Quiz problem 1: The entropy is an extensive state function for weakly interacting large systems and both the internal energy and the entropy are homogeneous functions of order 1

$$U(\lambda S, \lambda \mathcal{V}, \lambda N) = \lambda U(S, \mathcal{V}, N) \tag{1}$$

$$S(\lambda U, \lambda \mathcal{V}, \lambda N) = \lambda S(U, \mathcal{V}, N) \tag{2}$$

From the entropy equation directly derive the Euler relation (a)

$$S = \frac{1}{T}U + \frac{p}{T}\mathcal{V} - \frac{\mu}{T}N \tag{3}$$

and the corresponding form of the Gibbs-Duhem relation (b):

$$d\left(\frac{\mu}{T}\right) = \left(\frac{U}{N}\right) d\left(\frac{1}{T}\right) + \left(\frac{\mathcal{V}}{N}\right) d\left(\frac{p}{T}\right) \tag{4}$$

(a) **Solution:**

First we can start by differentiating equation 2 with respect to λ

$$S(U, \mathcal{V}, N) = \frac{\partial S(\lambda U, \lambda \mathcal{V}, \lambda N)}{\partial \lambda U}\frac{\partial \lambda U}{\partial \lambda} + \frac{\partial S(\lambda U, \lambda \mathcal{V}, \lambda N)}{\partial \lambda \mathcal{V}}\frac{\partial \lambda \mathcal{V}}{\partial \lambda} + \frac{\partial S(\lambda U, \lambda \mathcal{V}, \lambda N)}{\partial \lambda N}\frac{\partial \lambda N}{\partial \lambda} \tag{5}$$

Now, assume λ has a constant value of 1.

$$S(U, \mathcal{V}, N) = \frac{\partial S(U, \mathcal{V}, N)}{\partial \mathcal{V}}U + \frac{\partial S(U, \mathcal{V}, N)}{\partial \mathcal{V}}\mathcal{V} + \frac{\partial S(U, \mathcal{V}, N)}{\partial N}N \tag{6}$$

Recall TD equation,

$$dU = Tds - pd\mathcal{V}\mu dN \tag{7}$$

Solving for dS we get

$$dS = \frac{dU}{T} + \frac{pd\mathcal{V}}{T} - \frac{\mu dN}{T} \tag{8}$$

From this we know that

$$\left(\frac{\partial S}{\partial U}\right)_{\mathcal{V},N} = \frac{1}{T}; \quad \left(\frac{\partial S}{\partial \mathcal{V}}\right)_{U,N} = -\frac{p}{T}; \quad \left(\frac{\partial S}{\partial N}\right)_{U,\mathcal{V}} = \frac{\mu}{T} \tag{9}$$

Combining equation 9 with equation 6, we get

$$\boxed{S = \frac{1}{T}U + \frac{p}{T}\mathcal{V} - \frac{\mu}{T}N} \tag{10}$$

(b) **Solution:**

Now to get the Gibbs-Duhem relation, first we take the total differential of the Euler Relation: $U = TS - p\not{V} + \mu N$ and solve for dS.

$$dU = TdS + SdT - pd\not{V} - \not{V}dp + \mu dN + Nd\mu \tag{11}$$

$$dS = \frac{dU}{T} - \frac{SdT}{T} + \frac{pd\not{V}}{T} + \frac{\not{V}dp}{T} - \frac{\mu dN}{T} - \frac{Nd\mu}{T} \tag{12}$$

We can also take the total derivative of equation 10 and get

$$dS = \frac{1}{T}dU + Ud\frac{1}{T} + \frac{p}{T}d\not{V} + \not{V}d\frac{p}{T} - \frac{\mu}{T}dN - Nd\frac{\mu}{T} \tag{13}$$

Now combining our two expressions for dS we get,

$$-\frac{SdT}{T} + \frac{\not{V}dp}{T} - \frac{Nd\mu}{T} = Ud\frac{1}{T} + \not{V}d\frac{p}{T} - Nd\frac{\mu}{T} \tag{14}$$

AAANNNNDDDDDD... for some reason I have extra terms in this expression and I am unsure about how to remove them.