are positive and

$$\sum_{k=0}^{n} a_k < 2\pi. \tag{10}$$

Furthermore

$$\sum_{m=-\infty}^{\infty} f_0(a_0 m) f_1(a_1 m) \cdot \dots \cdot f_n(a_n m) e^{-im\pi}$$

$$= \int_{-\infty}^{\infty} 2\cos(\pi t) f_0(a_0 t) f_1(a_1 t) \cdot \dots \cdot f_n(a_n t) dt$$
(11)

if a_0, a_1, \ldots, a_n are positive and

$$\sum_{k=0}^{n} a_k < 3\pi. \tag{12}$$

Proof. If f(t) is a function with a Fourier transform with support in [-1, 1] then f(at) has a Fourier transform with support in [-a, a]. Put $f(t) = f_1(a_1t)f_2(a_2t) \cdot \ldots \cdot f_n(a_nt)$. The Fourier transform \hat{f} is given by a convolution with support in $[-(a_0 + a_1 + \ldots + a_n), a_0 + a_1 + \ldots + a_n]$. Thus (9) is a consequence of (7). In a similar way we get (11) from (8).

Remark. We used (7) and (9) already in [1].

The idea of the next proposition can be found in [4].

Proposition 1. If f(t) is a function in $L_2(\mathbf{R})$ with a Fourier transform with support in [-a, a] and if f is continuous at t = 0 then

$$\int_{-\infty}^{\infty} f(t) \frac{\sin(bt)}{t} dt = \pi f(0)$$
(13)

if 0 < a < b.

Proof. We observe that $\frac{\sin(bt)}{t}$ has the Fourier transform $\pi(H(\omega+b)-H(\omega-b))$, where H is the Heaviside function given by

$$H(\omega) = \begin{cases} 1, & \text{if } \omega > 0 \\ 0, & \text{if } \omega < 0. \end{cases}$$

Put $f_{\varepsilon}(t) = f(t)\operatorname{sinc}(\varepsilon t)$ where $a < a + \varepsilon < b$. Then f_{ε} is in $L_1(\mathbf{R}) \cap L_2(\mathbf{R})$. According to Parseval's theorem \hat{f}_{ε} has its support in $[-a - \varepsilon, a + \varepsilon]$.

A version of Fourier inversion formula states that

$$g(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} e^{i\omega t} \hat{g}(\omega) d\omega$$

if g and \hat{g} are in $L_1(\mathbf{R})$ and g is continuous at t.

If we combine Parseval's theorem and the Fourier inversion formula we get

$$\int_{-\infty}^{\infty} f_{\varepsilon}(t) \frac{\sin(bt)}{t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_{\varepsilon}(\omega) \pi (H(\omega + b) - H(\omega - b)) d\omega$$
$$= \frac{\pi}{2\pi} \int_{-a-\varepsilon}^{a+\varepsilon} \hat{f}_{\varepsilon}(\omega) d\omega = \pi \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i0\omega} \hat{f}_{\varepsilon}(\omega) d\omega = \pi f_{\varepsilon}(0) = \pi f(0).$$

But since

$$\left| f_{\varepsilon}(t) \frac{\sin(bt)}{t} \right| \le \left| f(t) \frac{\sin(bt)}{t} \right| \in L_1(\mathbf{R})$$

we can use Lebesgue's dominated convergence theorem and let $\varepsilon \to 0$. This will give us (13).

Applications or examples.

Example 1. We will here study (1). If we change the variable t to πt and observe that the integrand is even we have to prove that

$$\int_{-\infty}^{\infty} \prod_{k=0}^{n} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt = 1, \quad n = 0, 1, 2, 3, 4, 5, 6$$
(14)

and < 1 if n > 6.

The function $\operatorname{sinc}(t)$ has the Fourier transform $\pi(H(\omega+1)-H(\omega-1))$. Let the functions f_0, f_1, \ldots, f_n in (9) all be $\operatorname{sinc}(t)$. Consequently $a_k = \frac{\pi}{2k+1}$. The condition (10)

corresponds to $\sum_{k=0}^{n} \frac{1}{2k+1} < 2$. Since

$$\sum_{k=0}^{6} \frac{1}{2k+1} = \frac{88069}{45045} \approx 1.9551$$

but

$$\sum_{k=0}^{7} \frac{1}{2k+1} = \frac{91072}{45045} \approx 2.0218$$

(10) is fulfilled for $n = 0, 1, \dots, 6$ but not for n > 6.

The value of the integral in (14) will now be given by the series on the left hand side of (9). Since

$$f_0(a_0m) = \operatorname{sinc}(\pi m) = \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{otherwise} \end{cases}$$

the sum of the series will be 1. If n > 6 the integral in (14) will be

$$1 - 2\sum_{k=1}^{\infty} \hat{f}(2k\pi) < 1.$$

Cf. (5) and remember that $\hat{f}(\omega)$ is a convolution which is positive for $\omega \in [-(a_0 + a_1 + \ldots + a_n), a_0 + a_1 + \ldots + a_n]$

Example 2. Here we prove (2) and calculate the integral when n = 56. It is difficult to test the calculation with a computer. A straightforward calculation of the integral by means of Maple was no success.

We copy the method in Example 1. Via (11) we find that

$$\int_{-\infty}^{\infty} 2\cos(\pi t) \prod_{k=0}^{n} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt = 1$$

if
$$\sum_{k=0}^{n} \frac{1}{2k+1} < 3$$
. But

$$\sum_{k=0}^{55} \frac{1}{2k+1} \approx 2.994437501$$

and

$$\sum_{k=0}^{56} \frac{1}{2k+1} \approx 3.003287059.$$

and we have proved (2). In order to calculate

$$\int_{-\infty}^{\infty} 2\cos(\pi t) \prod_{k=0}^{56} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt$$

we use (6) with

$$f(t) = \prod_{k=0}^{56} \operatorname{sinc}(\frac{\pi t}{2k+1}).$$

Since f(m) = 0 for $m \neq 0$ and f(0) = 1 we get that

$$\int_{-\infty}^{\infty} 2\cos(\pi t) \prod_{k=0}^{56} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt = 1 - 2 \sum_{k=1}^{\infty} \hat{f}(\pi + 2k\pi).$$

But the support of \hat{f} is [-a, a], where

$$a = \sum_{k=0}^{56} \frac{\pi}{2k+1} \approx 9.435104562.$$

Thus $3\pi < a < 5\pi$ and

$$\int_{-\infty}^{\infty} 2\cos(\pi t) \prod_{k=0}^{56} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt = 1 - 2\hat{f}(3\pi).$$

We have to investigate the Fourier transform \hat{f} , which we also denote by F. We can rewrite f(t) as

$$f(t) = \prod_{k=0}^{56} \operatorname{sinc}(\frac{\pi t}{2k+1}) = \frac{c}{t^{57} \pi^{57}} \prod_{k=0}^{56} \sin(\frac{\pi t}{2k+1})$$

where $c = \prod_{k=1}^{56} (2k+1)$. Via Euler's formulae we continue to

$$t^{57}f(t) = \frac{c}{(2\pi)^{57}i} \prod_{k=0}^{56} \left(e^{i\frac{\pi t}{2k+1}} - e^{-i\frac{\pi t}{2k+1}} \right) = \frac{c}{(2\pi)^{57}i} \sum_{k} e_k e^{id_k t}$$
 (15)

where e_k are coefficients and the d_k are numbers between -a and a. The largest d_k is a and the next largest is $\sum_{k=0}^{55} \frac{\pi}{2k+1} - \frac{\pi}{113} \approx 9.379501153$. In the interval between that number and a we find 3π . On that interval $F(\omega)$ will be a polynomial.

The Fourier transform of tf(t) is $iF'(\omega)$ (at least in the distributional sense) and the Fourier transform of $t^{57}f(t)$ is $i^{57}F^{(57)}(\omega) = iF^{(57)}(\omega)$. Thus if we apply the Fourier transformation to both sides in (15) we get

$$iF^{(57)}(\omega) = \frac{c}{(2\pi)^{57}i}(2\pi\delta(\omega - a) + \text{other } \delta \text{terms})$$

where δ is the Dirac "function". Notice that the Fourier transform of e^{id_kt} is $2\pi\delta(\omega-d_k)$. From

$$F^{(57)}(\omega) = -\frac{c}{(2\pi)^{56}} (\delta(\omega - a) + \text{other } \delta \text{terms })$$

we conclude that

$$F^{(56)}(\omega) = \frac{c}{(2\pi)^{56}}$$
 if $\sum_{k=0}^{55} \frac{\pi}{2k+1} - \frac{\pi}{113} < \omega < a$.

F is built up of 57 convolution factors. Thus F is 55 times differentiable. But $F(\omega) \equiv 0$ if $\omega > a$. All together gives us that

$$F(\omega) = \frac{c}{2^{56}\pi^{56}56!}(\omega - a)^{56} \quad \text{if} \quad \sum_{k=0}^{55} \frac{\pi}{2k+1} - \frac{\pi}{113} < \omega < a.$$
 (16)

Introduce $b = \sum_{k=0}^{56} \frac{1}{2k+1}$. Thus $a = b\pi$ and we can write (16) as

$$F(\omega) = \frac{c}{2^{56}\pi^{56}56!}(\omega - b\pi)^{56}.$$

Now we get

$$2F(3\pi) = \frac{c}{2^{55}\pi^{56}56!}(3\pi - b\pi)^{56} = \frac{c}{2^{55}56!}(3 - b)^{56} = \frac{N}{D}$$

where

$$N = 347^{56} \cdot 39608671351^{56} \cdot 1786013712647720237751897933348037^{56}$$

and

$$D = 2^{53} \cdot 3^{222} \cdot 5^{112} \cdot 13^{56} \cdot 41^{56} \cdot 59^{55} \cdot 61^{55} \cdot 67^{55} \cdot 71^{55} \cdot 73^{55} \cdot 79^{55} \cdot 11^{56} \cdot 17^{56} \cdot 71^{112} \cdot 47^{56} \cdot 29^{55} \cdot 31^{55} \cdot 37^{55} \cdot 43^{56} \cdot 53^{56} \cdot 23^{56} \cdot 19^{55} \cdot 83^{55} \cdot 89^{55} \cdot 97^{55} \cdot 101^{55} \cdot 103^{55} \cdot 107^{55} \cdot 109^{55} \cdot 113^{55}.$$

Finally we get

$$\int_{-\infty}^{\infty} 2\cos(\pi t) \prod_{k=0}^{56} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt = 1 - 2F(3\pi) = 1 - \frac{N}{D}$$
$$\approx 1 - 1.484870809 \cdot 10^{-138}.$$

The integral is apparently only slightly smaller than 1.

Example 3. In order to prove (3) we prove that

$$\int_{-\infty}^{\infty} (2\cos(\pi t) + 2\cos(3\pi t)) \prod_{k=0}^{n} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt = 1, \quad n = 0, 1, 2, \dots, 3090.$$

We write (6) as

$$\sum_{m=-\infty}^{\infty} f(m)e^{-im\pi} = \int_{-\infty}^{\infty} (2\cos(\pi t) + 2\cos(3\pi t))f(t) dt + 2\sum_{k=2}^{\infty} \hat{f}(\pi + 2k\pi).$$

To finish the argument we observe that

$$\sum_{k=0}^{3090} \frac{\pi}{2k+1} \approx 15.70758624 < 5\pi \approx 15.70796327 < \sum_{k=0}^{3091} \frac{\pi}{2k+1} \approx 15.70809434.$$

With the same technique we get

$$\int_{-\infty}^{\infty} (2\cos(\pi t) + 2\cos(3\pi t) + 2\cos(5\pi t)) \prod_{k=0}^{n} \operatorname{sinc}(\frac{\pi t}{2k+1}) dt = 1,$$

$$n = 0, 1, 2, \dots, 168802.$$

Example 4. It is not quite easy to find a pattern for "the breaking points" n = 6,3090 and 168802, which we have met above. But if we change $\prod_{k=0}^{n} \operatorname{sinc}(\frac{\pi t}{2k+1})$ to $\operatorname{sinc}^{n}(\pi t)$ this problem can be solved.

With experience from the other examples we now rewrite (6) as

$$\sum_{m=-\infty}^{\infty} f(m)e^{-im\pi} = \int_{-\infty}^{\infty} 2\sum_{k=0}^{m} \cos((2k+1)\pi t)f(t) dt + 2\sum_{k=m+1}^{\infty} \hat{f}((2k+1)\pi).$$
 (17)

Put $f(t) = \operatorname{sinc}^n(\pi t)$. Then the left hand side in (17) will be 1 and the points $(2k+1)\pi$, $k = m+1, m+2, \ldots$ will all be outside the support of \hat{f} if $n\pi \leq (2m+3)\pi$. Thus

$$\int_{-\infty}^{\infty} 2\sum_{k=0}^{m} \cos((2k+1)\pi t) \operatorname{sinc}^{n}(\pi t) dt = \begin{cases} 1, & \text{if } n = 0, 1, 2, \dots, 2m+3 \\ < 1, & \text{if } n = 2m+4, 2m+5, \dots \end{cases}$$

Example 5. Let us introduce a function which is not a sinc function. Put

$$f(t) = \frac{t\sin(t) - \cos(t) + e}{(1 + t^2)(e - 1)}.$$

Then f(0) = 1 and

$$\hat{f}(\omega) = \begin{cases} \frac{\pi}{1 - e^{-1}} e^{-|\omega|}, & \text{if } |\omega| < 1\\ 0, \text{otherwise.} \end{cases}$$

Now we get another curious integral

$$\int_{-\infty}^{\infty} f(a_0 t) f(a_1 t) \cdot \dots \cdot f(a_n t) \frac{\sin(bt)}{t} dt = \pi, \quad \text{if } a_0 + a_1 + \dots + a_n < b.$$

We also assume that all $a_k > 0$ and that b > 0.

Proof. Alternative 1. Proposition 1 delivers a proof directly.

Alternative 2. If we change t to $\frac{\pi}{h}t$ we get

$$\int_{-\infty}^{\infty} f(a_0 t) f(a_1 t) \cdot \dots \cdot f(a_n t) \frac{\sin(bt)}{t} dt$$

$$= \int_{-\infty}^{\infty} f(\frac{\pi a_0}{b} t) f(\frac{\pi a_1}{b} t) \cdot \dots \cdot f(\frac{\pi a_n}{b} t) \frac{\sin(\pi t)}{t} dt = 1$$

if (see (10))

$$\frac{\pi a_0}{b} + \frac{\pi a_1}{b} + \ldots + \frac{\pi a_n}{b} + \pi < 2\pi \Leftrightarrow a_0 + a_1 + \ldots + a_n < b.$$

Remark. With the same technique as in alternative 2 we can use Theorem 1 to handle the last curious integrals $I_n(b)$ in [3].

Example 6. In [2] a "Lower Bound" result is proved. If $a_0 \ge a_k$ for k = 1, 2, ..., n then

$$\int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k x) \, dx \ge \int_0^\infty \operatorname{sinc}^{n+1}(a_0 x) \, dx. \tag{18}$$

If $(n+1)a_0 < 2\pi$ we can use Theorem 1 to translate the two integrals to sums. Thus

$$\sum_{m=0}^{\infty} \prod_{k=0}^{n} \operatorname{sinc}(a_k m) \ge \sum_{m=0}^{\infty} \operatorname{sinc}^{n+1}(a_0 m).$$

which is an analogous version to (18) for sums. Since

the condition $(n+1)a_0 < 2\pi$ cannot be omitted.

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