Reachability in Vector Addition Systems is Ackermann-complete

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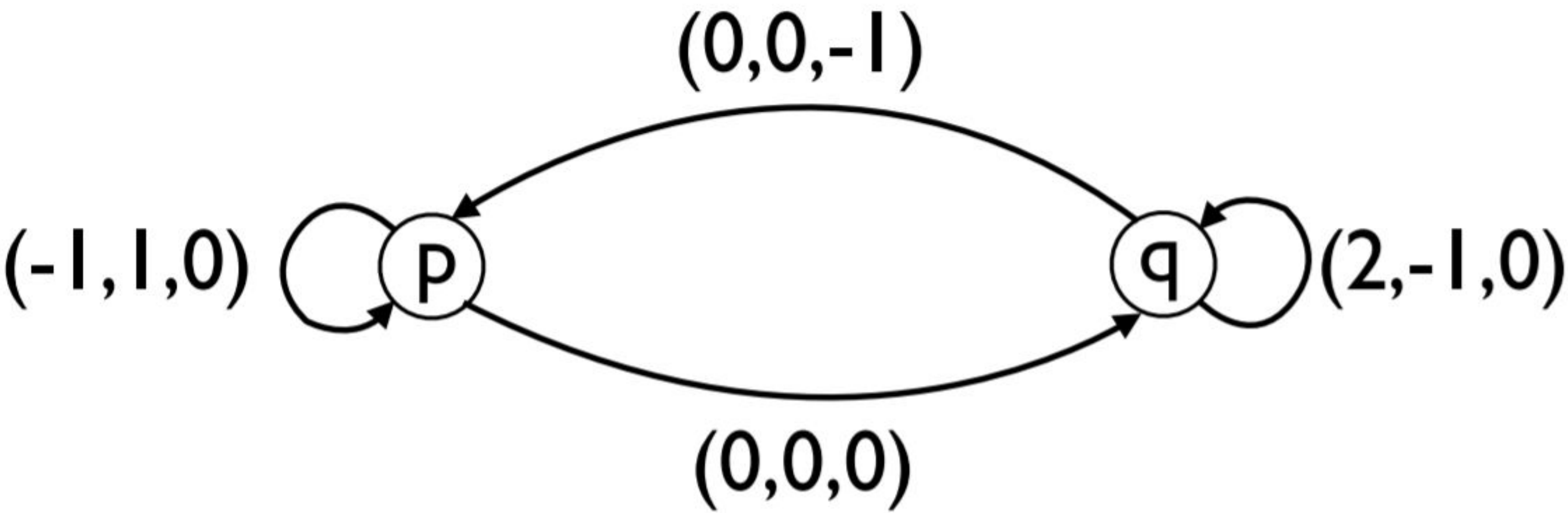
Presentation plan

- History and definition of the problem
- Main result
- Techniques

Vector Addition Systems with States



Vector Addition Systems with States

```
x += 1
loop
    x −= 1    y += 1
loop
    x += 2    y −= 1
loop
    x −= 1    y += 1
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```
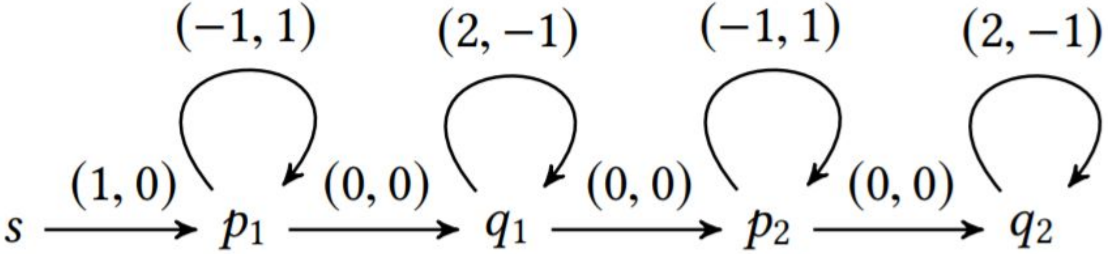
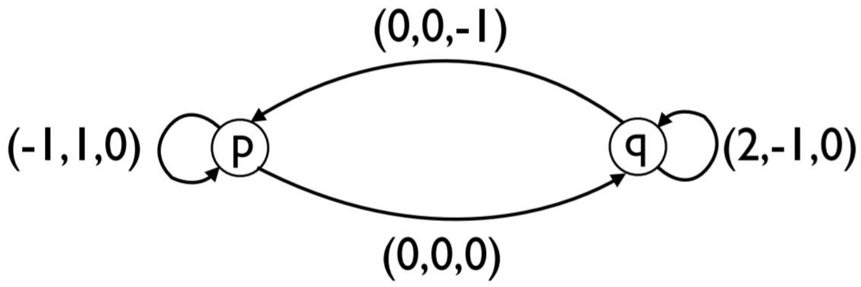



Reachability Problem

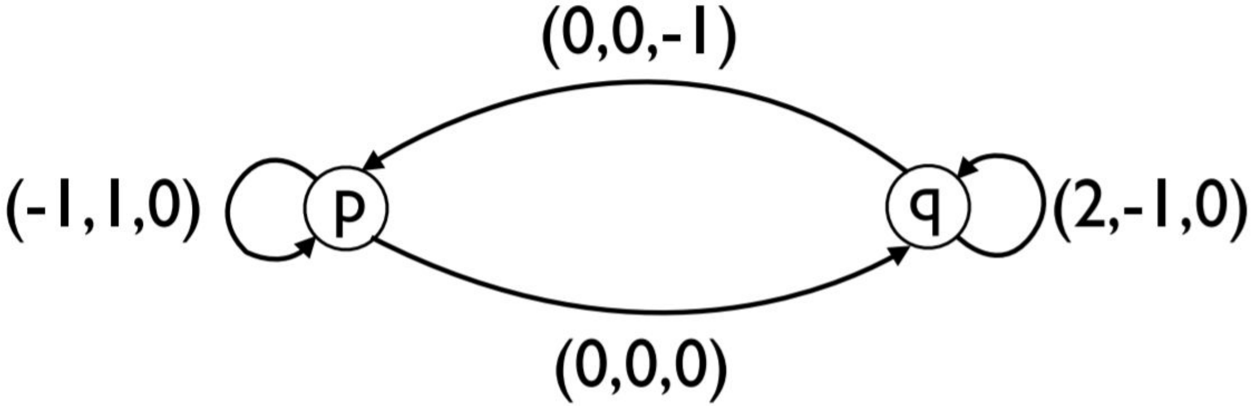
Given: a Vector Addition System with States (VASS) V, two configurations s and t

Question: is there a run from s to t in V?



Example of run

$$p(1,0,2) \to p(0,1,2) \to q(0,1,2) \to q(2,0,2) \to p(2,0,1)$$



Fast growing functions

$$F_1(n) = 2n$$

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$\text{Ackermann}(n) = F_n(n)$

Short History

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Our contribution

Theorem:

The Reachability Problem for Vector Addition Systems is Ackermann-hard.

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The Reachability Problem for 6d-VASSes is F_d-hard

Other works

- Jerome Leroux independent proof

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- Jerome Leroux independent proof
    - $4d+O(1)$ first version
    - $2d+O(1)$ last version
- Sławomir Lasota follow-up ($3d + O(1)$)

Techniques

Controlling counter

$$c_0 \xrightarrow{\rho_1} c_1 \xrightarrow{\rho_2} \ldots \xrightarrow{\rho_{n-1}} c_{n-1} \xrightarrow{\rho_n} c_n.$$

- x_i value of counter x in state c_i

Controlling counter

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- x_i value of counter x in state c_i
- x'_i be the effect of run ρ_i on counter x
- How to check if $x_i = 0$ for all i?
- Introduce new counter $c = x_1+x_2+\dots+x_n = nx'_1 + \dots+x'_{n+1}$

Controlling counter - example

```
x += 1    c += n
for i := 1 to n do
    loop
        x −= 1    y += 1
    loop
        x += 2    y −= 1    c += n − i − 1
```

Multiplication triples technique

- Let’s have a triple (b,2c, 2bc)
- Allows for c zero-tests on b-bounded counters

Multiplication triples technique

flush(x,y,z):

loop

$$x \mathrel{-}= 1 \qquad y \mathrel{+}= 1 \qquad z \mathrel{-}= 1$$

Multiplication triples technique

Let's have triple (b, y, z) = (B, 2C, 2BC)

Zero-test(x):

$\textbf{flush}(b, x, z)$

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$y \mathrel{-}= 2$

Multiplication triples technique

Let's have triple (b, y, z) = (B, 2C, 2BC)

Zero-test(x):

flush(b, x, z)
flush(x, b, z)
$y \mathrel{-}= 2$

Invariant (b+x)y=z kept only if x was indeed zero!

Minsky machine

- counter program with zero-tests and three counters

Minsky machine

- counter program with zero-tests and three counters
- in general reachability problem is undecidable

F_d bounded reachability problem

Given: a Minsky machine M and two configurations s and t

Question: is there a run from s to t in M where all counters are bounded by $F_d(|M|)$?

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F_d bounded reachability problem

Given: a Minsky machine M and two configurations s and t

Question: is there a run from s to t in M where all counters are bounded by $F_d(|M|)$?

This problem is **F_d-hard**

We can simulate it by VASS using multiplication triples technique ($F_d(|M|)$, C, $F_d(|M|)$C)

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- Possible due to controlling counter technique

Thank you!