Lecture 4: MKE

$$\frac{\partial}{\partial t}\overline{\left(\tfrac{1}{2}U_iU_i\right)} + U_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}U_iU_i\right)$$

$$= \frac{\partial}{\partial x_j}\Big(\underbrace{-\frac{P}{\rho}U_j}_{\text{pressure work}} + \underbrace{2\nu U_i S_{ij}}_{\text{transport by viscous stresses}} - \underbrace{\overline{u_iu_j}\,U_i}_{\text{transpor by Reynolds stresses}}\Big) - \underbrace{2\nu S_{ij}S_{ij}}_{\text{viscous dissipation}} + \underbrace{\overline{u_iu_j}S_{ij}}_{P}$$

Scaling: $\dfrac{\partial U_i}{\partial x_j} \sim \dfrac{u}{\ell}$, $S_{ij} \sim u/\ell$

$\overline{u_iu_j} \sim u^2$

Compare viscous terms:

$$P = \overline{u_iu_j}\,S_{ij} \sim u^2\cdot u/\ell \sim u^3/\ell \sim u\ell S_{ij}S_{ij}$$

$$\frac{2\nu S_{ij}S_{ij}}{u\ell S_{ij}S_{ij}} = \frac{\text{dissipation at }\ell}{P} = \frac{2}{Re_\ell}$$

so diss $\ll P$ when Re_ℓ is high

$$\frac{\text{viscous transport}}{\text{turb transport}} = \frac{2\nu U_iS_{ij}}{\overline{u_iu_j}\,U_i}$$

$\overline{u_iu_j}\,U_i \sim u^2U_i$
$= u\ell\, U_iS_{ij}$

$$\sim \frac{2\nu U_iS_{ij}}{u\ell\, U_iS_{ij}} = \frac{2}{Re_\ell}$$

$\Rightarrow$ by scaling, we conclude ℓ eddies not directly affected by ν

Prescription for TKE: $\begin{cases} \cdot\ v_i \times \text{momentum eq} \\ \cdot\ \text{time average} \\ \cdot\ \text{subtract MKE} \end{cases}$

define $q^2 = \frac{1}{2} u_i u_i = \frac{1}{2}(u^2+v^2+w^2) = \frac{1}{2}(u_1^2+u_2^2+u_3^2)$

$\uparrow$ tke

TKE

$$\frac{\partial}{\partial t}\overline{q^2} + U_j \frac{\partial}{\partial x_j}\overline{q^2} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_i p} + \frac{1}{2}\overline{u_j q^2} - 2\nu\overline{u_i \Delta_{ij}}\right)$$
$$\underbrace{-\overline{u_i u_j}\, S_{ij}}_{P} \underbrace{- 2\nu\overline{\Delta_{ij}\Delta_{ij}}}_{\epsilon}$$

Quick look:

1) LHS - time + advective rates of change of $\overline{q^2}$
2) 1st 3 terms on RHS $\rightarrow$ transport terms
$\rightarrow$ not sources / sinks
3) last 2 terms are sources / sinks $\rightarrow$ buoyancy term.

P $\overline{u_i u_j} S_{ij}$ is a source of tke
and it comes from MKE

$\epsilon = 2\nu\overline{\Delta_{ij}\Delta_{ij}}$, since $\overline{\Delta_{ij}\Delta_{ij}} > 0$, $\epsilon > 0$

ϵ must a sink of tke $\rightarrow$ heat

while we neglect $2\nu S_{ij} S_{ij}$ in MKE, ϵ is critical to the dynamics of turbulence if sources are removed, turbulence decays via ϵ

<u>Closer look at tke:</u>

consider steady, homogeneous shear flow. $S_{ij}=0$

$\int dV$ so ignore effects of transport (divergence) terms:

only sources/sinks; $-\overline{u_i u_j}\, S_{ij} = 2\nu \overline{s_{ij} s_{ij}}$

$$P = \epsilon$$

$S_{ij} \sim u/l \qquad \overline{u_i u_j} \sim u^2$

$\overline{u_i u_j}\, S_{ij} \sim u^2 \cdot u/l \sim u\,l\, S_{ij} S_{ij}$

$$\frac{P}{\epsilon} \sim \frac{u\,l\, S_{ij} S_{ij}}{\cancel{2}\nu\, \overline{s_{ij} s_{ij}}} = R_l \frac{S_{ij} S_{ij}}{\overline{s_{ij} s_{ij}}}$$

if $P=\epsilon$ $\quad \overline{s_{ij} s_{ij}} \gg S_{ij} S_{ij}$ $\qquad S_{ij}$ - units s^{-1}

fluctuating strain rates $\gg$ mean strain rates

$$T_{\text{small eddies}} \ll T_{\text{energy-containing eddies}} \to l$$

if $P = \epsilon$ and $P \sim u^3/l$, then $\epsilon \sim u^3/l$

or another way to think about this ...

time scale for viscous decay

$\frac{TKE}{\epsilon} \sim \frac{u^2}{\epsilon}$ | we previously said l/u was this time scale

$\frac{u^2}{\epsilon} \sim \frac{l}{u} \Rightarrow \epsilon \sim u^3/l$

Since $T_{\text{small scales}} \ll T_{\text{large eddies}}$, the small scales can evolve rapidly & independent of l

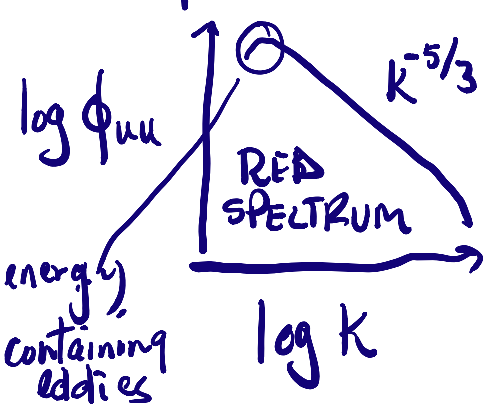
they don't strongly interact as they are well separated in k, f

length scale → k; timescale → f

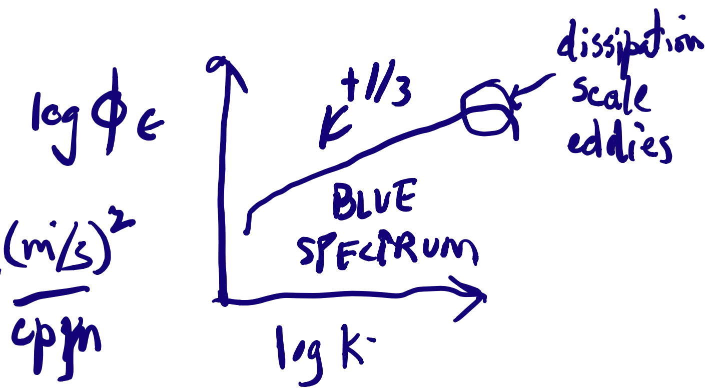
the operation of $\partial/\partial x$ translates to k^2 in Fourier space

$\phi_{\Delta ij} \sim k^2 \phi_u$

Energy Spectrum
(spectrum of velocity



$\phi_{uu} \frac{(m/s)^2}{cpm}$



the larger the Re, the greater is the separation of scales between the peak of ϕ_u & the peak of ϕ_ϵ

$\phi_{\Delta ij}, \phi_{\partial u_i/\partial x_j}$

$\phi_u, \phi_{uu} \rightarrow \frac{m^2/s^2}{cpm}$ | $\phi_{\Delta ij} \rightarrow s^{-2}/cpm$

$\int \phi_{\Delta ij}\, dk = \overline{\Delta_{ij}}$

$\int \Phi_u \, dk = u^2$ | $\int \Phi_{ij} \, dk = ...$

$\nu \, s_{ij} s_{ij} = \frac{m^2}{s} s^{-2} \left(\frac{m^2}{s^3}\right)$

$u \rightarrow$ rms velocity

$u \rightarrow$ scaling length scale

<u>Scale s_{ij}</u> $\overline{s_{ij} s_{ij}} >> S_{ij} S_{ij}$

length Scale to scale s_{ij} must not be l

define Taylor microscale $s_{ij} \sim u/\lambda$ $\leftarrow \delta$

$\epsilon \sim \nu u^2/\lambda^2$ $\leftarrow$ viscous scaling of ϵ

compare $\epsilon \sim u^3/l$

λ not characteristic length scale for dissipation $\rightarrow \eta$

u is not proper length scale for dissipation

u/λ to scale s_{ij} only $\rightarrow \lambda/u$ is the time scale of strain rate fluctuations

$\eta << \lambda << l$

<u>Practical Assessment</u>

$\epsilon = 2\nu \overline{s_{ij} s_{ij}}$ is a 12-term scalar.

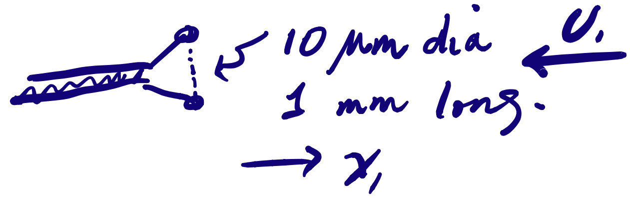
$\overline{(\partial u_1/\partial x_1)^2}$, $\overline{(\partial u_1/\partial x_3)^2}$, $\overline{\frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1}}$

assume homogeneity & isotropy at small scales ref Taylor 1935 "Statistical theory of turbulence" Proc Roy Soc.

Hinze p219

these weird terms are related thru homo. & isotr.

① hot wire anemometer



time series of $u_1(t)$

if we assume that turbulent eddies are advected by mean flow without distortion,

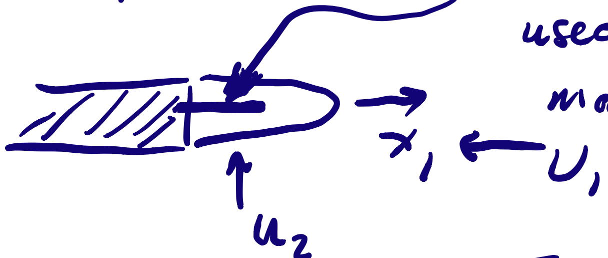
then $x_1 = U_1 t$ ← Taylor's frozen flow hypothesis

$u_1(x_1)$, $\frac{\partial u_1}{\partial x_1} = \frac{1}{U_1}\frac{\partial u_1}{\partial t}$

equivalent to $f = U_1 K_1$ → $\epsilon \simeq 15\nu\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}$

② airfoil probe — piezoceramic beam

used in profiling / towed mode where U_1 is induced



$\frac{\partial u_2}{\partial x_1}$ → $\epsilon \simeq \frac{15}{2}\nu\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2}$

③ PIV particle image velocimetry

④ structure function

$$D(x,r) = \overline{[u(x) - u(x+r)]^2} \leftarrow \text{reference.}$$

is mean square of u between 2 points separated by r.

Kolmogorov's original works done as structure functions

$$D(x,r) = C_r^2 \epsilon^{2/3} r^{2/3} \leftarrow$$ reinterpreted in Fourier space $\propto k^{-5/3}$

Scale TKE transport terms

- $-\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j p}\right) \sim \frac{1}{l}\frac{1}{\rho}u \cdot \rho u^2 \sim u^3/l$ $\qquad p \sim \rho u^2$ $\quad$ ✓
- $-\frac{\partial}{\partial x_j}\left(\overline{u_j q^2}\right) \sim u^3/l$ $\quad$ ✓
- $2\nu \frac{\partial}{\partial x_j}\overline{u_i A_{ij}} \sim \frac{\nu}{l} \cdot u \cdot \frac{u}{\lambda} \sim \frac{1}{R_\lambda} \cdot u^3/l \leftarrow$ small

typically ignore transport terms.

Caveat - a point measurement is not a volume average.

Role of Pressure as a Redistribution Term

$$-\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j p}\right) = \overline{\frac{u_j}{\rho}\frac{\partial p}{\partial x_j}} + \overline{\frac{p}{\rho}\frac{\partial u_j}{\partial x_j}}$$

assume $\epsilon/3$ is lost equally in each component

$$u_1: \frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{u_1^2}\right) + U_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\overline{u_1^2}\right) = -\overline{u_1 u_j}\frac{\partial U_1}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\overline{u_j u_1^2}}{2}$$
$$-\overline{u_1\frac{\partial}{\partial x_1}(p/\rho)} + \overline{\frac{p}{\rho}\frac{\partial u_1}{\partial x_1}} - \epsilon/3$$

$$u_2: \frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{u_2^2}\right) + U_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\overline{u_2^2}\right) = -\overline{u_2 u_j}\frac{\partial U_2}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\overline{u_j u_2^2}}{2}$$
$$-\overline{u_2\frac{\partial}{\partial x_2}\frac{p}{\rho}} + \overline{\frac{p}{\rho}\frac{\partial u_2}{\partial x_2}} - \epsilon/3$$

$$u_3: \frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{u_3^2}\right) + U_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\overline{u_3^2}\right) = \;? \quad -\overline{u_3 u_j}\frac{\partial U_3}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{\overline{u_j u_3^2}}{2}\right)$$
$$-\overline{u_3\frac{\partial}{\partial x_3}(p/\rho)} + \overline{\frac{p}{\rho}\frac{\partial u_3}{\partial x_3}} - \epsilon/3$$

Consider a simple shear flow. only $\frac{\partial U_1}{\partial x_3} \neq 0$

$$\frac{\partial}{\partial x_1} = 0, \quad \frac{\partial}{\partial x_2} = 0 \quad \frac{\partial}{\partial t} = 0$$

if Re_ℓ is large. the eq'n becomes

$$0 = -\overline{u_1 u_3}\frac{\partial U_1}{\partial x_3} - \frac{\partial}{\partial x_3}\left(\frac{1}{\rho}\overline{u_3 p} + \frac{1}{2}\overline{u_i u_i u_3}\right) - \epsilon$$

$$1: 0 = -\overline{u_1 u_3}\frac{\partial U_1}{\partial x_3} + \frac{1}{\rho}\overline{p\frac{\partial u_1}{\partial x_1}} - \frac{\partial}{\partial x_3}\left(\frac{1}{2}\overline{u_1^2 u_3}\right) - \epsilon/3$$

$$2: 0 = +\frac{1}{\rho}\overline{p\frac{\partial u_2}{\partial x_2}} - \frac{\partial}{\partial x_3}\left(\frac{1}{2}\overline{u_2^2 u_3}\right) - \epsilon/3$$

$$3: 0 = +\frac{1}{\rho}\overline{p\frac{\partial u_3}{\partial x_3}} - \frac{\partial}{\partial x_3}\left(\frac{1}{2}\overline{u_3^2 u_3}\right) - \epsilon/3$$

P only appears in u_1 component eq'n

how does it transformed to u_2, u_3

since $\frac{\partial u_i}{\partial x_i} = 0$ and only u_1^2 is created by P

then expect $\frac{\partial u_1}{\partial x_1} < 0$ so $\overline{u_1^2}$ is lost

$\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} > 0$, $\overline{u_2^2}, \overline{u_3^2}$ is generated.

this means that $\frac{\partial u_1}{\partial x_1} \neq \partial u_2/\partial x_2, \partial u_3/\partial x_3$

So not isotropic at least at large scales

measurements in shear flows → $\overline{u_1^2} \sim 2(\overline{u_2^2}, \overline{u_3^2})$