Ejercicio 1: Sea $(M, g)$ v.r. Dada $f: M \to \mathbb{R}$

se define el gradiente como la 1-forma dada por

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Siendo $(x^i)$ coordenadas arbitrarias. Considera el campo grad $f$ como el único que satisface

$$\langle \text{grad } f, X \rangle = df(X) \qquad \forall X \in \mathfrak{X}(M)$$

Expresa grad $f$ en las coordenadas $(x^i)$. Este campo se llama "gradiente métrico". ¿Cuándo coinciden las coordenadas del campo grad $f$ y las de la 1-forma $df$?

Dem:

Por definición de gradiente de $f$, para todo campo se cumple que

$$\langle \text{grad } f, X \rangle = df(X)$$

En particular se cumple para el campo $X = \partial x^j$

$$df(\partial x^j) = \langle \text{grad } f, \partial x^j \rangle$$

Escribimos el grad $f$ en (~~coordenadas~~) coordenadas:

$$\text{grad } f = G^i \partial x^i$$

Por tanto

$$\frac{\partial f}{\partial x^j} =: df(\partial x^j) = \langle G^i \partial x^i, \partial x^j \rangle = G^i \langle \partial x^i, \partial x^j \rangle =$$

$$= G^i g_{ij}$$

$$df(v) := v(f)$$

Multiplicando a ambos lados por $g^{kj}$ tenemos

$$\frac{\partial f}{\partial x^j} g^{kj} = G^i g_{ij} g^{kj}$$

$$\frac{\partial f}{\partial x^j} g^{kj} = G^i \delta_i^k .$$

$$\Rightarrow G^i = g^{ij} \frac{\partial f}{\partial x^j}$$

Nos preguntamos cuando coinciden las coordenadas del campo grad $f$ con las de la 1-forma $df$, es decir:

$$\circledast \quad ¿ \; G^i = \left.\frac{\partial}{\partial x^i}\right|_p (f) \; ? \qquad i = 1, 2, \ldots, n$$

$$\Longleftrightarrow$$

$$\underbrace{g^{ij} \frac{\partial f}{\partial x^j}}_{\text{Suma en } j} = \left.\frac{\partial}{\partial x^i}\right|_p (f) \qquad \forall i$$

$$\Longleftrightarrow$$

$$\begin{cases} g^{ii} = 1 \\ g^{ij} = 0 \quad \forall i \neq j \end{cases}$$

Es decir, se cumple $\circledast$ si y solo si $(g^{ij}) \equiv Id_n$ o sea $g(\cdot, \cdot) =: \langle \cdot , \cdot \rangle$ es la métrica euclídea

**Ejercicio 2:** Con la notación del ejercicio anterior, supongamos que $|\text{grad } f| = 1$. Demuestra entonces que $\nabla_X X = 0$ siendo $X = \text{grad } f$.

Indicación: Prueba que $\langle \nabla_X X, Y \rangle = 0$ $\forall Y$ usando $|X| = 1$, definición de grad $f$ y la simetría de la conexión.

**Dem:**

Hacemos la demostración en 2 pasos:

1. Pruebo que $\forall Y, Z \in \mathfrak{X}(M)$ se verifica:

$$\langle \nabla_Z X, Y \rangle = \langle \nabla_Y X, Z \rangle$$

2. Demuestro que $\langle \nabla_X X, Y \rangle = 0$ $\forall Y \in \mathfrak{X}(M)$

En efecto usando 1. y poniendo $Z = X$ tenemos

$$\langle \nabla_X X, Y \rangle = \langle \nabla_Y X, X \rangle = 0$$

Pruebo la igualdad ANTERIOR

$$0 = Y\langle X, X \rangle = \langle \nabla_Y X, X \rangle + \langle X, \nabla_Y X \rangle =$$

$$= 2\langle \nabla_Y X, X \rangle$$

$1 = |X|^2 = \langle X, X \rangle$ Compatibilidad

$\Rightarrow \langle \nabla_Y X, X \rangle = 0$ como queríamos probar

Por tanto si demuestro 1 tendríamos el resultado.

Sean $Y, Z \in \mathfrak{X}(M)$ campos arbitrarios y $X = \operatorname{grad} f$

$$\langle \nabla_Y X, Z \rangle - \langle \nabla_Z X, Y \rangle =$$

$$= \langle \nabla_Y X, Z \rangle - \langle \nabla_Z X, Y \rangle + \langle X, \overbrace{\nabla_Y Z - \nabla_Z Y - [Y, Z]}^{=0 \text{ (simetría)}} \rangle$$

$$= \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle - \langle \nabla_Z X, Y \rangle - \langle X, \nabla_Z Y \rangle - \langle X, [Y, Z] \rangle$$

(compatibilidad)

$$\hookrightarrow = Y \langle X, Z \rangle - Z \langle X, Y \rangle - \langle X, [Y, Z] \rangle$$

(def. de gradiente)

$$\hookrightarrow = Y\, df(Z) - Z\, df(Y) - df\,[Y, Z]$$

$$= Y df(Z) - Z\, df(Y) - df\, YZ + df\, ZY = 0$$

$$\Rightarrow \langle \nabla_Z X, Y \rangle = \langle \nabla_Y X, Z \rangle \quad \forall Y, Z \in \mathfrak{X}(M)$$

Como queríamos demostrar.

$\square$

Ejercicio 3: Sea $M = \{(x, y) \mid y > 0\} \subset \mathbb{R}^2$ con la métrica (——) dada por $g = e^{2y}(dx^2 + dy^2)$

Sea $\alpha(t) = (0, -t)$ con $-2 \leq t \leq -1$.

a) Calcula $L(\alpha)$ entre $-2$ y $-1$. ¿Es una geodésica?

b) Define $s(t) = \int |\alpha'(u)| \, du$ y considera $\beta(s) = \alpha(t(s))$

Demuestra que $\beta$ es geodésica.

c) Sea $\gamma$ una curva arbitraria uniendo $(0,2)$ y $(0,1)$

Prueba que $L(\gamma) \geq L(\beta) = L(\alpha)$

d) Demuestra que $F(x, y) = (-x, y)$ es una isometría de $(M, g)$. Utiliza este hecho para demostrar que $\beta$ es geodésica

e) ¿Alcanza $\beta$ la frontera $y = 0$ en "tiempo finito"?

f) Calcula la curvatura seccional de este espacio. ¿Puede ser isométrico al semiplano hiperbólico? Justifica tu respuesta.

## Apartado a)

Por definición $L_a^b(\alpha) = \int_a^b |\alpha'| \, dt$

Calculamos la derivada y modulo de $\alpha(t) = (0, -t)$

$\alpha'(t) = (0, -1)$

$|\alpha'(t)|^2 = |g(\alpha', \alpha')| = |e^{2y}(\alpha_x'^2 + \alpha_y'^2)| =$

$= |e^{2(-t)}(0^2 + (-1)^2)| = |-e^{-2t}| = e^{-2t}$

$\uparrow$
y es la función coordenada

$\Rightarrow |\alpha'(t)| = e^{-t}$, luego $\alpha$ no es geodésica ya que hemos demostrado que su modulo no es constante.

Calculamos

$$L_{-2}^{-1}(\alpha) = \int_{-2}^{-1} |\alpha'(t)| \, dt = \int_{-2}^{-1} e^{-t} \, dt = -[e^{-t}|_{-2}^{-1}$$

$$= -(e - e^2) = e^2 - e = e(e-1)$$

## Apartado b)

La función $S(t)$ se define por

$$s(t) = \int |\alpha'(u)| \, du = \int e^{-u} \, du = -e^{-u} + \underset{\text{cte}}{K}$$

Determinamos la función $t(s)$

$s(t) = -e^{-u} + K \iff$ ( ~~$\log(s(t)) = -u + C \iff$~~ )

$\iff e^{-u} = K - s(t) \iff -u = \log(K - s(t)) \iff$

$\Leftrightarrow t(s) = u = -\log(K - s(t))$

Por tanto

$$\beta(s) = \alpha(t(s)) = \alpha(-\log(K - s(t))) = (0, \log(K - s(t)))$$

$\uparrow$ $\boxed{\text{definición de } \alpha}$

A continuación calculamos los <u>símbolos de Christoffel</u> para comprobar que $\beta$ es geodésica

$$(g_{ij}) = \begin{pmatrix} e^{2y} & 0 \\ 0 & e^{2y} \end{pmatrix} \longrightarrow (g^{ij}) = \begin{pmatrix} e^{-2y} & 0 \\ 0 & e^{-2y} \end{pmatrix}$$

$$\Gamma_{11}^{1} = \frac{1}{2} g^{11} \left( \frac{\partial g_{11}}{\partial x} + \cancel{\frac{\partial g_{11}}{\partial x}} - \cancel{\frac{\partial g_{11}}{\partial x}} \right) = \frac{1}{2} e^{-2y} \underbrace{\cancel{\frac{\partial}{\partial x}(e^{2y})}}_{0} = 0$$

$$\Gamma_{21}^{1} = \Gamma_{12}^{1} = \frac{1}{2} g^{11} \left( \frac{\partial g_{11}}{\partial y} + \overset{0}{\overset{\shortparallel}{\frac{\partial g_{21}}{\partial x}}} - \overset{0}{\overset{\shortparallel}{\frac{\partial g_{12}}{\partial x}}} \right) = \frac{1}{2} e^{-2y} \frac{\partial}{\partial y}(e^{2y})$$

$$= \frac{1}{2} e^{-2y}\, 2 e^{2y} = 1$$

$$\Gamma_{22}^{1} = \frac{1}{2} g^{11} \left( \underbrace{\cancel{\frac{\partial g_{21}}{\partial y}}}_{0} + \underbrace{\cancel{\frac{\partial g_{21}}{\partial y}}}_{0} - \underbrace{\cancel{\frac{\partial g_{12}}{\partial x}}}_{0} \right) = 0$$

$$\Gamma_{11}^{2} = \frac{1}{2} g^{22} \left( \underbrace{\cancel{\frac{\partial g_{12}}{\partial x}}}_{0} + \underbrace{\cancel{\frac{\partial g_{12}}{\partial y}}}_{0} - \frac{\partial g_{11}}{\partial y} \right) = -\frac{1}{2} e^{-2y} \frac{\partial}{\partial y}(e^{2y}) =$$

$$= -\frac{1}{2} e^{-2y} \cdot 2 \cdot e^{2y} = -1$$

$$\Gamma_{21}^{2} = \Gamma_{12}^{2} = \frac{1}{2} g^{22} \left( \overset{0}{\overset{\shortparallel}{\frac{\partial g_{12}}{\partial y}}} + \underbrace{\cancel{\frac{\partial g_{12}}{\partial x}}}_{0} - \overset{0}{\overset{\shortparallel}{\frac{\partial g_{12}}{\partial y}}} \right) = 0$$

$$\Gamma_{22}^{2} = \frac{1}{2} g^{22} \left( \frac{\partial g_{22}}{\partial y} + \cancel{\frac{\partial g_{y2}}{\partial y}} - \cancel{\frac{\partial g_{y2}}{\partial y}} \right) =$$

$$= \frac{1}{2} e^{-2y} \frac{\partial}{\partial y} \left( e^{2y} \right) = \frac{1}{2} e^{-2y} \, 2 e^{2y} = 1$$

Por tanto, $\beta(s) = (0, \log(k-s))$ es geodésica si:

$$= (\beta_1(s), \beta_2(s))$$

$$\frac{d^2 \beta^k}{ds^2} + \frac{d\beta^i}{ds} \frac{d\beta^j}{ds} \Gamma_{ij}^{k} = 0 \quad \text{para } k = 1, 2$$

$$\begin{cases} \dfrac{d^2\beta_1}{ds^2} + \dfrac{d\beta_1}{ds} \dfrac{d\beta_2}{ds} \cdot 1 + \dfrac{d\beta_2}{ds} \dfrac{d\beta_1}{ds} = 0 \\[2ex] \dfrac{d^2\beta^2}{ds^2} + \left( \dfrac{d\beta_1}{ds} \right)^2 (-1) + \left( \dfrac{d\beta_2}{ds} \right)^2 (1) = 0 \end{cases}$$

$$\begin{cases} \underbrace{\cancel{\dfrac{d^2\beta_1}{ds^2}}}_{0} + 2 \underbrace{\cancel{\dfrac{d\beta_1}{ds}}}_{0} \dfrac{d\beta_2}{ds} = 0 \quad \checkmark \\[2ex] \dfrac{d^2\beta_2}{ds^2} - \underbrace{\left( \cancel{\dfrac{d\beta_1}{ds}} \right)^2}_{0} + \left( \dfrac{d\beta_2}{ds} \right)^2 = 0 \end{cases}$$

$\Rightarrow$ Se debe cumplir que

$$\frac{d^2\beta_2}{ds^2} + \left( \frac{d\beta_2}{ds} \right)^2 = 0$$

$$\iff \frac{d^2}{ds^2}(\log(K-s)) + \left(\frac{d}{ds}(\log(K-s))\right)^2 = 0$$

$$\iff \frac{d}{ds}\left(\frac{-1}{K-s}\right) + \left(-\frac{1}{K-s}\right)^2 =$$

$$-\frac{(-1)(-1)}{(K-s)^2} + \frac{1}{(K-s)^2} = \frac{-1}{(K-s)^2} + \frac{1}{(K-s)^2} = 0 \checkmark$$

$\Rightarrow \beta(s)$ es geodésica, ya que verifica las ecuaciones.

__Apartado c)__

Sea $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ uniendo $(0,2)$ y $(0,1)$.

[redacted] [redacted]

Observamos que $\beta(s) = (0, \log(K-s))$, $K = cte$

es una geodésica que también une los puntos $(0,2)$ y $(0,1)$.

En efecto

- $\beta(s) = (0, \log(K-s)) = (0,2) \iff \log(K-s) = 2 \iff s = K - e^2$
- $\beta(s) = (0, \log(K-s)) = (0,1) \iff \log(K-s) = 1 \iff s = K - e$

Puesto que las geodésicas son curvas que localmente minimizan la distancia tenemos que

$$L(\gamma) \geq L(\beta)$$

A continuación demuestro que $L(\beta) = L(\alpha)$

- $\alpha(t) = (0, -t)$ con $-2 \leq t \leq -1$

$$\left.\begin{array}{l} t=-2 \ , \ \alpha(-2) = (0,2) \\ t=-1 \ , \ \alpha(-1) = (0,1) \end{array}\right| \Rightarrow$$ $\alpha$ es una curva uniendo $(0,2)$ y $(0,1)$

- $\beta(s) = (0, \log(k-s))$

$$\left.\begin{array}{l} s = k-e^2 \ , \ \beta(k-e^2) = (0,2) \\ s = k-e \ , \ \beta(k-e) = (0,1) \end{array}\right| \Rightarrow$$ $\beta$ es una curva uniendo $(0,2)$ y $(0,1)$

Calculamos sus longitudes

$$\boxed{L_{-2}^{-1}(\alpha)} = \int_{-2}^{-1} |\alpha'(t)| \, dt \underset{\text{Apartado (a)}}{=} \boxed{e(e-1)}$$

$$\boxed{L_{k-e^2}^{k-e}(\beta)} = \int_{k-e^2}^{k-e} |\beta'(s)| \, ds \underset{(*)}{=} \int_{k-e^2}^{k-e} 1 \, ds = s \Big|_{k-e^2}^{k-e} = k-e-(k-e^2)$$

$$= -e + e^2$$

$$= \boxed{e(e-1)}$$

$(*)$ $\beta'(s) = -\frac{1}{k-s}$

$$|\beta'(s)|^2 = \left| e^{2y}\left(0^2 + \left(\frac{1}{k-s}\right)^2\right)\right| = \left| e^{2\log(k-s)} \frac{1}{(k-s)^2}\right|$$

$$= \left|(k-s)^2 \frac{1}{(k-s)^2}\right| = 1 \Rightarrow \boxed{|\beta'(s)| = 1}$$

$$\Rightarrow \quad L(\alpha) = e(e-1) = L(\beta)$$

Apartado d)

$$F(x,y) := (-x,y)$$

Tenemos el diagrama

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "Id"'] & M \arrow[d, "Id"] \\ (x,y) \arrow[r, dotted] & (-x,y) \end{tikzcd}$$

$$dF_{(x,y)}(v) = -v_x\,\partial x + v_y\,\partial y \in T_{(x,y)}M$$

$\forall\, v \in T_{(x,y)}M$ se cumple que

$$\|v\|^2 = g(v,v) = e^{2y}(v_x^2+v_y^2) = e^{2y}\overbrace{((-v_x)^2+v_y^2) = \|dF(v)\|^2}^{= g(dF(v),dF(v)) =}$$

$$\Rightarrow \quad \|v\| = \|dF(v)\| \quad \forall\, v \in T_{(x,y)}M$$

$$\Rightarrow \quad g_{(x,y)}(v,w) = g_{F(x,y)}(dF(v), dF(w))$$

Es decir, $F$ es una isometría

$\square$

## Apartado e)

Sí, en efecto como $\beta(s) = (0, \log(k-s)) = (\beta_1(s), \beta_2(s))$

$$\beta_2(s) = \log(k-s) = 0 \iff k-s = 1 \iff s = -1+k$$

## Apartado f)

Consideramos $\{\partial u, \partial v\}$ siendo $X$ parametrización de $M$
$$\begin{array}{cc} \partial u & \partial v \\ \parallel & \parallel \\ X_u & X_v \end{array}$$

Tenemos que

$$g(\partial u, \partial u) = e^{2v}$$
$$g(\partial u, \partial v) = g(\partial v, \partial u) = 0$$
$$g(\partial v, \partial v) = e^{2v}$$

Matriz de la métrica

$$\begin{pmatrix} e^{2v} & 0 \\ 0 & e^{2v} \end{pmatrix}$$

La curvatura seccional viene dada por

$$K = \frac{R_m(\partial u, \partial v, \partial u, \partial v)}{Q(\partial u, \partial v)}$$

Donde $Q(\partial u, \partial v) = \langle \partial u, \partial u \rangle \langle \partial v, \partial v \rangle - \langle \partial u, \partial v \rangle^2$

Calculamos el numerador

$$R_m(\partial_u, \partial_v, \partial_u, \partial_v) = \langle \partial_u, R(\partial_v, \partial_u, \partial_v) \rangle$$

donde

$$R(\partial_v, \partial_u, \partial_v) = \nabla_{\partial_u} \nabla_{\partial_v} \partial_v - \nabla_{\partial_v} \nabla_{\partial_u} \partial_v - \cancel{\nabla_{[\partial_u, \partial_v]} \partial_v}$$

$0$

$$= \nabla_{\partial_u} \nabla_{\partial_v} \partial_v - \nabla_{\partial_v} \nabla_{\partial_u} \partial_v$$

Usando los S. Christ. del apartado (a):

$$\nabla_{\partial_u} \partial_u = \Gamma^1_{11} \partial_u + \Gamma^2_{11} \partial_v = -\partial_v$$

$$\nabla_{\partial_u} \partial_v = \Gamma^1_{21} \partial_u + \Gamma^2_{21} \partial_v = \partial_u$$

$$\nabla_{\partial_v} \partial_u = \Gamma^1_{12} \partial_u + \Gamma^1_{12} \partial_v = \partial_u$$

$$\nabla_{\partial_v} \partial_v = \Gamma^1_{22} \partial_u + \Gamma^2_{22} \partial_v = \partial_v$$

$$\Rightarrow R(\partial_v, \partial_u, \partial_v) = \nabla_{\partial_u} \partial_v - \nabla_{\partial_v} \partial_u = \partial_u - \partial_u = 0$$

$$\Rightarrow K \equiv 0 \quad \text{(curvatura seccional nula)}$$

M no puede ser isométrico al semiplano hiperbólico, ya que al ser superficie Riemanniana, <u>su curvatura seccional coincide con su curvatura de Gauss</u>, es decir $K \equiv 0$ también. Mientras que el semiplano hiperbólico tiene curvatura de Gauss constante negativa. Como la c. de Gauss se conserva por isometrías, M no puede ser isométrica al semiplano hiperbólico.

ej. 5

Ejercicio 4: Sea $\mathbb{H}^2$ el semiplano de Poincaré, i.e., el semiplano $\{(x,y) \mid y>0\}$ con la métrica

$$\breve{g} = \frac{1}{y^2}(dx^2 + dy^2)$$

Calcula los símbolos de Christoffel y la curvatura seccional de $\mathbb{H}^2$. Sea ahora $\vec{v} = (0,1) \in T_{(0,1)}\mathbb{H}^2$ y sea $V(t)$ el transporte paralelo de $\vec{v}$ a lo largo de $\alpha(t) = (t,1)$. Demuestra

a) Si $V(t) = (a(t), b(t))$, entonces $a(t)^2 + b(t)^2 = 1$ por lo que $\exists\, \rho(t)$ tq $V(t) = (\cos\rho(t), \operatorname{sen}\rho(t))$

b) Plantea las ecuaciones del transporte paralelo y concluye que $\rho'(t) \equiv 1$

c) Demuestra finalmente que $\rho(t) = \frac{\pi}{2} - t$

**Dem**

$$(g_{ij}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \longrightarrow g^{ij} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$$

Símbolos de Christoffel

$$\boxed{\Gamma^1_{11}} = \frac{1}{2} \underset{\substack{\uparrow \\ m=1}}{g^{1m}} \left( \underset{0}{\cancel{\frac{\partial g_{11}}{\partial x}}} + \underset{0}{\cancel{\frac{\partial g_{11}}{\partial x}}} - \underset{0}{\cancel{\frac{\partial g_{11}}{\partial x}}} \right) = \boxed{0}$$

$$\boxed{\Gamma^1_{21} = \Gamma^1_{12}} = \frac{1}{2} \underset{\substack{\uparrow \\ m=1}}{g^{11}} \left( \frac{\partial g_{11}}{\partial y} + \underset{0}{\cancel{\frac{\partial g_{21}}{\partial x}}} - \overset{0}{\overset{\parallel}{\frac{\partial g_{12}}{\partial x}}} \right) = \frac{1}{2} y^2 \frac{\partial}{\partial y}\left(\frac{1}{y^2}\right)$$

$$= \boxed{-\frac{1}{y}}$$

$$\boxed{\Gamma^1_{22}} = \frac{1}{2} \underset{\substack{\downarrow \\ m=1}}{g^{11}} \left( \overset{0}{\overset{\shortparallel}{\frac{\partial g_{21}}{\partial y}}} + \overset{0}{\overset{\shortparallel}{\frac{\partial g_{21}}{\partial y}}} - \underset{0}{\cancel{\frac{\partial g_{22}}{\partial x}}} \right) = \boxed{0}$$

$$\boxed{\Gamma^2_{11}} = \frac{1}{2} \underset{\substack{\downarrow \\ m=2}}{g^{22}} \left( \overset{0}{\overset{\shortparallel}{\frac{\partial g_{12}}{\partial x}}} + \overset{0}{\overset{\shortparallel}{\frac{\partial g_{12}}{\partial x}}} - \frac{\partial g_{11}}{\partial y} \right) =$$

$$= \frac{1}{2} y^2 \left( - \frac{\partial}{\partial y} \left( \frac{1}{y^2} \right) \right) = - \frac{1}{\cancel{2}} y^2 (-\cancel{2}) \frac{1}{y^3} =$$

$$= \boxed{\frac{1}{y}}$$

$$\boxed{\Gamma^2_{21} = \Gamma^2_{12}} = \frac{1}{2} \underset{\substack{\downarrow \\ m=2}}{g^{22}} \left( \overset{0}{\overset{\shortparallel}{\frac{\partial g_{12}}{\partial y}}} + \underset{0}{\cancel{\frac{\partial g_{22}}{\partial x}}} - \overset{\partial}{\overset{\shortparallel}{\frac{\partial g_{12}}{\partial y}}} \right) = \boxed{0}$$

$$\boxed{\Gamma^2_{22}} = \frac{1}{2} \underset{\substack{\downarrow \\ m=2}}{g^{22}} \left( \frac{\partial g_{22}}{\partial y} + \cancel{\frac{\partial g_{22}}{\partial y}} - \cancel{\frac{\partial g_{22}}{\partial y}} \right) = \frac{1}{2} y^2 \frac{\partial}{y} \left( \frac{1}{y^2} \right)$$

$$= \frac{1}{\cancel{2}} y^2 (-\cancel{2}) \frac{1}{y^3} = \boxed{-\frac{1}{y}}$$

Calculamos la curvatura seccional de $\mathbb{H}^2$.

Tomamos $(x, y)$ coordenadas en $(\mathbb{H}^2, \breve{g})$. Por tanto

$$K(p) = \frac{R_m(\partial x, \partial y, \partial x, \partial y)}{Q(\partial x, \partial y)}$$

$$R_m(\partial x, \partial y, \partial x, \partial y) = \langle \partial x, R(\partial y, \partial x, \partial y) \rangle$$

donde $R(\partial y, \partial x, \partial y) = \nabla_{\partial x} \nabla_{\partial y} \partial y - \nabla_{\partial y} \nabla_{\partial x} \partial y -$

$$- \underbrace{\nabla_{[\partial x, \partial y]} \partial y}_{=0} =$$

$$= \nabla_{\partial x} \nabla_{\partial y} \partial y - \nabla_{\partial y} \nabla_{\partial x} \partial y \ \textcircled{=}$$

$$\left\{ \begin{array}{l}
\nabla_{\partial x} \partial x = \underset{\underset{0}{\parallel}}{\Gamma_{11}^1} \partial x + \underset{\underset{\frac{1}{y}}{\parallel}}{\Gamma_{11}^2} \partial y = \frac{1}{y} \partial y \\[2ex]
\nabla_{\partial x} \partial y = \underset{\underset{-\frac{1}{y}}{\parallel}}{\Gamma_{21}^1} \partial x + \underset{\underset{0}{\parallel}}{\Gamma_{21}^2} \partial y = -\frac{1}{y} \partial x \\[2ex]
\nabla_{\partial y} \partial x = \underset{\underset{-\frac{1}{y}}{\parallel}}{\Gamma_{12}^1} \partial x + \underset{\underset{0}{\parallel}}{\Gamma_{12}^2} \partial y = -\frac{1}{y} \partial x \\[2ex]
\nabla_{\partial y} \partial y = \underset{\underset{0}{\parallel}}{\Gamma_{22}^1} \partial x + \underset{\underset{-\frac{1}{y}}{\parallel}}{\Gamma_{22}^2} \partial y = -\frac{1}{y} \partial y
\end{array} \right.$$

$\boxed{\text{Leibniz}}$

$$\textcircled{=} \nabla_{\partial x}\left(-\frac{1}{y}\right)\partial y - \nabla_{\partial y}\left(-\frac{1}{y}\right)\partial x =$$

$$= \underbrace{\cancel{\frac{\partial}{\partial x}\left(-\frac{1}{y}\right)}}_{0}\partial y + \left(-\frac{1}{y}\right)\nabla_{\partial x} \partial y - \frac{\partial}{\partial y}\left(-\frac{1}{y}\right)\partial x - \left(-\frac{1}{y}\right)\nabla_{\partial y} \partial x$$

$$= -\cancel{\frac{1}{y}\left(-\frac{1}{y}\right)\partial x} + \cancel{(-1)\frac{1}{y^2}\partial x} + \frac{1}{y}\left(-\frac{1}{y}\right)\partial x = \boxed{-\frac{1}{y^2}\partial x}$$

$$\Rightarrow R_m(\partial x, \partial y, \partial x, \partial y) = \langle \partial x, -\frac{1}{y^2}\partial x \rangle =$$

$$= -\frac{1}{y^2} g_{11} = -\frac{1}{y^4}$$

Por otro lado

$$Q(\partial x, \partial y) = \langle \partial x, \partial x\rangle \langle \partial y, \partial y\rangle - \underbrace{\langle \partial x, \partial y\rangle^2}_{(g_{12})^2 = 0} =$$

$$= g_{11} \cdot g_{22} = \frac{1}{y^2}\cdot\frac{1}{y^2} = \frac{1}{y^4}$$

$$\Rightarrow \boxed{K(p) = \frac{-\frac{1}{y^4}}{\frac{1}{y^4}} = -1} \text{ curvatura seccional}$$

Apartado a) Tenemos la siguiente cadena de igualdades

$$V(\alpha(t)) = V(t) = (a(t), b(t)) = (a(\alpha(t)), b(\alpha(t)))$$
$$= (a(t,1), b(t,1))$$

$$|V(t)|^2 = \breve{g}(V(t), V(t)) = \frac{1}{1^2}(a^2(t) + b^2(t)) \overset{(*)_1}{=} 1$$

$(*)_1$ Por ser $V$ campo paralelo (definición de transporte paralelo) entonces $|V| = \text{cte}$. Puesto que $V$ es el transporte paralelo del vector $\vec{v} = (0,1)$, por definición tiene que ser $V(t_0) = \vec{v} = (0,1)$ donde $|\vec{v}| = 1$

$$\left\{\begin{array}{l} |V(t)| = \text{cte} \\ |V(t_0)| = 1 \end{array}\right. \Bigg| \Rightarrow |V(t)| = 1$$

$$\Rightarrow \boxed{1} = |V(t)|^2 = \frac{1}{\underset{= 1^2 \text{ (función coordenada)}}{y^2}}(a^2(t) + b^2(t)) = \boxed{a^2(t) + b^2(t)}$$

Demostramos la existencia de una función $\theta(t)$ tal que

$$(a(t), b(t)) = V(t) = (\cos\theta(t), \operatorname{sen}\theta(t))$$

$\overset{\vee}{0}$ en el semiplano $\{y>0\}$.

Es claro que

$(*_2)$ $a^2(t) + b^2(t) = 1 = \cos^2\theta(t) + \operatorname{sen}^2\theta(t)$

$\uparrow$ relación fundamental

Ponemos

$(*_3)$ $a(t) = \cos\theta(t) \Rightarrow \theta(t) = \arccos a(t)$

$$¿\ V(t) = (\underbrace{\cos\arccos a(t)}_{=a(t)}, \underbrace{\operatorname{sen}\arccos a(t)}_{\pm b(t) \text{ por } (*_2) \text{ y } (*_3)})?$$

$a(t) = \cos\theta(t) \in [-1,1] \Rightarrow \arccos([-1,1]) \in [0,\pi]$

$\Rightarrow \operatorname{sen}([0,\pi]) > 0 \Rightarrow \operatorname{sen}\arccos a(t) = +b(t) > 0$

$\boxed{\text{ya que estamos trabajando sobre } \mathbb{H}^2 = \{(x,y) \mid y>0\}}$

Por tanto la función $\theta(t)$ existe y verifica

$$(a(t), b(t)) = V(t) = (\cos\theta(t), \operatorname{sen}\theta(t)) \checkmark$$

Apartado b) Como $V(t)$ es campo paralelo along $\alpha$, por definición es:

$$\nabla_{\alpha'(t)} V =: \frac{DV}{dt} = 0$$

Por tanto $V$ es paralelo si:

$$\frac{dV^k}{dt} + \frac{d\gamma^i}{dt} V^j \Gamma^k_{ji} = 0 \qquad K = 1,2$$

Usando los símbolos de Christoffel calculados antes tenemos que las ecuaciones del transporte paralelo son:

$$(K=1) \qquad a'(t) + \frac{d\alpha^1}{dt}(t) V^1(t) \Gamma^1_{11} + \frac{d\alpha^1}{dt}(t) V^2(t) \Gamma^1_{21}$$

$$+ \underbrace{\frac{d\alpha^2}{dt}}_{0}(t) V^1(t) \Gamma^1_{12} + \underbrace{\frac{d\alpha^2}{dt}}_{0}(t) V^2(t) \Gamma^1_{22} =$$

$$= a'(t) + 1 \cdot a(t) \cdot 0 + 1 \cdot b(t) \cdot \left(-\frac{1}{y}\right) =$$

$$= \boxed{a'(t) - \frac{1}{y} b(t) = 0}$$

$$(K=2) \qquad b'(t) + \frac{d\alpha^1}{dt}(t) V^1(t) \Gamma^2_{11} + \frac{d\alpha^1}{dt}(t) V^2(t) \overbrace{\Gamma^2_{21}}^{0}$$

$$+ \underbrace{\frac{d\alpha^2}{dt}}_{0}(t) V^1(t) \Gamma^2_{12} + \underbrace{\frac{d\alpha^2}{dt}}_{0}(t) V^2(t) \Gamma^2_{22} =$$

$$= b'(t) + a(t) \frac{1}{y}$$

$$= \boxed{b'(t) + \frac{1}{y} a(t) = 0}$$

$$\begin{cases} a'(t) - \frac{1}{y}\, b(t) = 0 \\ b'(t) + \frac{1}{y}\, a(t) = 0 \end{cases}$$ Ecuaciones del transporte paralelo

Donde $y \equiv 1$ (función coordenada)

Teniendo en cuenta que $(a(t), b(t)) = V(t) = (\cos\theta(t), \operatorname{sen}\theta(t))$ y las ecuaciones del transporte paralelo tenemos que:

$$\left.\begin{cases} -\operatorname{sen}\theta(t) \cdot \theta'(t) - \operatorname{sen}\theta(t) = 0 \\ \cos\theta(t) \cdot \theta'(t) - \cos\theta(t) = 0 \end{cases}\right| \Rightarrow \boxed{\theta'(t) \equiv -1}$$

Apartado c) Por el apartado anterior, tenemos que

$\theta'(t) = 1 \iff \boxed{\theta(t) = \int \theta'(t)\,dt = \int -1\,dt = \boxed{-t + C_1}}$

Por ser $V(t)$ el transporte paralelo de $\vec{v} = (0,1)$ a lo largo de $\alpha$, se verifica que $V(t_0) = \vec{v} = (0,1)$

$$\iff (\cos\theta(t_0), \operatorname{sen}\theta(t_0)) = (0,1) \qquad (*_3)$$

Además por el apartado (a) es

$\underline{\underline{\theta(t)}} = \arccos(a(t)) \in \underline{\underline{(0,\pi)}} =$ Imagen de la función $\arccos(x)$

$(*_4)$

De $(*_3)$ y $(*_4)$ tenemos que tiene que ser

$$\theta(t_0) = \frac{\pi}{2}$$

Sin pérdida de generalidad podemos suponer $t_0 = 0$

Por tanto $\theta(0) = \frac{\pi}{2} \Rightarrow \boxed{\theta(t) = -t + \frac{\pi}{2}}$ $\square$

Ejercicio 5: Demuestra que en una 2- variedad riemanniana, $(M^2, g)$, la curvatura seccional coincide con la curvatura de Gauss ¿Qué relación hay entre la curvatura escalar y la curvatura de Gauss en este caso?

Dem:

~~(Q.v)~~

Tomamos $(u, v)$ coordenadas en $(M^2, g)$

Sin pérdida de generalidad podemos suponer que $g_{12} = 0$ $(\equiv F)$ (es decir, la parametrización es ortogonal)

Queremos probar que:

$$\frac{Rm(\partial_u, \partial_v, \partial_u, \partial_v)}{Q(\partial_u, \partial_v)} = K(p) \overset{?}{=} K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

Curvatura seccional (para $K(p)$) — Curvatura de Gauss (para $K$)

Por un lado en la curvatura seccional, el término

$$Q(\partial u, \partial v) = \underbrace{\langle \partial u, \partial u\rangle}_{E} \underbrace{\langle \partial v, \partial v\rangle}_{G} - \underbrace{\langle \partial u, \partial v\rangle^2}_{F^2} = EG$$

Siendo $E = \langle \partial u, \partial u\rangle$, $G = \langle \partial v, \partial v\rangle$, $F = \langle \partial u, \partial v\rangle$ los coeficientes de la primera forma fundamental

Notación: $\partial u = X_u$, $\partial v = X_v$, siendo $X$ parametrización de $M^2$, $X_u$ = derivada de $X$ respecto a $u$

$X_v$ = " " " " $v$

Por otro lado,

$$R_m(\partial u, \partial v, \partial u, \partial v) = \langle \partial u, R(\partial v, \partial u, \partial v) \rangle$$

donde

$$R(\partial v, \partial u, \partial v) = \nabla_{\partial u} \nabla_{\partial v} \partial v - \nabla_{\partial v} \nabla_{\partial u} \partial v - \underbrace{\nabla_{[\partial u, \partial v]}}_{=0} \partial v =$$

$$= \nabla_{\partial u} \nabla_{\partial v} \partial v - \nabla_{\partial v} \nabla_{\partial u} \partial v$$

La matriz de la métrica, es en este caso

$$(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \longrightarrow \begin{pmatrix} 1/E & 0 \\ 0 & 1/G \end{pmatrix} = (g^{ij})$$

Los (~~coeficientes~~ símbolos) de Christofel se calculan según la fórmula

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left( \frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right)$$

$$\boxed{\Gamma^1_{11}} = \frac{1}{2} g^{11} \left( \frac{\partial g_{11}}{\partial u} + \cancel{\frac{\partial g_{11}}{\partial u}} - \cancel{\frac{\partial g_{11}}{\partial u}} \right) = \frac{1}{2} \frac{1}{E} E_u = \boxed{\frac{E_u}{2E}}$$

$$\boxed{\Gamma^1_{21} = \Gamma^1_{12}} = \frac{1}{2} g^{11} \left( \frac{\partial g_{11}}{\partial v} + \underbrace{\frac{\partial g_{21}}{\partial u}}_{0} - \underbrace{\frac{\partial g_{12}}{\partial u}}_{0} \right) = \frac{1}{2} \frac{1}{E} E_v = \boxed{\frac{E_v}{2E}}$$

$$\boxed{\Gamma^1_{22}} = \frac{1}{2} g^{11} \left( \underbrace{\cancel{\frac{\partial g_{21}}{\partial v}}}_{0} + \underbrace{\cancel{\frac{\partial g_{21}}{\partial v}}}_{0} - \frac{\partial g_{22}}{\partial u} \right) = \frac{1}{2} \frac{1}{E} \cdot -G_u = \boxed{\frac{-G_u}{2E}}$$

$$\boxed{\Gamma^2_{11}} = \frac{1}{2} g^{22} \left( \underbrace{\cancel{\frac{\partial g_{12}}{\partial u}}}_{0} + \underbrace{\cancel{\frac{\partial g_{12}}{\partial u}}}_{0} - \frac{\partial g_{11}}{\partial v} \right) = \frac{1}{2} \frac{1}{G} \cdot -E_v = \boxed{\frac{-E_v}{2G}}$$

$$\boxed{\Gamma^2_{21} = \Gamma^2_{12}} = \frac{1}{2} g^{22} \left( \underbrace{\cancel{\frac{\partial g_{12}}{\partial v}}}_{0} + \frac{\partial g_{22}}{\partial u} - \underbrace{\cancel{\frac{\partial g_{12}}{\partial v}}}_{0} \right) = \frac{1}{2} \frac{1}{G} G_u = \boxed{\frac{G_u}{2G}}$$

$$\boxed{\Gamma^2_{22}} = \frac{1}{2} g^{22} \left( \frac{\partial g_{22}}{\partial v} + \cancel{\frac{\partial g_{22}}{\partial v}} - \cancel{\frac{\partial g_{22}}{\partial v}} \right) = \frac{1}{2} \frac{1}{G} G_v = \boxed{\frac{G_v}{2G}}$$

$$\nabla_{\partial u} \partial u = \Gamma^1_{11} \partial u + \Gamma^2_{11} \partial v = \frac{E_u}{2E} \partial u - \frac{E_v}{2G} \partial v$$

$$\nabla_{\partial u} \partial v = \Gamma^1_{21} \partial u + \Gamma^2_{21} \partial v = \frac{E_v}{2E} \partial u + \frac{G_u}{2G} \partial v$$

$$\nabla_{\partial v} \partial u = \Gamma^1_{12} \partial u + \Gamma^2_{12} \partial v = \frac{E_v}{2E} \partial u + \frac{G_u}{2G} \partial v$$

$$\nabla_{\partial v} \partial v = \Gamma^1_{22} \partial u + \Gamma^2_{22} \partial v = \frac{-G_u}{2E} \partial u + \frac{G_v}{2G} \partial v$$

$$\Rightarrow \nabla_{\partial u} \nabla_{\partial v} \partial v = \nabla_{\partial u} \left( \frac{-G_u}{2E} \partial u + \frac{G_v}{2G} \partial v \right) =$$

$$\boxed{\text{Leibniz}} \rightarrow = \frac{\partial}{\partial u} \left( \frac{-G_u}{2E} \right) \partial u + \frac{-G_u}{2E} \nabla_{\partial u} \partial u$$

$$+ \frac{\partial}{\partial u} \left( \frac{G_v}{2G} \right) \partial v + \frac{G_v}{2G} \nabla_{\partial u} \partial v$$

$$= \left( \frac{-G_u}{2E} \right)_u \partial u + \frac{-G_u}{2E} \left( \frac{E_u}{2E} \partial u + \frac{-E_v}{2G} \partial v \right)$$

$$\left( \frac{G_v}{2G} \right)_u \partial v + \frac{G_v}{2G} \left( \frac{E_v}{2E} \partial u + \frac{G_u}{2G} \partial v \right)$$

$$= \left( -\left( \frac{G_u}{2E} \right)_u + \frac{-G_u}{2E} \frac{E_u}{2E} + \frac{G_v}{2G} \frac{E_v}{2E} \right) \partial u$$

$$+ \left( \left( \frac{G_v}{2G} \right)_u + \frac{G_u}{2E} \frac{E_v}{2G} + \frac{G_v}{2G} \frac{G_u}{2G} \right) \partial v$$

• $$\nabla_{\partial v}\nabla_{\partial u}\partial v = \nabla_{\partial v}\left(\frac{E_v}{2E}\partial u + \frac{G_u}{2G}\partial v\right)$$

(leibniz) $$= \frac{\partial}{\partial v}\left(\frac{E_v}{2E}\right)\partial u + \frac{E_v}{2E}\nabla_{\partial v}\partial u$$

$$+ \frac{\partial}{\partial v}\left(\frac{G_u}{2G}\right)\partial v + \frac{G_u}{2G}\nabla_{\partial v}\partial v$$

$$= \left(\frac{E_v}{2E}\right)_v \partial u + \frac{E_v}{2E}\left(\frac{E_v}{2E}\partial u + \frac{G_u}{2G}\partial v\right)$$

$$+ \left(\frac{G_u}{2G}\right)_v\partial v + \frac{G_u}{2G}\left(-\frac{G_u}{2E}\partial u + \frac{G_v}{2G}\partial v\right)$$

$$= \left(\left(\frac{E_v}{2E}\right)_v + \frac{E_v}{2E}\frac{E_v}{2E} + \frac{-G_u}{2G}\frac{G_u}{2E}\right)\partial u$$

$$\left(\left(\frac{G_u}{2G}\right)_v + \cancel{\frac{E_v}{2E}\frac{G_u}{2G}} + \cancel{\frac{G_u}{2G}\frac{G_v}{2G}}\right)\partial v$$

$$\Rightarrow \nabla_{\partial u}\nabla_{\partial v}\partial v - \nabla_{\partial v}\nabla_{\partial u}\partial v =$$

$$= \Big[\left(-\frac{G_u}{2E}\right)_u + \frac{-G_u}{2E}\frac{E_u}{2E} + \frac{G_v}{2G}\frac{E_v}{2E} -$$

$$-\left(\left(\frac{E_v}{2E}\right)_v + \frac{E_v}{2E}\frac{E_v}{2E} - \frac{G_u}{2G}\frac{G_u}{2E}\right)\Big]\partial u$$

$$\left(\left(\frac{G_v}{2G}\right)_u - \left(\frac{G_u}{2G}\right)_v\right)\partial v$$

$$R_m(\partial u, \partial v, \partial u, \partial v) = \langle \partial u, R(\partial v, \partial u, \partial v) \rangle =$$

$$= \left( \left( \frac{-G_u}{2E} \right)_u - \left( \frac{E_v}{2E} \right)_v + \frac{-G_u}{2E} \frac{E_u}{2E} + \frac{G_v}{2G} \frac{E_v}{2E} - \left( \frac{E_v}{2E} \right)^2 + \left( \frac{G_u}{2E} \right) \frac{G_u}{2G} \right) \langle \partial u, \partial u \rangle$$

$$+ \left( \left( \frac{G_v}{2G} \right)_u - \left( \frac{G_u}{2G} \right)_v \right) \underbrace{\langle \partial u, \partial v \rangle}_{= 0}$$

$$= E \overbrace{\left( \left( \frac{-G_u}{2E} \right)_u - \left( \frac{E_v}{2E} \right)_v + \frac{-G_u}{2E} \frac{E_u}{2E} + \frac{G_v}{2G} \frac{E_v}{2E} - \frac{E_v}{2E} \frac{E_v}{2E} + \frac{G_u}{2E} \frac{G_u}{2G} \right)}^{(*)}$$

$$K(p) = \frac{R_m(\partial u, \partial v, \partial u, \partial v)}{Q(\partial u, \partial v)} = \frac{\cancel{E} \, (*)}{\cancel{E} G} = \frac{(*)}{G} =$$

$$= \frac{1}{G} \left[ - \frac{2E G_{uu} - 2 E_u G_u}{4E^2} - \frac{2E E_{vv} - 2E_v E_v}{4E^2} - \frac{E_u G_u}{4E^2} + \frac{E_v G_v}{4EG} - \frac{E_v^2}{4E^2} + \frac{G_u^2}{4EG} \right]$$

$=$

$$= -\frac{G_{uu}}{2EG} - \frac{E_{vv}}{2EG} + \frac{E_u G_u}{4E^2G} + \frac{E_v^2}{4E^2G}$$

$$+ \frac{E_v G_v}{4EG^2} + \frac{G_u^2}{4EG^2}$$

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) =$$

$$= -\frac{1}{2\sqrt{EG}}\left(\frac{E_{vv}\sqrt{EG} - E_v \frac{1}{2}(EG)^{-\frac{1}{2}}(E_vG + EG_v)}{EG}\right.$$

$$\left. + \frac{G_{uu}\sqrt{EG} - G_u\frac{1}{2}(EG)^{-\frac{1}{2}}(E_uG + EG_u)}{EG}\right)$$

$$= -\frac{1}{2\sqrt{EG}}\left[\frac{E_{vv}\sqrt{EG}}{EG} - \frac{E_v^2G + EE_vG_v}{2\sqrt{EG}\,EG}\right.$$

$$\left. + \frac{G_{uu}\sqrt{EG}}{EG} - \frac{E_uG_uG + EG_u^2}{2\sqrt{EG}\,(EG)}\right]$$

$$= -\frac{E_{vv}}{2EG} - \frac{G_{uu}}{2EG} + \frac{\cancel{G}E_uG_u}{4E^2G^{\cancel{2}}} + \frac{\cancel{G}E_v^2}{4E^2G^{\cancel{2}}}$$

$$+ \frac{\cancel{E}E_vG_v}{4E^{\cancel{2}}G^2} + \frac{\cancel{E}G_u^2}{4E^{\cancel{2}}G^2}$$

$$= -\frac{E_{vv}}{2EG} - \frac{G_{uu}}{2EG} + \frac{E_uG_u}{4E^2G} + \frac{E_v^2}{4E^2G} + \frac{E_vG_v}{4EG^2} + \frac{G_u^2}{4EG^2}$$

$\Rightarrow \boxed{K(p) = K}$ Como queríamos demostrar

¿Qué relación hay entre la curvatura escalar y la curvatura de Gauss en este caso?

Para superficies en $\mathbb{R}^3$ acabamos de demostrar que la curvatura de Gauss coincide con la curvatura seccional.

Tomamos span $\{\partial u, \partial v\} \subset T_pM$ parametrización ~~ortogonal~~ ortonormal de $M^2$.

Sabemos que la curvatura seccional de $T_pM$ viene dada por

$$K(T_pM) = \frac{R(\partial u, \partial v, \partial u, \partial v)}{Q(\partial u, \partial v)} = \frac{R_{1212}}{EG} = R_{1212}$$

Parametrización ortogonal $(F=0)$

Podemos suponer SPG que es param ortonormal

$E = \langle \partial u, \partial u \rangle = 1$

$G = \langle \partial v, \partial v \rangle = 1$

Por tanto

$$\boxed{K = K(T_p(M)) = R_{1212}}$$

Por otro lado, al ser

(*) $$(Ric)_{ij} = \sum_m R_{imjm}$$ (lo demostro después)

la curvatura escalar se define como:

$$S(x) = g^{ij}(Ric)_{ij} = g^{11}(Ric)_{11} + g^{22}(Ric)_{22} \overset{(*)}{=}$$

matriz de la métrica es diagonal

$$\overset{*}{=} g^{11}\left(\cancel{R_{1111}} + R_{1212}\right) + g^{22}\left(R_{2121} + \cancel{R_{2222}}\right)$$

$$\cancel{R_{1111}} = 0 \qquad \cancel{R_{2222}} = 0$$

$$= g^{11} R_{1212} + g^{22} R_{2121}$$

Propiedad antisimétrica de $Rm$:

$$\boxed{R_{2121} = -R_{1221} = --R_{1212} = R_{1212}}$$

$$= g^{11} R_{1212} + g^{22} R_{1212}$$

$$= (\underbrace{g^{11}}_{=1} + \underbrace{g^{22}}_{=1}) R_{1212}$$

(param. ortonormal)

$$= 2 R_{1212} = \boxed{2K} \Rightarrow$$

$$\Rightarrow \boxed{S = 2K}$$

La curvatura escalar es 2 veces la curvatura de Gauss

Demuestro $*$

$$(Ric)_{ij} := R^m_{imj} \left( = \sum_m R^m_{imj} \text{ es sumatorio en } m \right)$$

<u>Inciso</u>: Cuando demostramos que Ric es tensor simétrico, vimos que

$$g_{km} R^m_{imj} = \langle \partial x^k, \partial x^m \rangle R^m_{imj} = \langle \partial x^k, R^m_{imj} \partial x^m \rangle = R_{kimj}$$

Multiplicando por $g^{lm}$:

$$\delta^l_k R^m_{imj} = g^{lm} g_{km} R^m_{imj} = g^{lm} R_{kimj} \underset{\delta^l_k}{=} g^{km} R_{kimj}$$

Por ser $(g^{ij})$ matriz diagonal, tenemos que

$$g^{km} \neq 0 \iff k = m$$

$$\Rightarrow R^{m}_{imj} = g^{mm} R_{mimj} \underset{\substack{\uparrow \\ \text{(antisimetría)}}}{=} g^{mm} (-R_{immj}) =$$

$$\text{(antisimetría)} \rightarrow = g^{mm} (-- R_{imjm}) = g^{mm} R_{imjm}$$

$$\underset{\substack{\uparrow \\ \text{parametrización} \\ \text{ortonormal}}}{=} R_{imjm}$$

$$\Rightarrow (Ric)_{ij} = R_{imjm} \quad \text{(sumatorio en } m\text{)}$$

$\square$

**Ejercicio 15:** En $\mathbb{R}^3$ con esféricas

$$(r, \phi, \rho) \longmapsto (\underbrace{r\cos\phi \operatorname{sen}\rho}_{x}, \underbrace{r \operatorname{sen}\phi \operatorname{sen}\rho}_{y}, \underbrace{r\cos\rho}_{z}) \quad (*)$$

Calcula los símbolos de $(\longrightarrow)$ la conexión

**Dem:**

Consideramos la parametrización de la esfera

$$X(r, \phi, \rho) = (*)$$

Los campos $\partial r, \partial\phi, \partial\rho \in \mathcal{T}_0^1$

$$\partial r = dX_{(r,\phi,\rho)}(\partial r) = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}$$

$$= \cos\phi \operatorname{sen}\rho \frac{\partial}{\partial x} + \operatorname{sen}\phi \operatorname{sen}\rho \frac{\partial}{\partial y} + \cos\rho \frac{\partial}{\partial z}$$

$$= (\cos\phi \operatorname{sen}\rho, \operatorname{sen}\phi \operatorname{sen}\rho, \cos\rho)$$

$$\partial\phi = (-r \operatorname{sen}\phi \operatorname{sen}\rho, r\cos\phi \operatorname{sen}\rho, 0)$$

$$\partial\rho = (r\cos\phi\cos\rho, r\operatorname{sen}\phi\cos\rho, -r\operatorname{sen}\rho)$$

- $\Gamma^{r}_{rr}\,\partial r + \Gamma^{\phi}_{rr}\,\partial\phi + \Gamma^{\theta}_{rr}\,\partial\theta = \overline{\nabla}_{\partial r}\,\partial r =$

def $\rightarrow$ $= (\partial r\,(\cos\phi \operatorname{sen}\theta),\ \partial r\,(\operatorname{sen}\phi \operatorname{sen}\theta),\ \partial r\,(\cos\theta))$

$= (0, 0, 0)$

$\Rightarrow \boxed{\Gamma^{r}_{rr} = \Gamma^{\phi}_{rr} = \Gamma^{\theta}_{rr} = 0}$

- $\Gamma^{r}_{\phi\phi}\,\partial r + \Gamma^{\phi}_{\phi\phi}\,\partial\phi + \Gamma^{\theta}_{\phi\phi}\,\partial\theta = \overline{\nabla}_{\partial\phi}\,\partial\phi =$

def $\rightarrow$ $= (\partial\phi\,(-r\operatorname{sen}\phi \operatorname{sen}\theta),\ \partial\phi\,(r\cos\phi \operatorname{sen}\theta),\ \partial\phi\,(0))$

$= (-r\cos\phi \operatorname{sen}\theta,\ -r\operatorname{sen}\phi \operatorname{sen}\theta,\ 0) = -r\,\partial r$

$\Rightarrow \boxed{\Gamma^{r}_{\phi\phi} = -r\ ,\quad \Gamma^{\phi}_{\phi\phi} = \Gamma^{\theta}_{\phi\phi} = 0}$

- $\Gamma^{r}_{\theta\theta}\,\partial r + \Gamma^{\phi}_{\theta\theta}\,\partial\phi + \Gamma^{\theta}_{\theta\theta}\,\partial\theta = \overline{\nabla}_{\partial\theta}\,\partial\theta =$

$= (\partial\theta(\cos\phi \operatorname{sen}\theta),\ \partial\theta(\operatorname{sen}\phi \operatorname{sen}\theta),\ \partial\theta(\cos\theta))$

$= (\cos\phi\cos\theta,\ \operatorname{sen}\phi\cos\theta,\ -\operatorname{sen}\theta) = \frac{-1}{r}\,\partial r$

$\Rightarrow \boxed{\Gamma^{r}_{\theta\theta} = -\frac{1}{r}\ ,\quad \Gamma^{\phi}_{\theta\theta} = \Gamma^{\theta}_{\theta\theta} = 0.}$

- $\Gamma^{r}_{r\theta}\,\partial r + \Gamma^{\phi}_{r\theta}\,\partial\phi + \Gamma^{\theta}_{r\theta}\,\partial\theta = \overline{\nabla}_{\partial\theta}\,\partial r =$

$$= (\partial_\theta(\cos\phi \,\text{sen}\,\theta),\ \partial_\theta(\text{sen}\,\phi\,\text{sen}\,\theta),\ \partial_\theta(\cos\theta))$$

$$= (\cos\phi\cos\theta,\ \text{sen}\,\phi\cos\theta,\ -\text{sen}\,\theta) = \frac{1}{r}\partial_\theta$$

$$\Rightarrow \boxed{\Gamma^{r}_{r\theta} = \Gamma^{r}_{\theta r} = \Gamma^{\phi}_{r\theta} = \Gamma^{\phi}_{\theta r} = 0\ ,\quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}}$$

- $\Gamma^{r}_{r\phi}\,\partial r + \Gamma^{\phi}_{r\phi}\,\partial\phi + \Gamma^{\theta}_{r\phi}\,\partial\theta = \nabla_{\partial\phi}\,\partial r =$

$$= (\partial_\phi(\cos\phi\,\text{sen}\,\theta),\ \partial_\phi(\text{sen}\,\phi\,\text{sen}\,\theta),\ \partial_\phi(\cos\theta))$$

$$= (-\text{sen}\,\phi\,\text{sen}\,\theta,\ \cos\phi\,\text{sen}\,\theta,\ 0) = \frac{1}{r}\partial\phi$$

$$\Rightarrow \boxed{\Gamma^{r}_{r\phi} = \Gamma^{r}_{\phi r} = \Gamma^{\theta}_{\phi r} = \Gamma^{\theta}_{r\phi} = 0\ ,\quad \Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}}$$

- $\Gamma^{r}_{\phi\theta}\,\partial r + \Gamma^{\phi}_{\phi\theta}\,\partial\phi + \Gamma^{\theta}_{\phi\theta}\,\partial\theta = \overline{\nabla}_{\partial\theta}\,\partial\phi =$

$$= (\partial_\theta(-r\,\text{sen}\,\phi\,\text{sen}\,\theta),\ \partial_\theta(r\cos\phi\,\text{sen}\,\theta),\ 0) =$$

$$= (-r\,\text{sen}\,\phi\cos\theta,\ r\cos\phi\cos\theta) = \frac{\cos\theta}{\text{sen}\,\theta}\partial\phi$$

$$\Rightarrow \boxed{\Gamma^{r}_{\phi\theta} = \Gamma^{r}_{\theta\phi} = \Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = 0\ ,\quad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\text{sen}\,\theta}}$$