Turbulence Spectra – cont'd

we frequently infer wavenumber spectra from frequency spectra.

→ development of the universal spectra is in terms of k.

Taylor's ff hypothesis $f = kU$

correlation tensor:

$$R_{ij}(\underset{\sim}{r}) = \overline{u_i(\underset{\sim}{x},t)\,u_j(\underset{\sim}{x}+\underset{\sim}{r},t)}$$

if turbulence is homogeneous + isotropic, R_{ij} is a function of separation only (not a function of $\underset{\sim}{x}$ or t)

spectrum tensor ϕ_{ij} is the Fourier transform of R_{ij}

definitions

$$\phi_{ij}(\underset{\sim}{k}) \equiv \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{r})\,R_{ij}(\underset{\sim}{r})\,d\underset{\sim}{r}$$

$$R_{ij}(\underset{\sim}{r}) \equiv \iiint_{-\infty}^{\infty} \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r})\,\phi_{ij}(\underset{\sim}{k})\,d\underset{\sim}{k}$$

We are most interested in the sum of diagonal components $\phi_{ii} = \phi_{11} + \phi_{22} + \phi_{33}$ which represents the tke at some k.

off diagonal components $i \neq j \Rightarrow$ cospectra between u, v or v, w

shear stress →

or Re stress co-spectra.

momentum flux co-spectra.

$$R_{ii}(0) = \overline{u_i u_i} = 3u^2 = \iiint_{-\infty}^{\infty} \phi_{ii}(\underset{\sim}{k})\, d\underset{\sim}{k}$$

$\phi_{ii}(\underset{\sim}{k})$ is a directional waveno. spectrum

amplitude spectrum $k^2 = \underset{\sim}{k} \cdot \underset{\sim}{k}$

$\int_0^\infty E(k)\, dk$ = total tke

turb. vel. scale

$$\int_0^\infty E(k)\, dk = \frac{1}{2} \int_0^\infty \left[\oiint \phi_{ii}(\underset{\sim}{k})\, d\sigma \right] dk = \frac{1}{2} \overline{u_i u_i} \sim \frac{3}{2} u^2$$

Common 1D Spectra

$R_{11}(r, 0, 0)$ → longitudinal or downstream correlation in downstream direction

geometry →

this geometry achieved using 2 probes and changing r

Alternatively if $r = Ut$, move a single probe through r at speed U.

$R_{22}(r,0,0) \rightarrow$ transverse or cross-stream correlation in downstream direction

geometry 2 $\Rightarrow$

corresponding 1D spectra: F_{11}, F_{22}.

$$R_{11}(r,0,0) \equiv \int_{-\infty}^{\infty} \exp(ik_1 r) F_{11}(k_1)\, dk_1$$

$$R_{22}(r,0,0) \equiv \int_{-\infty}^{\infty} \exp(ik_1 r) F_{22}(k_1)\, dk_1$$

$$R_{11}(r,0,0) = \int_{-\infty}^{\infty} \exp(ik_1 r)\left(\iint \Phi_{11}(\underset{\sim}{k})\, dk_2\, dk_3\right) dk_1$$

$$F_{11}(k_1) = \iint_{-\infty}^{\infty} \Phi_{11}(\underset{\sim}{k})\, dk_2\, dk_3$$

Isotropic Relations

if turbulence is isotropic, at least at large k then there are simplified relations between $E(k)$, $F_{11}(k_1)$, $F_{22}(k_1)$

derived in Batchelor Ch3 Theory of Homogeneous Turbulence.

Mathieu & Scott p274

if $E(k) \propto k^{-5/3}$, then $F_{11} \propto k_1^{-5/3}$

$F_{22} \propto k_1^{-5/3}$

$$\frac{F_{22}(k_1)}{F_{11}(k_1)} = \frac{4}{3}$$ in the inertial subrange

this is considered a strong test of isotropy

Kolmogorov's Theories of the Spectrum of Turbulence

1) Spectral forms predicted via dimensional reasoning
2) Spectrum of TKE
3) spectrum of dissipation, or strain rate $D(k)$
4) Spectrum of $\overline{\theta^2}(k)$ ← Batchelor

large eddies produce successively smaller eddies through vortex stretching → NL term in momentum eq.

∴ net energy transfer to small scales where ν is important

high-k time scales are short

$$D(k) = k^2 E(k) \propto k^{+1/3}, \quad I(k) \propto k^{-1/3}$$

short time scales at small spatial scales

the greater the separation in spatial scales,
the greater the separation in time scales

→ if Re is sufficiently large, there ought to be a range of high k whose time scales are v. small compared to energy-containing scales

→ these will adjust rapidly to changes in external flow conditions

recall $l/\eta \propto R_l^{3/4}$

the requirement for local isotropy is $l \gg \eta$
→ large R_l

This leads to Kolmogorov's 1st hypothesis:

"At sufficiently high values of R_l there is a range of k where the turbulence is statistically in equilibrium & determined by ϵ, ν. This state is universal."

We can separate out the viscous-dominated scales by restricting k range. At $k < \sim k_s/20$, ν is not so important. Further if this range is separated from energy-containing scales, then all that matters is ϵ.

consider $\quad \kappa_\ell \ll \kappa \ll \kappa_s$

$$\ell \gg 2\pi/\kappa \gg \eta$$

the turbulence in this range is independent of both energy-containing eddies & those dominated by ν

This leads to Kolmogorov's 2nd hypothesis:

"If Re is infinitely large, the energy spectrum in the intermediate subrange satisfying $\kappa_\ell \ll \kappa \ll \kappa_s$ is independent of ν & determined solely by ϵ"

This range is the inertial subrange

this is where inertial energy transfers dominate

Spectral Forms

① $\kappa_\ell \lll \kappa_s \qquad E = E(\kappa, \epsilon, \nu)$

the only ND form is

$$\frac{E(\kappa)}{(\epsilon\nu^5)^{1/4}} = \frac{E(\kappa)}{v^2\eta} = f(\kappa/\kappa_s)$$

$$v = (\nu\epsilon)^{1/4} \qquad \kappa_s = (\epsilon/\nu^3)^{1/4} = \eta^{-1}$$

$f(k/k_s)$ is some universal form

② Inertial Subrange $\quad (k_\ell \ll k \ll k_s)$

$$E = E(k, \epsilon)$$

$$E(k) = \alpha\, \epsilon^{2/3} k^{-5/3} \qquad \alpha = 1.5, \text{ experiments}$$

then $f(k/k_s) = \alpha (k/k_s)^{-5/3}$ in the inertial subrange.

The greater the scale separation between ℓ & η then the greater is the wavenumber extent of the inertial subrange

Effects of Stratification on Turbulence Spectra

Gargett et al 1984 — J. Fluid Mech

buoyancy length scale $\ell_b = (\epsilon/N^3)^{1/2}$, $k_b = 2\pi/\ell_b$

if ℓ_b is the appropriate energy-containing length scale, then $\epsilon \sim u^3/\ell_b \Rightarrow u_b = (\epsilon/N)^{1/2}$

Can also define a buoyancy Reynolds #

$$R_b = \frac{u_b \ell_b}{\nu} = \epsilon/\nu N^2$$

Can use this to classify flows in terms of weak / strong influence by stratification, N^2

GON84 define an isotropy index $I = \frac{k_s}{k_b} = \frac{l_b}{l_s} \propto R_b^{3/4}$

k_s – Kolmogorov, $l_s = 2\pi/k_s$
(η)

large values of R_b or I imply large scale separation between smallest (dissipation) scales & buoyancy-modified scales (energy-containing scales)

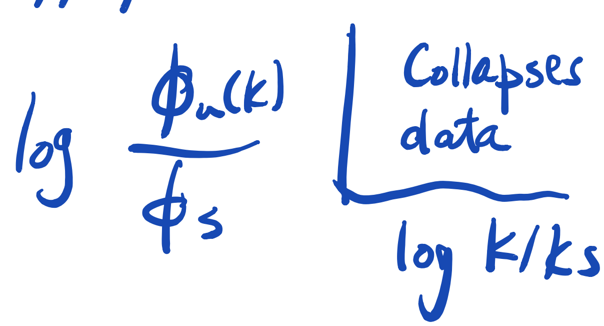
Gon84 look at local isotropy (at high k) in a range of stratification conditions

local isotropy requires → large R_b, I

predicts: $\phi_{11}(k_1) = A_1 \epsilon^{2/3} k^{-5/3}$

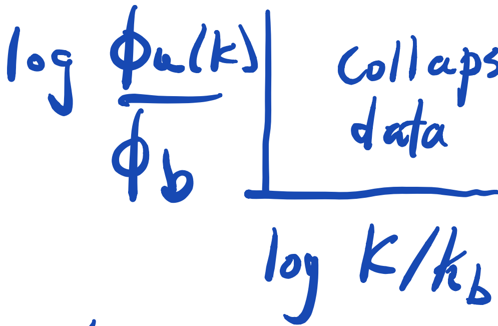
$A_2/A_1 = 4/3 \Leftarrow \phi_{22}(k_1) = A_2 \epsilon^{2/3} k^{-5/3}$

in inertial subrange $\phi_s = (\epsilon \nu^5)^{1/4}$ is the appropriate non-dimensionalization for $\phi_u(k)$



in buoyancy-modified range,

dimensionally $\phi_b = \frac{u_b^2}{l_b} = \frac{1}{\sqrt{2\pi}}\, \epsilon/N$



$\Leftarrow$ buoyancy modified turbulence.

Discuss GON84