Category Theory Introduction

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1 Introduction

This note covers basic definitions and results of category theory. It mostly follows [**riehl˙2016**], but also contains notes from [**simmons˙2011**], as well as special topics from [**nourani˙2014**].

Definition 1: /category

A category C consists of

- A class $\mathbf{Ob}(C)$ consisting of objects
- A class $\mathbf{Hom}(C)$ of morphisms.

  Definition 2: /morphism

  A morphism is any object that has a source object $A \in \mathbf{Ob}(C)$ and a target $B \in \mathbf{Ob}(C)$. Morphisms are sometimes called arrows.
  If f is a morphism with source $A \in \mathbf{Ob}(C)$ and target $B \in \mathbf{Ob}(C)$, then this is usually written as $f : A \to B$.

- A binary operation $\circ : M \to M$, called composition, which satisfies:
  1. $\circ$ is associative
  2. $\mathbf{Hom}(A)$ has an idenity morphism for every $X \in \mathbf{Ob}(C)$

     Definition 3: [morphism]/ identity_morphism

     For every objecrmt $X \in \mathbf{Ob}(C)$, there exists an identity morphism $\mathrm{id}_X : X \to X$, such that for every morphism $f : X \to Y$:

     $$\boxed{f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f} \tag{1}$$

     Or as a diagram:

     $$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}_X"] \arrow[r, "f"] & Y \arrow[loop right, "\mathrm{id}_Y"] \end{tikzcd}$$

Example 1.1: [category]/ common_categories

Some common categories are:

- **Set** has objects consisting of all sets, and morphisms consisting of all functions between sets.
- **Top** has objects consisting of all topological spaces, and morphisms consisting of all continuous functions between these spaces.
- **Group** has objects consisting of all groups, and morphisms consisting of all homomorphisms between groups.
- $\mathbf{Mod}_R$ for a fixed ring R (with identity), is the category of left R-modules and R-module homomorphisms. If R is a field, then we call this
- **Graph** has objects consisting of all graphs, and morphisms consisting of graph homomorphisms.
- $\mathbf{Model}_T$ for any language $\mathcal{L}$ and first order $\mathcal{L}$-theory T is a category with objects as $[\mathcal{L}, T]$-structures (i.e. $\mathcal{L}$-structures $\mathcal{M}$ that model T, so $\mathcal{M} \models T$).

Result 1: [category]/ unique_identity

Identity morphisms in a category are unique.

Proof. Consider an object A with two identity morphisms $f, g : A \to A$. Then note $f = f \circ g = g$. Thus $f = g$ and identity morphisms are unique. □

Definition 4: [category]/ hom_class

Let C be a category. Let $A, B \in (C)$ be two objects. Denote $C(A, B) = \{f \in \mathbf{Hom}(C) | f : A \to B\}$, i.e. the class containing all morphisms with source A and target B. This is called the Hom-class, and is sometimes written as $\mathbf{Hom}(A, B)$.

Definition 5: [morphism]/ isomorphism

A morphism $f : X \to Y$ is an isomorphism if and only if it is invertible, i.e there exists some $f^{-1} : Y \to X$ such that:

$$\boxed{\begin{aligned} f^{-1} \circ f &= \mathrm{id}_X & (2) \\ f \circ f^{-1} &= \mathrm{id}_Y & (3) \end{aligned}}$$

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}_X"] \arrow[r, bend left, "f"] & Y \arrow[loop right, "\mathrm{id}_Y"] \arrow[l, bend left, dotted, "\exists f^{-1}"] \end{tikzcd}$$

We then say two objects X, Y are isomorphic.

Definition 6: [morphism]/ endomorphism

An endomorphism is a morphism whose domain is the same as the codomain, i.e. $f : X \to X$ is an endomorphism. a set of all endomorphisms of an object X is denoted $\mathbf{End}(X)$.

Definition 7: [morphism]/ automorphism

A automorphism is a morphism which is both an isomorphism and an endomorphism.

Example 1.2: [isomorphism]/ category_isomorphisms

Note that morphisms are technically binary relations, (if they arent a set then they can be though of as a relation of a class) but this sometimes is not the right way of looking at them. This is true in the following example:

1. For any ring R, define the category C:
   - $\mathbf{Ob}(C) \stackrel{\text{def}}{=} \mathbb{Z}_+$
   - $\mathbf{Hom}(C) \stackrel{\text{def}}{=}$ the set of $C(n, m) = R^{n \times m}$, i.e. all n by m matrices.
   - $\circ \stackrel{\text{def}}{=}$ matrix multiplication

   To check this forms a category, note that:

   - $\circ$ is associative because matrix multiplication is associative
   - Every object has an identity, namely for any $n \in \mathbf{Ob}(C)$ there is the $n \times n$ identity matrix I_n, which has the property that for any morphism $f : m \to n$ (i.e. for every $n \times m$ matrix) we have $I_n \circ f = f \circ I_m = f$.

   Thus C is a category.
   Note that while technically $\mathbf{Hom}(C)$ consists of relations, (i.e. you have a relation for each $n \times m$ matrix) it is not productive to think of morphisms this way, so you should rather think of morphisms as some new object, i.e. an arrow.
2. For any monoid $\mathcal{M} = (M, *)$, define the category $C = \mathbf{B}_M$:
   - $\mathbf{Ob}(C)$ consists of some single object (could be anything, let's call it o)
   - For every monoid element $m \in M$, define a morphism $f_m : o \to o$.
   - Define $\circ$ as the binary operation $f_m \circ f_n \mapsto f_{m*n}$.

   Note that monoids have identity elements and associative binary operation.

Definition 8: [category]/ small_category

A category is small if both $\mathbf{Ob}(C)$ and $\mathbf{Hom}(C)$ are sets.