Eaqmac

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1 Introduction

Here we give a couple of notes for the EAQMAC, i.e. a quantum multi access channel with entanglement assist.

2 Prerequisites

Recall that the capacity of any CQMAC channel $\sigma_{(.,.)}$ under message distribution P_{s_x,s_z} is:

$$\boxed{\begin{aligned} \chi(\sigma_{(.,.)}; P_{s_x,s_z}) &\stackrel{\text{def}}{=} H\left(\sum_{s_x}\sum_{s_z} P_{s_x,s_z}(s_x,s_z)\sigma_{(s_x,s_z)}\right) && (1)\\ &- \sum_{s_x}\sum_{s_z} P_{s_x,s_z}(s_x,s_z) H\left(\sigma_{(s_x,s_z)}\right) && (2)\end{aligned}}$$

These are the assumptions for the rest of this document:

(A1) X_A, X_B, Z_A, Z_B are i.i.d.

(A2) X_A, X_B, Z_A, Z_B are uniform.

(A1) $\wedge$ (A2) $\implies$ $P_{s_x,s_z}(s_x,s_z) = \frac{1}{d^2}$, call this (R1).

3 Analysis

3.1 ??

Start with **??**:

$$?? = H\left[\sum_{s_x}\sum_{s_z}\frac{1}{d^2}\sum_{x_A,x_B}\sum_{z_A,z_B} \Pr\{(x_A,z_A,x_B,z_B)|s_x,s_z\}\rho_{\mathcal{N}}(x_A,z_A,x_B,z_B)\right] \tag{3}$$

When you fix (x_A, z_A), this implies $x_B = -x_A, z_B = -z_A$. Now note:

$$\begin{aligned} \Pr\{(x_A,z_A,x_B,z_B)|s_x,s_z\} &= \mathbb{1}_{x_B=s_x-x_A}\mathbb{1}_{z_B=s_z-z_A}\Pr\{(x_A,z_A)|s_x,s_z\} && (4)\\ &\stackrel{(A1)}{=} \mathbb{1}_{x_B=s_x-x_A}\mathbb{1}_{z_B=s_z-z_A}\Pr\{x_A|s_x\}\Pr\{z_A|s_z\} && (5)\\ &\stackrel{(R1)}{=} \mathbb{1}_{x_B=s_x-x_A}\mathbb{1}_{z_B=s_z-z_A}\frac{1}{d^2} && (6)\end{aligned}$$

So now:

$$\begin{aligned} ?? &= H\left[\sum_{s_x}\sum_{s_z}\frac{1}{d^2}\sum_{\substack{x_A,x_B:\\ x_A\oplus x_B=s_x}}\sum_{\substack{z_A,z_B:\\ z_A\oplus z_B=s_z}}\frac{1}{d^2}\rho_{\mathcal{N}}(x_A,z_A,x_B,z_B)\right] && (7)\\ &= H\left[\sum_{s_x}\sum_{s_z}\frac{1}{d^2}\sum_{x_A}\sum_{x_B}\frac{1}{d^2}\rho_{\mathcal{N}}(x_A,z_A,s_x-x_A,s_z-z_A)\right] && (8)\\ &= H\left[\sum_{s_x}\sum_{s_z}\frac{1}{d^2}\sum_{x_A}\sum_{x_B}\frac{1}{d^2}(I\otimes\mathcal{N}\otimes I)\circ(I\otimes W_{x_A,z_A}\otimes W_{s_x-x_A,s_z-z_A}\otimes I)\psi_{\max}\otimes\psi_{\max}\right] && (9)\\ &= H\left[(I\otimes\mathcal{N}\otimes I)\circ\sum_{s_x}\sum_{s_z}\sum_{x_A}\sum_{x_B}\frac{1}{d^4}(I\otimes W_{x_A,z_A}\otimes W_{s_x-x_A,s_z-z_A}\otimes I)\psi_{\max}\otimes\psi_{\max}\right] && (10)\end{aligned}$$

Result 1: [weyl_operators]/ shifted_weyl_operators_are_orthogonal

Let W_{x_A,z_A} be the d dimensional Weyl operators on a space , for $x_A, z_A \in [0,d)_{\mathbb{Z}}$. Let $s_x, s_z \in [0,d)_{\mathbb{Z}}$. Then:

$$\boxed{\begin{aligned} \sum_x\sum_z\frac{1}{d^2}W_{x,z} &= I/d && (11)\\ \sum_x\sum_z\frac{1}{d^2}W_{s_x-x,s_z-z} &= I/d && (12)\end{aligned}}$$

This result implies:

Probably not true, can you actually sparate these out because of the s_x, s_z also being summed?

$$\begin{aligned} ?? &= H\left[(I\otimes\mathcal{N}\otimes I)\sum_{s_x}\sum_{s_z}\frac{1}{d^2}I/d^2\otimes I/d^2\right] && (13)\\ &= H[(I\otimes\mathcal{N}\otimes I)I/d^2\otimes I/d^2] && (14)\\ &= H[I/d\otimes\mathcal{N}(I^{\otimes 2}/d^2)\otimes I/d] && (15)\\ &= 2H[I/d] + H[\mathcal{N}(I^{\otimes 2}/d^2)] && (16)\end{aligned}$$

3.2 ??

$$\begin{aligned} ?? &= \sum_{s_x}\sum_{s_z}\Pr_{s_x,s_z}(s_x,s_z)H\left[\sigma_{(s_x,s_z)}\right] && (17)\\ &\stackrel{(R1)}{=} \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[\sigma_{(s_x,s_z)}\right] && (18)\\ &= \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[\sum_{\substack{x_A,x_B:\\ x_A\oplus x_B=s_x}}\sum_{\substack{z_A,z_B:\\ z_A\oplus z_B=s_z}}\frac{1}{d^2}\rho_{\mathcal{N}}(x_A,z_A,x_B,z_B)\right] && (19)\\ &= \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[\sum_{x_A}\sum_{z_A}\frac{1}{d^2}\rho_{\mathcal{N}}(x_A,z_A,s_x-x_A,s_z-z_A)\right] && (20)\\ &= \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[\sum_{x_A}\sum_{z_A}\frac{1}{d^2}(I\otimes\mathcal{N}\otimes I)\circ(I\otimes W_{x_A,z_A}\otimes W_{s_x-x_A,s_z-z_A}\otimes I)\psi_{\max}\otimes\psi_{\max}\right] && (21)\\ &\stackrel{\dagger\text{-trick}}{=} \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[\sum_{x_A}\sum_{z_A}\frac{1}{d^2}(I\otimes\mathcal{N}\otimes I)\circ(W^\dagger_{x_A,z_A}\otimes I\otimes I\otimes W^\dagger_{s_x-x_A,s_z-z_A})\psi_{\max}\otimes\psi_{\max}\right] && (22)\\ &= \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[(I\otimes\mathcal{N}\otimes I)\circ\sum_{x_A}\sum_{z_A}\frac{1}{d^2}(W^\dagger_{x_A,z_A}\otimes I\otimes I\otimes W^\dagger_{s_x-x_A,s_z-z_A})\psi_{\max}\otimes\psi_{\max}\right] && (23)\\ & && (24)\end{aligned}$$

Now focus on the weyl operators:

$$\begin{aligned} &\sum_{x_A}\sum_{z_A}\frac{1}{d^2}(W^\dagger_{x_A,z_A}\otimes I\otimes I\otimes W^\dagger_{s_x-x_A,s_z-z_A}) && (25)\\ &= \sum_{x_A}\sum_{z_A}\frac{1}{d^2}(\omega_d^{x_Az_A}W_{-x_A,-z_A}\otimes I\otimes I\otimes\omega_d^{(s_x-x_A)(s_z-z_A)}W_{x_A-s_x,z_A-s_z}) && (26)\\ &= \sum_{x_A}\sum_{z_A}\frac{1}{d^2}\omega_d^{x_Az_A+(s_x-x_A)(s_z-z_A)}(W_{-x_A,-z_A}\otimes I\otimes I\otimes W_{x_A-s_x,z_A-s_z}) && (27)\\ &\text{Now we can apply } I\otimes I\otimes I\otimes W_{s_x,s_z} \text{ because } H \text{ is invariant under unitaries.} && (28)\\ &\mapsto \sum_{x_A}\sum_{z_A}\frac{1}{d^2}\omega_d^{x_Az_A+(s_x-x_A)(s_z-z_A)}(W_{-x_A,-z_A}\otimes I\otimes I\otimes W_{s_x,s_z}\circ W_{x_A-s_x,z_A-s_z}) && (29)\\ &= \sum_{x_A}\sum_{z_A}\frac{1}{d^2}\omega_d^{x_Az_A+(s_x-x_A)(s_z-z_A)}(W_{-x_A,-z_A}\otimes I\otimes I\otimes\omega_d^{s_x(z_A-s_z)}W_{x_A,z_A}) && (30)\\ &= \sum_{x_A}\sum_{z_A}\frac{1}{d^2}\omega_d^{x_Az_A+(s_x-x_A)(s_z-z_A)+s_x(z_A-s_z)}(W_{-x_A,-z_A}\otimes I\otimes I\otimes W_{x_A,z_A}) && (31)\end{aligned}$$

Now focus on the relative phase:

$$\begin{aligned} &x_Az_A+(s_x-x_A)(s_z-z_A)+s_x(z_A-s_z) && (32)\\ &= x_Az_A+s_x(s_z-z_A)-x_A(s_z-z_A)+s_x(z_A-s_z) && (33)\\ &= x_Az_A-x_A(s_z-z_A) && (34)\\ &= x_A(z_a-(s_z-z_a)) && (35)\\ &= -x_As_z && (36)\end{aligned}$$

This makes:

$$\begin{aligned} ?? &= \sum_{x_A}\sum_{z_A}\frac{1}{d^2}\omega_d^{-x_As_z}(W_{-x_A,-z_A}\otimes I\otimes I\otimes W_{x_A,z_A}) && (37)\\ & && (38)\end{aligned}$$

Now going back to **??**:

$$\begin{aligned} ?? &= \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[(I\otimes\mathcal{N}\otimes I)\circ\sum_{x_A}\sum_{z_A}\frac{1}{d^2}\omega_d^{-x_As_z}(W_{-x_A,-z_A}\otimes I\otimes I\otimes W_{x_A,z_A})\psi_{\max}\otimes\psi_{\max}\right] && (39)\\ &= \sum_{s_x}\sum_{s_z}\frac{1}{d^2}H\left[(I\otimes\mathcal{N}\otimes I)\circ\sum_{x_A}\sum_{z_A}\frac{1}{d^2}\omega_d^{-x_As_z}\Phi_{-x_A,-z_A}\otimes\Phi_{x_A,z_A}\right] && (40)\\ & && (41)\end{aligned}$$