ASSUMPTION :

ENSEMBLE AVERAGES =
TIME AVERAGES

TRUE, IF SYSTEM HAS TO BE
ERGODIC.

DEF: IN AN ERGODIC SYSTEM THE
THE TRAJECTORY OF ALMOST ANY
POINT IN PHASE SPACE (INITIAL COND)
PASSES ARBITRARILY CLOSE TO EVERY
OTHER POINT ON THE SURFACE OF
CONSTANT ENERGY.

DEF II:
NOTATION: S : ENERGY SURFACE
R : ERGODIC COMPONENT
OF S : $r(0) \in R$
$\rightarrow r(t) \in R$ forever

TIME EVOLUTION IN S IS ERGODIC
IF ALL ERGODIC COMPONENTS R
IN S HAVE EITHER 0 VOLUME
OR THE SAME VOLUME AS S.

1) TIME AVERAGES ARE CONST ON A TRAJECTORY

$$\overline{O(\mathbb{P}(t_0), \mathbb{Q}(t_0))} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{T} O(\mathbb{P}(t), \mathbb{Q}(t))\,dt$$

FINITE TIME INTERVAL $(0, t_0)$ DOES NOT AFFECT LONG TERM TIME AVERAGE.

$$\overline{O(\mathbb{P}(t), \mathbb{Q}(t))} = \text{const}$$

2) TIME AVERAGES ARE CONST ON THE MICROCANICAL ENERGY SURFACE

ASSUME R_a WHERE $\overline{O} < a$

$\overline{O}$ = const ALONG TRAJECTORY (1)

$\rightarrow$ POINT IN R_a $\xrightarrow[\text{ENDS UP}]{}$ ANOTHER POINT IN R_a.

ERGODIC : R_a HAS VOLUME OF S OR ZERO VOLUME.

$\Rightarrow$ $\overline{O}$ IS CONSTANT ON ENERGY SURFACE WITH VALUE a^*.

3) TIME AVERAGES EQUAL MICROCANONICAL ENSEMBLE AVERAGES

$$\langle O \rangle_S = \int \underset{\text{Observable}}{O} \, \rho \, d\mathbb{P} \, d\mathbb{Q} = \frac{1}{\Omega(E)} \int O \, d\mathbb{P} d\mathbb{Q}$$

use $\frac{d\rho}{dt} = 0$ FOLLOWS LIOUVILLES THEOREM

$$\langle O \rangle_S = \langle O(\mathbb{P}(t), \mathbb{Q}(t)) \rangle_S \cdot \frac{T}{T}$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \langle O(\mathbb{P}(t), \mathbb{Q}(t)) \rangle_S \, dt$$

$$\langle \quad \rangle_S = \int \quad d\mathbb{P} d\mathbb{Q}$$

CHANGE ORDER OF INTEGRATION

$$= \left\langle \lim_{T \to \infty} \frac{1}{T} \int_0^T O(\mathbb{P}(t), \mathbb{Q}(t)) \, dt \right\rangle_S$$

$$= \langle \overline{O(\mathbb{P}, \mathbb{Q})} \rangle_S$$

$$= \langle a^* \rangle_S = a^*$$

$$\langle O \rangle_S = a^* = \overline{O}$$

$\square$

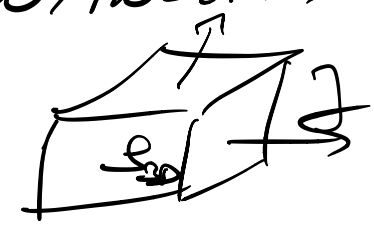
LIOUVILLE'S THEOREM

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

LOOK AT $\rho(P, Q)$:

$$\int \rho(P, Q)\, dP\, dQ = 1$$

CONSERVED QUANTITY →
CONTINUITY EQUATION



$$\frac{\partial \rho}{\partial t} = -\nabla J$$

$$J = \rho v$$

PHASE SPACE PROBABILITY CURRENT
$(\rho \dot{P}, \rho \dot{Q})$

$$0 = \frac{\partial \rho}{\partial t} + \nabla J$$

$$= \frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3N}\left(\frac{\partial(\rho \dot{q}_\alpha)}{\partial q_\alpha} + \frac{\partial(\rho \dot{p}_\alpha)}{\partial p_\alpha}\right)$$

$$= \frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3N}\left(\frac{\partial \rho}{\partial q_\alpha}\dot{q}_\alpha + \rho \frac{\partial \dot{q}_\alpha}{\partial q_\alpha} + \frac{\partial \rho}{\partial p_\alpha}\dot{p}_\alpha + \rho \frac{\partial \dot{p}_\alpha}{\partial p_\alpha}\right)$$

$$\frac{\partial \dot{q}_\alpha}{\partial q_\alpha} = \frac{\partial}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} = \frac{\partial^2 H}{\partial p_\alpha \partial q_\alpha}$$

$$\frac{\partial \dot{p}_\alpha}{\partial p_\alpha} = \frac{\partial}{\partial p_\alpha}\left(-\frac{\partial H}{\partial q_\alpha}\right) = -\frac{\partial^2 H}{\partial p_\alpha \partial q_\alpha}$$

$$0 = \frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3N}\left(\frac{\partial \rho}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \rho}{\partial p_\alpha}\cdot\dot{p}_\alpha\right)$$

$$\overbrace{\frac{d\rho}{dt}}^{\rho(P(t),\, Q(t),\, t)} = \frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3N}\left(\frac{\partial \rho}{\partial q_\alpha}\cdot\dot{q}_\alpha + \frac{\partial \rho}{\partial p_\alpha}\dot{p}_\alpha\right)$$

$$\overset{!}{=} 0$$

$\Rightarrow$ ρ = const IN MICROCANONICAL ENSEMBLE.

$\longrightarrow$ ρ stays = const AS A FUNCTION OF TIME

STRANGE CASES (0 VOLUME)
CRYSTAL CASES

NOT ERGODIC IN STRICT SENSE
(SYSTEM WON'T SHIFT IN POS.
AND ROTATE)
=> SOLUTION: RESTRICT PHASE
SPACE AND THEORIES TO
THE SYSTEM (HERE CRYSTAL)