Reachability in 3-VASS is Elementary

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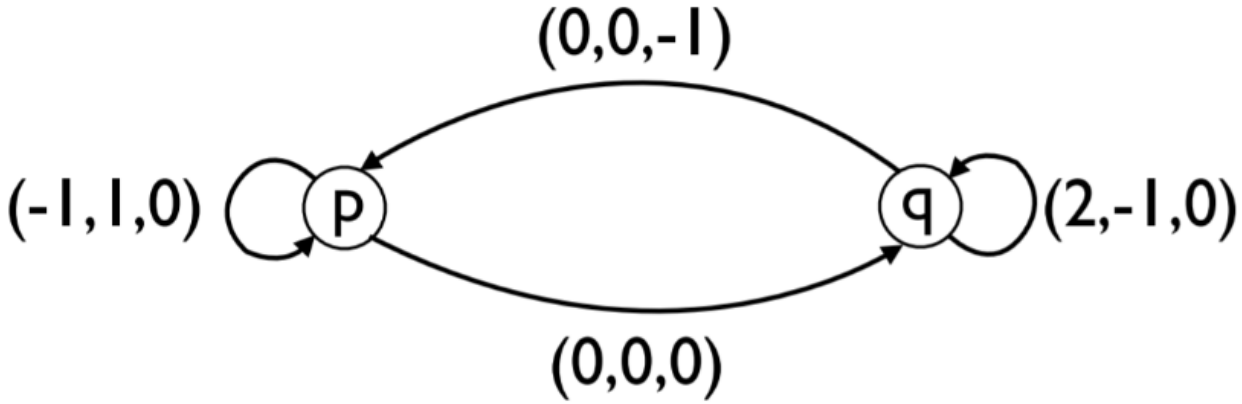
Vector Additions Systems with States



Reachability Problem

Given: A Vector Addition System with States (VASS) V, two configurations s and t.

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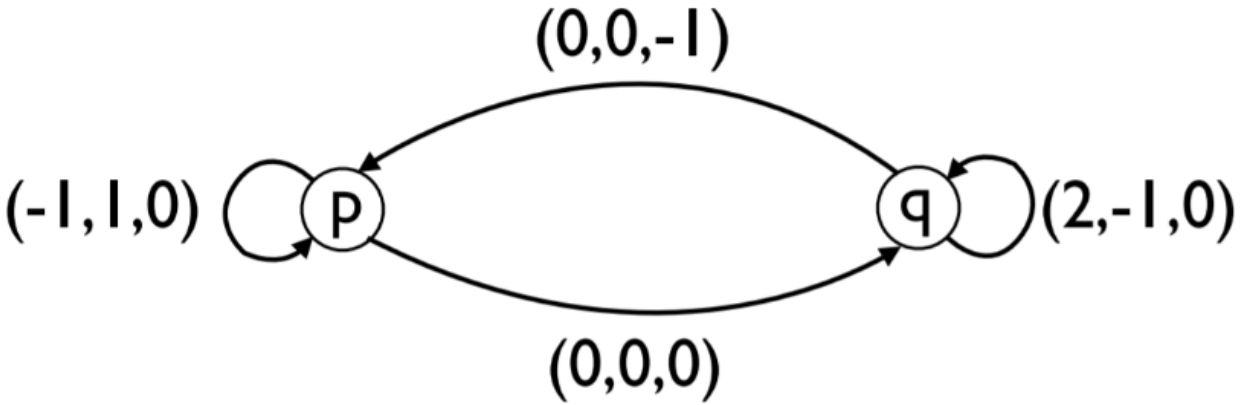
Question: Is there a run from s to t in V?



Short history

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- Leroux and Czerwiński, O. ‘21: Ackermann-hardness

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- Hope for elementary complexity bounds

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Theorem

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The reachability problem in 3-VASS is in 2-ExpSpace, under binary encoding.

- We prove this by induction on the number of strongly connected components
- We show, that length of the shortest run is $n^{2^{2^{\mathcal{O}(k)}}}$ where n is size of VASS and k is the number of strongly connected components

In the previous episode...(HIGHLIGHTS 2024)

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15h09-15h18 Henry Sinclair-Banks
The Tractability Border of Reachability in Simple Vector Addition Systems with States

15h18-15h27 Shrisha Rao
Continuous Pushdown VASS in One Dimension are Easy

15h27-15h36 A. R. Balasubramanian
Decidability and Complexity of Decision Problems for Affine Continuous VASS

15h36-15h45 Łukasz Orlikowski
Languages of unambiguous vector addition systems with states

15h45-15h54 Roland Guttenberg
Reachability in Priority Vector Addition Systems

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Reachability in Two-Dimensional Branching VASS is decidable

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  - We use $\mathbb{Z}$-reachability
- Harder case: non-diagonal VASS
  - Zigzag characterisation of 2-VASS runs
  - **Challenge**: finite reachability sets of up to Tower size
  - **Solution**: over- and under-approximating a big reachability set by similarly behaving small semilinear sets

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Thank You!