The connection between the Riemann zeta function and the stable distributions of Cauchy and Gauss

Lorenzo Fassina, PhD

Dept. of Electrical, Computer and Biomedical Engineering, University of Pavia, Italy
E-mail: lorenzo.fassina@unipv.it

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Abstract

In this work, I try to explore further connection between the Riemann $\zeta(s)$ function and Statistics. In particular, such connection is shown in terms of the stable distributions of Cauchy and Gauss. In addition, given the connection, I try to rephrase the Riemann hypothesis and to obtain a prime number theorem via the Tchebychev's inequality.[1]

Introduction

It is well known that the non-trivial zeros of the Riemann $\zeta(s)$ function (the simplest L-function) precisely determine the position of the prime numbers and of their powers. In particular, there are both prime gaps equal to 2 (for the twin primes) and very large prime gaps (for the isolated primes), supporting the idea that the prime numbers are randomly distributed (but unlike according a Poisson process [1]).

On the other hand, given the non-trivial solutions of $\zeta(s) = 0$ and their pairs [2]-[3], the spacing between the imaginary parts of consecutive solutions is in connection with the Gaussian unitary ensemble (GUE) distribution from random matrix theory [4]. To this regard, Odlyzko [3] made an intensive computation about the Montgomery's pair correlation conjecture [2] and showed that the non-trivial zeros of the Riemann $\zeta(s)$ function behave almost like the eigenvalues of random Hermitian matrices (these matrices are of importance also in Physics).

More recently, Keating and Snaith showed the connection of the random matrix theory with $\zeta(s)$ at the critical line [5] and also with the L-functions at $s = 1/2$ [6].

In this work, I try to explore further connection between the Riemann $\zeta(s)$ function and Statistics. In particular, such connection is shown in terms of the stable distributions of Cauchy and Gauss.

1 The connection

Let me recall the characteristic function $\varphi_X(t; x_0, \gamma)$ of the Cauchy probability density function $g_X(x; x_0, \gamma)$ of a random variable X, where x_0 is the location parameter, specifying the position of g_X's maximum, whereas γ is the scale parameter, i.e. the half-width at half-maximum (HWHM) of g_X:

$$\varphi_X(t; x_0, \gamma) = e^{ix_0 t - \gamma|t|}, \tag{1}$$

$$g_X(x; x_0, \gamma) = \frac{1}{\pi\gamma\left[1 + \left(\frac{x - x_0}{\gamma}\right)^2\right]}. \tag{2}$$

In other words, from the Eq. 2, the maximum (or amplitude) of the probability density has value $A_{Cauchy} = 1/(\pi\gamma)$ and is located in $x = x_0$; moreover, alternatively, 2γ is called full width at half

[1] See here for code and this digitally signed PDF.

maximum (FWHM). The Cauchy characteristic function (CCF, φ_X in Eq. 1) is the Fourier transform of the Cauchy probability density function (CPDF, g_X in Eq. 2); to this regard, in the following discussion, I will use g_X and its Cauchy cumulative distribution function (CCDF, κ_X in Eq. 3) instead of φ_X:

$$\mathrm{CCDF}(x;x_0,\gamma) = \kappa_X(x;x_0,\gamma) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x-x_0}{\gamma}\right). \tag{3}$$

Definition 1.1. I will consider only specific Cauchy distributions: $\forall n \in \mathbb{N}^+$, $\forall s \in \mathbb{C}$ such that $0 < \mathrm{Re}(s) < 1$, i.e. $\forall s$ in the critical strip of the complex plane, I define $\mathrm{RE} \coloneqq \mathrm{Re}(s)$, $\mathrm{MI} \coloneqq \mathrm{Im}(s)$, $\gamma \coloneqq \mathrm{RE}\log(n)$, and $x_0 \coloneqq -\mathrm{MI}\log(n)$.

Remark 1.1. $\mathrm{RE} \in \mathbb{R}$ and $0 < \mathrm{RE} < 1$, $\mathrm{MI} \in \mathbb{R}$, $\gamma \in \mathbb{R}$ and $\gamma \geq 0$, $x_0 \in \mathbb{R}$. The case of $n = 1$ produces a Dirac's δ distribution (also known as the unit impulse with $x_0 = 0$) since $\gamma = 0$.

Definition 1.2. I define the Hippias radius $R_{Hippias} \coloneqq \frac{\pi}{2}A_{Cauchy} = 1/(2\gamma)$.

Remark 1.2. For the Riemann hypothesis, the $\pi/3$ angle is related to the critical line $x = 1/2$ because $\cos(\pi/3) = 1/2$. As a consequence, I consider the angle trisection problem (trisectrix of Hippias), in particular the trisection of the π angle (Remark 1.11). In the trisectrix method, the quarter circle of radius $R_{Hippias}$ is used to divide the radius in two parts, one of whom is $\frac{2}{\pi}R_{Hippias} = A_{Cauchy}$.

Let me recall that, for the Lambert W function, the image of the real axis is based on the trisectrix of Hippias, and that the Lambert W function can also be used to gain a new perspective on the distribution of the prime numbers [7].

Remark 1.3. The definition of $R_{Hippias}$ is an "invitation" to think about the prime number theorem (PNT): $\mathrm{RE} = 1/2$ if and only if $R_{Hippias} = 1/\log(n)$, which approximately is, for large enough $n \geq 2$, the probability that a random and positive integer ≥ 2 and $\leq n$ is prime.

Lemma 1.1. *Let $n \in \mathbb{N}^+$, $s \in \mathbb{C}$ and in the critical strip $(0 < \mathrm{RE} < 1)$, the Cauchy characteristic function's value $\varphi_X(t=1;x_0,\gamma)$ is equal to the Riemann addend $1/n^s$.*

Proof. Given the definition 1.1, I can write

$$\begin{aligned}\varphi_X(t=1;x_0,\gamma) &= e^{ix_0-\gamma} = e^{-(\mathrm{RE}+i\,\mathrm{MI})\log(n)} = e^{-s\log(n)} = \left(e^{\log(n)}\right)^{-s} = \frac{1}{n^s} \\ &= \varphi_X(t=1;n,s) = \varphi_X(t=1;n,\mathrm{RE},\mathrm{MI}).\end{aligned} \tag{4}$$

□

Lemma 1.2. *To study the non-trivial solutions of $\zeta(s) = 0$, I can use a series of Cauchy characteristic function's values $\varphi_X(t=1;n,\mathrm{RE},\mathrm{MI})$.*

Proof. In the critical strip $(0 < \mathrm{RE} < 1)$, the Riemann zeta function is

$$\begin{aligned}\zeta(s) = \frac{1}{1-2^{(1-s)}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} &= \frac{1}{1-2^{(1-s)}}\eta(s) \\ &= \frac{1}{1-2^{(1-s)}}\sum_{n=1}^{\infty}(-1)^{n+1}\varphi_X(t=1;n,\mathrm{RE},\mathrm{MI}),\end{aligned} \tag{5}$$

where $\eta(s)$ is the Dirichlet eta function (or alternating zeta function). Because of the equations $\zeta(s) = 0$ and $\eta(s) = 0$ have the same solutions in the critical strip, in order to study the non-trivial solutions of $\zeta(s) = 0$, from the Eq. 5, I can write

$$\eta(s) = \sum_{n=1}^{\infty}(-1)^{n+1}\varphi_X(t=1;n,\mathrm{RE},\mathrm{MI}) = 0. \tag{6}$$

□

Remark 1.4. The Eq. 5 is a functional equation linking s and $(1-s)$, which are symmetric with respect to the reflection point $O' \equiv (1/2, 0)$ belonging to the critical line $x = 1/2$ (Fig. 1); to this regard, in the following discussion on the critical strip, I will study s and $(1-s)$ together.

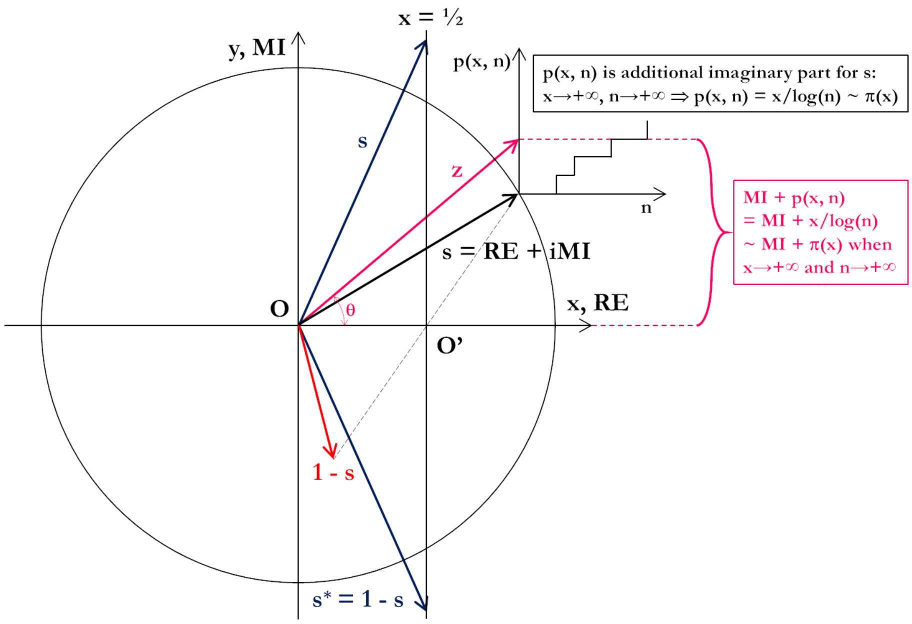

Figure 1: Complex plane.

Remark 1.5. In the Eq. 6 there is an alternating sum of $\varphi_X(t = 1; n, \mathrm{RE}, \mathrm{MI})$ which are Fourier transforms of alternating Cauchy probability density functions $g_X(x; x_0, \gamma) = g_X(x; n, s) = g_X(x; n, \mathrm{RE}, \mathrm{MI})$ with $x \in \mathbb{R}$, $n \in \mathbb{N}^+$, $\mathrm{RE} \in \mathbb{R}$ and $0 < \mathrm{RE} < 1$, $\mathrm{MI} \in \mathbb{R}$ (Fig. 2). In this figure, an odd n gives rise to Cauchy probability density functions with $y > 0$, while an even n brings $y < 0$.

Lemma 1.3. *Given $n \in \mathbb{N}^+$, the line $x = 0$ is axis of symmetry for the Cauchy probability density functions $g_X(x; n, \mathrm{RE}, \mathrm{MI})$ generated by s and $(1 - s)$ if and only if $\mathrm{RE}_s = \mathrm{RE}_{1-s} = 1/2$.*

Proof. The symmetry respect to the line $x = 0$ is equivalent to have, $\forall n \in \mathbb{N}^+$, $x_{0,s} = -x_{0,1-s}$, $\gamma_s = \gamma_{1-s}$, and $A_{Cauchy,s} = A_{Cauchy,1-s}$ (Fig. 2). Moreover, $(1 - s) = 1 - 1/2 - i\,\mathrm{MI}_s = 1/2 - i\,\mathrm{MI}_s = s^*$ (Fig. 1), in other words, $\mathrm{MI}_{1-s} = -\mathrm{MI}_s$. As a consequence, $\forall n \in \mathbb{N}^+$, I obtain

$$x_{0,s} = -\mathrm{MI}_s \log(n) = \mathrm{MI}_{1-s} \log(n) = -x_{0,1-s}\,, \tag{7}$$

$$\gamma_s = \mathrm{RE}_s \log(n) = \frac{1}{2} \log(n) = \mathrm{RE}_{1-s} \log(n) = \gamma_{1-s}\,, \tag{8}$$

$$A_{Cauchy,s} = \frac{1}{\pi\gamma_s} = \frac{1}{\pi\gamma_{1-s}} = A_{Cauchy,1-s}\,. \tag{9}$$

□

Remark 1.6. When $\mathrm{RE}_s = \mathrm{RE}_{1-s} = 1/2$, the line $x = 0$ is axis of symmetry for the alternating Cauchy probability density functions generated by s and $s^* = (1 - s)$, which belong to the critical line, as well as O' is reflection point for them in the complex plane (Fig. 1). In other words, the previous two symmetries appear hidden in the functional equation 5, but, when $s^* \neq (1 - s)$, the line $x = 0$ is not axis of symmetry (Fig. 3, Fig. 4).

Remark 1.7. In general, the characteristic function $\varphi_X(t) = E\left[e^{itX}\right]$ completely determines the features of the probability distribution of a random variable X. The characteristic function is similar to the cumulative distribution function $\kappa_X(x) = E\left[\mathbf{1}_{\{X \le x\}}\right]$, where $\mathbf{1}_{\{X \le x\}}$ is the indicator function. The Cauchy cumulative distribution function $\kappa_X(x; x_0, \gamma)$ (Eq. 3) completely determines the features of the probability distribution of a random variable X just as $\varphi_X(t; x_0, \gamma)$ does. The two approaches are equivalent, yet they provide different insights; to this regard, in the following discussion on the critical

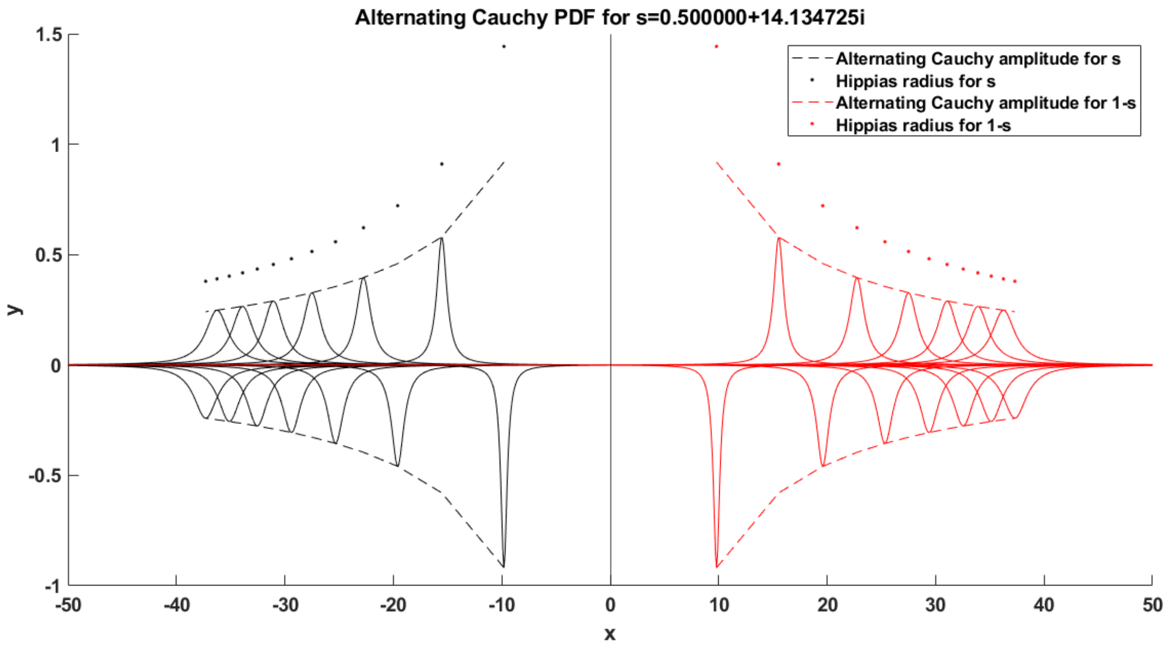

Figure 2: Alternating Cauchy probability density functions generated by s and $s^* = (1-s)$, which belong to the critical line and are non-trivial solutions of $\zeta(s) = 0$. The functions move away from the line $x = 0$ as n increases.

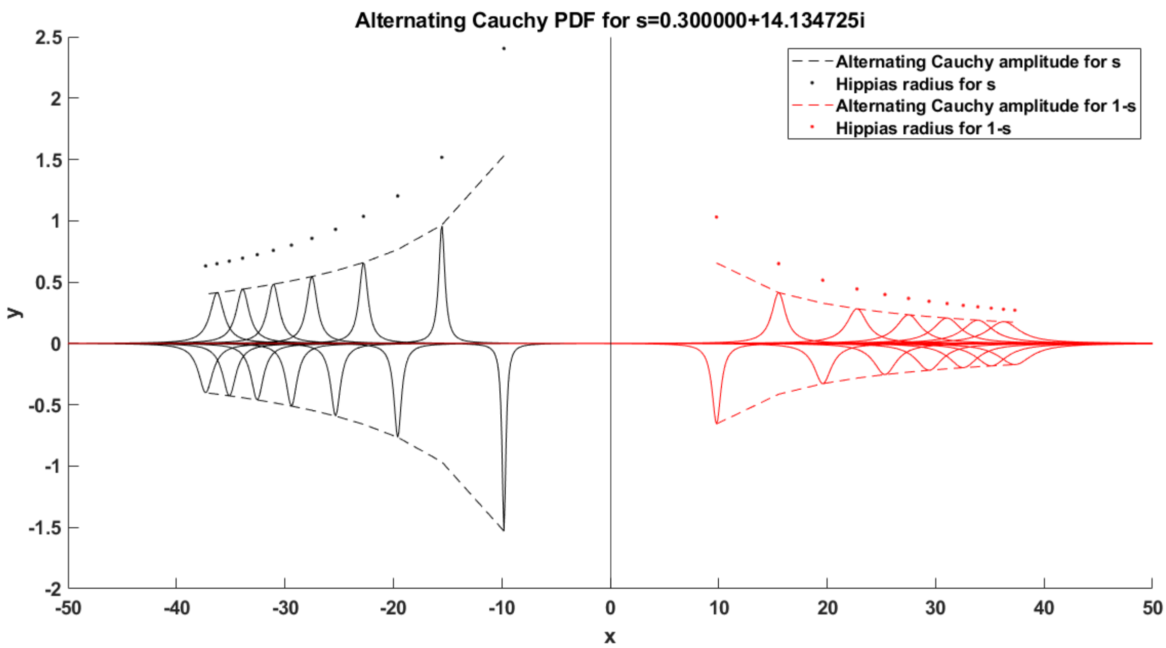

Figure 3: Alternating Cauchy probability density functions generated by s and $(1-s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$. The functions move away from the line $x = 0$ as n increases.

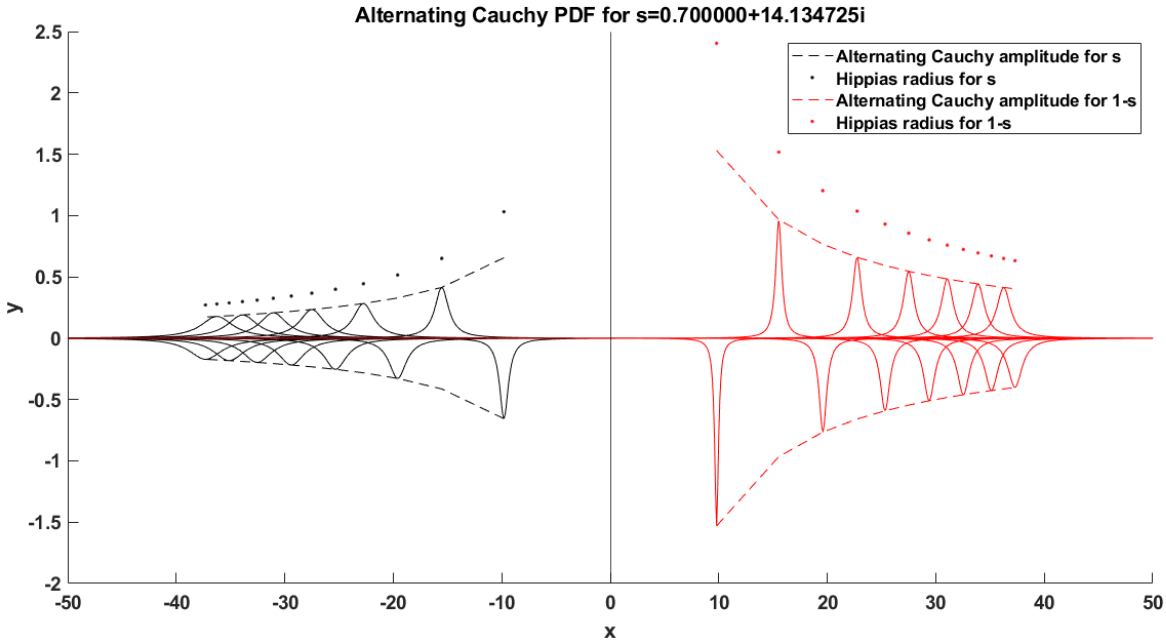

Figure 4: Alternating Cauchy probability density functions generated by s and $(1-s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$. The functions move away from the line $x = 0$ as n increases.

strip, I will study κ_X (e.g., Fig. 5, Fig. 6, Fig. 7) instead of φ_X. In other words, in the critical strip, the equation $\eta(s) = 0$ (Eq. 6) is equivalent to the null alternating sum

$$\sum_{n=1}^{\infty}(-1)^{n+1}\kappa_X(x;x_0,\gamma) = \sum_{n=1}^{\infty}(-1)^{n+1}\kappa_X(x;n,\mathrm{RE},\mathrm{MI}) = 0\,. \tag{10}$$

Lemma 1.4. *The value $\kappa_X(x=0;x_0,\gamma)$ is invariant respect to $n \in \mathbb{N}^+$.*

Proof. From Eq. 3, I obtain

$$\kappa_X(x=0;x_0,\gamma) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(-\frac{x_0}{\gamma}\right) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\mathrm{MI}}{\mathrm{RE}}\right). \tag{11}$$

□

Remark 1.8. The point $P \equiv (0, \kappa_X(x=0;x_0,\gamma))$, belonging to the line $x = 0$ (e.g., Fig. 8), could be intended as the "common starting point" for the alternating Cauchy cumulative distribution functions $\kappa_X(x;x_0,\gamma)$ toward their asymptotic convergence to the lines $y = \pm 1$ when $x \to +\infty$, that is, to $y = 1$ for odd n or to $y = -1$ for even n (Fig. 5, Fig. 6, Fig. 7). This convergence will be shown to be interesting in the perspective of studying the prime numbers. By comparing Fig. 5, Fig. 6, and Fig. 7, for $x \to +\infty$, the asymptotic convergence can differ depending on $n \in \mathbb{N}^+$ and s. In other words, given n and MI, the functions κ_X with $0 < \mathrm{RE} < 1/2$ are anticipated (i.e. they converge before) respect to those with $\mathrm{RE} = 1/2$, whereas the functions κ_X with $1/2 < \mathrm{RE} < 1$ are delayed (i.e. they converge after) respect to those with $\mathrm{RE} = 1/2$. On the other hand, given s, the functions κ_X converge more rapidly as n decreases (Fig. 8). Given n and RE, the functions κ_X move away from the line $x = 0$ as $|\mathrm{MI}|$ increases. So, it is important to build a metrics of the gap to fill in order to approach the horizontal asymptotes (Lemma 1.5).

Lemma 1.5. *The asymptotic convergence of the Cauchy cumulative distribution functions $\kappa_X(x;x_0,\gamma)$ to the line $y = 1$ depends on $s = \mathrm{RE} + i\,\mathrm{MI}$ and on the function $p(x,n) \coloneqq x/\log(n)$ with $x \in \mathbb{R}$, $n \in \mathbb{N}^+$ and $n \geq 2$.*

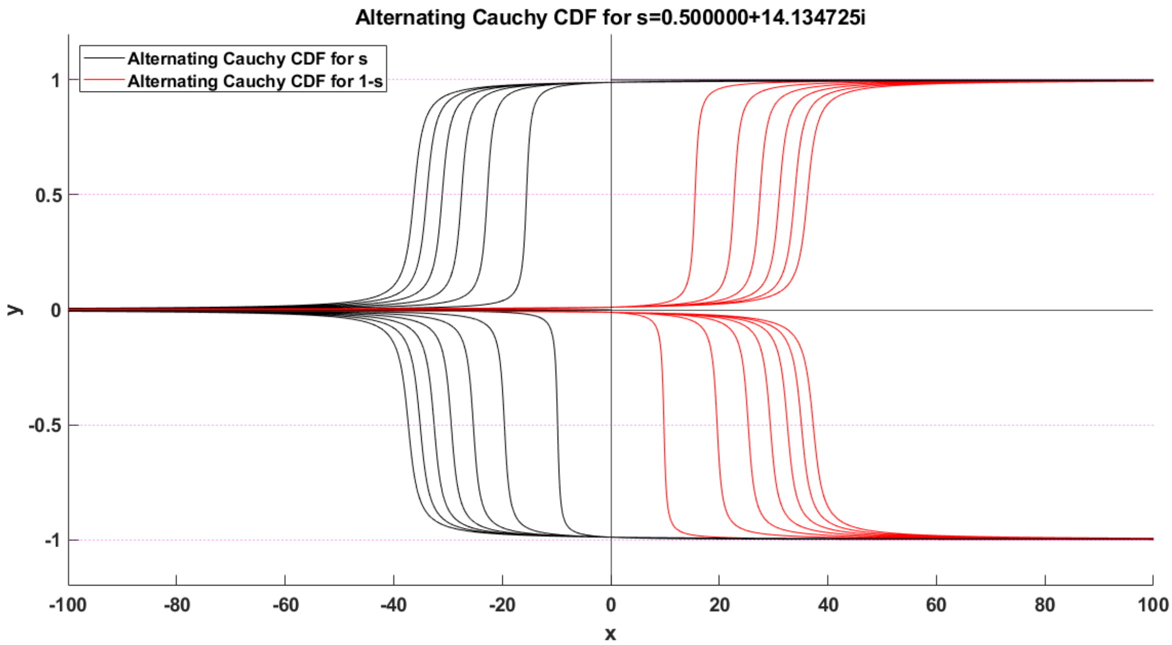

Figure 5: Alternating Cauchy cumulative distribution functions generated by s and $s^* = (1 - s)$, which belong to the critical line and are non-trivial solutions of $\zeta(s) = 0$. The functions move away from the line $x = 0$ as n increases.

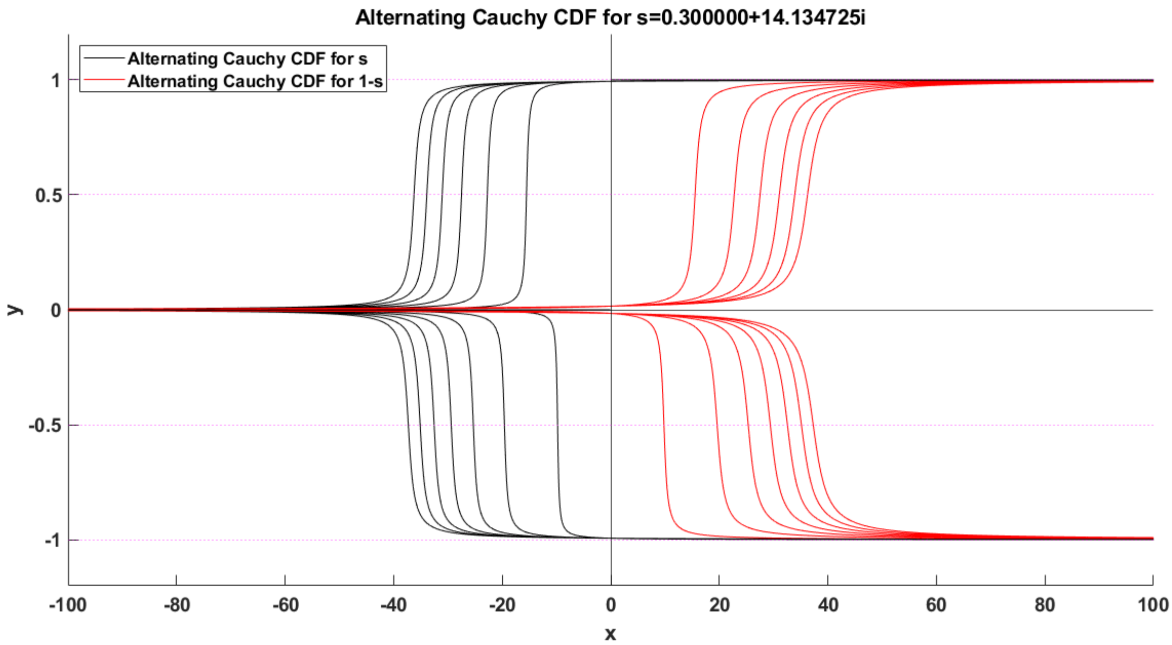

Figure 6: Alternating Cauchy cumulative distribution functions generated by s and $(1 - s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$. The functions move away from the line $x = 0$ as n increases.

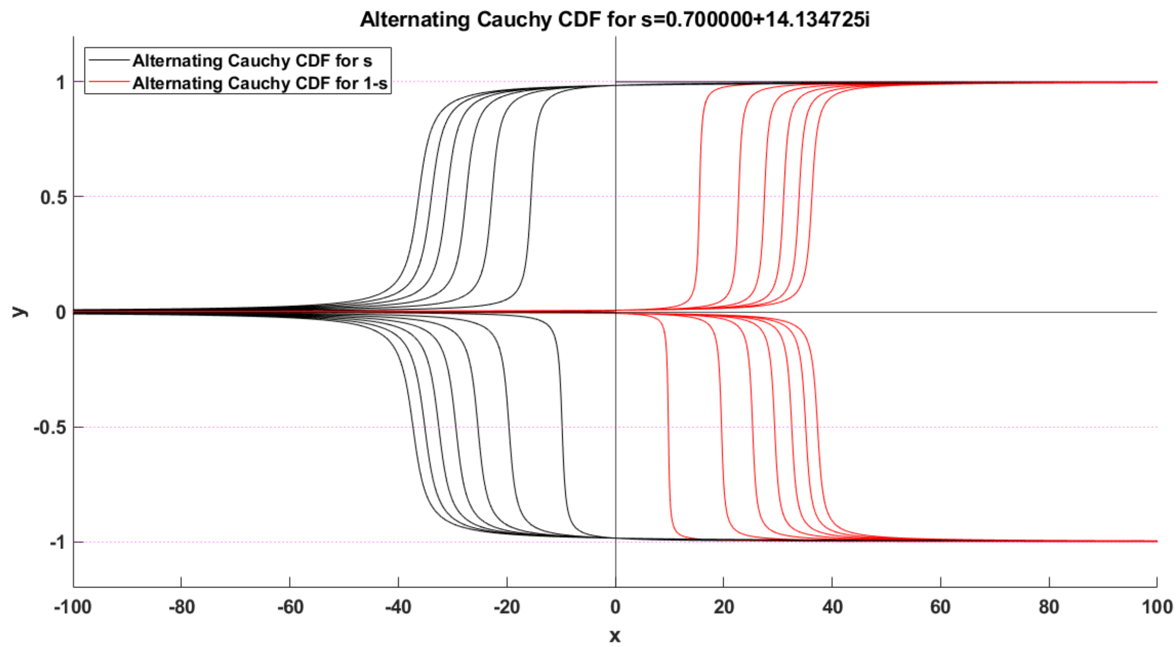

Figure 7: Alternating Cauchy cumulative distribution functions generated by s and $(1-s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$. The functions move away from the line $x = 0$ as n increases.

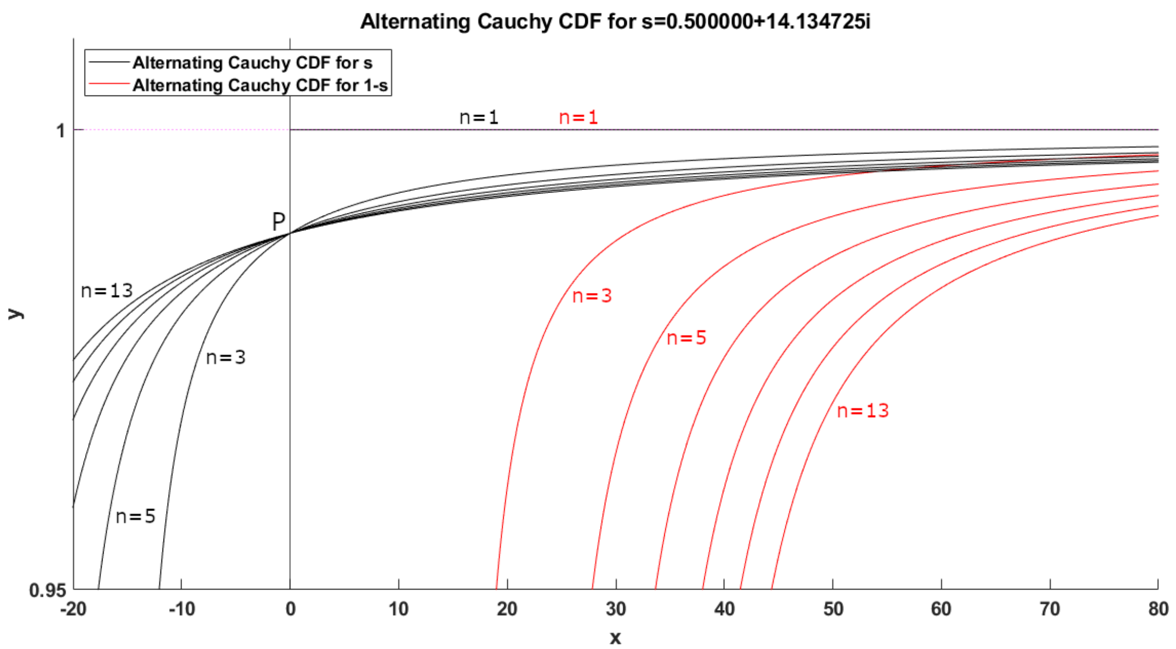

Figure 8: Detail of alternating Cauchy cumulative distribution functions generated by s and $s^* = (1-s)$, which belong to the critical line and are non-trivial solutions of $\zeta(s) = 0$.

Proof. Let me recall that the Laurent series $L(x)$ of $\arctan(x)$ for $x \to +\infty$ is

$$L(x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + O\left(\frac{1}{x^6}\right). \tag{12}$$

From Eq. 3 and by using only the first two terms of $L(x)$, I obtain

$$\begin{aligned}\lim_{x\to+\infty} \kappa_X(x; x_0, \gamma) &\approx \lim_{x\to+\infty}\left[\frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\gamma}{x - x_0}\right)\right] = \lim_{x\to+\infty}\left[1 - \frac{\gamma}{\pi(x - x_0)}\right] \\ &= \lim_{x\to+\infty}\left[1 - \frac{\mathrm{RE}\log(n)}{\pi(x + \mathrm{MI}\log(n))}\right] = \lim_{x\to+\infty}\left[1 - \frac{\mathrm{RE}}{\pi\left(\mathrm{MI} + \frac{x}{\log(n)}\right)}\right].\end{aligned} \tag{13}$$

□

Remark 1.9. The Eq. 13 can not be used for $n = 1$, nevertheless $n = 1$ produces a Dirac's δ distribution (Remark 1.1) and $\lim_{x\to+\infty} \kappa_X(x; x_0 = 0, \gamma = 0) = 1$ (Fig. 8). The limit in Eq. 13 is interesting in the perspective of studying the prime numbers because the asymptotic convergence depends on $s = \mathrm{RE} + i\,\mathrm{MI}$ and, for $x \to +\infty$ and $n \to +\infty$, on the function $p(x, n) = x/\log(n) \sim \pi(x)$, which is the prime-counting function, in the sense that the relative error of this approximation approaches 0 (asymptotic law of the distribution of the prime numbers).

Moreover, the asymptotic convergence in Eq. 13 is anticipated with $0 < \mathrm{RE} < 1/2$ respect to that with $\mathrm{RE} = 1/2$, whereas is delayed with $1/2 < \mathrm{RE} < 1$ respect to that with $\mathrm{RE} = 1/2$. In particular, when $\mathrm{RE} = 1/2$, the Eq. 13 becomes

$$\lim_{x\to+\infty} \kappa_X(x; x_0, \gamma) \approx \lim_{x\to+\infty}\left[1 - \frac{1}{2\pi\left(\mathrm{MI} + \frac{x}{\log(n)}\right)}\right], \tag{14}$$

where the 2π angle, let me write in advance, corresponds to punctuality (Remark 1.12) and the term $(\mathrm{MI} + x/\log(n))$ is the extended imaginary part of s (Fig. 1, Eq. 15).

The Eq. 13 shows that the Cauchy cumulative distribution functions $\kappa_X(x; x_0, \gamma)$ asymptotically converge according to the asymptotic law of the distribution of the prime numbers and that such convergence is modulated by $s = \mathrm{RE} + i\,\mathrm{MI}$.

Lemma 1.6. *The value of the Cauchy cumulative distribution functions $\kappa_X(x; x_0, \gamma)$ depends on $s = \mathrm{RE} + i\,\mathrm{MI}$ and on the function $p(x, n) = x/\log(n)$ with $x \in \mathbb{R}$, $n \in \mathbb{N}^+$ and $n \geq 2$.*

Proof. Using the functional equation $\arctan(x) = \frac{1}{2}i\log(1 - ix) - \frac{1}{2}i\log(1 + ix)$, I rewrite the Eq. 3:

$$\begin{aligned}\kappa_X(x; x_0, \gamma) &= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x - x_0}{\gamma}\right) \\ &= \frac{1}{2} + \frac{i}{2\pi}\log\left(\frac{1 - i\frac{x - x_0}{\gamma}}{1 + i\frac{x - x_0}{\gamma}}\right) \\ &= \frac{1}{2} + \frac{i}{2\pi}\log\left(\frac{\gamma - i(x - x_0)}{\gamma + i(x - x_0)}\right) \\ &= \frac{1}{2} + \frac{i}{2\pi}\log\left(\frac{\mathrm{RE}\log(n) - i\,\mathrm{MI}\log(n) - ix}{\mathrm{RE}\log(n) + i\,\mathrm{MI}\log(n) + ix}\right) \\ &= \frac{1}{2} + \frac{i}{2\pi}\log\left(\frac{\mathrm{RE} - i\,\mathrm{MI} - i\frac{x}{\log(n)}}{\mathrm{RE} + i\,\mathrm{MI} + i\frac{x}{\log(n)}}\right) \\ &= \frac{1}{2} + \frac{i}{2\pi}\log\left(\frac{s^* - i\frac{x}{\log(n)}}{s + i\frac{x}{\log(n)}}\right) \\ &= \frac{1}{2} + \frac{i}{2\pi}\log\left(\frac{s^* - ip}{s + ip}\right) = \kappa_X(x; n, \mathrm{RE}, \mathrm{MI}) = \kappa_X(x; n, s).\end{aligned} \tag{15}$$

□

Corollary 1.7. *In the critical strip, the equation $\eta(s) = 0$ is equivalent to the null alternating sum of the Cauchy cumulative distribution functions $\kappa_X(x; x_0, \gamma)$.*

Proof. From the Eqs. 10 and 15, I obtain

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\kappa_X(x;x_0,\gamma) &= \sum_{n=1}^{\infty}(-1)^{n+1}\kappa_X(x;n,\mathrm{RE},\mathrm{MI}) \\ = \sum_{n=1}^{\infty}(-1)^{n+1}\kappa_X(x;n,s) &= \sum_{n=1}^{\infty}(-1)^{n+1}\left[\frac{1}{2}+\frac{i}{2\pi}\log\left(\frac{s^*-ip}{s+ip}\right)\right] = 0\,.\end{aligned} \tag{16}$$

□

Remark 1.10. From the Eq. 15, I get

$$\kappa_X(x;n,(1-s)) = \frac{1}{2}+\frac{i}{2\pi}\log\left(\frac{(1-s)^*-ip}{(1-s)+ip}\right), \tag{17}$$

$$\kappa_X(x;n,s^*) = \frac{1}{2}+\frac{i}{2\pi}\log\left(\frac{s-ip}{s^*+ip}\right), \tag{18}$$

$$\kappa_X(x;n,(1-s)^*) = \frac{1}{2}+\frac{i}{2\pi}\log\left(\frac{(1-s)-ip}{(1-s)^*+ip}\right). \tag{19}$$

In Eqs. 15-19, $p(x,n) = x/\log(n)$ is additional imaginary part for s, $(1-s)$, s^*, and $(1-s)^*$ (Fig. 1, Fig. 9, Fig. 10). Given $x \to +\infty$, p is $O(\delta)$ for $n \to 1^+$, p is $O(x)$ for $n \in [2, +\infty)$, and $p \sim \pi(x)$ for $n \to +\infty$, in other words, $\pi(x)$ is attracting the asymptotically stable p.

Remark 1.11. The Eq. 15 is a functional equation that can be rewritten as follows:

$$\begin{aligned}\kappa_X(x;x_0,\gamma) &= \frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x-x_0}{\gamma}\right) \\ &= \frac{1}{2}+\frac{i}{2\pi}\log\left(\frac{s^*-ip}{s+ip}\right) \\ &= \frac{1}{2}+\frac{i}{2\pi}\log\left(\frac{z^*}{z}\right) = \frac{1}{2}+\frac{i}{2\pi}\log\left(e^{-i2\theta}\right) \\ &= \frac{1}{2}+\frac{i}{2\pi}(-i2\theta) = \frac{1}{2}\left(1+\frac{2}{\pi}\theta\right),\end{aligned} \tag{20}$$

where $z \coloneqq s + ip$ is the extension of s, $z^* = s^* - ip$ is the extension of s^* (Fig. 9, Fig. 10), and where $\theta = \theta(x; x_0, \gamma)$ is the argument of z, whose imaginary part is $[\mathrm{MI} + p(x,n)]$ (Fig. 1). When $x \to +\infty$ and $n \to +\infty$, I obtain that $\theta \to (\pi/2)^-$, i.e. $z \sim s + i\pi(x)$ and, as a consequence, $z^* \sim s^* - i\pi(x)$.

Moreover, in Eq. 20, there is the ratio $2/\pi$, which recalls, again, the angle trisection problem or the trisectrix of Hippias (Remark 1.2). In Eq. 20, when $\theta \to (\pi/2)^-$, I obtain that $\mathrm{Arg}(z^*/z) = -2\theta \to (-\pi)^-$; in other words, I can study the convergence of the Cauchy cumulative distribution functions $\kappa_X(x; x_0, \gamma)$ in terms of $\mathrm{Arg}(z^*/z) = -2\theta$, which approaches the $(-\pi)^-$ angle in different fashions based on RE (Fig. 11).

In Fig. 11, when RE = 1/2, the $\widehat{QA}$ arc comes from the trisection of the $\widehat{QR}$ semi circumference and $\widehat{A'R}$ has double the length of $\widehat{QA}$: the A and A' points meet together and I define this meeting as punctuality ($s_p \in \mathbb{C}$ is an example of punctuality); moreover, in the A'' point, there is the complex number $(z^*/z)_p$ for punctuality with argument equal to $-2\theta_p$.

In similar way, $\widehat{BA}$ has half the length of $\widehat{A'B'}$, $s_d \in \mathbb{C}$ is an example of delay, in the B'' point there is the complex number $(z^*/z)_d$ for delay with argument equal to $-2\theta_d$. Furthermore, $\widehat{AC}$ has half the length of $\widehat{C'A'}$, $s_a \in \mathbb{C}$ is an example of anticipation, in the C'' point there is the complex number $(z^*/z)_a$ for anticipation with argument equal to $-2\theta_a$.

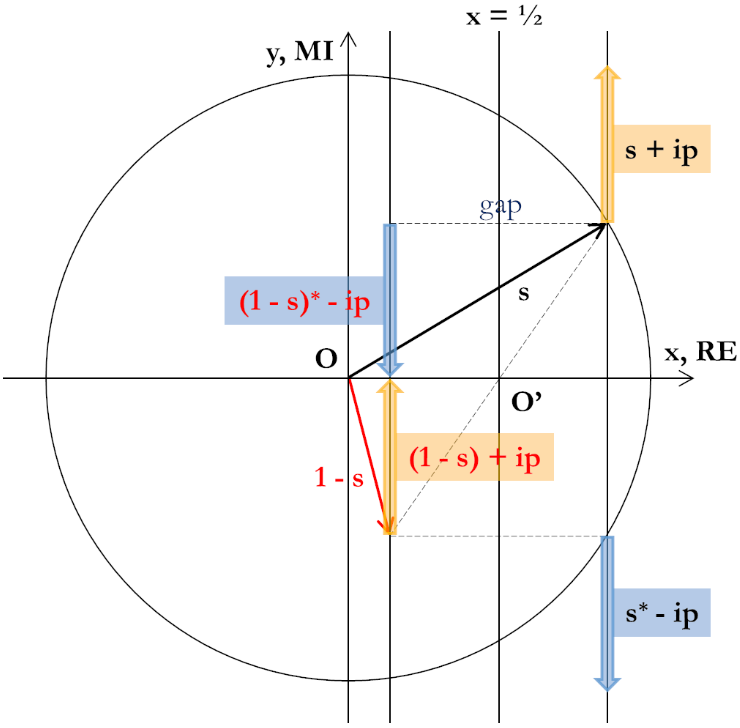

Figure 9: Additional imaginary part for s, $(1-s)$, s^*, and $(1-s)^*$; the colored arrows indicate the location of the extended complex numbers when RE ≠ 1/2.

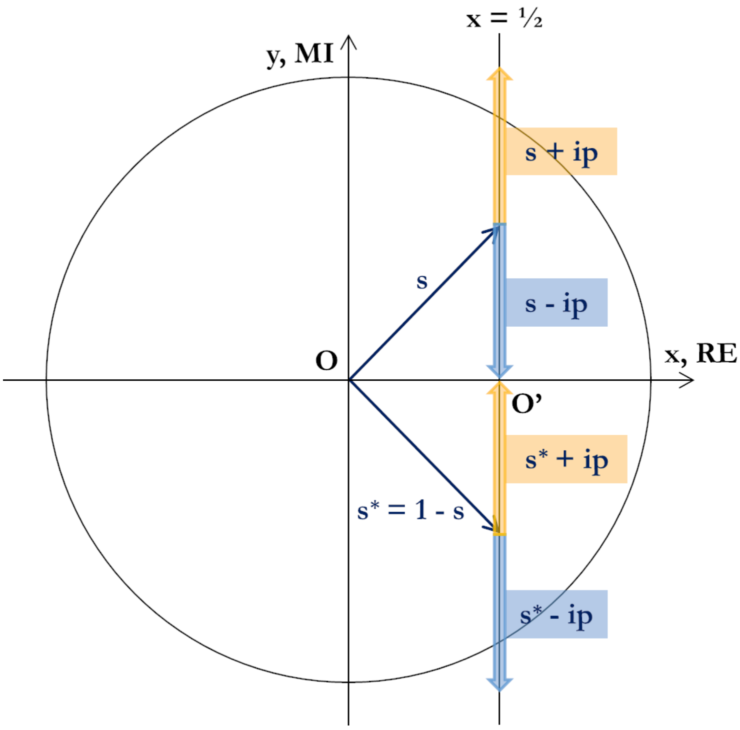

Figure 10: Additional imaginary part for s and s^*; the colored arrows indicate the location of the extended complex numbers when RE = 1/2.

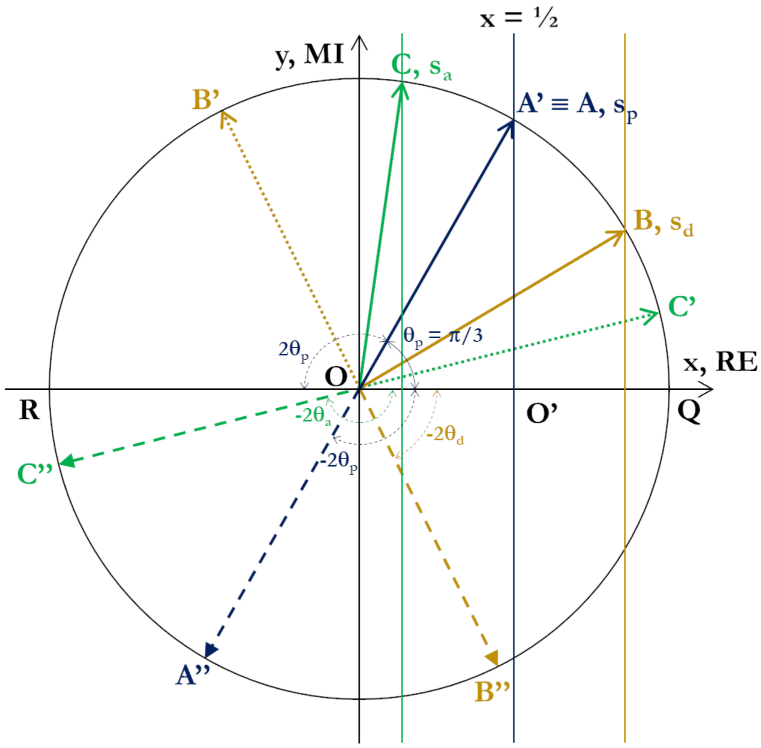

Figure 11: Complex plane for delay, punctuality, and anticipation.

Remark 1.12. I rewrite the Eq. 13:

$$\lim_{x\to+\infty} \kappa_X(x;x_0,\gamma) \approx \lim_{x\to+\infty}\left[1-\frac{1}{\frac{\pi}{\mathrm{RE}}\left(\mathrm{MI}+\frac{x}{\log(n)}\right)}\right], \tag{21}$$

where $\alpha \coloneqq \pi/\mathrm{RE}$ is an angle. Given the preceding discussion, when $\mathrm{RE} = 1/2$, then $\alpha = 2\pi$, which corresponds to punctuality in the critical strip (Remark 1.9, Remark 1.11). When $\mathrm{RE} \to 1^-$, then $\alpha \to \pi^+$ (π is inferior limit angle of delay), whereas, when $\mathrm{RE} \to 0^+$, then $\alpha \to +\infty$ (angle of anticipation or, better, the angle of anticipation is greater than 2π) [the concepts of delay, punctuality, and anticipation angle refer to the asymptotic convergence of the Cauchy cumulative distribution functions $\kappa_X(x;x_0,\gamma)$].

Theorem 1.8. *Let $N \in \mathbb{N}^+$ and odd, $s \in \mathbb{C}$ and in the critical strip* $(0 < \mathrm{RE} < 1)$, *there exists a Cauchy distribution whose $x_{0,sum}$ (location parameter) and γ_{sum} (scale parameter) depend on $\zeta(-N)$ and $\zeta(N+1)$.*

Proof. Let me recall that, taking the n^{th} sample, each one independently, from the n^{th} Cauchy distribution up to N distributions in total, there exists a Cauchy distribution whose $x_{0,sum}$ (location parameter) and γ_{sum} (scale parameter) are the sum of the N specific location parameters and of the N specific scale parameters, respectively, that is,

$$x_{0,sum} = \sum_{n=1}^{N}(-\mathrm{MI}\log(n)) = -\mathrm{MI}\sum_{n=1}^{N}\log(n) = -\mathrm{MI}\log(N!) = -\mathrm{MI}\log\left(\Gamma(N+1)\right), \tag{22}$$

$$\gamma_{sum} = \sum_{n=1}^{N}(\mathrm{RE}\log(n)) = \mathrm{RE}\sum_{n=1}^{N}\log(n) = \mathrm{RE}\log(N!) = \mathrm{RE}\log\left(\Gamma(N+1)\right). \tag{23}$$

Using the Riemann's functional equation $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s)$ with $s = -N$, I obtain for odd N the following equations:

$$\Gamma(N+1) = -\frac{2^N\pi^{N+1}\zeta(-N)}{\sin\left(\frac{\pi}{2}N\right)\zeta(N+1)} > 0, \tag{24}$$

$$x_{0,sum} = -\,\mathrm{MI}\log\left(-\frac{2^N\pi^{N+1}\zeta(-N)}{\sin\left(\frac{\pi}{2}N\right)\zeta(N+1)}\right), \tag{25}$$

$$\gamma_{sum} = \mathrm{RE}\log\left(-\frac{2^N\pi^{N+1}\zeta(-N)}{\sin\left(\frac{\pi}{2}N\right)\zeta(N+1)}\right). \tag{26}$$

The last two equations show the connection between the Riemann $\zeta(s)$ function and the Cauchy distribution. In other words, given odd N and $s = \mathrm{RE} + i\,\mathrm{MI}$, the Riemann $\zeta(s)$ function generates a new, Cauchy distributed, random variable X. □

Corollary 1.9. *Let $N \in \mathbb{N}^+$, large enough and even or odd, $s \in \mathbb{C}$ and in the critical strip $(0 < \mathrm{RE} < 1)$, $x_{0,sum}$ (location parameter) and γ_{sum} (scale parameter) are approximated by the N^{th} prime number p_N.*

Proof. Let me recall the Stirling's approximation for factorials: $N! = \sqrt{2\pi N}(N/e)^N(1 + O(1/N))$. From the Eqs. 22 and 23, using the PNT (i.e. $p_N \sim N\log(N)$), I get

$$x_{0,sum} \sim -\,\mathrm{MI}\left(N\log(N) - N + \frac{1}{2}\log(N) + \frac{1}{2}\log(2\pi)\right) \sim -\,\mathrm{MI}\left(p_N - N + \frac{1}{2}\log(N) + \frac{1}{2}\log(2\pi)\right), \tag{27}$$

$$\gamma_{sum} \sim \mathrm{RE}\left(N\log(N) - N + \frac{1}{2}\log(N) + \frac{1}{2}\log(2\pi)\right) \sim \mathrm{RE}\left(p_N - N + \frac{1}{2}\log(N) + \frac{1}{2}\log(2\pi)\right). \tag{28}$$

□

2 Riemann hypothesis

Theorem 2.1. *Let $n \in \mathbb{N}^+$, $s \in \mathbb{C}$ and in the critical strip $(0 < \mathrm{RE} < 1)$, all Riemann addends $1/n^s$ are solution of a unique differential equation describing an under-damped harmonic oscillator with unique initial conditions, s is a dimensionless angular frequency, and the eigenvalues are $-s^*$ and $-s$.*

Proof. Let me recall the differential equation of the under-damped harmonic oscillator:

$$\ddot{x} + 2\mu\omega_0\dot{x} + \omega_0^2 x = 0\,, \tag{29}$$

where ω_0 is the undamped angular frequency of the oscillator and μ is the damping ratio $(0 < \mu < 1)$. I rewrite the Eq. 29 according to [8]:

$$\ddot{x} + \frac{2}{\tau}\dot{x} + \omega_0^2 x = 0\,, \tag{30}$$

where $\tau = 1/(\mu\omega_0) = 1/\lambda$ is the time constant and $\lambda = \mu\omega_0$ is the decay rate. The solution of the Eq. 30, the initial conditions, the angular frequency ω, and the eigenvalues $\rho_\pm$ are

$$x(t, c_1, c_2) = e^{-\frac{t}{\tau}}(c_1\cos\omega t + c_2\sin\omega t)\,, \tag{31}$$

$$x(t = 0, c_1, c_2) = c_1\,, \tag{32}$$

$$\dot{x}(t = 0, c_1, c_2) = -\frac{c_1}{\tau} + c_2\omega \text{ with } \omega < \omega_0\,, \tag{33}$$

$$\omega = \sqrt{\omega_0^2 - \frac{1}{\tau^2}} \text{ with } \lim_{\tau\to+\infty}\omega = \omega_0\,, \tag{34}$$

$$\rho_\pm = -\frac{1}{\tau} \pm i\sqrt{\omega_0^2 - \frac{1}{\tau^2}} = -\frac{1}{\tau} \pm i\omega\,. \tag{35}$$

The Eq. 31 can be compared with the Cauchy characteristic function's value $\varphi_X(x_0, \gamma) = e^{-\gamma}e^{ix_0} = e^{-\gamma}(\cos x_0 + i\sin x_0)$, which is equal to the Riemann addend $1/n^s$ (Lemma 1.1), giving the following identifications:

1. $c_1 = 1$,

2. $c_2 = i$,

3. $\frac{t}{\tau} \longleftrightarrow \gamma = \mathrm{RE}\log(n)$,

4. $\omega t \longleftrightarrow x_0 = -\mathrm{MI}\log(n)$ and, as a consequence,

5. $t \longleftrightarrow \log(n)$,

6. $\frac{1}{\tau} = \lambda = \mu\omega_0 \longleftrightarrow \mathrm{RE}$,

7. $\omega \longleftrightarrow -\mathrm{MI}$.

As a consequence, for each Riemann addend $1/n^s$, using the Eqs. 29 and 30, defining the "dimensionless" time $t \coloneqq \log(n)$, I can suppose and assign the equation of an under-damped harmonic oscillator:

$$\ddot{\beta} + 2\,\mathrm{RE}\,\dot{\beta} + \omega_0^2\beta = 0\,, \tag{36}$$

where $\mathrm{RE} = \frac{1}{\tau} = \lambda = \mu\omega_0$ is the dimensionless decay rate, τ is the dimensionless time constant, ω_0 is the dimensionless undamped angular frequency of the oscillator, and μ is the damping ratio ($0 < \mu < 1$). The solution of the Eq. 36, the initial conditions, the dimensionless angular frequency ω, and the eigenvalues $\rho_\pm$ are

$$\beta(t, c_1 = 1, c_2 = i) = \beta(t) = e^{-\mathrm{RE}\,t}(\cos\omega t + i\sin\omega t)\,, \tag{37}$$

$$\beta(0) = 1\,, \tag{38}$$

$$\dot{\beta}(0) = -\mathrm{RE} + i\omega = -\mathrm{RE} - i\,\mathrm{MI} = -s \text{ with } \omega = -\mathrm{MI} < \omega_0\,, \tag{39}$$

$$|\omega| = |\,\mathrm{MI}\,| = \sqrt{\omega_0^2 - \mathrm{RE}^2} \Leftrightarrow \omega_0^2 = \mathrm{RE}^2 + \mathrm{MI}^2 \Leftrightarrow \omega_0 = |s| \text{ for all Riemann addends } 1/n^s\,, \tag{40}$$

$$\rho_\pm = -\mathrm{RE} \pm i\sqrt{\omega_0^2 - \mathrm{RE}^2} = -\mathrm{RE} \pm i|\,\mathrm{MI}\,|\,. \tag{41}$$

The Eqs. 36-39 can be rewritten as

$$\ddot{\beta} + 2\,\mathrm{RE}\,\dot{\beta} + |s|^2\beta = 0\,, \tag{42}$$

$$\begin{aligned}\beta(t; s) &= e^{-\mathrm{RE}\,t}(\cos\omega t + i\sin\omega t) = e^{-\mathrm{RE}\,t}e^{i\omega t} = e^{-\mathrm{RE}\,t}e^{-i\,\mathrm{MI}\,t} = e^{-st}\\ &= e^{-s\log(n)} = \left(e^{\log(n)}\right)^{-s} = \frac{1}{n^s}\,,\end{aligned} \tag{43}$$

$$\beta(t = \log 1 = 0; s) = 1\,, \tag{44}$$

$$\dot{\beta}(t = \log 1 = 0; s) = -s\,. \tag{45}$$

□

Remark 2.1. The Eq. 42 is invariant respect to $n \in \mathbb{N}^+$ and all Riemann addends $1/n^s$ are solution of the same differential Eq. 42 with the same initial conditions (Eqs. 44 and 45). The Eq. 43 can be interpreted as an under-damped harmonic oscillator and RE is the dimensionless decay rate damping the oscillations; moreover, the dimensionless undamped angular frequency ω_0 is equal to $|s|$ (Eq. 40), in other words, ω_0 depends on RE and MI, and s is a dimensionless angular frequency.

On the other hand, because the dimensionless time $t = \log(n)$ (Eq. 43), the set containing all Riemann addends $1/n^s$ can be thought as the codomain of the function $\beta(t; s)$ (Fig. 12, Fig. 13, Fig. 14); in these figures, for $n \to +\infty$, I recognize a spiral sink, which is due to the eigenvalues $-s^*$ and $-s$, and the origin is an asymptotically stable orbit.

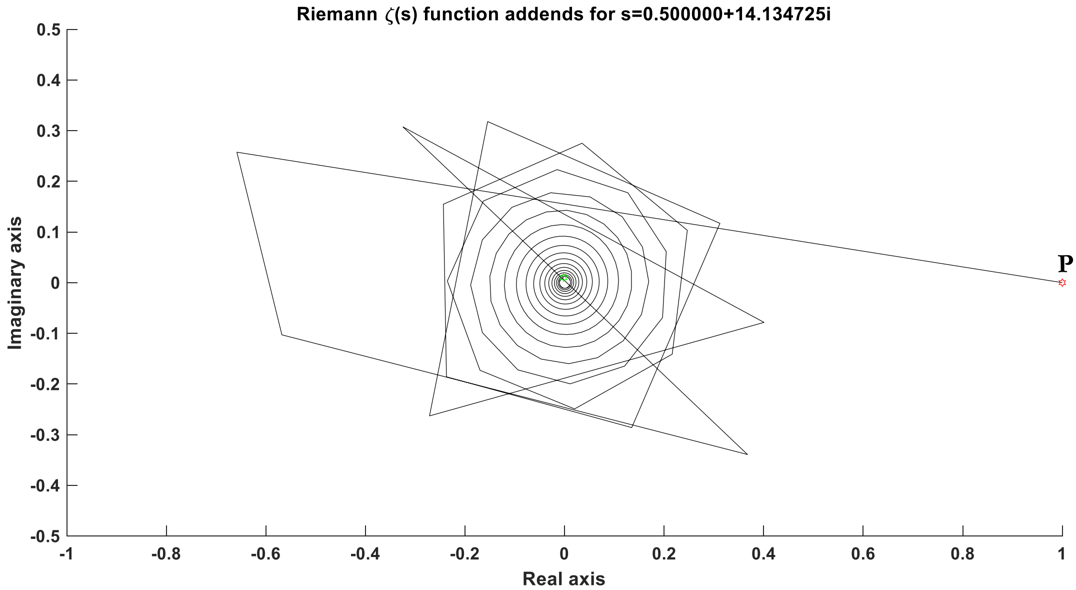

Figure 12: Riemann addends generated by s, which belongs to the critical line and is non-trivial solution of $\zeta(s) = 0$. The point P is the first initial condition $\beta(t = \log 1 = 0; s) = 1$.

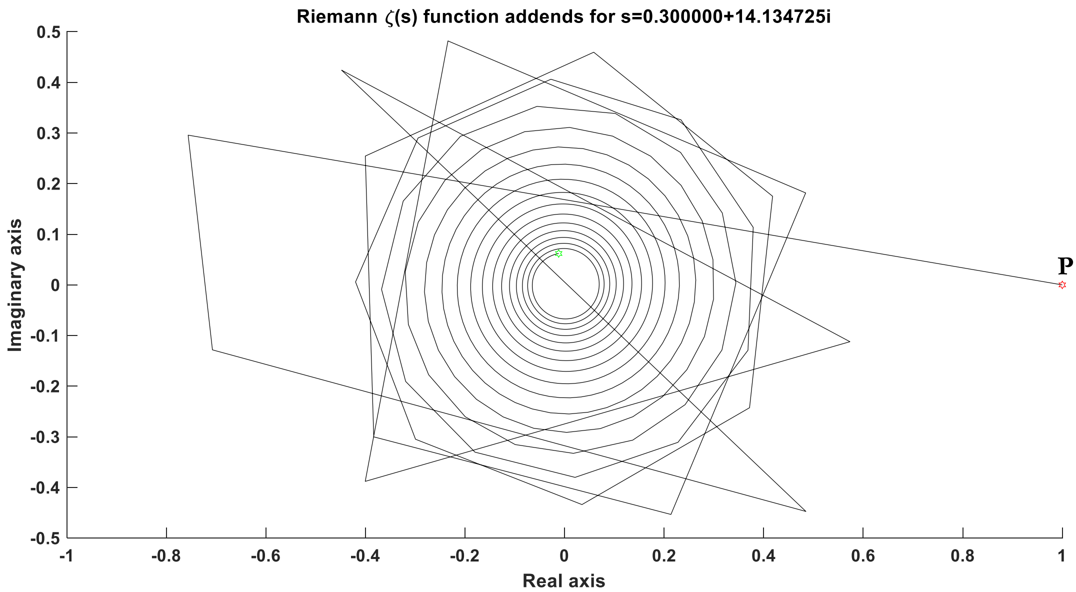

Figure 13: Riemann addends generated by s, which does not belong to the critical line and is not solution of $\zeta(s) = 0$. The point P is the first initial condition $\beta(t = \log 1 = 0; s) = 1$.

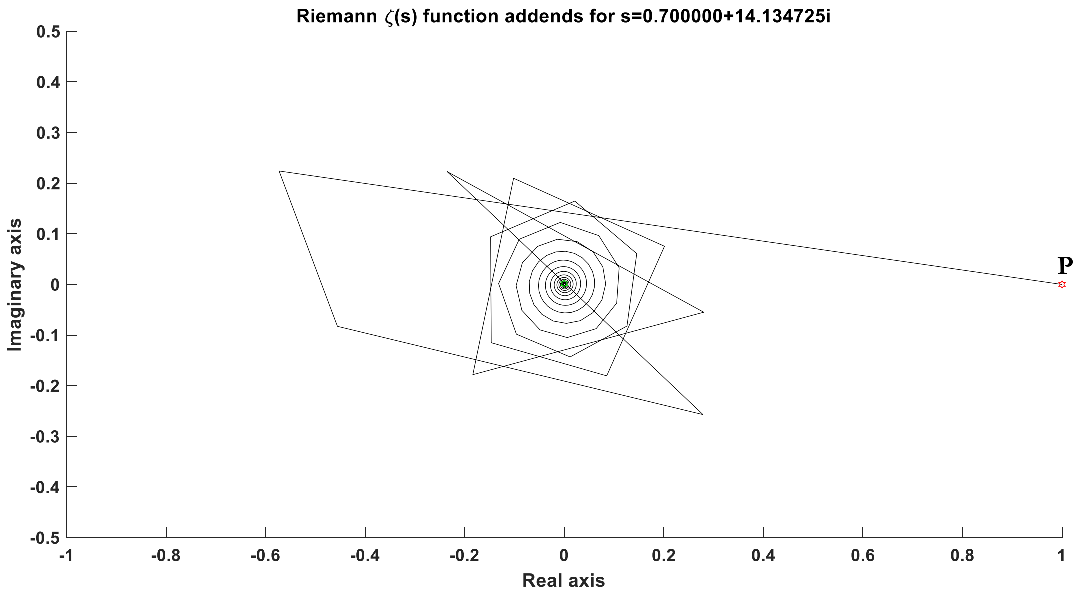

Figure 14: Riemann addends generated by s, which does not belong to the critical line and is not solution of $\zeta(s) = 0$. The point P is the first initial condition $\beta(t = \log 1 = 0; s) = 1$.

Lemma 2.2. *The function $\beta(t;s)$ describes a spiral sink for $s \in \mathbb{C}$ and in the critical strip* $(0 < \mathrm{RE} < 1)$ *and such that* $|\mathrm{MI}| \neq 0$.

Proof. Using the eigenvalues $-s^*$ and $-s$ in the matrix $A = \begin{pmatrix} -s^* & 0 \\ 0 & -s \end{pmatrix}$, I obtain

$$\operatorname{tr} A = -s^* - s = -2\,\mathrm{RE} < 0\,, \tag{46}$$

$$4 \det A - (\operatorname{tr} A)^2 = 4 s^* s - 4\,\mathrm{RE}^2 = 4|s| - 4\,\mathrm{RE}^2 = 4\,\mathrm{RE}^2 + 4\,\mathrm{MI}^2 - 4\,\mathrm{RE}^2 = 4\,\mathrm{MI}^2 > 0\,. \tag{47}$$

□

Remark 2.2. The first initial condition is $\beta(t = \log 1 = 0; s) = 1$, which is invariant respect to $s \in \mathbb{C}$ and is highlighted by a "common starting point" P (Fig. 12, Fig. 13, Fig. 14). There is a "common starting point" P also for the alternating Cauchy cumulative distribution functions $\kappa_X(x; x_0, \gamma)$ toward their asymptotic convergence to the lines $y = \pm 1$ when $x \to +\infty$, that is, to $y = 1$ for odd n or to $y = -1$ for even n (Figs. 5-8, Remark 1.8, Lemma 1.5): this other starting point P is invariant respect to $n \in \mathbb{N}^+$.

The second initial condition is the dimensionless complex angular frequency $-s = -\mathrm{RE} - i\,\mathrm{MI}$; for $t = \log 1 = 0$, $\omega_0 = |s|$ imparts an anticlockwise rotation to the first Riemann addends (the first five, comprising the first addend in P, in Figs. 12-14), whereas, for $t = \log(n) > 0$ and $\mathrm{MI} > 0$, the clockwise rotation effect of $-s$ starts to act and the Riemann addends form a clockwise spiral sink when n is large enough.

For $n \to +\infty$, the n^{th} Riemann addend $1/n^s$ is asymptotically sunk into the origin, the alternating Cauchy cumulative distribution functions $\kappa_X(x; x_0, \gamma)$ move from the "common starting point" P (Fig. 8) via $x \to +\infty$: as a consequence, according to the stability theory, the given orbit $\pi(x)$ is attracting the asymptotically stable orbit $p(x, n) = x/\log(n)$.

In order to have $\eta(s) = 0$ (Corollary 1.7), I will sum the alternating Cauchy cumulative distribution functions $\kappa_X(x; x_0, \gamma)$ for a given value of RE (Fig. 9, Fig. 10), trying to find a balance between complex numbers (e.g., $(s + ip)$ vs. $(s^* - ip)$ in Remark 2.6), which are in the domain of the functions $\kappa_X(x; n, s)$ (Eq. 16).

There is an interesting particular: the dimensionless angular frequency ω immediately goes from the ω_0 value to the $-\mathrm{MI}$ value (Eq. 39) (this recalls the concept of act of movement [9] in Classical Mechanics), showing a jump discontinuity in $t = 0$; this observation can be done not only when $n = 1$, which produces a Dirac's δ distribution (Remark 1.1), but also $\forall n \in \mathbb{N}^+$.

Furthermore, there is a comprehensive effect of an infinite number of the unique under-damped harmonic oscillator $\beta(t;s)$ (Theorem 1.8, Corollary 1.9).

Remark 2.3. Because of s is a dimensionless angular frequency, the angle $\alpha = \pi/\mathrm{RE}$ can be interpreted also as a dimensionless time (Eq. 21). As a consequence, π is the dimensionless inferior limit time of delay, 2π corresponds to punctuality, and the dimensionless time of anticipation is greater than 2π (the concepts of delay, punctuality, and anticipation dimensionless time refer to the asymptotic convergence of the Cauchy cumulative distribution functions $\kappa_X(x;x_0,\gamma)$).

Remark 2.4. The extensions of MI, that is, $p(x,n) = x/\log(n)$ and $\pi(x)$ can be thought as dimensionless angular frequencies. For the Riemann hypothesis, there are four dimensionless angular frequencies: RE, MI, $p(x,n) = x/\log(n)$, and $\pi(x)$; they determine the asymptotic convergence of the alternating Cauchy cumulative distribution functions $\kappa_X(x;x_0,\gamma)$ to the lines $y = \pm 1$ (Figs. 5-8, Remark 1.8, Lemma 1.5). As a consequence, from Remark 1.7, the Riemann hypothesis can be expressed as: only for RE = 1/2, there is the possibility to obtain the null alternating sum

$$\sum_{n=1}^{\infty}(-1)^{n+1}\kappa_X(x \to +\infty; n \to +\infty, \mathrm{RE} = \frac{1}{2}, \mathrm{MI}) = 0\,. \tag{48}$$

Remark 2.5. The damping ratio μ $(0 < \mu < 1)$ is equal to $1/(2Q)$, where Q is the quality factor describing how under-damped an oscillator or a pendulum is. Because of $2\,\mathrm{RE} = 2\mu\omega_0$, I obtain

$$2\,\mathrm{RE} = \frac{\omega_0}{Q}\,, \tag{49}$$

which is the coefficient of $\dot{\beta}$ in Eq. 42. For an under-damped pendulum, the coefficient $2\,\mathrm{RE}$ is equal to F/M, where F is the frictional damping force per unit velocity and M is the mass of the bob. In other words, I can associate a pendulum to RE; for example, in Fig. 9, I see four pendulums characterized by an infinite length (i.e. with the pivot at $x \to -\infty$):

1. pendulum with dimensionless angular frequency equal to $s + ip$,
2. pendulum with dimensionless angular frequency equal to $s^* - ip$,
3. pendulum with dimensionless angular frequency equal to $(1-s) + ip$,
4. pendulum with dimensionless angular frequency equal to $(1-s)^* - ip$.

I think the puzzle is here, because Riemann and Landau felt the need for symmetry and balance in the functional equation

$$\xi(1-s) = \xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\,. \tag{50}$$

To this regard, it is convenient to normalize the preceding dimensionless angular frequencies respect to their real part:

1. pendulum with normalized dimensionless angular frequency equal to $1 + i\,\mathrm{MI}/\mathrm{RE} + ip/\mathrm{RE}$,
2. pendulum with normalized dimensionless angular frequency equal to $1 - i\,\mathrm{MI}/\mathrm{RE} - ip/\mathrm{RE}$,
3. pendulum with normalized dimensionless angular frequency equal to $1 - i\,\mathrm{MI}/(1-\mathrm{RE}) + ip/(1-\mathrm{RE})$,
4. pendulum with normalized dimensionless angular frequency equal to $1 + i\,\mathrm{MI}/(1-\mathrm{RE}) - ip/(1-\mathrm{RE})$.

In Fig. 15, I show this normalization, in particular, the dashed arrows indicate the normalized imaginary part before the extension via ip. The pendulums generated by s and $(1-s)$ are balanced and in punctuality if and only if RE = 1/2, that is, when $(1-s) = s^*$ and $\mathrm{MI}/\mathrm{RE} = 2\,\mathrm{MI}$. On the contrary, when s causes anticipation, $(1-s)$ gives delay and, viceversa, when s causes delay, $(1-s)$ gives anticipation. Furthermore, when $\mathrm{RE} \to 0^+$, $(1-s)$ causes the asymptotic convergence to the superior limit delay $-\mathrm{MI}$, whereas, when $\mathrm{RE} \to 1^-$, s gives the asymptotic convergence to the inferior limit delay MI.

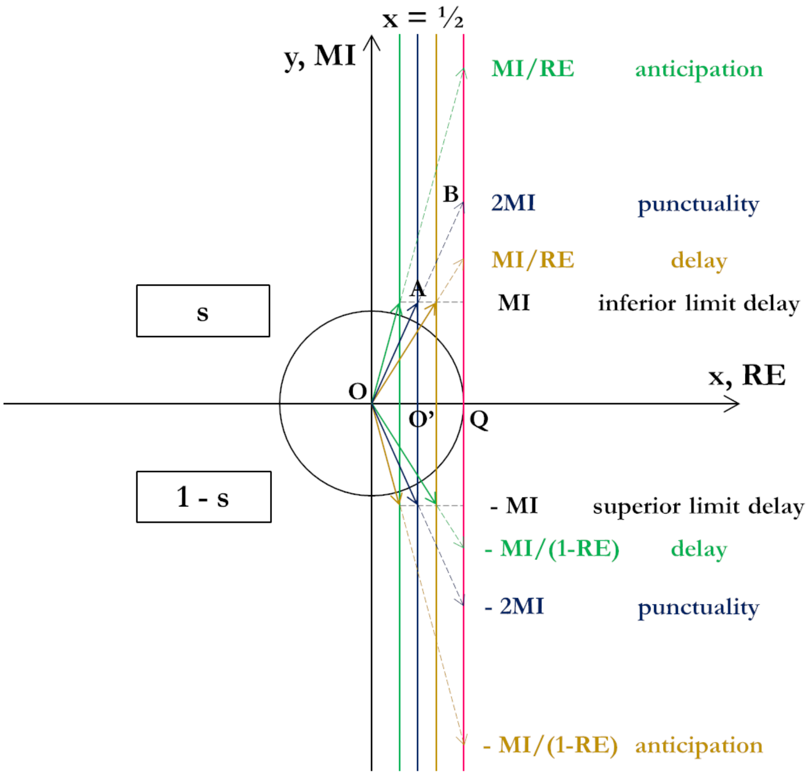

Figure 15: Complex plane for normalized delay, punctuality, and anticipation.

Remark 2.6. It is natural to think that the pendulums' balance and punctuality for RE = 1/2 should give $\zeta(s) = 0$ for all s on the critical line: in fact, for RE = 1/2, I have $(1-s) = s^*$ and, using the first and the second normalized dimensionless angular frequencies,

$$\mathrm{Im}\left(1 + i\frac{\mathrm{MI}}{\mathrm{RE}} + i\frac{p}{\mathrm{RE}}\right) + \mathrm{Im}\left(1 - i\frac{\mathrm{MI}}{\mathrm{RE}} - i\frac{p}{\mathrm{RE}}\right) = \mathrm{Im}(1 + 2i\,\mathrm{MI} + 2ip) + \mathrm{Im}(1 - 2i\,\mathrm{MI} - 2ip) = 0\,. \tag{51}$$

Indeed, the Cauchy distribution does not obey to the strong law of large numbers, whereas it obeys to the weak law of large numbers because of the heavy tails of the probability density function $g_X(x; x_0, \gamma)$ (Eq. 2) [10]. As a consequence, there is the possibility that $\zeta(s) \neq 0$ happens an infinite number of times on the critical line.

Theorem 2.3. *For* RE $\neq 1/2$*, the pendulums generated by* s *and* $(1-s)$ *are neither in balance nor in punctuality, that is,* $\zeta(s) \neq 0$*.*

Proof. I suppose that the first and the third normalized dimensionless angular frequencies are in balance and cause punctuality, that is,

$$\begin{aligned}\mathrm{Im}\left(1 + i\frac{\mathrm{MI}}{\mathrm{RE}} + i\frac{p}{\mathrm{RE}}\right) + \mathrm{Im}\left(1 - i\frac{\mathrm{MI}}{1-\mathrm{RE}} + i\frac{p}{1-\mathrm{RE}}\right) &= \frac{(p+\mathrm{MI})(1-\mathrm{RE}) + (p-\mathrm{MI})\,\mathrm{RE}}{\mathrm{RE}(1-\mathrm{RE})} \\ &= \frac{p + \mathrm{MI}(1-2\,\mathrm{RE})}{\mathrm{RE}(1-\mathrm{RE})} = 0 \\ &\Leftrightarrow p = \mathrm{MI}(2\,\mathrm{RE}-1)\,.\end{aligned} \tag{52}$$

As a consequence, for $0 < \mathrm{RE} < 1/2$ and for $1/2 < \mathrm{RE} < 1$, I obtain $-|\mathrm{MI}| < p < 0$ and $0 < p < |\mathrm{MI}|$, respectively, which are contradictions because p is unbound. ☐

Remark 2.7. The precision, to which I can estimate a parameter χ of the Cauchy distribution, is fundamentally limited by the Fisher information of the likelihood function. For the Cauchy distribution, the Fisher information $\mathrm{I}(\chi)$ is

$$\mathrm{I}(\chi) = \frac{1}{2\gamma^2} = \frac{1}{2\,\mathrm{RE}^2 \log^2(n)}\,. \tag{53}$$

According to the Cramér-Rao bound [11]-[12], given $\gamma_{sum} \sim \mathrm{RE}\, p_N$ (Eq. 28, p_N is the N^{th} prime number), the variance $\mathrm{Var}(\hat{\chi})$ and the standard deviation $\mathrm{STD}(\hat{\chi})$ of an unbiased estimator of χ (distribution parameter) are

$$\mathrm{Var}(\hat{\chi}) \geq \frac{1}{\mathrm{I}(\chi)} = 2\gamma_{sum}^2 \sim 2\,\mathrm{RE}^2\, p_N^2\,, \tag{54}$$

$$\mathrm{STD}(\hat{\chi}) = \sqrt{\mathrm{Var}(\hat{\chi})} \geq \sqrt{2}\gamma_{sum} \sim \sqrt{2}\,\mathrm{RE}\, p_N\,. \tag{55}$$

The standard deviation gives the precision to estimate both parameters of the Cauchy distribution: $x_{0,sum}$ and γ_{sum} (Eqs. 27 and 28); such precision depends on RE ($0 < \mathrm{RE} < 1$) and on p_N itself (Eq. 55). I note that the precision is inversely proportional to p_N; moreover, the anticipation ($0 < \mathrm{RE} < 1/2$) enhances the precision, the delay ($1/2 < \mathrm{RE} < 1$) worsens the precision, whereas the punctuality ($\mathrm{RE} = 1/2$) causes a "precision quadrature":

$$\mathrm{STD}(\hat{\chi}) \geq \sqrt{2}\gamma_{sum} \sim \frac{\sqrt{2}}{2} p_N\,. \tag{56}$$

Remark 2.8. If the Theorem 2.3 is right, then the Riemann hypothesis is proven. As a consequence, I can use a result of Cramér [13]: the prime number theorem implies that, on average, the gap between the prime number p_N and its prime successor p_{N+1} is $\log(p_N)$, however, some gaps may be much larger than $\log(p_N)$. In particular, Cramér [13] proved that, assuming the Riemann hypothesis as true, each prime gap is $O\left(\sqrt{p_N}\log(p_N)\right)$.

The preceding result of Cramér is consistent with the Cauchy distribution's $\mathrm{Var}(\hat{n}) \geq \frac{1}{\mathrm{I}(n)} = 2\gamma^2$, that is, with $\mathrm{STD}(\hat{n}) \geq \sqrt{2}\gamma$ or, better, with $\mathrm{STD}(\hat{n}) = \sqrt{n}\gamma = \sqrt{n}\,\mathrm{RE}\log(n)$, where $n \in \mathbb{N}^+$ and $n \geq 2$: in particular, $\hat{n}$ is an unbiased estimator of natural numbers. In other words, $\forall n \in \mathbb{N}^+$ and $n \geq 2$, there is a standard deviation and, if n is prime, its prime successor can be reached via gap fill using $\mathrm{STD}(\hat{n}) = \sqrt{n}\gamma = \sqrt{n}\,\mathrm{RE}\log(n)$. In the Figs. 16, 17, and 18, I plot the variance lower bound $\mathrm{Var}(\hat{n}) = 2\gamma^2$ and the standard deviation lower bound $\mathrm{STD}(\hat{n}) = \sqrt{2}\gamma$ (in these figures, the point in the origin is related to the Dirac's δ distribution since $\gamma = 0$).

The use of $\mathrm{STD}(\hat{n}) \geq \sqrt{2}\gamma$ appears justified considering also a result of Dudek [14], that is, if the Riemann hypothesis is true, $\forall n \in \mathbb{N}^+$ and $n \geq 2$, there is a prime number p satisfying: $n - \frac{4}{\pi}\sqrt{n}\log(n) < p \leq n$; the left-hand side of the preceding inequality contains a standard deviation proportional to $\mathrm{STD}(\hat{n}) = \sqrt{n}\gamma = \sqrt{n}\,\mathrm{RE}\log(n)$, whereas the right-hand side is analog to the line $y = x$.

Because of the prime numbers determine the morphology of the $\pi(x)$ function and the prime gaps can be estimated via the $\mathrm{STD}(\hat{n}) \geq \sqrt{2}\gamma$, because of $y = f(x) = x/\log(x)$ estimates the mean cardinality of the prime numbers less than the x magnitude, given the preceding inequality of Dudek [14], there is the same gap not only between prime numbers, but also between $y = x$ (all $n \geq 1$ are prime) and $y = f(x) = x/\log(x)$ (not all $n \geq 2$ are prime), that is, there is the same standard deviation also for the $f(x)$ function itself:

$$\mathrm{STD}\left(\hat{f}(x)\right) \geq \sqrt{2}\gamma = \sqrt{2}\,\mathrm{RE}\log(x) \tag{57}$$

or, better,

$$\mathrm{STD}\left(\hat{f}(x)\right) = \sqrt{n}\gamma = \sqrt{n}\,\mathrm{RE}\log(x)\,, \tag{58}$$

where $n \in \mathbb{N}^+$ and $n \geq 2$.

Theorem 2.4. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = x/\log(x) + O\left(\sqrt{x}\log(x)\right)$.*

Proof. Let me recall from Dusart [15] that, for $x \in \mathbb{R}^+$ and $x \geq 88789$,

$$\pi(x) > \frac{x}{\log(x)}\left(1 + \frac{1}{\log(x)} + \frac{2}{\log^2(x)}\right). \tag{59}$$

As a consequence,

$$\pi(x) - \frac{x}{\log(x)} > \frac{x}{\log(x)}\left(\frac{1}{\log(x)} + \frac{2}{\log^2(x)}\right) = \sqrt{x}\log(x)\underbrace{\left(\frac{\sqrt{x}}{\log^3(x)} + \frac{2\sqrt{x}}{\log^4(x)}\right)}_{\mathrm{A}(x)}, \tag{60}$$

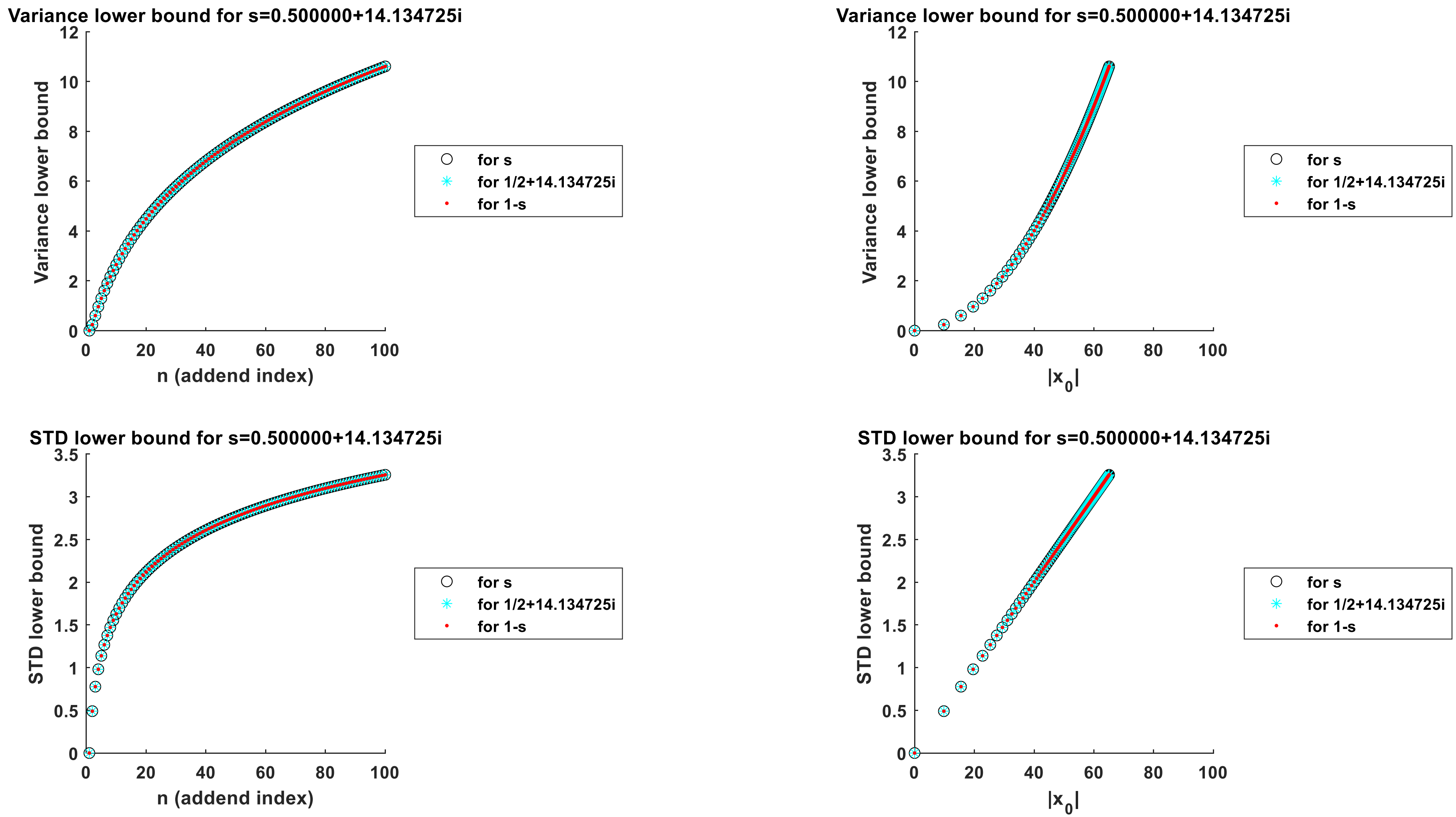

Figure 16: Variance lower bound and STD lower bound of $\hat{n}$ or $\hat{f}(x)$ generated by s and $s^* = (1 - s)$, which belong to the critical line and are non-trivial solutions of $\zeta(s) = 0$.

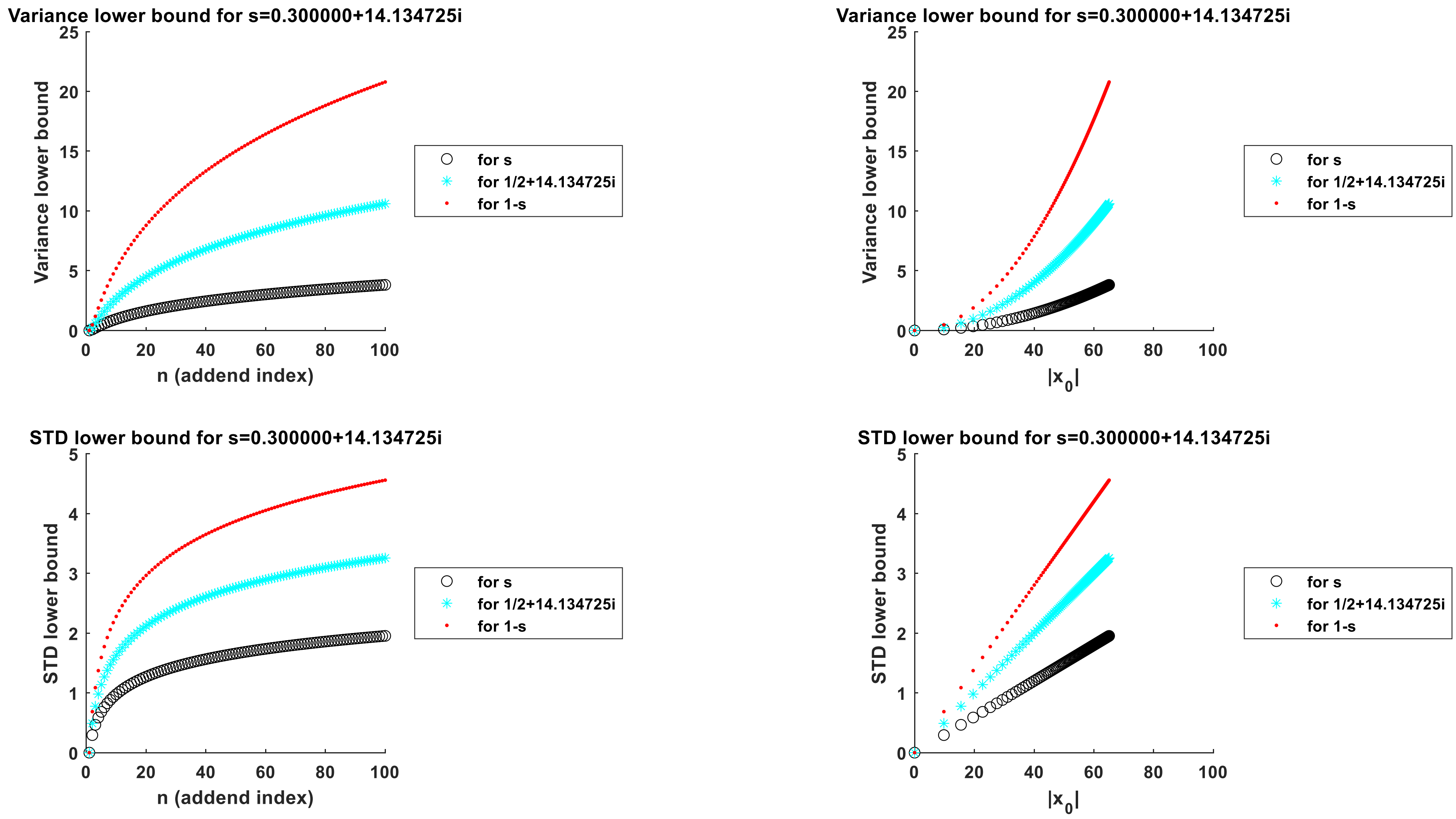

Figure 17: Variance lower bound and STD lower bound of $\hat{n}$ or $\hat{f}(x)$ generated by s and $(1-s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$.

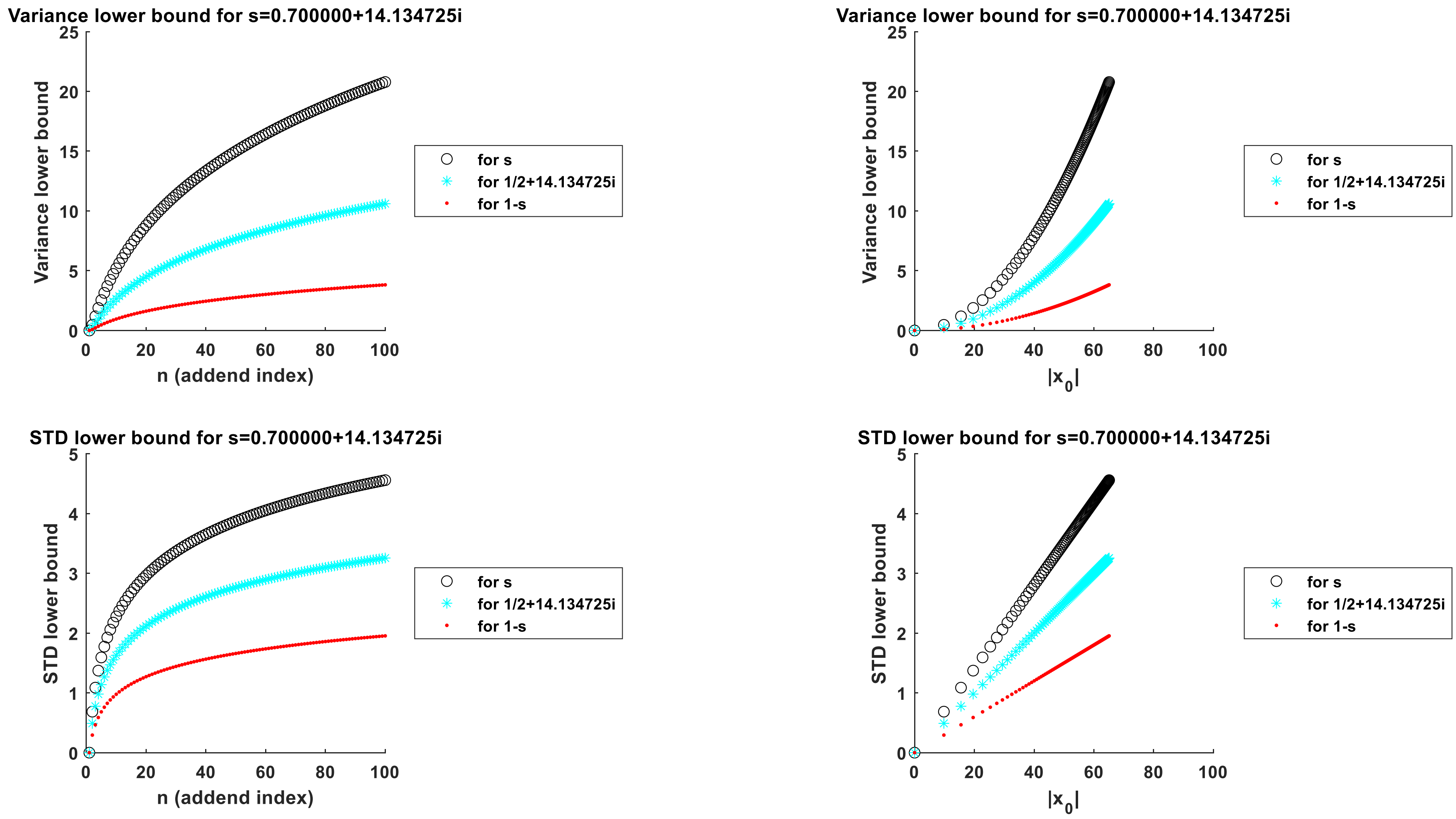

Figure 18: Variance lower bound and STD lower bound of $\hat{n}$ or $\hat{f}(x)$ generated by s and $(1-s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$.

where $\mathrm{A}(x)$ is monotone increasing for circa $x \geq 646.87$. Given $n \in \mathbb{N}^+$ and $n \geq 2$, given $0 < \mathrm{RE} < 1$, for large enough $x \in \mathbb{R}^+$ and $x \geq 88789$, using the Eq. 58, I obtain

$$\sqrt{x}\log(x)\,\mathrm{A}(x) \geq \sqrt{x}\log(x)\sqrt{n}\,\mathrm{RE} = \sqrt{x}\,\mathrm{STD}\left(\hat{f}(x)\right)\,, \tag{61}$$

then, via the Tchebychev's inequality and the Eqs. 60 and 61,

$$\mathrm{Prob}\left(\pi(x) - \underbrace{\frac{x}{\log(x)}}_{mean} > \sqrt{x}\log(x)\,\mathrm{A}(x) \geq \sqrt{x}\log(x)\sqrt{n}\,\mathrm{RE} = \underbrace{\sqrt{x}}_{k}\,\underbrace{\mathrm{STD}\left(\hat{f}(x)\right)}_{\substack{standard\\deviation}}\right) \leq \underbrace{\frac{1}{x}}_{1/k^2} \to 0^+\,, \tag{62}$$

that is, with probability $\to 1^-$ ($\approx$ certainty),

$$\pi(x) - \frac{x}{\log(x)} \leq \sqrt{n}\,\mathrm{RE}\,\sqrt{x}\log(x)\,, \tag{63}$$

where $\sqrt{n}\,\mathrm{RE} = \epsilon$, $n \in \mathbb{N}^+$ and $n \geq 2$, and $0 < \mathrm{RE} < 1$. □

Corollary 2.5. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\log(x)\right)$.*

Proof. For large enough $x \in \mathbb{R}^+$ and $x \geq 88789$, $x/\log(x) \sim \mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 63, I obtain

$$|\pi(x) - \mathrm{Li}(x)| \leq \sqrt{n}\,\mathrm{RE}\,\sqrt{x}\log(x)\,, \tag{64}$$

where $\sqrt{n}\,\mathrm{RE} = \epsilon$, $n \in \mathbb{N}^+$ and $n \geq 2$, and $0 < \mathrm{RE} < 1$. □

Remark 2.9. The inequalities 63 and 64 can be intended as prime number theorems (a prime number theorem describes the asymptotic distribution of the prime numbers and also states that the prime numbers are less common as they become larger). Furthermore, to show the lower bound of the standard deviation $\mathrm{STD}(\hat{n})$, a convenient way is to plot the half of an Agnesi versiera, which is the parametric curve

$$\begin{cases} x = \gamma\tan(\theta) \\ y = \gamma\cos^2(\theta)\,, \end{cases} \tag{65}$$

where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (Figs. 19, 20, and 21): in these figures, the maximum of the versiera is in the point $(0,\gamma)$ and the magenta diagonal of the square has length equal to the standard deviation lower bound $\mathrm{STD}(\hat{n}) = \sqrt{2}\gamma$.

Theorem 2.6. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = x/\log(x) + o\left(\sqrt{x}\log(x)\right)$.*

Proof. In Eq. 63, $\sqrt{n}\,\mathrm{RE} = \epsilon \in \mathbb{R}^+$ and ϵ can have every positive value, however small. □

Corollary 2.7. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = \mathrm{Li}(x) + o\left(\sqrt{x}\log(x)\right)$.*

Proof. For large enough $x \in \mathbb{R}^+$ and $x \geq 88789$, $x/\log(x) \sim \mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 64, I observe that $\sqrt{n}\,\mathrm{RE} = \epsilon \in \mathbb{R}^+$ and that ϵ can have every positive value, however small. □

Lemma 2.8. *Let $n \in \mathbb{N}^+$, $s \in \mathbb{C}$ and in the critical strip $(0 < \mathrm{RE} < 1)$, the Gauss characteristic function's value $\varphi_X(t = 1; x_0, \sigma^2)$ is equal to the Riemann addend $1/n^s$.*

Proof. Let me recall the characteristic function $\varphi_X(t; x_0, \sigma^2)$ of the Gauss distribution of a random variable X, where x_0 is the mean and σ^2 is the variance:

$$\varphi_X(t; x_0, \sigma^2) = e^{ix_0 t - \frac{1}{2}\sigma^2 t^2}\,, \tag{66}$$

then, defining $\sigma^2 \coloneqq 2\,\mathrm{RE}\log(n)$ and $x_0 \coloneqq -\mathrm{MI}\log(n)$, I obtain

$$\begin{aligned} \varphi_X(t = 1; x_0, \sigma^2) &= e^{ix_0 - \frac{1}{2}\sigma^2} = e^{-(\mathrm{RE} + i\,\mathrm{MI})\log(n)} = e^{-s\log(n)} = \left(e^{\log(n)}\right)^{-s} = \frac{1}{n^s} \\ &= \varphi_X(t = 1; n, s) = \varphi_X(t = 1; n, \mathrm{RE}, \mathrm{MI})\,. \end{aligned} \tag{67}$$

□

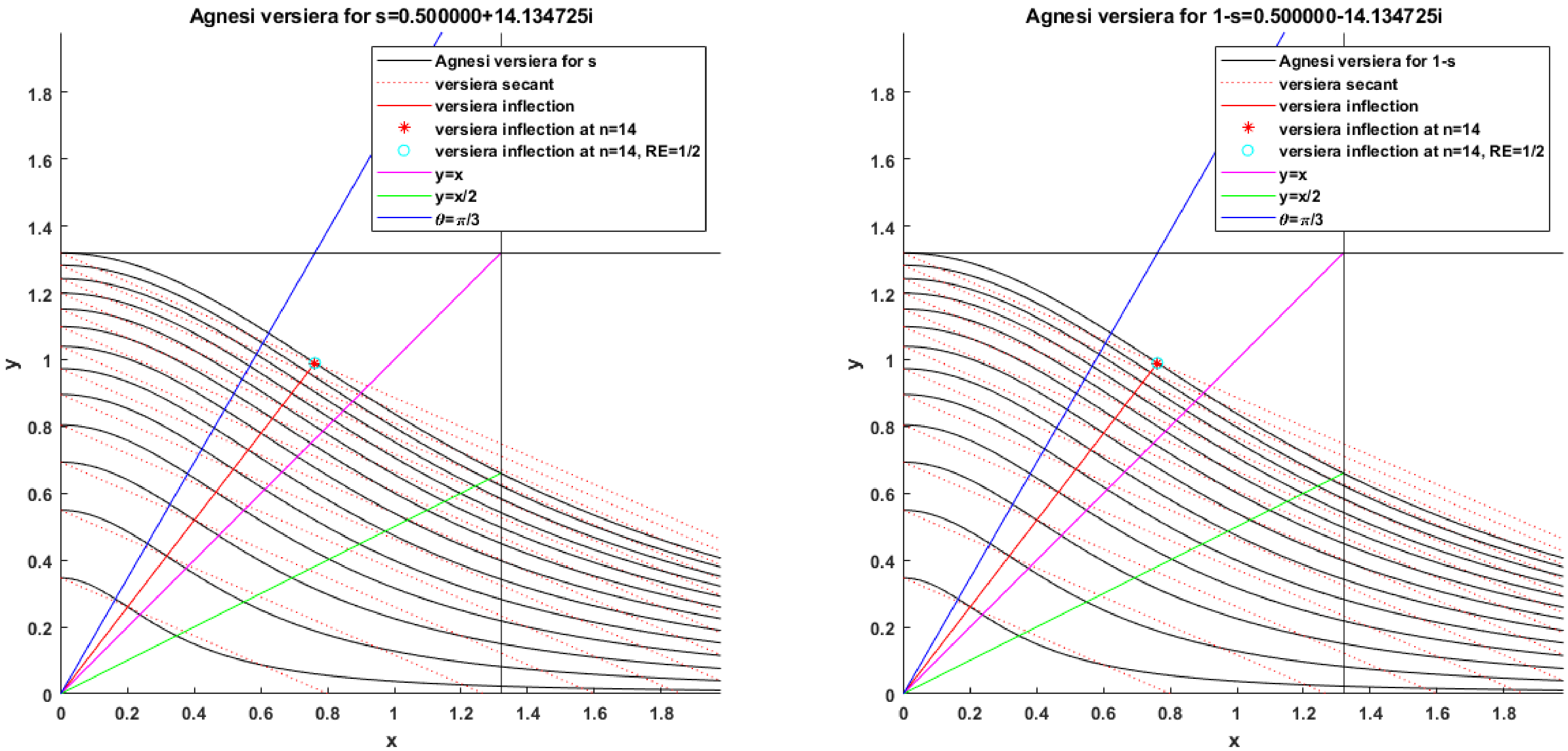

Figure 19: Agnesi versiera generated by s and $s^* = (1 - s)$, which belong to the critical line and are non-trivial solutions of $\zeta(s) = 0$.

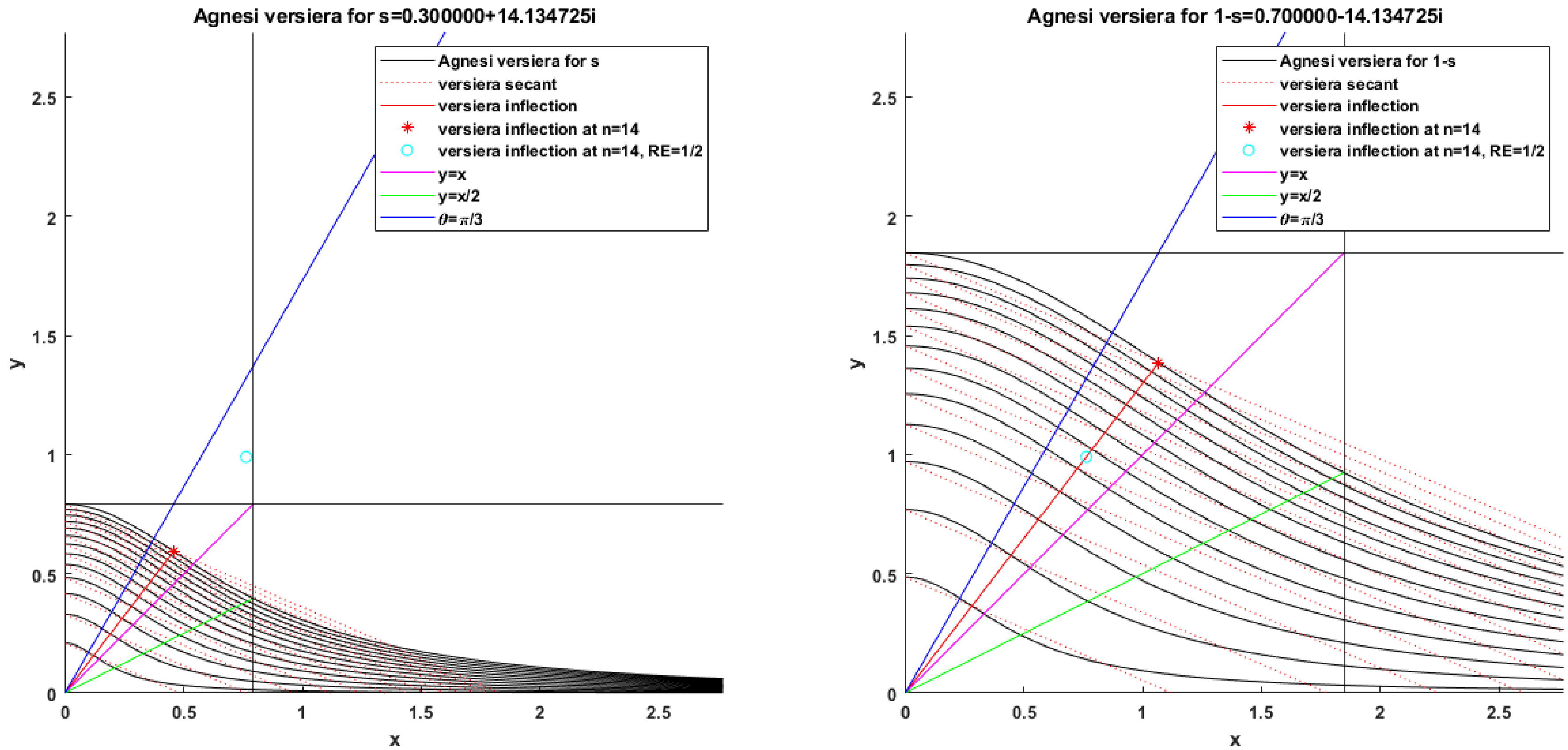

Figure 20: Agnesi versiera generated by s and $(1-s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$.

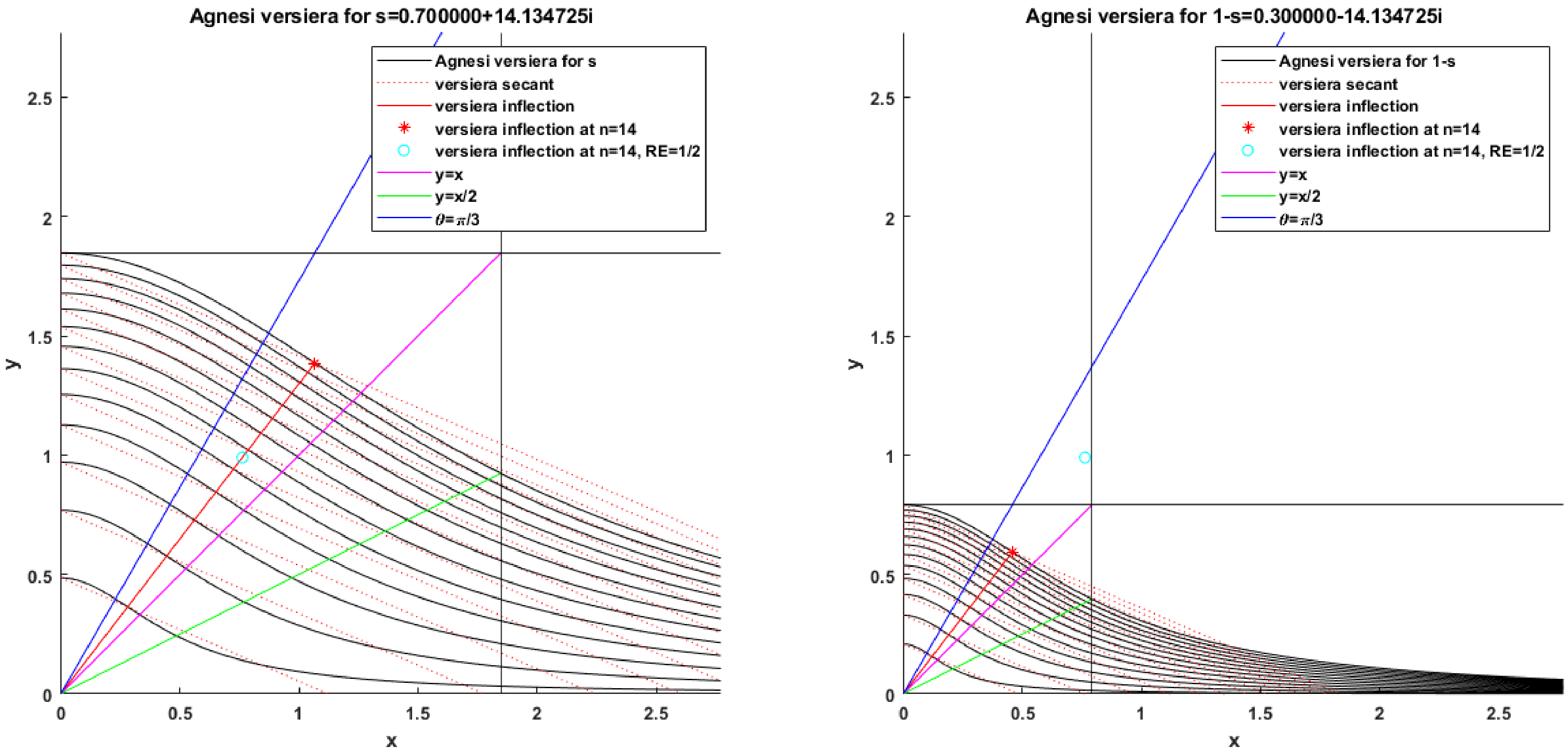

Figure 21: Agnesi versiera generated by s and $(1 - s)$, which do not belong to the critical line and are not solutions of $\zeta(s) = 0$.

Remark 2.10. Let me recall that, for the Gauss distribution, the Fisher information $\mathrm{I}(x_0, \sigma^2)$ is

$$\mathrm{I}(x_0, \sigma^2) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix} ; \tag{68}$$

as a consequence, in the diagonal of $\mathrm{I}^{-1}(x_0, \sigma^2)$, there are the lower bounds (LBs) of the variances of two unbiased estimators:

$$\begin{aligned} \mathrm{I}^{-1}(x_0, \sigma^2) = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix} &= \begin{pmatrix} \mathrm{Var}\left(\hat{f}_1(n)\right)_{\mathrm{LB}} & 0 \\ 0 & \mathrm{Var}\left(\hat{f}_2(n)\right)_{\mathrm{LB}} \end{pmatrix} \\ &= \begin{pmatrix} \left[\mathrm{STD}\left(\hat{f}_1(n)\right)_{\mathrm{LB}}\right]^2 & 0 \\ 0 & \left[\mathrm{STD}\left(\hat{f}_2(n)\right)_{\mathrm{LB}}\right]^2 \end{pmatrix} . \end{aligned} \tag{69}$$

From the Eq. 69, in analogy to Eqs. 57 and 58, I obtain

$$\mathrm{STD}\left(\hat{f}_1(x)\right) = \sqrt{n\,\mathrm{RE}}\sqrt{\log(x)} \geq \sqrt{2\,\mathrm{RE}}\sqrt{\log(x)} \,, \tag{70}$$

$$\mathrm{STD}\left(\hat{f}_2(x)\right) = n\sqrt{n}\,\mathrm{RE}\log(x) \geq 2\sqrt{2}\,\mathrm{RE}\log(x) \,, \tag{71}$$

where $n \in \mathbb{N}^+$ and $n \geq 2$.

The only difference between $\mathrm{STD}\left(\hat{f}_2(x)\right) = n\sqrt{n}\,\mathrm{RE}\log(x)$ and $\mathrm{STD}\left(\hat{f}(x)\right) = \sqrt{n}\,\mathrm{RE}\log(x)$ (Eq. 58) is the multiplication by n, so that Theorem 2.4, Corollary 2.5, Theorem 2.6, and Corollary 2.7 hold.

Theorem 2.9. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = x/\log(x) + O\left(\sqrt{x}\sqrt{\log(x)}\right)$.*

Proof. Using the Eq. 70, I rewrite the Eq. 62:

$$\begin{aligned} &\mathrm{Prob}\left(\pi(x) - \underbrace{\frac{x}{\log(x)}}_{mean} > \sqrt{x}\log(x)\,\mathrm{A}(x) \geq \sqrt{x}\log(x)\sqrt{n}\,\mathrm{RE} \geq \sqrt{x}\sqrt{\log(x)}\sqrt{n}\sqrt{\mathrm{RE}} = \underbrace{\sqrt{x}}_{k}\underbrace{\mathrm{STD}\left(\hat{f}_1(x)\right)}_{\substack{standard \\ deviation}}\right) \\ &\quad\leq \underbrace{\frac{1}{x}}_{1/k^2} \to 0^+ \,, \end{aligned} \tag{72}$$

where $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, that is, with probability $\to 1^-$ ($\approx$ certainty),

$$\pi(x) - \frac{x}{\log(x)} \leq \sqrt{n\,\mathrm{RE}}\sqrt{x}\sqrt{\log(x)} \,, \tag{73}$$

where $\sqrt{n\,\mathrm{RE}} = \epsilon$, $n \in \mathbb{N}^+$ and $n \geq 2$, and $0 < \mathrm{RE} < 1$. ☐

Corollary 2.10. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\sqrt{\log(x)}\right)$.*

Proof. For large enough $x \in \mathbb{R}^+$ and $x \geq 88789$, $x/\log(x) \sim \mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 73, I obtain

$$|\pi(x) - \mathrm{Li}(x)| \leq \sqrt{n\,\mathrm{RE}}\sqrt{x}\sqrt{\log(x)} \,, \tag{74}$$

where $\sqrt{n\,\mathrm{RE}} = \epsilon$, $n \in \mathbb{N}^+$ and $n \geq 2$, and $0 < \mathrm{RE} < 1$. ☐

Theorem 2.11. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = x/\log(x) + o\left(\sqrt{x}\sqrt{\log(x)}\right)$.*

Proof. In Eq. 73, $\sqrt{n\,\mathrm{RE}} = \epsilon \in \mathbb{R}^+$ and ϵ can have every positive value, however small. ☐

Corollary 2.12. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = \mathrm{Li}(x) + o\left(\sqrt{x}\sqrt{\log(x)}\right)$.*

Proof. For large enough $x \in \mathbb{R}^+$ and $x \geq 88789$, $x/\log(x) \sim \mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 74, I observe that $\sqrt{n}\,\mathrm{RE} = \epsilon \in \mathbb{R}^+$ and that ϵ can have every positive value, however small. □

Remark 2.11. According to Cramér [13], if the Theorem 2.3 is right (or assuming the Riemann hypothesis as true), also given the previous discussion, the STD is a method to estimate both the prime gaps and $\pi(x)$. In particular, in the Theorem 2.11 and in the Corollary 2.12, the STD $= o\left(\sqrt{x}\sqrt{\log(x)}\right)$: this result is consistent with that of Heath-Brown and Goldston [16] who found that, assuming the Riemann hypothesis as true, the gap between the prime number p_N and its prime successor p_{N+1} is $o\left(\sqrt{p_N}\sqrt{\log(p_N)}\right)$.

Theorem 2.13. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = x/\log(x) + O\left(\log^2(x)\right)$.*

Proof. I rewrite the Eq. 62:

$$\mathrm{Prob}\left(\pi(x) - \underbrace{\frac{x}{\log(x)}}_{mean} > \sqrt{x}\log(x)\,\mathrm{A}(x) > \log^2(x)\,\mathrm{A}(x) \geq \log^2(x)\sqrt{n}\,\mathrm{RE} = \underbrace{\log(x)}_{k}\,\underbrace{\mathrm{STD}\left(\hat{f}(x)\right)}_{\substack{standard\\ deviation}}\right) \leq \underbrace{\frac{1}{\log^2(x)}}_{1/k^2} \to 0^+\,, \tag{75}$$

where $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, that is, with probability $\to 1^-$ ($\approx$ certainty),

$$\pi(x) - \frac{x}{\log(x)} \leq \sqrt{n}\,\mathrm{RE}\log^2(x)\,, \tag{76}$$

where $\sqrt{n}\,\mathrm{RE} = \epsilon$, $n \in \mathbb{N}^+$ and $n \geq 2$, and $0 < \mathrm{RE} < 1$. □

Corollary 2.14. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = \mathrm{Li}(x) + O\left(\log^2(x)\right)$.*

Proof. For large enough $x \in \mathbb{R}^+$ and $x \geq 88789$, $x/\log(x) \sim \mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 76, I obtain

$$|\pi(x) - \mathrm{Li}(x)| \leq \sqrt{n}\,\mathrm{RE}\log^2(x)\,, \tag{77}$$

where $\sqrt{n}\,\mathrm{RE} = \epsilon$, $n \in \mathbb{N}^+$ and $n \geq 2$, and $0 < \mathrm{RE} < 1$. □

Theorem 2.15. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = x/\log(x) + o\left(\log^2(x)\right)$.*

Proof. In Eq. 76, $\sqrt{n}\,\mathrm{RE} = \epsilon \in \mathbb{R}^+$ and ϵ can have every positive value, however small. □

Corollary 2.16. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = \mathrm{Li}(x) + o\left(\log^2(x)\right)$.*

Proof. For large enough $x \in \mathbb{R}^+$ and $x \geq 88789$, $x/\log(x) \sim \mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 77, I observe that $\sqrt{n}\,\mathrm{RE} = \epsilon \in \mathbb{R}^+$ and that ϵ can have every positive value, however small. □

Remark 2.12. Given the previous discussion, the STD is a method to estimate both the prime gaps and $\pi(x)$. In particular, in the Theorem 2.13 and in the Corollary 2.14, the STD $= O\left(\log^2(x)\right)$: this result is consistent with a conjecture of Banks, Ford, and Tao [17], where, in asymptotic notation, the largest prime gap is $G_{\mathcal{P}}(x) \sim K\log^2(x)$ for $x \to +\infty$, with $K \coloneqq 2e^{-\gamma} = 1.1229\cdots$ (OEIS: A125313; γ is the Euler–Mascheroni constant); moreover, $G_{\mathcal{P}}(x)$ matches the lower bound for the prime gap in the Granville model $\mathcal{G}$ [18], where $G_{\mathcal{G}}(x) \gtrsim K\log^2(x)$ for $x \to +\infty$.

Theorem 2.17. *Let $x \in \mathbb{R}^+$, $x \geq 88789$ and large enough, $\pi(x) = x/\log(x) + O\left(\sqrt{x}\log^2(x)\right)$.*

Proof. I rewrite the Eq. 62:

$$\operatorname{Prob}\left(\pi(x)-\underbrace{\frac{x}{\log(x)}}_{mean}>\sqrt{x}\log(x)\,\mathrm{A}(x)\geq\sqrt{x}\log^2(x)\sqrt{n}\,\mathrm{RE}=\underbrace{\sqrt{x}\log(x)}_{k}\underbrace{\operatorname{STD}\left(\hat{f}(x)\right)}_{\substack{standard\\ deviation}}\right)$$
$$\leq\underbrace{\frac{1}{x\log^2(x)}}_{1/k^2}\to 0^+\,,\tag{78}$$

where $x\in\mathbb{R}^+$, $x\geq 88789$ and large enough, that is, with probability $\to 1^-$ ($\approx$ certainty),

$$\pi(x)-\frac{x}{\log(x)}\leq\sqrt{n}\,\mathrm{RE}\,\sqrt{x}\log^2(x)\,,\tag{79}$$

where $\sqrt{n}\,\mathrm{RE}=\epsilon$, $n\in\mathbb{N}^+$ and $n\geq 2$, and $0<\mathrm{RE}<1$. ☐

Corollary 2.18. *Let $x\in\mathbb{R}^+$, $x\geq 88789$ and large enough, $\pi(x)=\mathrm{Li}(x)+O\left(\sqrt{x}\log^2(x)\right)$.*

Proof. For large enough $x\in\mathbb{R}^+$ and $x\geq 88789$, $x/\log(x)\sim\mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 79, I obtain

$$|\pi(x)-\mathrm{Li}(x)|\leq\sqrt{n}\,\mathrm{RE}\,\sqrt{x}\log^2(x)\,,\tag{80}$$

where $\sqrt{n}\,\mathrm{RE}=\epsilon$, $n\in\mathbb{N}^+$ and $n\geq 2$, and $0<\mathrm{RE}<1$. ☐

Theorem 2.19. *Let $x\in\mathbb{R}^+$, $x\geq 88789$ and large enough, $\pi(x)=x/\log(x)+o\left(\sqrt{x}\log^2(x)\right)$.*

Proof. In Eq. 79, $\sqrt{n}\,\mathrm{RE}=\epsilon\in\mathbb{R}^+$ and ϵ can have every positive value, however small. ☐

Corollary 2.20. *Let $x\in\mathbb{R}^+$, $x\geq 88789$ and large enough, $\pi(x)=\mathrm{Li}(x)+o\left(\sqrt{x}\log^2(x)\right)$.*

Proof. For large enough $x\in\mathbb{R}^+$ and $x\geq 88789$, $x/\log(x)\sim\mathrm{Li}(x)$ in asymptotic notation and, using the Eq. 80, I observe that $\sqrt{n}\,\mathrm{RE}=\epsilon\in\mathbb{R}^+$ and that ϵ can have every positive value, however small. ☐

Remark 2.13. In the Theorem 2.17 and in the Corollary 2.18, the $\mathrm{STD}=O\left(\sqrt{x}\log^2(x)\right)$: this result is consistent with a finding, reported by Cramér [19], concerning the prime gaps.

Moreover, in the Theorem 2.19 and in the Corollary 2.20, the $\mathrm{STD}=o\left(\sqrt{x}\log^2(x)\right)$: this result is consistent with a property of the Tchebychev's function $\psi(x)$ satisfying, under the Riemann hypothesis, $\psi(x)=x+E(x)$ with $E(x)\ll\sqrt{x}\log^2(x)$ [20].

Remark 2.14. The standard deviation $\operatorname{STD}\left(\hat{f}(x)\right)$ can be interpreted as a "rest value" around which there are the oscillations of the difference $[\pi(x)-x/\log(x)]$. To this regard, I rewrite the Eq. 62:

$$\operatorname{Prob}\left(\pi(x)-\underbrace{\frac{x}{\log(x)}}_{mean}>\sqrt{x}\log(x)\,\mathrm{A}(x)\geq\sqrt{x}\log(x)\sqrt{n}\,\mathrm{RE}>1\cdot\log(x)\sqrt{n}\,\mathrm{RE}=\underbrace{1}_{k}\cdot\underbrace{\operatorname{STD}\left(\hat{f}(x)\right)}_{\substack{standard\\ deviation}}\right)\tag{81}$$
$$\leq\underbrace{1}_{1/k^2}\,,$$

where $x\in\mathbb{R}^+$, $x\geq 88789$ and large enough, that is, with tautology,

$$\pi(x)-\frac{x}{\log(x)}\gtreqless\operatorname{STD}\left(\hat{f}(x)\right)=\sqrt{n}\,\mathrm{RE}\log(x)\,,\tag{82}$$

where $n\in\mathbb{N}^+$ and $n\geq 2$, and $0<\mathrm{RE}<1$.

3 Discussion

The Cauchy distribution is a stable distribution that can be analytically expressed. It was geometrically studied by Fermat and then by Agnesi, assuming the name "versiera", where the parameter $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (Eq. 65): in particular, the coordinate x is proportional to $\tan(\theta)$. Pitman and Williams [21] showed that a Cauchy distributed, random variable X can be obtained via the formula

$$x = \tan\left(\pi\left(u - \frac{1}{2}\right)\right), \tag{83}$$

where u is uniformly sampled from the interval $(0, 1)$ and x is a sample of X. In other words, there is a connection between the function $\tan(\theta)$ and the Cauchy distribution; such connection with trigonometry becomes more evident by considering the cumulative distribution function, which is based on the function $\arctan(x)$ (Eq. 3). In addition, the Cauchy distribution has no mean and no variance, whereas it has mode and median [10]; moreover, the Cauchy distribution does not obey to the strong law of large numbers, whereas it obeys to the weak law of large numbers because of the heavy tails of the probability density function (Eq. 2) [10].

As a consequence, it is reasonable to expect some intriguing behaviours of the mathematical objects connected to the Cauchy distribution.

In this work, I show a connection between the Cauchy distribution and the Riemann $\zeta(s)$ function because the Riemann addend $1/n^s$ is also a value of the characteristic function of the Cauchy distribution (Eq. 4). Nevertheless, the reasoning is simplified by using the Cauchy cumulative distribution function instead of the Cauchy characteristic function with the advantage that the "hidden" term $x/\log(n)$ appears in the equations (e.g., in Eqs. 13-15). In other words, the Cauchy distribution is in connection to the distribution of the prime numbers and such connection becomes evident by using a particular representation of the $\arctan(x)$ function (Eq. 15).

On the other hand, the Eq. 15, rewritten as Eq. 20, permits to introduce the concepts of delay, punctuality, and anticipation, which are related to the real part RE of the complex number s and which determine the asymptotic convergence of the Cauchy cumulative distribution function (Eq. 21).

Finally, the connection between the Riemann $\zeta(s)$ function and the Cauchy distribution is shown by the Eqs. 25 and 26.

The zeros of the Riemann $\zeta(s)$ function determine the oscillations of the prime numbers around their expected position. Such oscillations can be evaluated via $\hat{n}$, an unbiased estimator of natural numbers, whose standard deviation is $\mathrm{STD}(\hat{n}) = \sqrt{n}\gamma = \sqrt{n}\,\mathrm{RE}\log(n)$ filling the gap between two consecutive prime numbers (Figs. 16, 17, and 18).

On the other hand, each Riemann addend $1/n^s$ is solution of a differential equation describing an under-damped harmonic oscillator (Eq. 43), where $\omega_0 = |s|$ is the dimensionless undamped angular frequency and $\mathrm{RE} = \mathrm{Re}(s)$ is the dimensionless decay rate damping the oscillations and shaping the spiral sink of the Riemann addends (Figs. 12, 13, and 14). In particular, the real part of s determines the asymptotic convergence of the Cauchy cumulative distribution function, giving delay, punctuality, or anticipation of convergence (Figs. 5, 6, and 7).

The oscillations can also be seen associating a pendulum of infinite length to RE and normalizing its dimensionless angular frequency (Fig. 15). Reasoning in terms of pendulums' balance and punctuality of asymptotic convergence, I show that the critical line $x = 1/2$ should be exclusively composed by solutions of $\zeta(s) = 0$, nevertheless, the connection between $\zeta(s)$ and the Cauchy distribution gives the possibility that $\zeta(s) \neq 0$ happens an infinite number of times on the critical line. Moreover, when $\mathrm{RE} \neq 1/2$, there are never balance nor punctuality of asymptotic convergence and, as a consequence, $\zeta(s) = 0$ never happens (Theorem 2.3).

The connection between $\zeta(s)$ and the Cauchy distribution gives, in my opinion, a new insight about the location of the prime numbers because this distribution has no expected value and the strong law of large numbers does not hold for it. In fact, the Cauchy distribution permits to build an unbiased estimator of natural numbers, whose standard deviation is proportional to $\sqrt{n}\log(n)$ and is able to locate a prime number p according to: $n - \frac{4}{\pi}\sqrt{n}\log(n) < p \leq n$ [14]; moreover, the same standard deviation is associated to $y = f(x) = x/\log(x)$, which estimates the mean cardinality of the prime numbers less than the x magnitude: as a consequence, it is possible to build a prime number theorem in probabilistic manner (Theorem 2.4).

Finally, the inequalities 63 and 64 can explain both the phenomena of twin primes and isolated primes because the coefficient for $\sqrt{x}\log(x)$ is $\epsilon = \sqrt{n}\,\mathrm{RE}$, where $n \in \mathbb{N}^{+}$ and $n \geq 2$, and $0 < \mathrm{RE} < 1$.

In Eq. 67, I also show a connection between the Gauss distribution and the Riemann $\zeta(s)$ function; it is well known that, via the classical central limit theorem, the Gauss distribution can be obtained from a large number of independent and identically distributed random variables (not necessarily with continuous uniform distribution), which have the same expected value and the same finite variance.

In my opinion, the connection between the Riemann $\zeta(s)$ function and the stable distributions of Cauchy and Gauss makes the distribution of the prime numbers clearer. In addition, it would be worthy of further study to note that the characteristic function of a stable distribution has the term $2/\pi$ for the specific case of the non-skewed Cauchy distribution: the ratio $2/\pi$ recalls the trisectrix of Hippias and, as a consequence, the Lambert W function, which can be used to investigate the distribution of the prime numbers [7].

Acknowledgements

This study is dedicated to the People of Portugal and Russia according to the words of the Virgin Mary at Fatima.

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