Perfect Sets Isolated Points

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Definition 1: [topology]/ isolated_point

Let

1. (X, τ_X) be a topological space
2. $x \in X$
3. $S \subseteq X$

Then x is an isolated point of S iff

$$\boxed{\{x\} \in \tau_{S \subseteq X}} \tag{1}$$

Definition 2: [topology, closed_set, isolated_point]/ perfect_space

Let

1. (X, τ_X) be a topological space.
2. $S \subseteq X$.

Then (X, τ_X) is perfect iff

1. S is closed
2. S has no isolated points

Result 1: [perfect_space, countable, real_numbers]/ simple_cantor_bendixson_theorem

Let

1. $X \subseteq \mathbb{R}$.

Then there exists a partition:

$$\boxed{X = C \sqcup K} \tag{2}$$

such that:

1. C is countable
2. K is perfect

Proof. This proof uses [Definition **??**]. We build a sequence $f : \mathbb{ON} \to \mathcal{P}(X)$ such that:

$$X_0 \supseteq X_1 \supseteq \ldots \supseteq X_\omega \supseteq \ldots$$

$$X_0 \overset{\text{def}}{=} X \tag{3}$$

$$X_{a+1} \overset{\text{def}}{=} X_a \backslash \{x \in X_a \mid x \text{ is an isolated point of } X_a\} \tag{4}$$

$$X_l \overset{\text{def}}{=} \bigcap_{a \in l} X_a \tag{5}$$

Note that $\mathbb{ON}$ is a proper class, but $\mathcal{P}(X)$ is a set, so this sequence is not injective, and thus must stop at some point. Let n be the least ordinal such that $X_n = X_{n+1}$. Then we have:

$$\boxed{X_n = X_{n+1} = X_{n+2} = \ldots} \tag{6}$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, this implies: $\forall_a^{\mathbb{ON}} \forall_x^{X_a \backslash X_{a+1}}$ you can find $q_1, q_2 \in \mathbb{Q}$ such that:

$$\boxed{X_a \cap (q_1, q_2)_{\mathbb{R}} = \{x\}} \tag{7}$$

Since x is the unique point in $X_a \cap (q_1, q_2)_{\mathbb{R}}$, it is an isolated point of X_a. Thus X_a is not perfect. Thus $\exists_g C \overset{g}{\rightarrowtail} \mathbb{Q}^2$, and so C is countable. $\square$