Statistical Measures — Turbulence

$\overline{u_i' u_j'}$ — N.S

$\overline{u_i u_i u_j}$ — tke

$\overline{u_j p}$

Deterministic $\longleftrightarrow$ Random

→ Spectrum (continuous) → frequency dependent.

- Mean time: $U = \frac{1}{T}\int_0^T \tilde{u}(t)\,dt \qquad T \to \infty$

$$\overline{u^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\tilde{u}-U)^2\,dt$$

PROBABILITY MEASURES

$$T_u = \sum_i \Delta t_u$$

$B(u) \qquad \int_{-\infty}^{+\infty} B(u)\,du = \underline{1} \qquad \bar{U} \equiv \int_{-\infty}^{\infty} \tilde{u}(B(u))\,d\tilde{u}$

2nd Mom. $\} \; \overline{u^2} = \int u^2 B(u)\,du$

"Skewness" 3rd Mom $\} \; \overline{u^3} = \int u^3 B(u)\,du$

$S = \overline{u^3}/\sigma^3$

→ lack of Symmetry $-(S \to 0)$

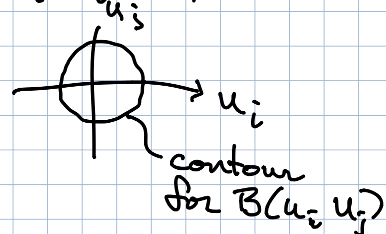
"Kurtosis" or "Flatness" $(\underline{\underline{K}})$ 4th moment $\quad \overline{u^4} = \int u^3 B(u)\,du$

Gaussian $K=3$

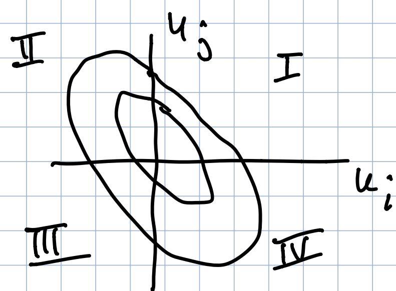
Joint Statistics: $u_i' u_j'$

$u_i \longrightarrow B(u_i) \qquad u_j \longrightarrow B(u_j)$

$B(u_i, u_j) \neq B(u_i)\, B(u_j)$ (only independent)

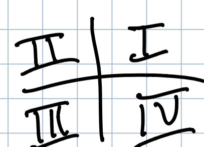


$\overline{u_i u_j} = 0$

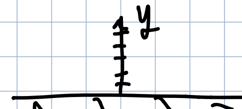


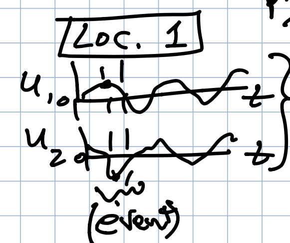
when $u_i > 0$, then $u_j < 0$ } IV
when $u_i < 0$, then $u_j > 0$ } II
} higher prob. of neg. prod.

Conditional Sampling.



time series:





Correlation Function (w.r.t. time)

$$R_{u_i u_j}(\bar{x}, \bar{r}, \tau) = \frac{1}{T}\int_0^T u_i(\bar{x}, t)\, u_j(\bar{x} + \bar{r}, t + \tau)\, dt$$

(τ time separation) $\quad T \to \infty$

$\bar{r}$: spatial shift; τ: temporal shift

Stationary $\to \neq f(t)$

homogeneous $\longrightarrow$ independent of $\bar{x}$

isotropic $\longrightarrow$ no preferred direction $\bar{r}$

Time dependent signals:

$$R_{u_i u_j}(\tau) = \frac{1}{T}\int_0^T u_i(t)\, u_j(t + \tau)\, dt$$

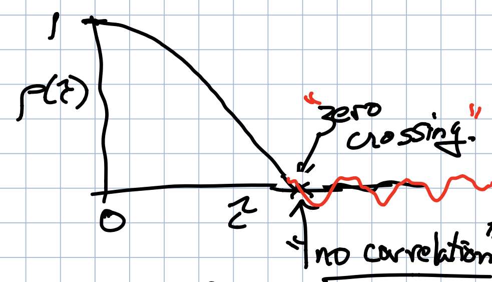
τ: time separation.

$i = j$ = "autocorrelation"

$i \neq j$ = "cross correlation"

$$\rho_{ij}(\tau) = \frac{R_{ij}(\tau)}{\left(\overline{u_i^2}\right)^{1/2}\left(\overline{u_j^2}\right)^{1/2}}$$

"correlation



(no longer existing correlation between variables)

Scales

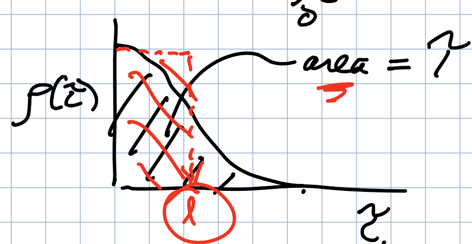
→ large

→ small → Taylor microscale

$\rho(\tau)$: → $\rho(0) = 1$

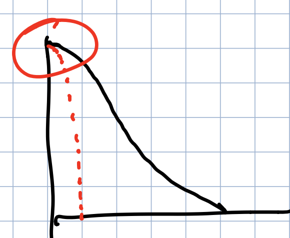
→ $\left.\frac{\partial \rho}{\partial \tau}\right|_{\tau=0} = 0$ symmetric about $\tau = 0$.

integral Time scale → $\int_0^\infty \rho(\tau)\, d\tau = \mathcal{T}$



$$\rho(\tau) = \rho(0) + \Delta\tau \cancel{\frac{\partial \rho}{\partial \tau}} + \frac{\Delta\tau^2}{2}\frac{\partial^2 \rho}{\partial \tau^2}$$

$$\rho \simeq 1 + \frac{\tau^2}{2}\frac{\partial^2 \rho}{\partial \tau^2} \simeq 1 + \frac{\tau^2}{\tau_{scale}^2} \quad ; \quad \frac{\partial^2 \rho}{\partial \tau^2} \simeq \frac{2}{\tau_{scale}^2}$$



Taylor's "Frozen" Turbulence Hypothesis.

$$\boxed{\Delta t = \frac{\Delta x}{V_{conv}}}$$

V_{conv}.

$$\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = \frac{\partial u}{\partial t}\frac{1}{V_{conv}} = \frac{\partial u}{\partial x}$$

$$x \leftrightarrow V_{conv}\, t$$

$$\frac{\partial^2 \rho_\tau}{\partial \tau^2} = V_{conv}^2 \frac{\partial^2 \rho_r}{\partial r^2} \qquad \frac{\partial^2 \rho}{\partial r^2} = \frac{2}{\delta_{scale}^2}$$

$$\delta_{scale} = V_{conv}\, \tau_{scale}$$

$$\overline{\frac{\partial u(t)}{\partial t}\frac{\partial u(t')}{\partial t'}} = -\overline{u^2}\frac{\partial^2 \rho}{\partial \tau^2}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t'}\left(\frac{u(t)\,u(t')}{\overline{u^2}}\right)\right] \qquad t' = t+\tau$$

$t' = t+\tau$ so $\partial t' = \partial\tau$

$$\frac{\partial u(t)}{\partial t'} = 0$$

$$\frac{\partial}{\partial t'}\left(u(t)u(t')\right) = u(t)\frac{\partial u(t')}{\partial t'} + u(t')\frac{\partial u(t)}{\partial t'}$$

↳ take $\frac{\partial}{\partial t}$; replace ∂t & $\partial t'$ with τ der.

· take time ave. whaa laa!

$$-\overline{u^2}\frac{\partial^2 \rho}{\partial \tau^2}$$

$$\overline{\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}} = \frac{1}{V_{conv}^2}\overline{\frac{\partial u(t)}{\partial t}\frac{\partial u(t')}{\partial t'}} = -\overline{u^2}\frac{\partial^2 \rho_r}{\partial r^2}$$

x near x' : $-\overline{u^2}\dfrac{\partial^2 \rho_r}{\partial r^2} = -\overline{u^2}\dfrac{2}{\delta_{scale}^2}$

at small scales: $\overline{\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial x}} \approx \dfrac{\varepsilon}{2\nu}$

$\varepsilon = 2\nu\,\overline{s_{ij}s_{ij}}$

$i=j$:

$$\varepsilon = 2\frac{\overline{u_2^2}}{\lambda^2} \Rightarrow \lambda^2 = \left(\frac{\delta_{scale}^2}{2}\right)$$

use $\rho(\tau) \longrightarrow \delta_{scale}^2 \longrightarrow \lambda^2$

$\rho(\tau)$ & $S(\omega)$ Power Spectral Density

$$\rho(\tau) = \int e^{i\tau\omega} S(\omega)\, d\omega \qquad \underline{S(\omega)} = \frac{1}{2\pi}\int e^{-\omega\tau} \underline{\rho(\tau)}\, d\tau$$

$$-\frac{d^2\rho}{d\tau^2} = \int e^{i\omega\tau} \omega^2 \underline{S(\omega)}\, d\omega$$

$$\longrightarrow \underline{\lambda}$$

<u>Integral time scale:</u> $\displaystyle \tilde{T} = \int \rho(\tau)\, d\tau$

$$S(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(\tau)\, d\tau = \frac{1}{\pi}\int_0^{\infty} \rho(\tau)\, d\tau$$

$$S(0) = \tilde{T}/\pi$$

Diffusion Process

diffusion over time

Position $\displaystyle x(t) = \int_0^t V(t')\, dt'$

average lots of particles: $\overline{x(t)} \xrightarrow[t]{} 0$

$$x(t)V(t) = x(t)\frac{dx}{dt} = \frac{d(\frac{1}{2}x^2)}{dt} = \int_0^t V(t)V(t')\, dt'$$

<u>ave</u> over lots particles

$$\frac{d(\overline{\frac{1}{2}x^2})}{dt} = \int \overline{V(t)V(t')}\, dt' \leftarrow \text{units: } L^2/t \text{ "diffusivity"}$$

$$\overline{V(t)V(t')} = R(\tau) = \overline{V^2}\rho(\tau)$$

$$\tau = t - t'$$

$$T = \int_0^{\infty} \rho(\tau)\, d\tau$$

$$\frac{d(\overline{\frac{1}{2}y^2})}{dt} = \int_0^t R(\tau)\, d\tau = \overline{V^2}\int_0^t \rho(\tau)\, d\tau$$

$= \text{turbulent diffusivity } \nu_t$

$$\rho(\tau) = \exp(-\tau/T)$$

$$\nu_T = \overline{V^2}\, T\left(1 - \exp(-\tau/T)\right)$$

$$\overline{y^2}(t) = 2\,\overline{v^2}\,\mathcal{T}\cdot t \qquad (\text{integrated } y^2)$$
$$= 2\,\nu_t\cdot t$$

Taylor's Diffusion Eqn:

$$\overline{y^2}(t) = 2\overline{v^2}\int_0^t \underbrace{\left(\int_0^{\tau}\rho(\xi)\,d\xi\right)}_{\mathcal{T} \text{ for } \tau \gg \mathcal{T}} dt$$

small times: $\overline{y^2}(t) = 2\overline{v^2}\,t^2/2 \qquad (\rho(\xi)\approx 1)$

large times: $\overline{y^2}(t) = 2\overline{v^2}\,\mathcal{T}\,t$

for $t > \mathcal{T}$ at larger times spreading decrease

$$\left[\begin{array}{l} \overline{v^2} \sim (.1\ m/s)^2 \\ \mathcal{T} \sim 1 \text{ millisec.} \end{array}\right] \qquad \overline{v^2}\,\mathcal{T} = 1\times 10^{-5}\ m^2/s$$
$$\nu_{air} = 1.6\times 10^{-5}$$

Turb. $\quad U = 20\ m/s \quad \sqrt{\overline{v^2}} \sim 10\%$ of U

$$\mathcal{T} = 0.1 \text{ sec.}$$
$$\overline{v^2}\,\mathcal{T} = .4\ m^2/s \sim 10^5\times \text{ larger dispersion.}$$