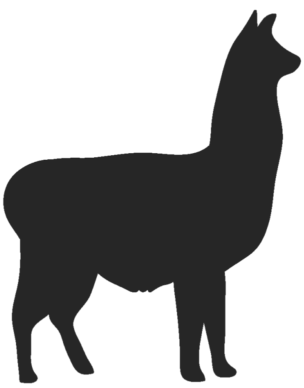
Foundation of Operations Research

Lorenzo Rossi and everyone who kindly helped!

2022/2023

Last update: 2022-10-12



no alpaca has been harmed while writing these notes

Contents

1 Introduction

1.1 Algorithm

An algorithm for a problem is a sequence of instructions that allows to solve any of its instances. The execution time of an algorithm depends on various factors, most notably the instance and the computer.

Properties:

- An algorithm is **exact** if it provides an optimal solution for every instance.
  - otherwise is **heuristic**
- A **greedy algorithm** constructs a feasible solution iteratively, by making at each step a *locally optimal* choice, without reconsidering previous choices
  - for most *discrete optimization problems*, greedy type algorithms yield a feasible solution with no guarantee of optimality

1.2 Dynamic Programming

Proposed by *Richard Bellman* in 1950, **dynamic programming** *(or DP)* is a method for solving optimization problems, composed of a sequence of decisions, by solving a set of recursive equations.
DP is applicable to any sequential decision problem, for which the optimality property is satisfied; as such, it has a wide range of applications, including scheduling, transportation, and assignment problems.

1.3 Complexity of algorithms

In order analyze an algorithm, it's necessary to consider its complexity as a function of the size of the instance *(the size of the input)*, independently of the computer; the complexity is defined as the number of elementary operations by assuming that each elementary operation takes a constant time.
Since it's hard to determine the exact number of elementary operations, an additional assumption is made: only the asymptotic number of elementary operations in the worst case (for the worst instances) is considered.
The complexity evaluation is then performed by looking for the function $f(n)$ that best approximates the upper bound on the number of elementary operations n for the worst instances.

1.3.1 Big-O notation

A function f if order of g, written $f(n) = \mathcal{O}(g(n))$ if exists a constant $c > 0$ and a constant $n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.
An illustration of the big-O notation is shown in Figure 1.

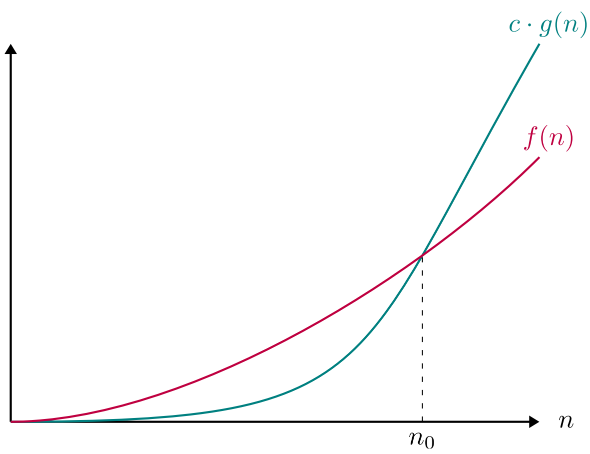


Figure 1: Big-O notation

n	n^2	2^n
1	$1\,\mu s$	$1\,\mu s$
10	$100\,\mu s$	$1.024\,ms$
20	$400\,\mu s$	$\approx 1.04\,s$
30	$900\,\mu s$	$\approx 18\,m$
40	$1.6\,ms$	$\approx 13\,d$
50	$2.5\,ms$	$\approx 36\,y$
60	$3.6\,ms$	$\approx 36535\,y$

Table 1: Complexity classes

1.4 Complexity classes

Two classes of algorithms are considered, according to their worst case order of complexity:

- **Polynomial**: $\mathcal{O}\left(n^d\right)$ for a constant $d > 0, d \in \mathbb{R}$
- **Exponential**: $\mathcal{O}\left(d^n\right)$ for a constant $d > 0, d \in \mathbb{R}$

Algorithms with a hight order Polynomial complexity are not considered efficient.
A comparison of the two classes, assuming that $1\,\mu s$ is needed for each elementary operation, is shown in Table 1.

2 Graph and Network Optimization

Many **decision making problems** can be formulated in terms of graphs and networks, such as:

- **transportation** and **distribution** problems
- **network design** problems
- **location** problem
- timetable **scheduling**
- ...

2.1 Graphs

A **graph** is a pair $G = (N, E)$ with:

- N a set of **nodes** or **vertices**
- $E \subseteq N \times N$ a set of **edges** or arcs connecting them pairwise
  - an edge connecting nodes i and j is represented by $\{i, j\}$ if the graph is **undirected**
  - an edge connecting nodes i and j is represented by (i, j) if the graph is **directed**

Properties:

- Two **nodes** are **adjacent** if they are connected by an edge
- An **edge** e is **incident** in a node v if v is an **endpoint** of e
  - **undirected** graphs: the degree of a node is the number of incident edges
  - **directed** graphs: the in-degree *(out-degree)* of a node is the number of arcs that have it as successor *(predecessor)*
- A **path** from $i \in N$ to $j \in N$ is a sequence of edges

$$p = \langle \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\} \rangle$$

  connecting nodes $v_1, \ldots, v_k$, with $\{v_i, v_{i+1} \in E\}$ for $i = 1, \ldots, k-1$
- A **directed path** $i \in N$ to $j \in N$ is a sequence of arcs

$$p = \langle (v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k) \rangle$$

  connecting nodes, with $(v_i, v_{i+1} \in E)$ for $i = 1, \ldots, k-1$
- Nodes u and v are **connected** if exists a path connecting them
- A graph (N, E) is **connecting** if u, v are connecting $\forall\, u, v \in N$
- A graph (N, E) is **strongly connected** if u, v are connected by a directed path $\forall\, u, v \in N$
- A graph is **bipartite** if there is a partition $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$ such that $\forall\, (u, v) \in E, u \in N_1$ and $v \in N_2$
- A graph is **complete** if $E = \{\{v_i, v_j\} \mid v_i, v_j \in N \land i \leq j\}$
- Given a directed graph $G = (N, A)$ and $S \subseteq N$, the **outgoing cut** induced by S is the set of arcs:

$$\delta^+(S) = \{(u, v) \in A \mid u \in S \land v \in N \setminus S\}$$

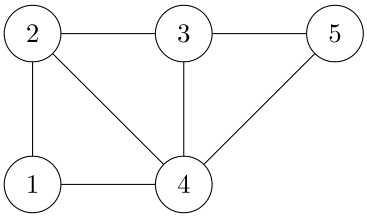
  the **incoming cut** induced by S is the set of arcs:

$$\delta^-(S) = \{(u, v) \in A \mid v \in S \land u \in N \setminus S\}$$

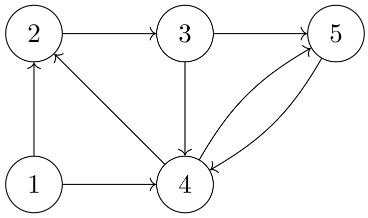
Some examples are shown in Figure 2.

Properties of graphs:

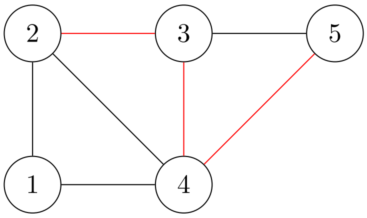
- A graph with n **nodes** has at most $m = \dfrac{n(n-1)}{2}$ **edges**
- A **directed** graph with n **nodes** has at most $m = n(n-1)$ **arcs**
  - a graph is **dense** if $m \approx n^2$
  - a graph is **sparse** if $m \ll n$



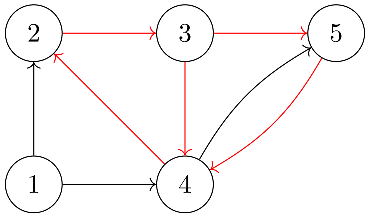
(a) Undirected graph



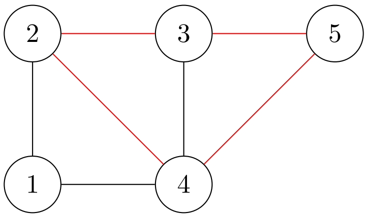
(b) Directed graph



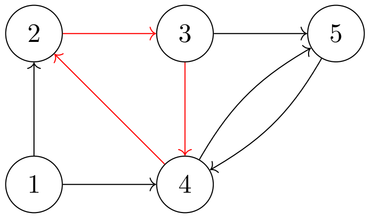
(c) Connected graph, nodes 2 and 5 are connected
$\langle\{2,3\},\{3,4\},\{4,5\}\rangle$ is a path



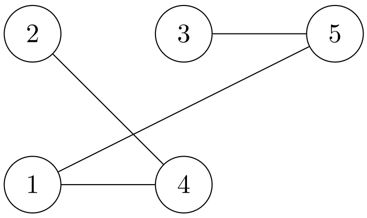
(d) Not strongly connected graph
$\langle\{3,5\},\{5,4\},\{4,2\},\{2,3\},\{3,4\}\rangle$ is a directed path



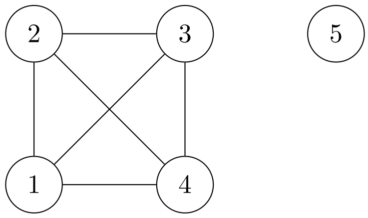
Cycle
(e) $\langle\{2,3\},\{3,5\},\{5,4\},\{4,2\}\rangle$ is a cycle



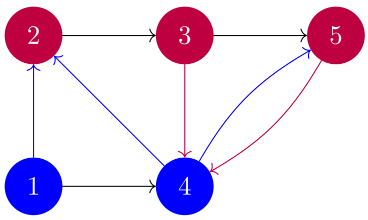
Circuit
(f) $\langle(2,3),(3,4),(4,2)$ is a circuit



Bipartite graph
(g) $N_1 = \{1,2,3\}\,, N_2 = \{4,5\}$

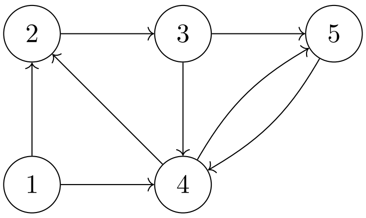


Complete graph
(h) $N = \{1,2,3,4\}$



(i) incoming (δ^+) and outgoing (δ^-) cuts of two sets of nodes *(purple and blue)*

Figure 2: Examples of graphs



(a) Graph

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) Adjacency matrix

$$S(1) = \{2, 4\}$$
$$S(2) = \{3\}$$
$$S(3) = \{4, 5\}$$
$$S(4) = \{2, 5\}$$
$$S(5) = \{4\}$$

(c) Adjacency list

Figure 3: Graph representation

2.1.1 Graphs representation

Graphs are represented by:

- Adjacency **matrix A** of size $n \times n$ if the graph is dense:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in A \\ a_{i_j} & \text{otherwise} \end{cases}$$

- Adjacency **list A** of size n if the graph is sparse

The same representation can be used for both directed and undirected graphs; the adjacency matrix for an undirected graph is **symmetric**.

An example of a graph representation is shown in Figure 3.

2.1.2 Graph reachability problem

Definition: given a directed graph $G = (N, A)$ and a node $s \in N$, find all nodes reachable from s

Goal

→ *Input*: graph $G = (N, A)$, described via successor lists, and a node $s \in N$
→ *Output*: subset $M \subseteq N$ of nodes of G reachable from s

The goal is reached by an *efficient* algorithm to solve the problem, with the following properties:

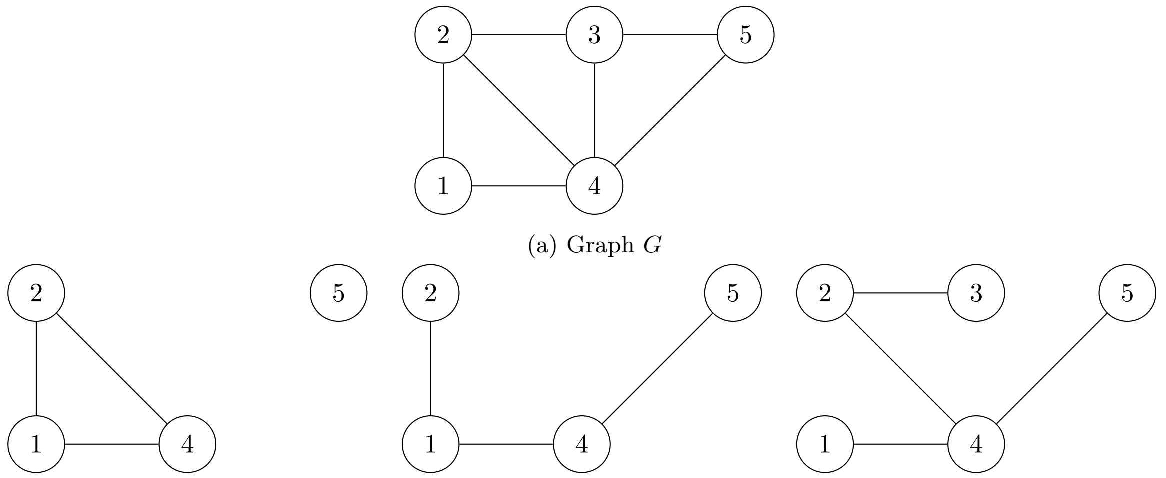
- a **queue** Q of nodes not yet processed is kept by the algorithm
- the queue uses a `FIFO` policy
- the nodes exploration is performed in a **breadth-first** manner

Algorithm The algorithm pseudocode is shown in Code 1.

```
Q := {s}
M := {}
while Q is not empty do
  u := node ∈ Q
  Q := Q \ {u}
  M := M ∪ {u}
  for (u, v) ∈ δ⁺(u) do
    if v ∉ M and v ∉ Q then
      Q := Q ∪ {v}
    end
  end
end
```

Code 1: Graph reachability

The algorithm stops when $\delta^+(M) = \emptyset$ *(when the outgoing cut of the set of nodes M is empty)*; $\delta^-(M)$ is the set of arcs with head node in M and tail in $N \setminus M$.



(a) Graph G

(b) Subgraph G' of G (c) Subgraph G'' of G and a tree (d) Subgraph G'' of G and a tree

Figure 4: Subgraphs and trees

2.1.2.1 Complexity analysis

At each iteration of the **`while`** loop:

1. A node u is **removed** from the queue Q and **added** to the set M
2. For all nodes v directly reachable from u and not already in M or Q, v is added to Q

Since each node u is inserted in Q at most once and each arch (u, v) is considered at most once, the overall complexity is:

$$\mathcal{O}(n+m) \quad n = |N|,\, m = |A|$$

For dense graphs, this value converges to $\mathcal{O}\left(n^2\right)$.

2.2 Subgraphs and Trees

Let $G = (N, E)$ be a graph. Then:

- $G' = (N', E')$ is a **subgraph** of G if $N' \subseteq N$ and $E' \subseteq E$
- A **tree** $G_T = (N', T)$ of G is a connected, acyclic, subgraph of G
- $G_T = (N', T)$ is a **spanning tree** of G if it contains all the nodes *($N' = N$)*
- The **leaves** of a tree are the nodes with degree 1

A representation of these concepts is shown in Figure 4.

2.3 Properties of trees

2.3.1 Property 1 - number of edges

Every tree with n nodes has $n-1$ edges.

2.3.1.1 Proof

- **Base case**: the claim holds for $n = 1$ *(a tree with a single node has no edges)*
- **Inductive steps**: show that the claim is valid for for any tree with $n+1$ nodes
  - let T_1 be a tree with $n+1$ and recall with any tree with $n \geq 2$ nodes has at least 2 leaves
  - by deleting one of the leaves and its incident edge, a tree T_2 with n nodes is obtained
  - by induction hypothesis, T_2 has $n-1$ edges; therefore, T_1 has $n - 1 + 1 = n$ edges

2.3.2 Property 2 - number of paths

Any pair of nodes in a tree is connected via a unique path.
Otherwise, the tree would contain a cycle.

2.3.3 Property 3 - new cycles

By adding a new edge to a tree, a new unique cycle is created. This cycle consists of the path in Property 2 - number of paths and the new edge.

2.3.4 Property 4 - exchange property

Let $G_T = (N, T)$ be a spanning tree of $G = (N, E)$. Consider an edge $e \notin T$ and the unique cycle C of $T \cup \{e\}$. For each edge $f \in C \setminus \{e\}$, the subgraph $T \cup \{e\} \setminus \{f\}$ is a spanning tree of G.

(a) Graph G_T, edge f is red, edge e is blue

(b) $T \cup \{e\} \setminus \{f\}$

Figure 5: Exchange property

2.3.5 Property 5 - cut property

Let F be a partial tree *(spanning nodes in $S \subseteq N$)* contained in a optimal spanning tree of $G = (N, E)$. Consider $e = \{u, v\} \in \delta(S)$ of minimum cost, then there exists a minimum cost spanning tree of G containing e.

2.3.5.1 Proof

By contradiction, assume $T^* \subseteq E$ is a minimum cost spanning tree with $F \subseteq T^*$ and $e \notin T^*$.
Adding an edge e to T^* creates the cycle C. Let $f \in \delta(S) \cap C$:

- If $c_e = c_f$, then $T^* \cup \{e\} \setminus \{f\}$ is a minimum cost spanning tree of G as it has the same cost as T^*
- If $c_e < c_f$, then $c\,(T^* \cup \{e\} \setminus \{f\}) < c\,(T^*)$, hence T^* is not optimal

2.4 Optimal cost spanning tree

Spanning trees have a number of applications:

- **network** design
- **IP network** protocols
- **compact memory** storage
- ...

Model: an undirected graph $G = (N, E)$, $n = |N|$, $m = |E|$ and a cost function $c : E \to \mathbb{R}$, that assigns a cost to each edge, with $e = \{u, v\} \in E$.

Required properties

1. Each **pair of nodes** must be in a **path** $\Rightarrow$ the output must be a **connected subgraph** containing all the nodes N of G

2. The **subgraph** must have **no cycles** $\Rightarrow$ the output must be a **tree**

Formalized problem: given an undirected graph $G = (N, E)$ and a cost function $c : E \rightarrow \mathbb{R}$, find a spanning tree $G_T(N, T)$ of G of minimum, total cost.
The objective is finding:

$$\min_{T \in X} \sum_{e \in T} c_e \qquad X = \text{ set of all spanning trees of } G$$

2.4.1 Theorem 1 - number of nodes in spanning trees

The **Theorem 1**, formulated by *Cayley* in 1889, states that:
A complete graph with n nodes ($n \geq 1$) has n^{n-2} spanning trees.

2.4.2 Prim's algorithm

Idea: iteratively build a spanning tree.

Method

1. Start from initial tree (S, T) with $S = \{u\}, S \subseteq N$ and $T = \emptyset$
2. At each ste, add to the current partial tree (S, T) an edge of minimum cost among those which connect a node in S to a node in $N \setminus S$

Goal

→ *Input*: connected graph $G = (N, E)$ with edge costs.
→ *Output*: subset $T \subseteq N$ of edges of G such that $G_T = (N, T)$ is a minimum cost spanning tree of G.

Complexity if all edges are scanned at each iteration, the complexity order is $\mathcal{O}(nm)$

Algorithm the pseudocode of the algorithm is shown in Code 2.

```
S := {u}
T := {}
while |T| < n - 1 do
  {u, v} := edge ∈ δ(S) with minimum cost // u ∈ S, v ∈ N \ S
  S := S ∪ {v}
  T := T ∪ {{u, v}}
end
```

Code 2: Prim's algorithm

Prim's algorithm is **greedy**: at each step a minimum cost edge is selected among those in the cut $\delta(S)$ induced by the current set of nodes S.

2.4.2.1 Correcteness of Prim's algorithm

Proposition: Prim's algorithm is exact.
The exactness does not depend on the choice of the first node nor on the selected edge of minimum cost $\delta(S)$. Each selected edge is part of the optimal solution as it belongs to a minimum spanning tree.

2.4.2.2 Optimality test

The optimality condition allows to verify whether a spanning tree T is optimal or not; it suffices to check that each $e \in E \setminus T$ is not a cost decreasing edge.

2.4.2.3 Implementation in quadratic time

The Prim's algorithm can be implemented in quadratic time ($\mathcal{O}\left(n^2\right)$).

Data structure

- k number of edges selected so far
- Subset $S \subseteq N$ of nodes incident to the selected edges
- Subset $T \subseteq E$ of selected edges
- $C_j = \begin{cases} \min\{c_{ij} \,|\, i \in S\} & j \notin S \\ +\infty & \text{otherwise} \end{cases}$
- $closest_j = \begin{cases} \arg\min\{c_{ij} \,|\, i \in S\} & j \notin S \\ \text{predecessor of } j \text{ in the minimum spanning tree} & j \in S \end{cases}$

An example of a step is shown in Figure 6.

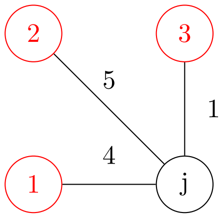


Figure 6: Data structure
nodes $1, 2, 3 \in S$, node $j \notin S$
$\text{closest}_j = 3 \quad c_{\text{closest}_j j} = 1$

The spanning tree is built is built by selecting the node j with minimum cost C_j and adding the edge $\{j, closest_j\}$ to the spanning tree.

The code for this algorithm is shown in Code 3.

```
T := {}
S := {u}
// initialization
for j ∉ N \ S do
  if {u, j} ∈ E then
    C_j := c_{u, j}
  else
    C_j := +\infty
  end
  closest_j := u
end
for k := 1 to n - 1 do
  min := +∞ // selection of min cost edge
  for j := 1, ..., n do
    if j ∉ S and C_j < min then
      min := C_j
      v := j
    end
  end
  S := S ∪ {v} // extend S
  T := T ∪ {{v, closest_v}} // extend T
  for j := 1 to n do
```

```
    if j ∉ S and c_vj < C_j then
      C_j := c_vj
      closest_j := v
    end
  end
end
```

Code 3: Prim's algorithm in quadratic time

The complexity of this algorithm is $\mathcal{O}\left(n^2\right)$. For sparse graphs, where $m \ll \dfrac{n(n-1)}{2}$, a more efficient implementation ($\mathcal{O}\left(m \log\left(2\right)\right)$) *(using priority queues)* is possible.

2.4.3 Optimality condition

Given a spanning tree T, an edge $e \notin T$ is cost decreasing if when added to T, it creates a cycle C with $C \subseteq T \cup \{e\}$ and $\exists\, f \in C \setminus \{e\}$ such that $c_e < c_f$.

2.4.4 Theorem 2 - Tree optimality condition

A tree T is of minimum total cost if and only if no cost decreasing edge exists.

2.4.4.1 Proof

- $\Rightarrow$ If a **cost decreasing edge exists**, then T is **not of minimum total cost**
- $\Leftarrow$ If **no cost decreasing edge exists**, then T is **of minimum total cost**
  - let T^* be a minimum cost spanning tree of graph G, found via by Prim's algorithm
  - it can be verified that T^* can be iteratively (changing one edge at a time) transformed into T without changing the total cost
  - thus, T is also optimal

2.5 Optimal paths

Optimal (shortest, longest, ...) paths have a wide range of applications:

- **Google Maps**, **GPS** navigators
- Planning and management of **transportation**, **electrical**, and **telecommunication networks**
- **Problem** planning
- ...

Model: Given a directed graph $G = (N, A)$ with a cost $c_{ij} \in \mathbb{R}$ associated to each arc $(i, j) \in A$, and two nodes s and t, determine a minimum cost *(shortest)* path from s to t.

- Each value $c_{i,j}$ represents the cost *(or length, travel time, ...)* of arc $(i, j) \in A$
- Node s is the origin *(or source)*, node t is the destination *(or sink)*

Properties of optimal paths:

- A path $\langle (i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k) \rangle$ is simple if no node is visited more than once

2.5.1 Property 6 - shortest path

If $c_{ij} \geq 0$ for all $(i, j) \in A$, there is at least one shortest path that is simple.

2.5.2 Dijkstra's algorithm

Idea: consider the nodes in increasing order of length *(cost)* of the shortest path from s to any one of the other nodes.

Method

- To each **node** $j \in N$, a **label** L_j is associated
  - $\Rightarrow$ at the end of the algorithm, this label will be the cost of the minimum cost path from s to j
- Another label $predecessor_j$ is associated to each node $j \in N$
  - $\Rightarrow$ at the end of the algorithm, this label will be the node that precedes j on the minimum cost path from s to j
- Make a **greedy** choice with respect to the paths from s to j
- A set of **shortest paths** from s to any node $j \notin s$ can be retrieved backwards from t to s iterating over the predecessors

Goal

→ *Input*: graph $G = (N, A)$, cost $c_{ij} \geq 0 \, \forall \, i, j$, origin $s \in N$
→ *Output*: shortest path from s to all other nodes in G

Data structure

- $S \subseteq N$: subset of nodes whose labels are permanent
- $X \subseteq N$: subset of nodes with temporary labels
- $L_j = \begin{cases} \text{cost of a shortest path from } s \text{ to } j & j \in S \\ \min\{L_i + c_{ij} \,|\, (i,j) \in \delta^+(S) & j \notin S\} \end{cases}$
  - → given a directed graph G and the current subset of nodes $S \subset N$, consider the outgoing cut $\delta^+(S)$ and select $(u, v) \in \delta^+(S)$ such that: $L_u + c_{uv} = \min\{L_i + c_{ij} \,|\, (i,j) \in \delta^+(S)\}$
  - → thus: $L_u + c_{uv} \leq L_i + c_{ij}, \forall \, (i,j) \in \delta^+(S)$
- $\text{predecessor}_j = \begin{cases} \text{predecessor of } j \text{ in the shortest path from } s \text{ to } j & j \in S \\ u \text{ such that } L_u + c_{uj} = \min\{L_i + c_{ij} \,|\, i \in S\} & j \notin S \end{cases}$

Complexity: the complexity of the algorithm depends on the how the arc (u, v) is selected among those of the current cut $\delta^+(u)$.

- If all m arcs are scanned, the overall complexity would be $\mathcal{O}(nm)$, hence $\mathcal{O}(n^3)$
- If all labels L_j are determined by appropriate updates (as in Prim's algorithm), only a single arc of $\delta^+(j)$ is scanned, hence the complexity is $\mathcal{O}(n^2)$

Notes:

- A set of shortest paths from s to all the nodes $j \in N$ can be retrieved backwards from t to s iterating over the predecessors
- The union of a set of shortest paths from node s to all the other nodes of G is an arborescence rooted at s
- Dijkstra's algorithm does not work when there are arcs with negative cost: if G contains a circuit of negative cost, the shortest path problem may not be well defined

The code for this algorithm is shown in Code 4.

```
S := {}
X := {s}
for u ∈ N do
  L_u := ∞
end
L_s := 0
while |S| < |N| do
  u := argmin{L_i | i ∈ X}
  X := X \ {u}
  S := S ∪ {u}
  for (u, v) ∈ δ+(u) do
    if L_v > L_u + c_uv then
      L_v := L_u + c_uv
      predecessor_v := u
      X := X ∪ {v}
    end
  end
end
```

Code 4: Dijkstra's algorithm

2.5.2.1 Correcteness of Dijkstra's algorithm

Dijkstra's algorithm is correct.
Proof:

1. A the k-th step:
   - $S = \{s, i_1, \dots, i_{k-1}\}$
   - $\begin{cases} \text{cost of a minimum cost path from } s \text{ to } j & j \in S \\ \text{cost of a minimum cost path with all intermediate nodes in } S & j \notin S \end{cases}$
2. By induction on the number k of steps:
   - base case: for $k = 1$ the statement holds, since

     $$S = \{s\}, \quad L_s = 0, \quad L_j = +\infty, \quad \forall\, j \notin S$$

   - inductive step: assume that the statement holds for $k + 1$
     - let $u \notin S$ be the node that is inserted in S and ϕ the path from s to u such that:

       $$L_v + c_{vu} \le L_i + c_{iu}, \quad \forall\, (i, v) \in \delta^+(S)$$

     - every path π from s to u has $c(\pi) \ge c(\phi)$, as there exists $i \in S$ and $j \notin S$ such that:

       $$\pi = \pi_1 \cup \{(i, j)\} \cup \pi_2$$

       where (i, j) is the first arc in $\pi \cap \delta^+(S)$
     - it holds that

       $$c(\pi) = c(\pi_1) + c_{ij} + c(\pi_2) \ge L_i + c_{ij}$$

       because $c_{ij} \ge 0 \Rightarrow c(\pi_2) \ge 0$ and by the choice of (v, u), $c(\pi_1) \ge L_i$
     - finally, by induction assumption:

       $$L_i + c_{ij} \ge L_v + c_{vu} = c(\phi)$$

     - a visualization of this step of the proof is shown in Figure 7

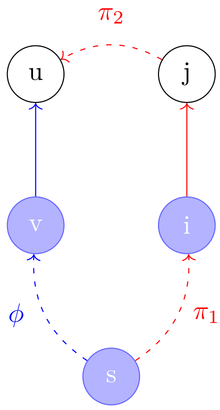


Figure 7: Proof of the induction step; nodes s, v, i are in cut S

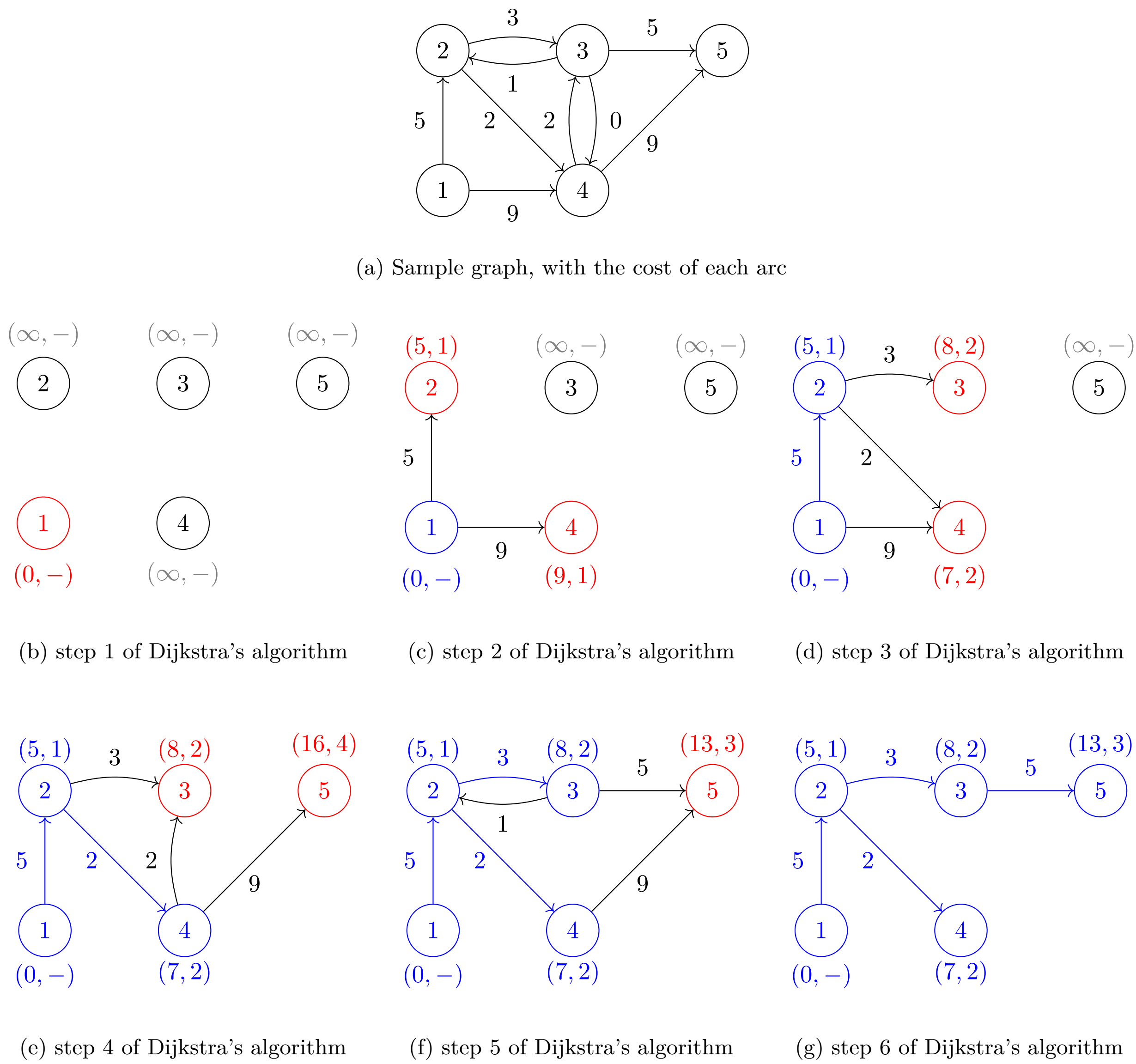


Figure 8: Example of Dijkstra's algorithm

2.5.2.2 Example of Dijkstra's algorithm

An example of Dijkstra's algorithm is shown in Figure 8.

2.5.3 Floyd-Warshall's algorithm

Goal

→ *Input*: a directed graph $G = (N, A)$ with an $n \times n$ cost matrix $C = [c_{ij}]$
→ *Output*: for each pair of nodes $i, j \in N$, the cost c_{ij} of the shortest path from i to j

Data structure

- Two $n \times n$ matrices D, P whose elements correspond, at the end of the algorithm, to:

  d_{ij} the cost of the shortest path from i to j
  p_{ij} the predecessor of j on the shortest path from i to j

Method

1. Initialization of D and P:
$$p_{ij} = i \quad \forall i$$
$$d_{ij} = \begin{cases} 0 & i = j \\ c_{ij} & i \neq j \land (i, j) \in A \\ +\infty & \text{otherwise} \end{cases}$$
2. Triangular operation: for each pair of nodes i, j, where $i \neq u, j \neq u$, check whether the path from i to j is shorter by going through u (i.e. $d_{iu} + d_{uj} < d_{ij}$)

Complexity

- Since in the worst case the triangular operation is executed for all nodes u ad for each pair of nodes i, j, the complexity is $\mathcal{O}\left(n^3\right)$

The code for this algorithm is shown in Code 5.

```
for i := 1 to n do
  for j := 1 to n do
    p_id := i

    if i = j then
      d_ij := 0
    else if (i, j) in A then
      d_ij := c_ij
    else
      d_ij := +∞
    end
  end
end
for u ∈ N do
  for i ∈ N \{ u } do
    for j ∈ N \{ u }
      if d_iu + d_uj < d_ij then
        p_ij := p_uj
        d_ij := d_iu + d_uj
      end
    end
```

```
  for i ∈ N do
    if d_ij < 0 then
      error "negative cycle"
    end
end
```

Code 5: Floyd-Warshall's algorithm

2.5.3.1 Correctness of Floyd-Warshall's algorithm

Floyd-Warshall's algorithm is correct.
Proof: assume that the nodes of G are numbered from 1 to n. Verify that, if the node index order is followed, after the u-th cycle the value d_{ij} *(for any i, j)* corresponds to the cost of a shortest path from i to j with at most u intermediate nodes ($\{1, \ldots, u\}$)

2.6 Optimal paths in directed, acyclic graphs

A directed graph $G = (N, A)$ is **acyclic** if it does not contain any circuit. A directed acyclic graph G is then referred to as a ***DAG***.
Property of *DAGs*: the nodes of any directed acyclic graph G can be ordered topologically, i.e. indexed so that for each arc $(i, j) \in A$ the index of i is less than the index of j *($i \leq j$)*. The topological order can be exploited by dynamic programming algorithms to compute efficiently the shortest paths in a *DAG*.
Problem: given a *DAG* $G = (N, A)$ with a cost $c_{ij} \in \mathbb{R}$ and nodes s, t, determine the shortest (or longest) path from s to t.

2.6.1 Topological ordering method

The method requires $G = (N, A)$ to be a *DAG* represented via the list of predecessors $\delta^-(v)$ and the list of successors $\delta^+(v)$ of each node $v \in N$. Then, it works as follows:

1. Assign the smallest positive integer not yet assigned to a node $v \in N$ with $\delta^-(v) = \emptyset$

   $\rightarrow$ such node always exists because G does not contain circuits
2. Delete the node v with all its incident arcs
3. Go to step (1) until all nodes have been assigned a number

(a) Topological ordering

(b) Node v with $\delta^-(v) = \emptyset$, as in step (1) of the algorithm

Figure 9: Topological ordering method

This algorithm has complexity $\mathcal{O}(|A|)$, because each node is assigned a number only once. Furthermore, all arcs incident to a node are deleted only once.

2.6.2 Dynamic programming for shortest path in *DAGs*

Any shortest path from 1 to t, called π_t, with at least 2 arcs can be subdivided into two parts:

- π_i, the shortest subpath from s to i
- (i, t), the remaining part

This decomposition is called the optimality principle of shortest paths in *DAGs*. An illustration of this decomposition is shown in Figure 10.

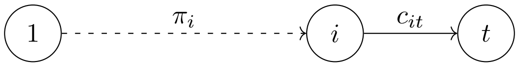


Figure 10: Shortest path from 1 to t

The strategy to find the shortest path is:

1. For each node $i = 1, \dots, t$ let L_i be the cost of a shortest path from 1 to i
   → $L_t = \min_{(i,t)\in\delta^-(t)} \{L_i + c_{it}\}$
   → the minimum is taken over all possible predecessors i of t
2. If G is topologically ordered *DAG*, then the only possible predecessors of t in a shortest path π_t from 1 to t are those with index $i < t$
   → $L_t = \min_{i<t} \{L_i + c_{it}\}$
   → in a graph with circuits, any node i can be a predecessor of t if $i \neq t$

For *DAGs* whose nodes are topologically ordered $L_{t-1}, \dots, L_1$ satisfy the same type of recursive relations:

$$L_{t-1} = \min_{i<t-1} \{L_i + c_{i,t-1}\}; \dots; L_2 = \min_{i=1} \{L_i + c_{i2}\} = L_1 + c_{12}; L_1 = 0$$

which can be solved in reversed order

$$L_1 = 0; L_2 = L_1 + c_{12}; \dots; L_t = \min_{i<t-1} \{L_i + c_t\}$$

Algorithm: finally, the algorithm is shown in pseudocode in Algorithm 6.

```
sort the nodes of G topologically
L_1 := 0
for j := 2 to n do
  L_j := min{L_i + c_{ij} | (i, j) ∈ δ⁻(j) ∧ i < j}
  pred_j := v such that (v, j) = argmin{L_i + c_{ij} | (i, j) ∈ δ⁻(j) ∧ i < j}
```

Code 6: Shortest path in *DAG*

Complexity of the algorithm is $\mathcal{O}(|A|)$:

- Topological ordering of the nodes: $\mathcal{O}(m)$ with $m = |A|$ (number of arcs)
- Each node/arc is processed only once: $\mathcal{O}(n + m)$

In order to find the longest path, the algorithm can be adapted as follows:

$$L_t = \max_{i<t} \{L_i + c_{it}\}$$

2.6.2.1 Optimality of the algorithm

The Dynamic Programming algorithm for finding shortest or longest paths in *DAGs* is exact. This is due to the optimality principle, already explored in the previous section.

2.7 Project planning

A project consists of a set of m activities with their (estimated) duration: activity A_i has duration $d_i \geq 0, i = 1, \ldots, m$. Some pair of activities allow a precedent constraint: $A_i \propto A_j$ indicated that A_i must be performed before A_j.
Model: a project can be represented by a directed graph $G = (N, A)$ where:

- each arc corresponds to an activity
- the arc length represent the duration of the corresponding activity

In order to account for precedence constraints, the arcs must be positioned such that for activities $A_i \propto A_j$ there exists a directed path where the arc associated to A_i precedes the arc associated to A_j. Such notation is shown in Figure 11.

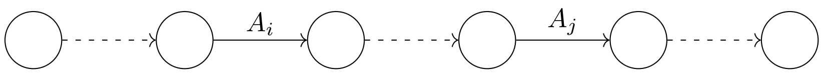


Figure 11: Precedence relation in project planning

Therefore, a nove v marks an event corresponding to the end fo all the activities $(i, v) \in \delta^-(v)$ and the (possible) start of all the activities $(v, j) \in \delta^+(v)$.

2.7.1 Property 7

The directed graph G representing a project is acyclic *(is a DAG)*.
Proof: by contradiction, if $A_{i1} \propto A_{12} \propto \ldots \propto A_{jk} \propto A_{kj}$ there would be a logical inconsistency.

2.7.2 Optimal paths

A graph G can be simplified by contracting some arcs, but it's important to not introduce unwanted precedence constraints. Artificial nodes or artificial arcs are introduced so that graph G:

- Contains a unique initial node s corresponding to the event *"beginning of the project"*
- Contains a unique final node t corresponding to the event *"end of the project"*
- does not contain multiple arcs with the same origin and destination

Problem: given a project (set of activities with duration and precedence constraints), schedule the activities in order to minimize the overall project duration (the time needed to complete all the activities).

2.7.2.1 Property 8

The minimum overall project duration is the length of a longest path from s to t in the graph G.
Proof: since any $s - t$ path represents a sequence of activities that must be executed in the specified order, its length provides a lower bound on the minimum overall project duration.

2.7.2.2 Critical path method - *CPM*

The critical path method (*CPM*) determines:

- A schedule *(a plan for executing the activities specifying the order and the assigned time)* that minimizes the overall project duration

- The slack of each activity *(the amount of time by which its execution can be delayed without affecting the overall minimum project duration)*

Initialization: construct the graph G representing the project.

Method:

1. Find a topological order of the nodes
2. Consider the nodes by increasing indices and for each $h \in N$ find the earliest time T_{min_h} at which the event associated to node h can occur

   $\rightarrow$ T_{min_h} corresponds to the minimum project duration
3. Consider the nodes by decreasing indices and for each $h \in N$ find the latest time T_{max_h} at which the event associated to node h can occur without delaying the project completion date beyond T_{min_n}
4. For each activity $(i, j) \in A$ find the slack

   $\rightarrow$ the slack is calculated as $\sigma ij = T_{max_j} - T_{min_i} - d_{ij}$

Input: graph $G = (N, A)$ with $n = |N|$ and the duration d_{ij} associated to each $(i, j) \in A/$
Output: $(T_{min_i}, T_{max_i}), i = 1, \ldots, n$

Algorithm: finally, the algorithm is shown in pseudocode in Algorithm 7.

```
sort the nodes topologically
T_min_i := 0
for j = 2 to n do
  T_min_j := max{T_min_i + d_ij | (i, j) ∈ δ⁻(j)}
end for
T_max_n := T_min_n // minimum project duration
for i = n-1 to 1 do
  T_max_i := min{T_max_j - d_ij | (i, j) ∈ δ⁺(i)}
end for
```

Code 7: Critical path method

Complexity: the overall complexity is $\mathcal{O}(n + m) \approx \mathcal{O}(m)$, due to the sum of

- complexity of the topological sort - $\mathcal{O}(n + m)$
- complexity of the first loop - $\mathcal{O}(n + m)$
- complexity of the second loop - $\mathcal{O}(n + m)$

2.7.2.3 Critical paths

An activity (i, j) with zero slack $\sigma_{ij} = T_{max_j} = T_{min_i} = d_{ij} = 0$ is called **critical**.
A critical path is a path in a $s - t$ composed uniquely by critical activities. At least one always exists.

2.7.2.4 Gantt charts

A Gantt chart is a graphical representation of a project schedule. It was introduced in 1896 by Henry Gantt, an American mechanical engineer and management consultant.
There are two types of Gantt charts:

- Gantt chart at **earliest** - each activity (i, j) starts at T_{min_i} and ends at $T_{min_i} + d_{ij}$
- Gantt chart at **latest** - each activity (i, j) starts at T_{max_i} and ends at $T_{max_i} + d_{ij}$

3 Linear Programming

A linear programming *(or LP)* problem is an optimization problem such as

$$
\begin{aligned}
&\min\ f(x) \\
&\text{s.t. } x \in X \subseteq \mathbb{R}^n \to \mathbb{R}
\end{aligned}
\tag{3.1}
$$

where:

- the objective function $f : X \to \mathbb{R}$ is linear
- the feasible ragion $X\,\{x \in \mathbb{R}^n \mid g_i(x)\, r_i\, 0 \wedge i \in \{1, \dots, m\}\}$ with $r_i \in \{=, \geq, \leq\}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are linear functions for $i \in \{1, \dots, m\}$
- $x^* \in \mathbb{R}^n$ is an optimal solution of the *LP* 3.1 if $f(x^*) \leq f(x)\,\forall\, x \in X$

A wide variety of decision making problems can be formulated or approximated as *LP*, as they often involve the optimal allocation of a given set of limited resources to different activities.

- General form of a linear programming problem:

$$
\begin{aligned}
\min\ & z = c_1 x^1 + \cdot + c_n x_n \\
\text{s.t. } & a_{11} x^1 + \cdot + a_{1n} x_n\, (\leq, =, \geq)\, b_1 \\
& \vdots \\
& a_{m1} x^1 + \cdot + a_{mn} x_n\, (\leq, =, \geq)\, b_m \\
& x^1, \dots, x_n \geq 0
\end{aligned}
\tag{3.2}
$$

- Matrix notation of a linear programming problem:

$$
\begin{aligned}
\min\ z = &\begin{bmatrix} c1 & c2 & \cdots & cn \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ \vdots \\ xn \end{bmatrix} \\
\text{s.t. } &\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x_n \end{bmatrix} (\leq, =, \geq) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\
&\begin{bmatrix} x^1 \\ \vdots \\ x_n \end{bmatrix} \geq 0
\end{aligned}
\tag{3.3}
$$

3.1 Assumptions of *LP* models

The *LP* model is based on the following assumptions:

- Linearity (proportionality and additivity) of the objective function and constraints
  - → proportionality: *contribution of each variable* $=$ *constant* $\times$ *variable.* It does not account for economies of scale
  - → additivity: *total contribution* $= \sum_i$ *contribution of each variable* i. It does not account for competing activities (their sum is not necessarily the total contribution)
- Divisibility of the variables, as they can assume fractional (rational) values

- Parameters are assumed to be constants which can be estimated with a sufficient degree of accuracy
  - → more complex mathematical programs are needed to account for uncertainty in the parameter values

LP **sensitivity analysis** allows to evaluate how *"sensitive"* an optimal solution is with respect to small changes in the parameters of the model.

3.2 Equivalent Forms

The general form (3.4) of a *LP* can be expressed in the equivalent standard form (3.5).

$$\begin{aligned} \min(\max)\ & z = c^T x & \\ \text{s.t. } & A_1 x \geq b_1 & \text{inequality constraints} \\ & A_2 x \leq b_2 & \text{inequality constraints} \\ & A_3 x = b_3 & \text{equality constraints} \\ & x_j \geq 0,\ j \in J \subseteq \{1, \dots, n\} & \text{non-negativity constraints} \\ & x_j \text{ free}\, j \in \{1, \dots, n\} \setminus J & \text{free variables} \end{aligned} \tag{3.4}$$

$$\begin{aligned} \min\ & z = c^T x & \\ \text{s.t. } & Ax \leq b & \text{equality constraints} \\ & x \geq 0 & \text{non negative variables} \end{aligned} \tag{3.5}$$

The two forms are equivalent, as simple transportation rules allow to pass from one form to the other; the transformation may involve adding and or deleting variables and constraints, as the next Section shows.

3.2.1 Transformation rules

- $\max c^T x \Rightarrow \min -c^T x$
- $a^T x \leq b \Rightarrow \begin{cases} a^T + x = b \\ s \geq 0 \end{cases}$ s is a slack variable
- $a^T x \geq b \Rightarrow \begin{cases} a^T + x = b \\ s \geq 0 \end{cases}$ s is a surplus variable
- x_j unrestricted in sign $\Rightarrow \begin{cases} x_j = x_j^+ - x_j^- \\ x_j^+ \geq 0 \\ x_j^- \geq 0 \end{cases}$
  - → after the substitution, x_j is deleted from the problem
- $a^T x \leq q \Leftrightarrow -a^T x \geq -b$
- $a^T x \geq q \Leftrightarrow -a^T x \leq -b$
- $a^T x = b \Leftrightarrow \begin{cases} a^T x \leq b \\ a^T x \geq b \end{cases} \Leftrightarrow \begin{cases} a^T x \leq b \geq b \\ -a^T x \geq -b \end{cases}$

3.3 Graphical solutions

A level curve of value z of a function f is the set of points in $\mathbb{R}^n$ where f is constant and takes value z.

Consider a *LP* with inequality constraints *(as it's easier to visualize)*.

- A hyperplane is the set of points that satisfies the constraint $H = \left\{x \in \mathbb{R}^n \mid a^T x = b\right\}$
- An affine half space is the set of points that satisfies the constraint $H = \left\{x \in \mathbb{R}^n \mid a^T x \leq b\right\}$
  - → each inequality constraint $a^T x \leq b$ defines an affine half space in the variable space

Definitions relative to the geometry of LP:

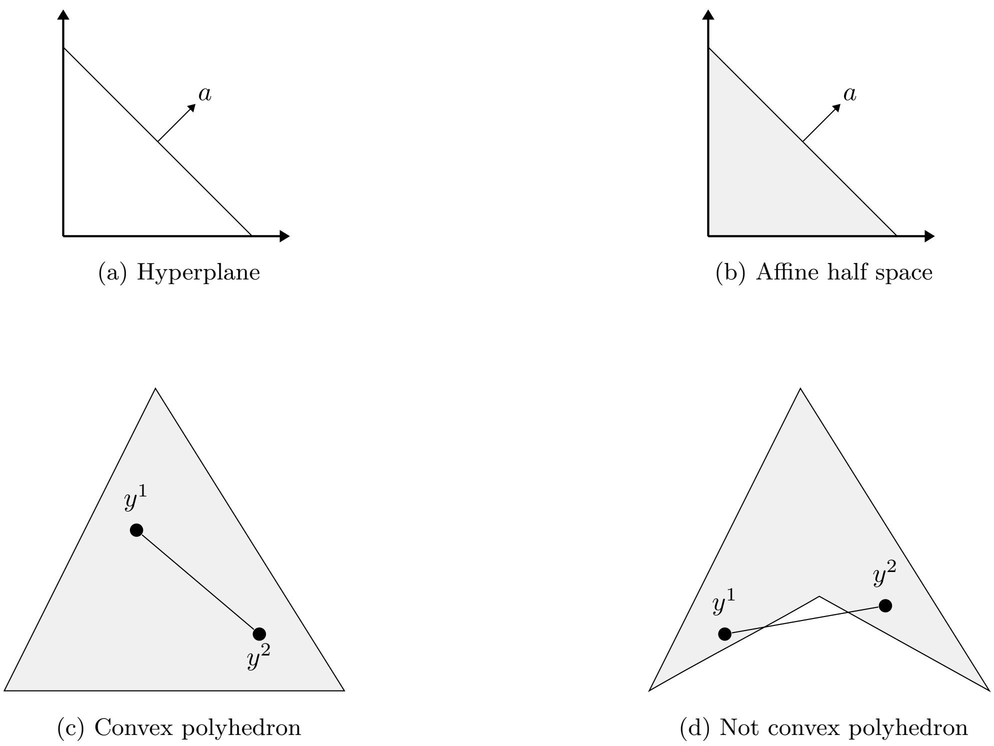
- The feasible region X of any LP is a polyhedron P defined by the intersection of a finite number of affine half spaces

  $\rightarrow$ P can be empty or unbounded
- A subset $S \subseteq \mathbb{R}^n$ is convex if for each pair of points $y^1, y^2 \in S$ the line segment between y^1 and y^2 is contained in S
- The segment defined by $y^1, y^2 \in S$ defined by all the convex combinations of y^1 and y^2 is called a convex hull

  $\rightarrow$ $\left[y^1, y^2\right] = \left\{x \in \mathbb{R}^n \,|\, x = \alpha y^1 + (1-\alpha)\, y^2 \wedge \alpha \in [0,1]\right\}$
- A polyhedron P is a convex set of $\mathbb{R}^n$
  - any half space is convex
  - the intersection of a finite number of convex sets is also a convex set
- A vertex of polyhedron P is a point op P that cannot be expressed as a convex combination of other points of P
  - mathematically, x is a vertex P iff

$$x = \alpha y^1 + (1-\alpha) y^2, \alpha \in [0,1] \quad y^1, y^2 \in P \Rightarrow x = y^1 \vee x = y^2$$

  - a non empty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ has a finite number $(n > 1)$ of vertices
- Given a polyhedron P, a vector $d \in \mathbb{R}^n, d \neq \overline{0}$ is an unbounded feasible direction of P if, for every point $x^0 \in P$, the ray $\left\{x \in \mathbb{R}^n \,|\, x = x^0 + \lambda d, \lambda \geq 0\right\}$ is contained in P



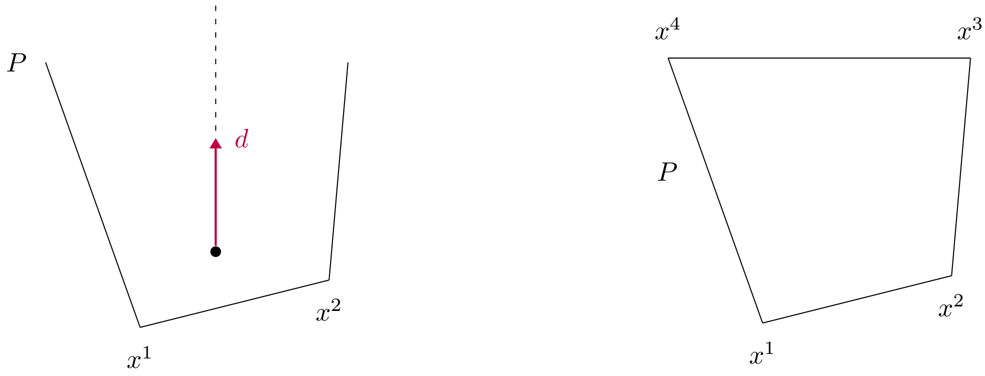
(a) Hyperplane

(b) Affine half space

(c) Convex polyhedron

(d) Not convex polyhedron

Figure 12: Illustrations of LP geometry definitions



(a) Unbounded polyhedron

(b) Bounded polyhedron

Figure 13: Illustration of the Weyl-Minkowski theorem

3.3.1 Theorem - representation of polyhedra

The *Weyl-Minkoswki* theorem on the representation of polyhedra states that:
Every point x of a polyhedron P can be expressed as a convex combination of its vertices $x^1, \ldots, x^k$ plus *(if needed)* an unbounded feasible direction d of P:

$$x = \alpha_1 x^1 + \ldots + \alpha_k x^k + d$$

where the multipliers $\alpha_1 \geq 0$ satisfy the constraint $\sum_{i=1} \alpha_i = 1$.

The unbounded feasible direction is needed if the polyhedron is unbounded; Figure 13 represent the cases of a bounded and unbounded polyhedron.

A polytope is a bounded polyhedron; it has the only unbounded feasible direction $d = 0$.
Consequence: every point x of polytope P can be expressed as a convex combination of its vertices.
Then the *Weyl-Minkowski theorem* can be used to describe any point. An example of this is shown in Figure 14.

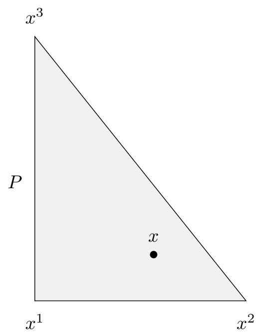


Figure 14: Example of polytope. $x = \alpha_1 x^1 + \alpha_1 x^2 + \alpha_3 x^3$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1, \alpha_i \geq 0, d = 0$

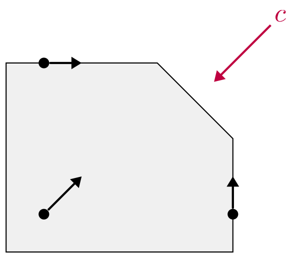
3.3.2 Geometry of *LP*

Geometrically:

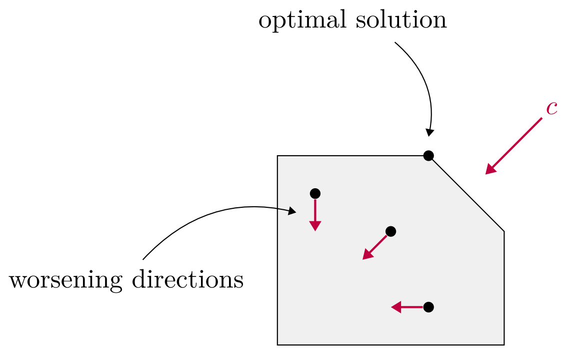
- An interior point $x \in P$ cannot be an optimal solution of the problem

- it always exists an improving direction
- consider Figure 15a where c represents the direction of fastest increase in z *(constant gradient)*

- In an optimal vertex, all feasible direction are worsening directions
  - consider Figure 15b where c represents the direction of fastest increase in z *(constant gradient)*
- The Weyl-Minkowski theorem implies that, although the variables can assume fractional values, LP can be seen as combinatorial problems
  - only the vertices of the polyhedron have to be considered in order to find the feasible solutions
  - the graphical method in only applicable for $n \leq 3$
  - the number of vertices often grows exponentially with respect to the number of variables



(a) Interior point of polyhedron P



(b) Worsening directions of polyhedron P

3.3.3 Four types of *LP*

There are four types of LP, depending on the number of solutions; all are illustrated in Figure 16. Since the objective of the problem is to minimize $f(x)$ *(as it's in form* $\min c^T x$*)*, better solutions are found by moving along the direction $-c$ *(the opposite of the gradient* $\nabla f(x)$*)*.

1. A unique optimal solution, Figure 16a
2. Multiple (infinitely many) optimal solutions, Figure 16b
3. Unbounded LP, Figure 16c
   - this type of problem has unbounded polyhedron and unlimited objective function values
4. Infeasible LP, Figure 16d
   - this type of problem has an empty polyhedron and no feasible solution

3.3.4 Basic feasible solutions and polihedra vertices

Due to the fundamental theorem of Linear Programming, to solve any LP problem it suffices to consider the (finitely many) vertices of the polyhedron P of feasible solutions. Since the geometrical definition of vertex cannot be exploited algorithmically, an algebraic definition is needed.
A vertex corresponds to the intersection of the hyperplanes associated to n inequalities. If a polyhedra is expressed in standard form

$$P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$$

it is possible to transform it into a inequality

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$$

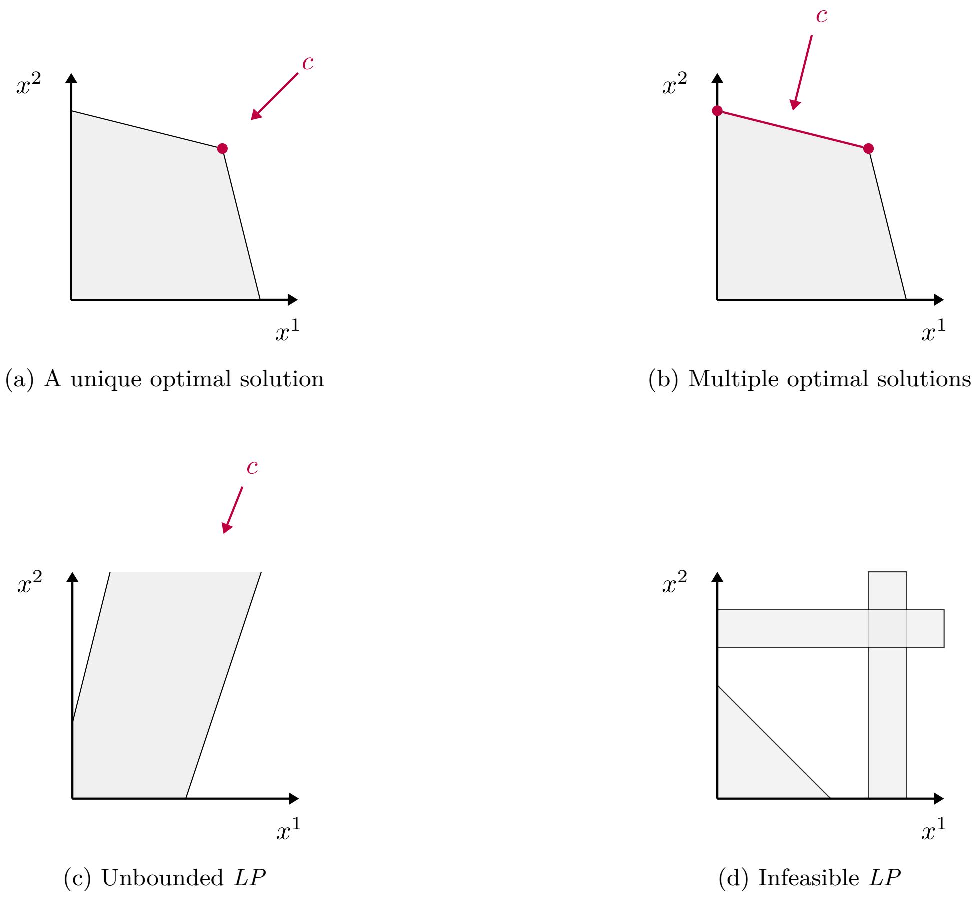


Figure 16: Four types of LP

and later transform it into standard form

$$P' = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$$

where P' is the polyhedron of feasible solutions of the original problem.
Finally, by renaming

$$A :- [A|I] \quad x :- \left(x^T|s^T\right)$$

the system of equation is represented in matrix form, where A has m rows.

3.3.4.1 Property - vertices of polyhedra

For any polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, where A has m rows:

- the facets (edges in $\mathbb{R}^2$) are obtained by setting one variable to 0
- the vertices are obtained by settings $n - m$ variables to 0

3.3.5 Algebraic characterization of vertices

Consider any polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, in standard form.
Assumption: $A \in \mathbb{R}^{m\times n}$ is such that $m \leq n$ of rank m (i.e. A is full rank). This is equivalent to assume that there are no redundant constraints.
Solutions:

- If $m = n$, there is a unique solution of $Ax = b$ *(*$x = A^{-1}b$)
- If $m \leq n$, there are ∞^{m-n} solutions of $Ax = b$
  - the system has $n - m$ degrees of freedom
  - by fixing the degrees of freedom to 0, a vertex is obtained

The basis of matrix A is a subset of m columns of A that are linearly independent and form an $m \times m$ non singular matrix B.

$$A = [\overbrace{B}^{n} \mid \overbrace{N}^{n-m}]$$

3.3.5.1 Basic solutions

Let $x^T = [\overbrace{x_B^T}^{m\ components} \mid \overbrace{x_N^T}^{n-m\ components}]$. Then any system $Ax = b$ can be written as $B_{x_B} + N_{x_N} = b$, and for any set of values of x_N, if B is not singular, then $x_B = B^{-1}(b - N_{x_N})$.
Definitions:

- A basic solution is a solution obtained by setting $x_N = 0$ and, consequently, $x_B = B^{-1}b$
- A basic solution with $x_B \geq 0$ is a basic feasible solution
- The variables in x_B are the basic variables and those in x_N are non basic variables
- By construction, $\left(x_B^T, x_N^T\right)$ satisfy $Ax = b$

3.3.5.2 Theorem of basic feasible solution

$x \in \mathbb{R}^n$ is a basic feasible solution if and only if x is a vertex of the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$

3.3.5.3 Number of basic feasible solutions

At most, there exists one basic feasible solution for each choice of the $n - m$ non basic variables out of the n variables:

$$\#\ \text{basic feasible solutions} \leq \begin{pmatrix} n \\ n-m \end{pmatrix} = \frac{n!}{(n-m)!\,(n-(n-m))!} = \begin{pmatrix} n \\ m \end{pmatrix}$$