Problem 2.3

Char Vel = u $T_p > T_a$ $l_p << l$

Char length = l $\Delta p = 0$ $\eta << l_p$

estimate the rate of spreading of the patch of hot fluid [A] & the rate at which the maximum temp difference decreases [B]

Recall from chapter 1 that

$$\frac{dr}{dt}(r) = \underbrace{\vec{V}(r) = [r\epsilon]^{1/3}}$$

$$\int_0^t \epsilon^{1/3} dt = \int_{r_i}^{r} r^{-1/3} dr$$

$$t\epsilon^{1/3} = \frac{3r^{2/3}}{2} - \frac{3r_i^{2/3}}{2}$$

$$r(t) = \left[\frac{2t\epsilon^{1/3}}{3} + r_i^{2/3}\right]^{3/2}$$

Recall that $\epsilon = \frac{u^3}{l}$ so

$$\boxed{r(t) = \left[\frac{2t}{3}\left[\frac{u}{l^{1/3}}\right] + r_i^{2/3}\right]^{3/2}}$$ [A]

and

$$\boxed{\frac{dr}{dt}(t) = \frac{u}{l^{1/3}}\left[\frac{2ut}{3l^{1/3}} + r_i^{2/3}\right]^{1/2}}$$

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Problem 2.3 continued

Let $\theta = T_p - T_a$ using eddy diffusivity means we have

$$\frac{\partial\theta}{\partial t} = K\frac{\partial^2\theta}{\partial x_i\partial x_i} = \frac{\partial^2\theta}{\partial x_i\partial x_i}\cdot[u\,l]$$

We can go to the expression,

$$\frac{d(V\cdot\theta)}{dt} = 0.$$

Recall that from the previous Part, $V \sim r(t)^3$ so we can say $\frac{d[r(t)^3\,\theta]}{dt} = 0$

where

$$\frac{d[r^3\theta]}{dt} = 3r^2\theta\frac{dr}{dt} + r^3\frac{d\theta}{dt}$$

$$\to \frac{3\,dr}{r} = -\frac{d\theta}{\theta} \xrightarrow{\sim} \frac{r(t)^3}{r_i^3} = \frac{\theta_i}{\theta}$$

recall from before, $r = \left[\frac{2t}{3}\left[\frac{u}{l^{1/3}}\right] + r_i^{2/3}\right]^{3/2}$

so $$\boxed{\theta(t) = \frac{\theta_i r_i^3}{\left[\frac{2t}{3}\left[\frac{u}{l^{1/3}}\right] + r_i^{2/3}\right]^{9/2}}}$$ B

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Problem 3.1

Estimate the characteristic velocity of eddies w/ size equal to the Taylor microscale λ. Show these eddies contribute little to total dissipation rate. A B

from P1.3, Recall

$$V_r = \epsilon^{1/3} r^{1/3} \quad \& \quad t_r = \epsilon^{-1/3} r^{2/3}$$

So,

$$V_\lambda \sim \epsilon^{1/3} \lambda^{1/3} \quad \& \quad t_\lambda = \epsilon^{-1/3} \lambda^{2/3}$$

if we include $\epsilon \sim u^3/l$ then we get

$$V_\lambda \sim u \frac{\lambda^{1/3}}{l^{1/3}} \sim \boxed{u\left[\lambda/l\right]^{1/3}} \quad \text{or} \quad \text{w/ } l/\lambda \sim Re^{1/2}$$

$$\boxed{\sim u\, Re^{-1/6}} \quad \boxed{A}$$

Now Recall $\epsilon \sim \nu \overline{(\partial u_i/\partial x_i)^2} \sim \nu (u^2/\lambda^2)$ (total) so for V_λ we'll have

$$\epsilon_\lambda \sim \frac{V_\lambda^2}{\lambda^2}(\nu) \quad \text{or} \sim \frac{\epsilon}{Re_l^{1/3}}$$

So if Re_l is large

$$\frac{\epsilon_T}{\epsilon_\lambda} \sim Re_l^{1/3} \text{ will be large so}$$

$$\boxed{\epsilon_T \gg \epsilon_\lambda} \; : \quad \boxed{B}$$

Problem 3.2

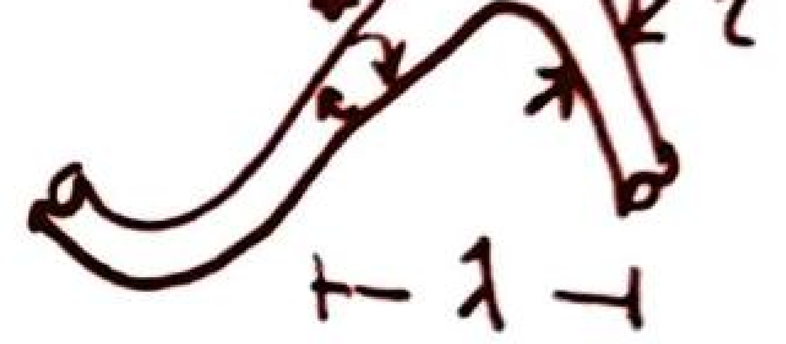
Use vortex tubes w/ dia = η and $v = \left[\frac{1}{3}\overline{u_i u_i}\right]^{1/2}$ to model dissapation rate what is the volume fraction occupied by these tubes? Verify if (3.3.42) holds for these tubes

Note: Eq 3.3.42 is $\omega_i \omega_i s_{ij} = \nu \overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}}$

Vortex tubes are of size η and are stretched by eddies of size λ. Estimate ϵ for tubes + use it to estimate a volume averaged ϵ

from our notes

$$\omega_i \omega_i s_{ij} \sim \cancel{\frac{u^3}{\lambda^3}} \sim \frac{u^3}{\eta^2 \lambda}$$



and $\nu \overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}} \sim \nu \frac{u^2}{\lambda^2 \delta^2}$ w/ $\delta \sim \eta$

The portion of volume the tube takes up is sim to

$$V_{Tube} \sim \eta^2 \lambda \quad \text{or} \quad \boxed{\frac{V_{Tube}}{V_{domain}} \sim \frac{\eta^2 \lambda}{\lambda^3} \sim \frac{\eta^2}{\lambda^2}}$$

so $\frac{u^3}{\lambda \eta^2} \sim \frac{u^2 \nu}{\lambda^2 \eta^2} \rightarrow \frac{u \eta^2}{\nu \lambda} \rightarrow \frac{\eta^2}{\lambda^2} Re_\lambda$

$$\boxed{\frac{\eta}{\lambda} \sim Re_\lambda^{-1/2}}$$

ϵ for the vortex tube:

$$\epsilon \sim \left[\frac{u}{\eta}\right]^2 \nu$$

$$\epsilon \cdot V_{frac} \sim \frac{u^2}{\cancel{\eta^2}} * \frac{\cancel{\eta^2}}{\lambda^2} \sim \boxed{\frac{u^2 \nu}{\lambda^2}}$$

$\boxed{\text{Problem 4}}$: Derive tke Equation

Starting from [redacted] the Kinetic Energy transport equation

$$\frac{\partial}{\partial t}\left[\frac{u_i u_i}{2}\right] + \frac{\partial}{\partial x_i}\left[u_j \frac{u_i u_i}{2}\right] = -\frac{1}{\rho}\frac{\partial u_j P}{\partial x_j} + \frac{\partial}{\partial x_j}\left[u_i 2\nu S_{ij}\right] - 2\nu S_{ij}\frac{\partial u_i}{\partial x_j}$$

Now let $u_i u_i = \bar{u}_i\bar{u}_i + 2\bar{u}_i u_i' + u_i' u_i'$ with $u_i = \bar{u}_i + u_i'$. so,

$$\frac{\partial}{\partial t}\left[\frac{\bar{u}_i\bar{u}_i}{2}\right] + \frac{\partial}{\partial t}\left[\frac{\overline{u_i' u_i'}}{2}\right] = -\frac{\partial}{\partial x_i}\bar{u}_i\left[\frac{\bar{P}}{\rho} + \frac{1}{2}\bar{u}_j\bar{u}_j\right] + \nu\frac{\partial}{\partial x_i}\left[\bar{u}_i 2\bar{S}_{ij}\right] - 2\nu S_{ij}\frac{\partial \bar{u}_j}{\partial x_i} - \frac{\partial}{\partial x_i}\overline{\left[u_i' \frac{P'}{\rho} + \frac{1}{2}u_j' u_j'\right]}$$
$$-\frac{\partial}{\partial x_i}\left[\bar{u}_j \overline{u_i' u_j'}\right] - \frac{1}{2}\frac{\partial}{\partial x_i}\left[\bar{u}_i \overline{u_j' u_j'}\right] + \nu\frac{\partial}{\partial x_i}\left[\overline{2u_j' S_{ij}}\right] - 2\nu S_{ij}\frac{\partial u_j'}{\partial x_i}$$

The MKE equation is given by

$$\frac{\partial}{\partial t}\left[\frac{\bar{u}_i\bar{u}_i}{2}\right] + \frac{\partial}{\partial x_i}\left[\bar{u}_i\left(\frac{\bar{P}}{\rho} + \frac{1}{2}\bar{u}_j\bar{u}_j\right)\right] = -\overline{u_j' u_i'}\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left[-\overline{u_i' u_j'}\bar{u}_i\right] + \nu\frac{\partial}{\partial x_i}\left[\bar{u}_j \bar{S}_{ij} 2\right] - 2\nu S_{ij}\frac{\partial \bar{u}_i}{\partial x_j}$$

We can get the tke equation by subtracting the ~~Total~~ MKE from the total KE.

$$\boxed{\frac{\partial}{\partial t}\left[\frac{\overline{u_i' u_i'}}{2}\right] + \bar{u}_j\frac{\partial}{\partial x_j}\left[\frac{\overline{u_i' u_i'}}{2}\right] = -\frac{\partial}{\partial x_i}\left[\overline{u_i'\left(\frac{P'}{\rho} + \frac{u_j' u_j'}{2}\right)}\right] - \overline{u_i' u_j'}\frac{\partial \bar{u}_j}{\partial x_i} + \nu\frac{\partial}{\partial x_i}\left[\overline{2S_{ij}u_j'}\right] - 2\nu S_{ij}\frac{\partial u_j'}{\partial x_i}}$$