Rational Numbers

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Definition 1: [integers]/ rational_numbers

The rational numbers are the smallest field (up to isomorphism) that contains the integers. More formally, given:

1. $\mathbb{Z}$ as the integers

Define the rational numbers as:

$$\boxed{(\mathbb{Q}, +^{\mathbb{Q}}, -^{\mathbb{Q}}, *^{\mathbb{Q}}, 0^{\mathbb{Q}}, 1^{\mathbb{Q}}, <^{\mathbb{Q}})} \tag{1}$$

where:

1. $\mathbb{Q}$ is the set of $\mathbb{Z}^2$ equivalance classes of the form:

$$\boxed{(x_1, y_1) \sim (x_2, y_2) \iff x_1 *^{\mathbb{Z}} y_2 = x_2 *^{\mathbb{Z}} y_1} \tag{2}$$

> **Note:** This is equivalent to saying $p_1/q_1 = p_2/q_2 \iff p_1q_2 = p_2q_1$. So from now on denote $\frac{x_1}{y_1}$ as (x_1, y_1).

2. $+^{\mathbb{Q}}$ is defined as: $\boxed{\frac{x_1}{y_1} +^{\mathbb{Q}} \frac{x_2}{y_2} \overset{\text{def}}{=} \frac{x_1 *^{\mathbb{Z}} y_2 +^{\mathbb{Z}} x_2 *^{\mathbb{Z}} y_1}{y_1 *^{\mathbb{Z}} y_2}}$
3. $-^{\mathbb{Q}}$ is defined as: $\boxed{-^{\mathbb{Q}}(\frac{x_1}{y_1}) \overset{\text{def}}{=} \frac{-^{\mathbb{Z}}(x_1)}{y_1}}$
4. $*^{\mathbb{Q}}$ is defined as: $\boxed{\frac{x_1}{y_1} *^{\mathbb{Q}} \frac{x_2}{y_2} \overset{\text{def}}{=} \frac{x_1 *^{\mathbb{Z}} x_2}{y_1 *^{\mathbb{Z}} y_2}}$
5. $0^{\mathbb{Q}}$ is defined as: $\boxed{\frac{0^{\mathbb{Z}}}{1^{\mathbb{Z}}}}$.
6. $1^{\mathbb{Q}}$ is defined as: $\boxed{\frac{1^{\mathbb{Z}}}{1^{\mathbb{Z}}}}$.
7. $<^{\mathbb{Q}}$ is defined as: $\boxed{\frac{x_1}{y_1} <^{\mathbb{Q}} \frac{x_2}{y_2} \iff x_1 *^{\mathbb{Z}} y_2 <^{\mathbb{Z}} x_2 *^{\mathbb{Z}} y_1}$

Result 1: [rational_numbers]/ every_rational_number_is_emptyset_definable

For any rational number $q = \frac{a}{b}$ for $a, b \in \mathbb{Z}$, $\{q\}$ is $\emptyset$-definable.

Proof.

$$\{q\} = \{c \in \mathbb{Q} | c *^{\mathbb{Q}} b = a\} \tag{3}$$
$$= \left\{ c \in \mathbb{Q} | c *^{\mathbb{Q}} \sum_{i=1}^{b} 1^{\mathbb{Q}} = \sum_{i=1}^{a} 1^{\mathbb{Q}} \right\} \tag{4}$$

□

Result 2: [characteristic, rational_numbers, field]/ rational_numbers_are_smallest_field_containing_the_integers

The rational numbers is the smallest field of characteristic zero that contains the integers.

Proof. Clearly the integers inject into $\mathbb{Q}$, so $\mathbb{Q}$ contains the integers. Because of this, $\mathbb{Q}$ has characteristic 0. Since every rational number is $\emptyset$ definable,

> Finish proof. Idea: Rational numbers are emptyset definable, this essentially means that there is nothing extra: Each rational number corresponds to the minimum amount of information needed to define it, i.e. each rational number exists only if it is guarenteed by the field axioms, which is the same as definable.

□