MKE / tke

<u>Vorticity:</u> $\tilde{\omega}_i = \epsilon_{ijk} \dfrac{\partial \tilde{u}_k}{\partial x_j}$

instantaneous.

permutation operator
switch sign changing order.
$\rightarrow 0$ if $i=j$ or $j=k$ $i=k$

$$\frac{\partial \tilde{u}_k}{\partial x_j} = (\tilde{S}_{kj} + \tilde{r}_{kj})$$

sym antisym.

$$\tilde{S}_{kj} = \frac{1}{2}\left(\frac{\partial \tilde{u}_k}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_k}\right)$$

$$\tilde{r}_{kj} = \frac{1}{2}\left(\frac{\partial \tilde{u}_k}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_k}\right)$$

$$\boxed{\tilde{\omega}_i = \epsilon_{ijk}\, \tilde{r}_{kj}}$$

$$\boxed{\tilde{r}_{ij} = \left(-\frac{1}{2}\epsilon_{ijk}\right)\tilde{\omega}_k}$$

$$\tilde{\omega}_k = \epsilon_{kpm} \frac{\partial \tilde{u}_m}{\partial x_p}$$

$$\boxed{\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}$$

$$\frac{\partial}{\partial x_j}(\tilde u_i \tilde u_j) = \tilde u_j\left(\frac{\partial \tilde u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \tilde u_j \frac{\partial \tilde u_j}{\partial x_i}$$

$$= 2\tilde u_j \tilde r_{ij} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} \tilde u_j \tilde u_j\right)$$

$$= \boxed{-\underbrace{\epsilon_{ijk}\tilde u_j \tilde\omega_k}_{\text{vorticity interaction}} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} \tilde u_j\tilde u_j\right)}$$

viscous term (N-S)

$$\nu \frac{\partial^2 \tilde u_i}{\partial x_j\partial x_j} = \nu\frac{\partial}{\partial x_j}\left(\frac{\partial\tilde u_i}{\partial x_j}-\frac{\partial \tilde u_j}{\partial x_i}\right) + \nu\frac{\partial}{\partial x_i}\underbrace{\left(\frac{\partial \tilde u_j}{\partial x_j}\right)}_{=\,0\ \text{incomp.}}$$

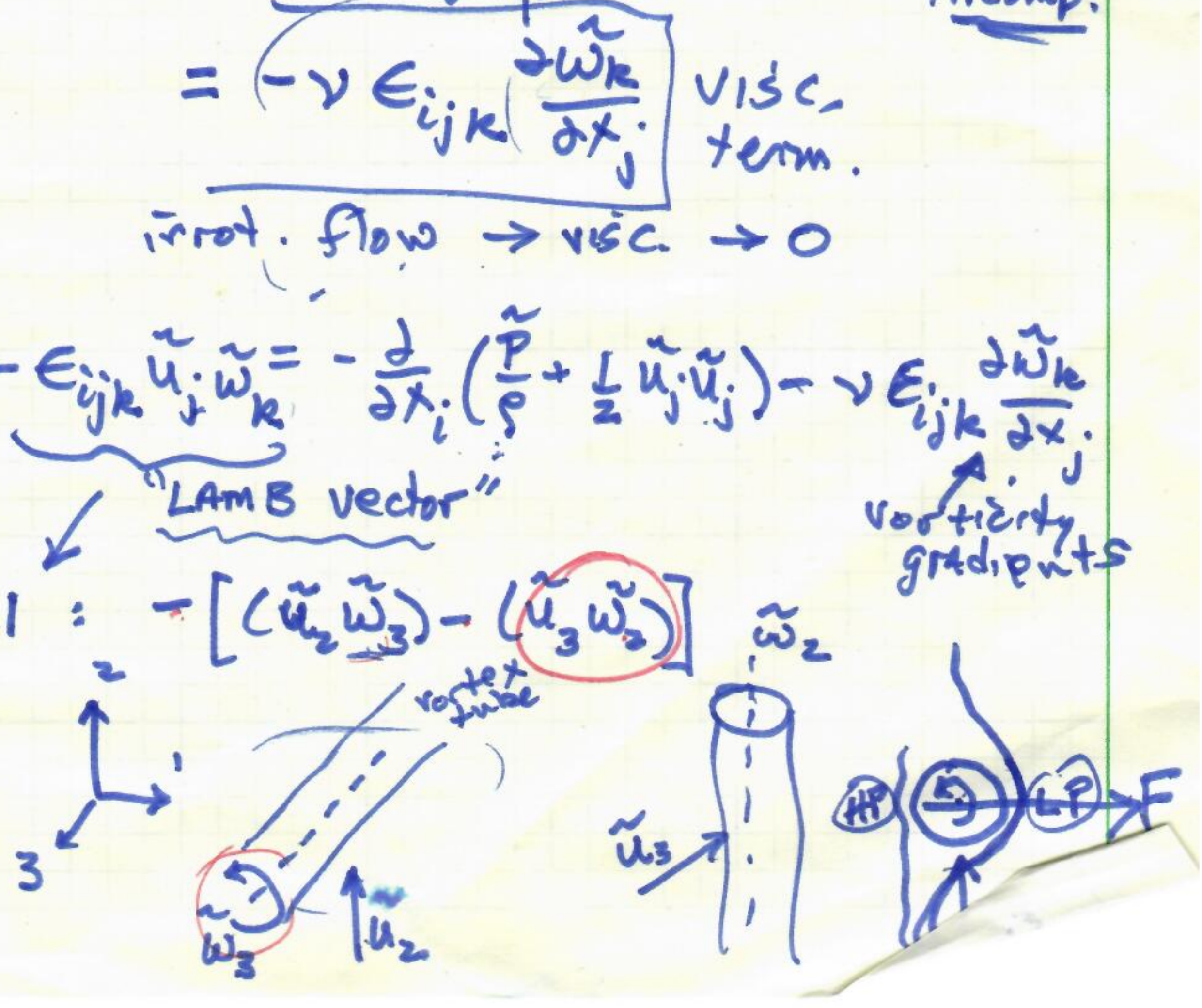
$$= 2\nu \frac{\partial}{\partial x_j}(\tilde r_{ij})$$

$$= \boxed{-\nu\,\epsilon_{ijk}\frac{\partial\tilde\omega_k}{\partial x_j}}\ \text{visc. term.}$$

irrot. flow → visc. → 0

$$\frac{\partial \tilde u_i}{\partial t} - \underbrace{\epsilon_{ijk}\tilde u_j\tilde\omega_k}_{\text{"LAMB vector"}} = -\frac{\partial}{\partial x_i}\left(\frac{\tilde p}{\rho}+\frac{1}{2}\tilde u_j\tilde u_j\right) - \nu\,\epsilon_{ijk}\underbrace{\frac{\partial\tilde\omega_k}{\partial x_j}}_{\text{vorticity gradients}}$$

$$i=1 :\ -\left[(\tilde u_2\tilde\omega_3)-(\tilde u_3\tilde\omega_2)\right]$$



$\nabla \cdot u = 0$

$\boxed{\nabla \cdot \omega = 0}$ vorticity field – divergence free.

$\tilde{\omega}_i \rightarrow$ dynamics. $\rightarrow$ vorticity-shear interactions (strain rate)

$\hookrightarrow$ consequence on $\underline{\underline{\omega}}$

Transport Eqn. for $\hat{\omega}_p$

N-S: $\rightarrow$ curl.

$$\frac{\partial \tilde{u}_i}{\partial t} = -\frac{\partial}{\partial x_i}\Big(\underbrace{\frac{\tilde{p}}{\rho} + \frac{1}{2}\tilde{u}_j \cdot \tilde{u}_j}_{\text{dynamic}}\Big) + \underbrace{\epsilon_{ijk}\tilde{u}_j\tilde{\omega}_k}_{\text{Lamb}} - \nu\,\epsilon_{ijk}\frac{\partial \tilde{\omega}_k}{\partial x_j}$$

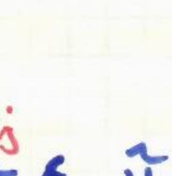
$-s_{ij}$

Curl of each term:

$$\frac{\partial}{\partial t}\Big(\epsilon_{pqi}\frac{\partial u_i}{\partial x_q}\Big) = \frac{\partial \tilde{\omega}_p}{\partial t}$$

$$\epsilon_{pqi}\frac{\partial}{\partial x_q}\frac{\partial}{\partial x_i}\Big(\frac{1}{2}\hat{u}_j\hat{u}_j\Big) = 0$$

antisym. symm.



$$\epsilon_{pqi}\frac{\partial^2 p}{\partial x_q \partial x_i} = 0$$

$$\underbrace{\epsilon_{pqi}\epsilon_{ijk}}_{\delta_{qj}\delta_{ik} - \delta_{qk}\delta_{ij}}\frac{\partial}{\partial x_q}(\tilde{u}_j \cdot \hat{\omega}_k) = -\frac{\partial}{\partial x_k}(\hat{\omega}_p u_k) + \frac{\partial}{\partial x_j}(\tilde{\omega}_j \cdot \tilde{u}_p)$$

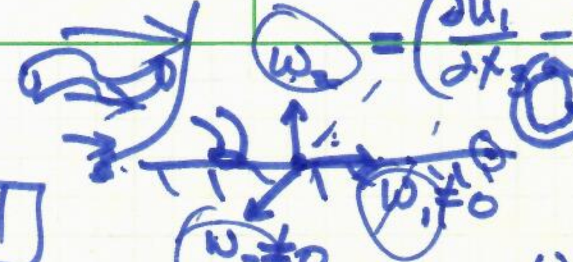
$$= -\hat{u}_k\frac{\partial \tilde{\omega}_p}{\partial x_k} + \hat{\omega}_j\frac{\partial \tilde{u}_p}{\partial x_j}$$

$$\boxed{\frac{\partial \tilde{\omega}_p}{\partial t} = \hat{\omega}_k\frac{\partial \hat{u}_p}{\partial x_k} - \tilde{u}_k\frac{\partial \tilde{\omega}_p}{\partial x_k} - \nu\frac{\partial^2 \tilde{\omega}_p}{\partial x_k \partial x_k}}$$

$$\frac{D\tilde{\omega}_p}{Dt} = \tilde{\omega}_k\frac{\partial \tilde{u}_p}{\partial x_k} - \nu\frac{\partial^2 \tilde{\omega}_p}{\partial x_k \partial x_k}$$

- viscous diff[illegible]
- deform[illegible]

Vortex tubes.



$$\omega_2 = \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = 0$$

$\omega_1 \neq 0$

$\omega_3 \neq 0$

$$\omega_1 = \left(\overset{=0}{\frac{\partial u_3}{\partial x_2}} - \overset{=0}{\frac{\partial u_2}{\partial x_3}}\right)$$

$$\omega_3 = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

$\boxed{\omega_2 = 0 \text{ at a wall}}$

Then $\vec{\omega}$ (vector)

→ a vortex tube

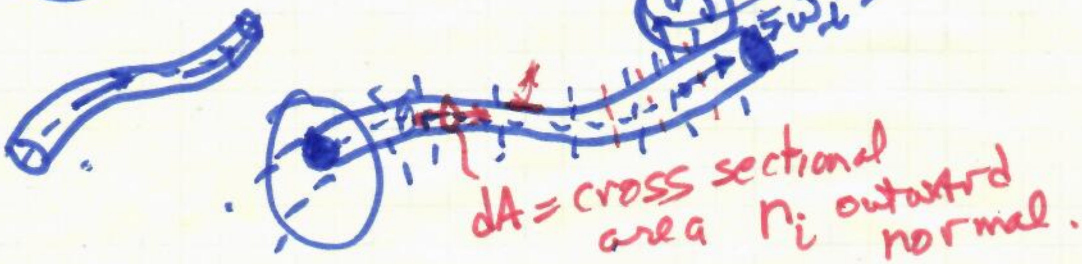
→ can't <u>end</u> on a wall

$$= \left(\underset{=0}{\frac{\partial u_2}{\partial x_1}} - \underset{\text{Something}}{\frac{\partial u_1}{\partial x_2}}\right)$$

$$\boxed{\Gamma = \text{circulation}} = \int \hat{n}_i \omega_i \, dA$$

$\omega_i = 0$

ω_i



dA = cross sectional area n_i outward normal.

$$\Gamma = \underbrace{\int_A n_i \omega_i \, dA}_{} = \oint u \cdot \hat{s} \, dl$$

ω_i

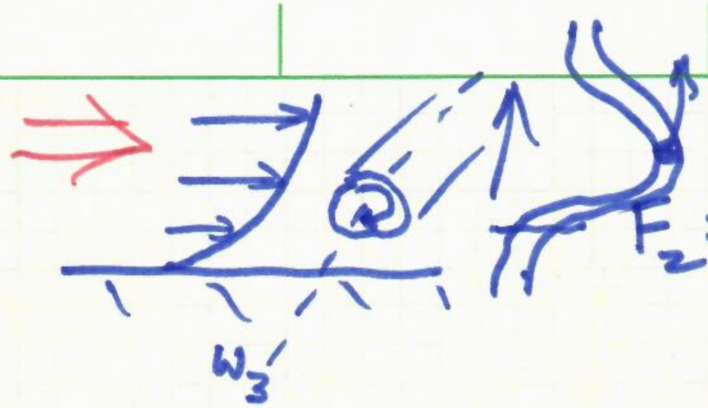
$u = \zeta \cdot r$, ζ = angular velocity

$dl = r\,d\theta$

$$\boxed{\Gamma = \oint \zeta \cdot r \, r \, d\theta = 2\pi \zeta r^2}$$

$\boxed{\text{as } r \downarrow \ \zeta \uparrow \ \text{ for } \Gamma = \text{const.}}$

as scales of length decrease; rot. rate & ω ↑



$F_2 = \tilde{u}_1 \tilde{\omega}_3$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

N-S: $\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}\right) \quad i = x, y, z$

$\frac{\partial U}{\partial y}$

neg

$(u'v') < 0$

positive

Prod. of tke

$$-\overline{u'v'}\,\frac{\partial U}{\partial y} = \text{Prod. of } \underline{\text{tke.}}$$

mean