Helmholtz's Theorems:

1. Γ = const. along vortex tube
2. Vortex tube can't end in a fluid (extend to boundary or form ~~closed~~ closed loop) (inviscid flow)
3. irrot. fluid remains irrot. if no viscous forces or other external forces.

Kelvin's Circulation Theorem: $\frac{D\Gamma}{Dt} = 0$

- $\rho = f(P)$ ~~$\rho = f(T)$~~ - barotropic P-ρ relationship
- inviscid
- no nonconservative bodyforces - Coriolis

Γ - around a mat'l contour (follow a contour in space/time by tracking fluid motion)

$\Gamma = \oint u \cdot dl$

$$\frac{D\Gamma}{Dt} = \oint \frac{Du}{Dt} \cdot ds + \oint u \cdot \frac{Dds}{Dt}$$

use N-S eqn. (no visc. term) → conv. to area integral $\int_A \underbrace{\left(\nabla \times \left(\frac{\nabla P}{\rho} + \nabla \overset{\text{body force}}{\Phi}\right)\right)}_{= 0} \cdot \hat{n}\, dA$

2nd Term: $\frac{Dds}{Dt}$ = (change of mat'l line element)

$$= (ds \cdot \nabla) u$$

so $\oint u \cdot \frac{Dds}{Dt} = \oint u \cdot (ds \cdot \nabla) u = \frac{1}{2} \oint \nabla(u^2) \cdot ds \xrightarrow{\text{apply Stokes}} \frac{1}{2} \int_A \nabla \times \nabla(u^2) \cdot dA$

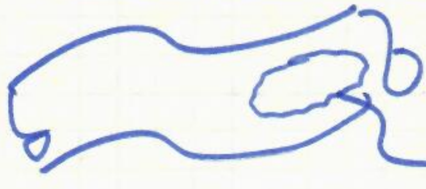
curl of grad = 0

so $\frac{D\Gamma}{Dt} = 0$

if viscous $\frac{D\Gamma}{Dt} = \oint \nu \frac{\partial u_i}{\partial x_j \partial x_j} ds$

FLUID in a moving vortex moves with the vortex

Why Γ const. along vortex tube



PATCH OF side wall of tube

- over this area $\Gamma = 0$ since $\omega_n = 0$

So any Γ inside tube can not be increased/decreased due to flux at the walls.

We know $\nabla \cdot \omega = 0$ (vorticity is divergence free)

Divergence Theorem: $\int_{\mathcal{V}} \nabla \cdot \omega \, d\mathcal{V} = \int_A \omega \cdot dA = 0$

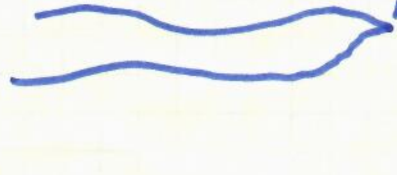
A surrounds $\mathcal{V}$

$\omega \cdot dA = 0$ on outside surface area

Left with A_1 & A_2 and $\rightarrow \omega_1 A_1 = \omega_2 A_2$

The magnitude of the "intensity" of the vorticity remains constant

Why vortex tubes can not end in the middle of a fluid:



end $\rightarrow A_2 \rightarrow 0$

so $\omega \rightarrow \infty$

Can Vortex tube end on a boundary?

FLUX of Vorticity thru any $\dot{A}_x$ is constant

$\Gamma \rightarrow$ "intensity":

if ends on solid boundaries:

$\omega = 0$ if no slip $\left\{ \begin{array}{l} \therefore \omega \text{ is tangent} \\ \text{to the surface} \end{array} \right.$

BUT if there is slip then we can have finite ω_z.

∴ 1. NO-slip condition: Vortex tubes can not end on surface

2. For inviscid flows vortex tubes can end on surface.