Monotone Set Operators and Rank

Loris Jautakas

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1 Monotone Set Operators

Definition 1: [function]/ monotone_set_operator

A monotone set operator $T : X \to X$ is a function such that for all $A, B \subseteq X$:

$$\boxed{A \subseteq B \implies T(A) \subseteq T(B)} \tag{1}$$

2 Rank

Definition 2: [preorder]/ rank

The rank of $(X, \prec)$ is defined as its Mostowski Collapse:

$$\boxed{\rho_\prec(x) \stackrel{\text{def}}{=} \rho_\prec(y) | y \prec x} \tag{2}$$

Definition 3: [monotone_set_operator]/ T_rank

Let T be a monotone set operator. The T-rank of $(X, \prec)$ is defined as its Mostowski Collapse:

$$\boxed{\rho^T_\prec(x) \stackrel{\text{def}}{=} \{\rho^T_\prec(y) | y \prec x\} \cup T(\{\rho^T_\prec(y) | y \prec x\})} \tag{3}$$

Result 1: [monotone_set_operator]/ knaster_tarski

Let

1. X be a set
2. $T : \mathcal{P}(X) \to \mathcal{P}(X)$ be a monotone set operator

Then

1. For each $A \subseteq X$ there is a smallest $\overline{T}(A) \subseteq X$ such that:
   (a) $A \subseteq \overline{T}(A) \subseteq T(\overline{T}(A))$

Proof. Just define:

$$\boxed{\overline{T}(A) \stackrel{\text{def}}{=} \bigcap \{B \subseteq A | A \subseteq B, T(B) \subseteq B\}} \tag{4}$$

□

Definition 4: [monotone_set_operator]/ finite_character

Let $T : \mathcal{P}(X) \to \mathcal{P}(Y)$ be a monotone set operator. We say T is of finite character if everything derived by A is also derived from a finite part of A, i.e.:

$$\boxed{T(A) = \bigcup_{\substack{F \subseteq A \\ F \text{ finite}}}} \tag{5}$$

Definition 5: [monotone_set_operator]/ countable_character

Let $T : \mathcal{P}(X) \to \mathcal{P}(Y)$ be a monotone set operator. We say T is of countable character if everything derived by A is also derived from a countable part of A, i.e.:

$$\boxed{T(A) = \bigcup_{\substack{F \subseteq A \\ F \text{ countable}}}} \tag{6}$$

Result 2: [monotone_set_operator, finite_character, countable_character]/ finite_and_countable_character_stabilization

Let:

1. $T : \mathcal{P}(X) \to \mathcal{P}(Y)$ be a monotone set operator that is of finite character.
2. $A \subseteq X$ be arbitrary.

Then the sequence:

$$\begin{aligned} (A_\alpha)_{\alpha \in \mathbb{ON}} : \mathbb{ON} &\mapsto \mathcal{P}(X) && (7) \\ 0 &\mapsto A && (8) \\ s+1 &\mapsto A_s \cup T(A_s) && (9) \\ \text{limit ordinal } l &\mapsto \bigcup_{\beta < l} A_\beta && (10) \end{aligned}$$

stabilizes at stage ω, meaning:

$$\boxed{A_\omega = A_{\omega+1} = \dots} \tag{11}$$

Likewise if T is of countable character, then:

$$\boxed{A_{\omega_1} = A_{\omega_1+1} = \dots} \tag{12}$$