Fractal Dimension and Understanding Brain Complexity

Math 6440: Nonlinear Dynamics and Chaos

Laurel Weber
Professor Elena Cherkaev

December 2025

Abstract

Fractal dimension provides a quantitative measure of how the complexity of a system changes with scale. Although originally developed for geometric sets, fractal methods are now used to analyze time series by characterizing how signal variability increases at finer temporal scales. This project introduces fractal dimension from the perspective of nonlinear dynamics and then applies these ideas to neural data, particularly electroencephalography (EEG). The emphasis is on numerical simulations illustrating how fractal dimension behaves under different dynamical regimes and why such measures are used to study real brain signals.

1 Introduction

A central theme in nonlinear dynamics is that simple deterministic systems can generate complex behavior. Chaotic attractors typically have non-integer (fractal) dimension, reflecting structure that is more intricate than a smooth curve but less than a full region in space. Fractal dimension formalizes this by quantifying how the effective size of a set changes as the observation scale decreases.

Neural systems also generate activity across many spatial and temporal scales. Recent work suggests that fractal dimension provides a useful link between theoretical dynamical systems concepts and empirical measures of brain complexity, particularly in electroencephalogram (EEG) analysis [2, 3, 4]. Highly regular EEG activity—such as strong rhythmic oscillations or periods of abnormal synchrony—produces lower fractal dimension because the signal changes minimally at finer scales. In contrast, normal awake EEG, bursty activity, or states with rich temporal variability yield higher fractal dimension.

This makes fractal dimension useful for comparing brain states, identifying pathology, and studying how neural dynamics become more or less complex under conditions such as sleep, anesthesia, or disorders.

This project aims to connect these perspectives through simulation and numerical estimation of fractal dimension.

2 Mathematical Background: Fractals and Fractal Dimension

2.1 Self-similarity and scaling

In Euclidean geometry, the dimension of an object is an integer. Fractals generalize this by allowing non-integer dimension. A set is fractal when it exhibits scale-dependent structure, meaning its measured size changes proportionally to a power of the observation scale.

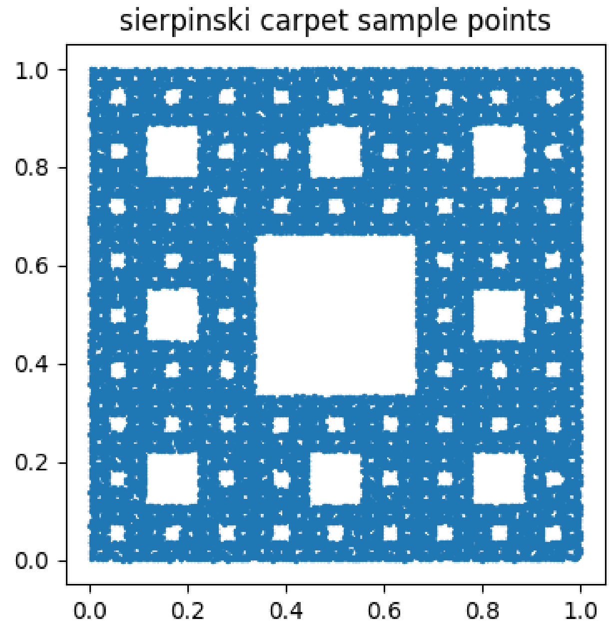

Figure 1: Simulation of the Sierpinski carpet. Estimated box-counting dimension for Sierpinski carpet is 1.89

A general form of fractal dimension is

$$D = \lim_{\epsilon \to 0} \frac{\log F(\epsilon)}{\log(1/\epsilon)}, \tag{1}$$

where $F(\epsilon)$ counts how many pieces of size ϵ are required to cover the object. Different choices of F lead to Hausdorff, box-counting, and correlation dimension [1].

2.2 Box-counting dimension

Let $N(\epsilon)$ be the number of boxes of side length ϵ required to cover a bounded set $S \subset \mathbb{R}^n$. The box-counting dimension is

$$D_B(S) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}. \tag{2}$$

In practical computation, we can use several scales ϵ_i and estimates D_B from the slope of $\log N(\epsilon_i)$ vs. $\log(1/\epsilon_i)$.

2.3 Correlation dimension and attractors

For time series embedded in state space, the correlation dimension D_2 is often used. Given points $\{\boldsymbol{x}_i\}$ on an attractor, define

$$C(r) = \frac{2}{N(N-1)} \sum_{i<j} \Theta(r - \|\boldsymbol{x}_i - \boldsymbol{x}_j\|), \tag{3}$$

where Θ is the Heaviside function. For fractal sets,

$$C(r) \propto r^{D_2}, \tag{4}$$

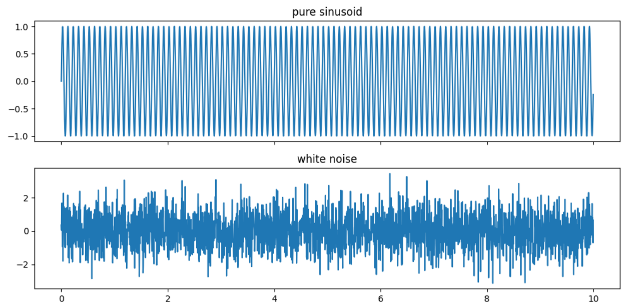

Figure 2: Sinusoid versus white noise. hfd sine: 0.075, hfd noise: 1.005

so

$$D_2 = \lim_{r \to 0} \frac{\log C(r)}{\log r}. \tag{5}$$

3 Fractal Dimension for Time Series Data

3.1 Higuchi fractal dimension

Higuchi's fractal dimension (HFD) estimates fractal dimension directly from a 1D time series without phase-space reconstruction. Given samples $X(1), \ldots, X(N)$, the algorithm constructs k-step subsequences, computes an effective length $L_m(k)$ for each, averages them to obtain $L(k)$, and estimates the slope of $\log L(k)$ versus $\log(1/k)$. This slope is the HFD!

The HFD method is used for EEG because it performs well on short, noisy, and nonstationary signals [3, 4].

3.2 Sanity check: sinusoid versus white noise

Before applying HFD to more complex data, it is useful to verify that the estimator behaves as expected on simple signals. A pure sinusoid is smooth and highly regular. Its geometric length changes very little under finer sampling, so its HFD is close to zero:

$$\mathrm{HFD}_{\mathrm{sine}} \approx 0.07.$$

In contrast, white noise fluctuates at all scales, producing rapid growth in effective length as the sampling step decreases. Its HFD is approximately one:

$$\mathrm{HFD}_{\mathrm{noise}} \approx 1.00.$$

This confirms that HFD assigns lower dimension to smooth signals and higher dimension to irregular, multiscale signals. This serves as a basic check of correctness before analyzing more complicated or neural-like data.

3.3 Synthetic signal comparison

In order to better understand EEG segments via fractal dimension, I provide a series of controlled reference signals to make the behavior of fractal dimension easier to interpret.

By comparing these signals, we can isolate how specific features (periodicity, noise, bandwidth, and intermittency) affect measures of scale-dependent complexity.

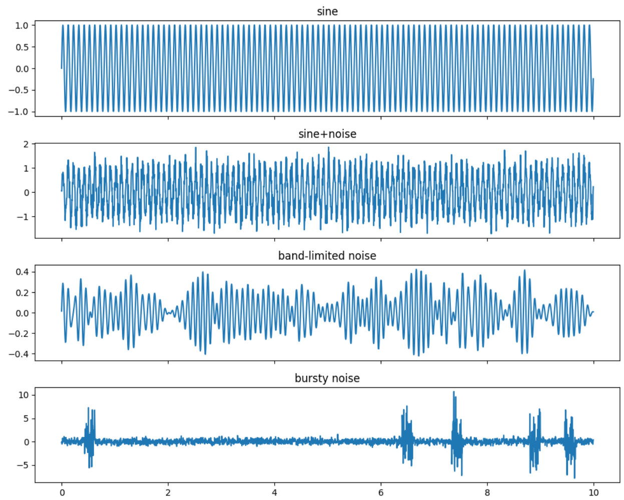

Figure 3: Synthetic signals to demonstrate different temporal structures.

The pure sinusoid represents an extreme case of regular dynamics. It is dominated by a single time scale and exhibits smooth, predictable behavior. As a result, minimal new structure appears when the signal is examined at finer temporal resolutions. The sinusoid with additive noise introduces small-scale irregularity while preserving an underlying periodic structure, illustrating how fractal dimension increases when fine-scale variability is superimposed on an otherwise regular signal.

The band-limited noise signal removes strict periodicity but retains structure over a restricted range of frequencies. This produces fluctuations on multiple time scales without the complete scale-free behavior of white noise. Finally, the burst noise signal combines low-amplitude background activity with intermittent high-amplitude events, creating strong multiscale structure and temporal heterogeneity.

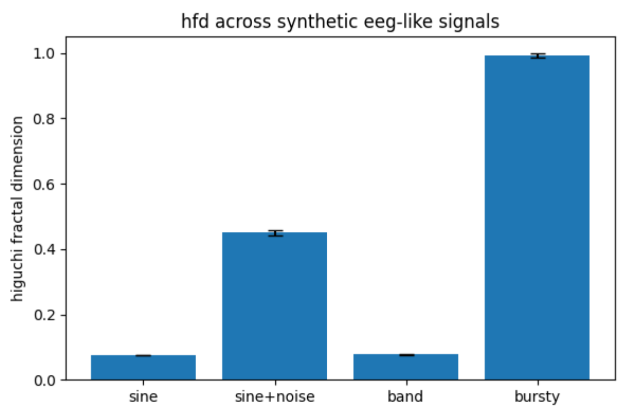

Figure 4: Comparison of HFD for different synthetic signals

Comparing these signals can clarify how variability accumulates across scales and how we can better understand that variability via fractal dimension. Signals that are smooth or dominated by a single scale yield lower fractal dimension, while signals that contain structure at many scales yield higher values.

3.4 Fractal dimension in clinical neurophysiology

Fractal dimension has been used to analyze pathological and normal EEG across many contexts, including seizure detection, aging, Alzheimer's disease, and levels of consciousness [5, 6, 7, 4]. In general, the hypothesis is that reduced fractal dimension is associated with overly synchronized or low-complexity neural activity, whereas higher values reflect rich temporal structure.

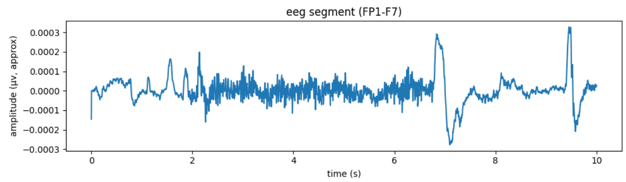

Figure 5: Real EEG segment analyzed using Higuchi fractal dimension; hfd eeg: 0.641.

The above figure shows a 10-second segment of real EEG recorded from the FP1–F7 channel. The signal exhibits a mixture of relatively low-amplitude background fluctuations and intermittent large-amplitude transients. This combination indicates that the dynamics are neither purely rhythmic nor purely stochastic. Instead the EEG displays structure across multiple time scales: short-term irregular fluctuations are superimposed onto slower variations and the occasional sharp events.

When Higuchi's fractal dimension was computed for this segment, the resulting value was $D_H = 0.641$. This value lies well above that of a smooth periodic signal and well below that of white noise, reflecting intermediate complexity. The HFD captures the fact that finer temporal sampling reveals additional structure, but not at the maximal rate characteristic of random processes. In this sense, the fractal dimension provides a quantitative summary of the visually apparent multiscale structure in the EEG, consistent with the interpretation of neural activity as structured yet irregular rather than fully synchronized or random.

4 Discussion

The simulations and analyses in this project showed that fractal dimension provides a fascinating way to quantify scale-dependent structure in both dynamical systems and time series. In classical fractals and chaotic attractors, non-integer dimension reflected how trajectories or sets filled space as parameters moved from regular to chaotic regimes. When applied to EEG time series, the resulting values lay between the extremes of smooth periodic signals and stochastic noise, consistent with the view that neural activity is structured but not fully regular. This aligns with findings in the clinical literature, where changes in fractal dimension have been associated with different brain states and pathologies, including epilepsy, anesthesia, and neurodegenerative disease. While this is all very exciting, it should, at the same time, be interpreted cautiously; fractal dimension is a descriptive summary of multiscale structure rather than a mechanistic explanation. Its value depends on factors such as signal length, noise level, frequency content, and the specific estimation method used. However, if used in conjunction with other analyses, fractal dimension could provide a useful measure of neural signal complexity.

5 Code Availability

My Python codebase for all of the simulations and images referenced in this project can be found here. You can also find more of my work on my GitHub.

References

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman, 1982.

[2] S. Moaveninejad, S. Cauzzo, and C. Porcaro, "Fractal dimension and clinical neurophysiology fusion to gain a deeper brain signal understanding: A systematic review," *Information Fusion*, vol. 118, 102936, 2025.

[3] S. Kesic and S. Spasic, "Application of Higuchi's fractal dimension from basic to clinical neurophysiology: A review," *Computer Methods and Programs in Biomedicine*, vol. 133, pp. 55–70, 2016.

[4] Z. J. Lau *et al.*, "Brain entropy, fractal dimensions and predictability: A review of complexity measures for EEG," *European Journal of Neuroscience*, vol. 55, pp. 2748–2789, 2022.

[5] R. Esteller *et al.*, "A comparison of waveform fractal dimension algorithms," *IEEE Transactions on Circuits and Systems I*, vol. 48, no. 2, pp. 177–183, 2001.

[6] T. Q. D. Khoa *et al.*, "Higuchi fractal properties of onset epilepsy EEG," *Computational and Mathematical Methods in Medicine*, 2012.

[7] F. M. Smits *et al.*, "EEG fractal dimension in healthy aging and Alzheimer's disease," *PLOS ONE*, vol. 11, e0149587, 2016.