Gradients Jacobians Hessian

Loris Jautakas

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1 Introduction

In this note we discuss the basic concepts of the gradient, jacobian, nad hessian.

2 Gradient

Definition 1: /gradient

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. The gradient of the function f is defined to be:

$$\boxed{\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right]^T} \tag{1}$$

Example 2.1: /simple-gradient

1. Let $f(x) = x_1^2 + 2x_2^2$. Then the gradient of f is:

$$\boxed{\nabla f(x) = [2x_1, 4x_2]^T} \tag{2}$$

Result 1: /basic-gradient-properties

1. ∇ is linear.
   - $\nabla r f(x) = r \nabla f(x)$ for $r \in \mathbb{R}$
   - $\nabla (f + g)(x) = \nabla f(x) + \nabla g(x)$
2. Product Rule: $\nabla (f * g)(x) = f(x) \nabla g(x) + g(x) \nabla f(x)$
3. Chain Rule: $\nabla (f \circ g)(x) = \nabla f(g(x)) \nabla g(x)$

3 Jacobian

Definition 2: /jacobian

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$. The jacobian of the function f is defined to be:

$$\boxed{J_f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}} \tag{3}$$

Assuming that all of the partial derivatives exist.

Just as the derivative allows us to approximatly linearize a single variable function

$$\boxed{f(x + \epsilon) \approx f(x) + f'(x) * \epsilon} \tag{4}$$

the Jacobian allows us to approximatly linearize a vector valued function

$$\boxed{f(x + \vec{\epsilon}) \approx f(x) + J_f(x) * \vec{\epsilon}} \tag{5}$$

4 Hessian

Definition 3: /hessian

Let $f : \mathbb{R}^n \to \mathbb{R}$. The Hessian of f, is defined as:

$$\boxed{\nabla^2 f \overset{\text{def}}{=} [\nabla^2 f]_{r,c} = \left[\frac{\partial^2 f}{\partial x_r x_c}\right]_{r,c}} \tag{6}$$

If f is twice continuously differentiable, then $\nabla^2 f$ is symmetric.