Transfinite Induction

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September 10, 2023

1 Introduction

In this document we give some results relating to transfinite induction. This includes defining the ordinal numbers, closure of monotone set operators, and principles of induction.

Definition 1: [total_order]/ ordinal_numbers

A set α is said to be an ordinal if and only if one of the following equivalent conditions hold:

1. α is transitive, and $\in \subseteq \alpha \times \alpha$ is a well-ordering on α.
2. α is transitive, and $\in \subseteq \alpha \times \alpha$ is a partial well-ordering on α.

Result 1: [relation, equivalance_relation]/ turning_any_binary_relation_into_equivalence_relation

Let

1. $R \subseteq X^2$ be a binary relation.

Then define:

1. $$\boxed{\begin{aligned} &T(X) : \mathcal{P}(X^2) \to \mathcal{P}(X^2) && (1)\\ &\quad T : S \mapsto S \cup \{(x,x) \in X^2 | x \in \operatorname{dom} R \cup \operatorname{ran} R\} && (2)\\ &\qquad \cup \{(x,y) \in X^2 | yRx\} && (3)\\ &\qquad \cup \{(x,z) \in X^2 | xRyRz\} && (4)\end{aligned}}$$

Then $\overline{T}(R)$ is an equivalence relation on X.

Proof. By definition of $\overline{T}(R)$, if there is nothing more to add to satisfy transitivity, reflexivity or symmetry, then $\overline{T}(R)$ must be transitive, reflexive and symmetric, and thus is a equivalance relation. $\square$