Review of Kinematics

→ expanded u, θ, p into mean + fluctuating components

$$u \cdot \nabla u \qquad u \cdot \nabla \theta$$

Reynolds stress $\tau_{ij} = -\rho \overline{u_i u_j}$ [N/m^2] Pa

turb heat flux $J_2 = -\rho C_p \overline{u_j \theta}$ [W/m^2]

density, heat capacity

parameterizing, $-\rho \overline{u_i u_j} = \rho K_m \frac{\partial U_i}{\partial x_j}$

eddy viscosity

$$-\rho C_p \overline{u_j \theta} = \rho K_h \frac{\partial \Theta}{\partial x_j}$$

eddy diffusivity

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[(K_m + \nu)\frac{\partial U_i}{\partial x_j}\right]$$

$$\frac{\partial \Theta}{\partial t} + U_j \frac{\partial \Theta}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(K_h + \gamma)\frac{\partial \Theta}{\partial x_j}\right]$$

ν, γ are properties of fluid → $\nu(p,\theta), \gamma(p,\theta)$

$\left.\begin{array}{l} K_m = K_m(x,y,z,t) \\ K_h = K_h(x,y,z,t) \end{array}\right\}$ properties of flow

$K_h = K_h(x, y, z, t)$

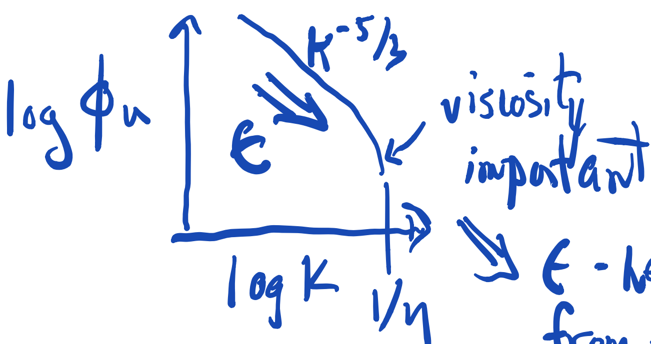
<u>Energetics</u> → Ch. 3. - gain insight by scaling

ϵ - viscous rate of dissipation of tke

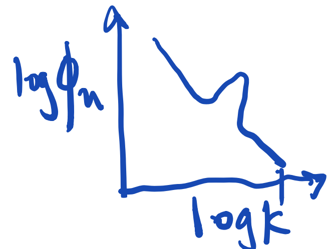
- rate at which energy is lost to heat

at small scales where motion is viscous $R_\eta = \frac{v\eta}{\nu} \approx 1$

heat lost to viscosity ~~large~~ velocity gradients (strain rates) at small scales.



ϵ - heat loss from energy in the flow



ϵ must also be the energy transfer rate down the spectrum to feed heat loss at small scales

<u>Dissipative heating</u> → ϵ is lost to heat

$$\rho C_p \frac{\partial \theta}{\partial t} = \frac{\partial J_q}{\partial x_j} + \underbrace{\rho \epsilon}_{\text{source term}}$$

$\frac{\epsilon}{C_p}$ is rate of change of θ units of ϵ $\left[\frac{m^2}{s^3} = \frac{W}{kg}\right]$

$\frac{\partial KE}{\partial t}$

ocean thermocline

$\epsilon \sim 10^{-9}\, m^2/s^3$

water $c_p \sim 4\times10^3 \frac{J}{kg\,K}$ $\rho \sim 1000 \frac{kg}{m^3}$

$\epsilon / c_p \sim .01\ K/1000y$ ← small cf. all else.

- contributes to tropical cyclone dynamics (TC) via intense turbulence in atmospheric boundary layer heat outer core of TC, $\uparrow \partial\theta/\partial x$, $\therefore \partial p/\partial x$ $\therefore$ winds by ~ 20%
- breaking of surface gravity waves - surf zone dissipative heating contribute $O(10)\ W/m^2$

<u>Inviscide estimate of ϵ</u> $\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$ $\frac{u}{T} \sim \frac{u^2}{l}$

KE of large eddy scales $\sim u^2$

$T \sim l/u$ ← eddy turnover time scale

also time scale for energy transfer to smaller scales

rate of energy supplied to small scales $\sim \frac{u^2}{T} = \frac{u^2}{l/u} = \frac{u^3}{l}$

$$\boxed{\epsilon = u^3/l}$$ ← energy transfer rate down the spectrum

get a time scale for viscous decay by

$$\frac{\partial u_i}{\partial t} = \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \Rightarrow \frac{u}{\tau} \sim \nu\frac{u}{l^2} \Rightarrow \boxed{\tau \sim l^2/\nu}$$

time scale for

viscous decay at length scale ℓ

$$\frac{u^2}{\tau} \sim \frac{u^2}{\ell^2/\nu} = \frac{\nu u^2}{\ell^2}$$ ← energy lost to heat at ℓ (large eddies)

$$(\nu u^2/\ell^2)/(u^3/\ell) = \frac{1}{Re_\ell}$$

$$\frac{\nu u^2}{\ell^2} \ll u^3/\ell$$ when Re_ℓ is large

The energy lost from large scales does not go directly to heat → goes exciting smaller scales

Suppose we introduce δ s.t. $\underset{NL}{\frac{u^2}{\ell}} \sim \underset{\text{viscous}}{\frac{\nu u}{\delta^2}}$

$$\frac{\delta}{\ell} \sim Re_\delta^{-1} \sim Re_\ell^{-1/2}$$

that is, viscous forces at small scales balance NL forces at large scales

but scale separation is large & viscous forces only important at small scale

time scale δ^2/ν is the correct time scale for viscous decay of turbulence

$$\epsilon \sim \frac{u^2}{\delta^2/\nu} \sim \frac{\nu u^2}{\delta^2}$$ ← viscous estimate for ϵ

$$\epsilon \sim u^3/\ell$$ ← inviscid estimate

$\delta \sim (\nu \ell)^{1/2}$: $\delta \sim (\nu\ell)^{1/2} (\epsilon)^{1/4} \sim Re_\ell^{1/4}$ with

$$l \gg \delta \gg \eta$$

$$\eta = \left(\nu^3/\epsilon\right)^{1/4} \qquad \tau = \left(\nu/\epsilon\right)^{1/2} \qquad v = (\nu\epsilon)^{1/4}$$

$$\eta/l \sim 1/Re_l^{3/4} \qquad v/u \sim 1/Re_l^{1/4}$$

$$\tau/(u/l) \sim 1/Re_l^{1/2}$$

$\therefore$ 1) length, time, velocity scales of smallest eddies $\ll$ those of largest eddies

2) Scale separation depends strongly on Re

Scaling example: ocean mixed layer $\epsilon \sim 10^{-6}\, m^2/s^3$
$l \sim 30\, m \quad \nu \sim 10^{-6}\, m^2/s$

then $\eta \simeq 0.001\, m$
$u \sim (\epsilon l)^{1/3} \sim 0.03\, m/s$

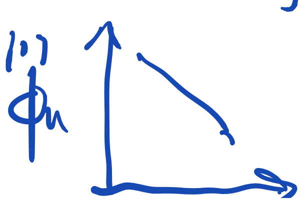
$$\eta/l \sim \frac{1}{30{,}000} \qquad Re = \frac{ul}{\nu} \sim 10^6$$

$$\#\text{ degrees of freedom} = \left(\frac{l}{\eta}\right)^3 = Re^{9/4} > 10^{13}$$

---

energy $\sim u^2$ vorticity $\sim 1/\text{time}$

$\therefore$ our scalings say
- vorticity at small scale $\gg$ vorticity of large scale
- energy of small scale $\ll$ energy at large scale



log k

Since time scales of small eddies small relative than of large eddies → they rapidly adjust to changes in mean flow ←

this leads to the notion of "local isotropy"

even though large scales may not be isotropic the small scales (in $k^{-5/3}$ range) may be

This is why high Re flow (or large scale separation) is needed to see $k^{-5/3}$.

Look at MKE equation

3 important simplifying assumptions

1. stationarity - invariant wrt time $\frac{\partial}{\partial t} = 0$
2. homogeneity - invariant wrt translations in space $\frac{\partial}{\partial x_j}(\;\;) = 0$, however $\overline{u_1^2} \neq \overline{u_2^2} \neq \overline{u_3^2}$, $\overline{u_1\theta} \neq \overline{u_2\theta} \neq \overline{u_3\theta}$
3. isotropy - invariant wrt rotation in 3D

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$$
$$\overline{u_1\theta} = \overline{u_2\theta} = \overline{u_3\theta}$$
$$\overline{\frac{\partial u_1}{\partial x_1}^2} = \overline{\frac{\partial u_2}{\partial x_2}^2} = \overline{\frac{\partial u_3}{\partial x_3}^2}$$

$$\overline{\frac{\partial u_1}{\partial x_2}^2} = \overline{\frac{\partial u_1}{\partial x_3}^3} = \ldots$$

flow may not be homogeneous or isotropic at large scale but we assume it is locally at ~~smallest~~ small scales these assumptions are violated by

Shear, Buoyancy, Boundary effects

MKE → start with momentum + continuity

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(T_{ij}/\rho\right) \quad ; \quad \frac{\partial U_i}{\partial x_i} = 0 \quad (1)$$

Stress tensor: $T_{ij} = -P\delta_{ij} + 2\mu S_{ij} - \rho\,\overline{u_i u_j}$

strain rate tensor: $S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$

$$MKE = \frac{1}{2} U_i U_i = \frac{1}{2}\left(U^2 + V^2 + W^2\right)$$

(1) × U_i

$$U_i \frac{\partial U_i}{\partial t} + U_i U_j \frac{\partial U_i}{\partial x_j} = U_i \frac{\partial}{\partial x_j}\left(T_{ij}/\rho\right)$$

rewrite as eq'n for $\frac{1}{2} U_i U_i$

$$U_i \frac{\partial U_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} U_i U_i\right)$$

$$U_i U_j \frac{\partial U_i}{\partial x_j} = U_j \frac{\partial}{\partial x_j}\left(\frac{1}{2} U_i U_i\right) - U_i U_i \cancelto{0}{\frac{\partial U_j}{\partial x_j}}$$

$$U_i \frac{\partial}{\partial x_j}\left(T_{ij}/\rho\right) = \frac{\partial}{\partial x_j}\left(U_i T_{ij}/\rho\right) - T_{ij}/\rho \frac{\partial U_i}{\partial x_j}$$

$$T_{ij} \frac{\partial U_i}{\partial x_j} = T_{ij} S_{ij} \qquad \frac{\partial U_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)}_{S_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)}_{A_{ij}}$$

symmetric tensor

anti-symm tensor

$$T_{ij}\frac{\partial U_i}{\partial x_j} = T_{ij}S_{ij} + \cancel{T_{ij}A_{ij}} \to 0$$

<u>MKE</u> $\to \underbrace{\rho\frac{\partial}{\partial t}\left(\tfrac{1}{2}U_iU_i\right)}_{\text{I}} + \underbrace{\rho U_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}U_iU_i\right)}_{\text{II}} = \underbrace{\frac{\partial}{\partial x_j}\left(T_{ij}U_i\right)}_{\text{III}} - \underbrace{T_{ij}S_{ij}}_{\text{IV}}$

I - time rate of change of MKE

II - advection of MKE by mean flow

III - divergence term

IV - deformation work

Gauss' theorem $\int_V \frac{\partial}{\partial x_j}(T_{ij}U_i)\,dV = \int_S n_j T_{ij} U_i \, dS$.

change in volume   flux thru surface

if flux thru $S = 0$, then the effect of

$\frac{\partial}{\partial x_j}(T_{ij}U_i) = 0$ ← <u>divergence term</u> are referred to as <u>transport terms</u>

primary function → move energy around within V but do not act as sources/sinks.

they neither create nor destroy MKE

only $T_{ij} S_{ij}$ changes MKE

$$T_{ij} S_{ij} = -P\delta_{ij} S_{ij} + 2\mu S_{ij} S_{ij} - \rho \overline{u_i u_j} S_{ij}$$

* pressure term : $-P\delta_{ij} S_{ij} = -P S_{ii} = -P \frac{\partial U_i}{\partial x_i} = 0$

for an incompressible fluid the pressure term does no work

* viscous term : $S_{ij} S_{ij} > 0$ $\qquad T_{ij} S_{ij}$ $\quad \frac{\partial MKE}{\partial t} \propto -T_{ij} S_{ij}$

then this must be a loss of energy MKE
viscous dissipation of MKE
we anticipate this small relative to dissipation of TKE

* Reynolds Stress: also represents loss of MKE

recall $\overline{u_i u_j} < 0$ when $\frac{\partial U_i}{\partial x_j} > 0$

so generally $\overline{u_i u_j} S_{ij} < 0$

represents interaction of turbulence w mean flow
and is a loss term in MKE

↳ where does it go?

$$\mathcal{P} = \rho \overline{u_i u_j} S_{ij} \text{ , TKE production}$$