Preorders and Alexandroff Topologies

Loris Jautakas

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1 Upwards Closed Sets

Definition 1: [preorder]/ upwards_closed_sets

Let

1. $\mathcal{X} = (X, \leq^{\mathcal{X}})$ be a preorder

Define the set of upwards closed sets of $\mathcal{X}$ as:

$$\boxed{\mathcal{U}(\mathcal{X}) = U((X, \leq^{\mathcal{X}})) \overset{\text{def}}{=} \{U \subseteq X | U \text{ is } \leq^{X}\text{-upwards closed}\}} \tag{1}$$

Definition 2: [family_of_subsets]/ specilization_preorder

Result 1: [preorder, alexandroff_topology]/ preorder_is_alexandroff_topology

Let:

1. X be any set.

Then there exists a bijection:

$$\boxed{\{\leq \subseteq X^2 | \leq \text{ is a preorder}\} \cong \{\mathcal{A} \subseteq \mathcal{P}(X) | \mathcal{A} \text{ is an Alexandroff topology}\}} \tag{2}$$

Given by:

$$U(X, \cdot) : \leq \mapsto \mathcal{U}((X, \leq)) \tag{3}$$

$$U^{-1}(X, \cdot) : \mathcal{A} \mapsto \leq^{\mathcal{A}} \tag{4}$$

where $\leq^{\mathcal{A}}$ is the specilization preorder of $\mathcal{A}$.