1 Delay Differential Equations

Differential equations are often used to describe the dynamics of a deterministic system, whose future behavior depends on the present state. For an *ordinary differential equation* (ODE), this state is an element of $\mathbb{R}^n$. The rate of change only depends on the current time instant.

In *delay differential equations* (DDEs) however, the system is influenced by the past through the appearance of a deviated time argument, the current state needs to contain the previous evolution. This leads to a functional state space, its elements are functions on a past time interval. For that reason, DDEs belong to the class of *functional differential equations* (FDEs).

DDEs often appear in automatic control, where a controller monitors the state of a system in order to make control decisions to adjust this state. If there is a delay between the observation and the control action, the differential equation describing the system not only depends on its current state, but also on its past. These previous values need to be specify in an initial condition, for at least the time of the longest delay.

Examples of phenomena which have been modeled using delay differential equations include epidemics, traffic flow and vibrations/chattering. See [5] and the references therein.

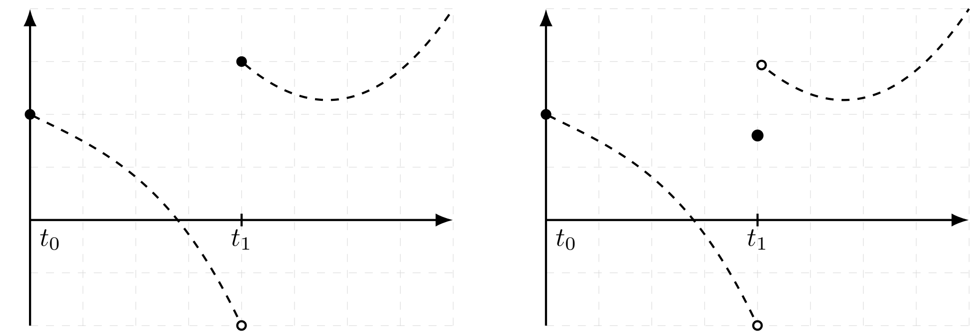
Some methods to solve basic DDEs analytically are presented in [5]. Numerical procedures are not as far developed as for ODEs. See [1] or [42] for a rigorous integration algorithm.

1.1 Piecewise Continuous Functions

The following definition is motivated by the character of evolution arising from hybrid systems. We define the main functional space of operation for the following chapters.

Definition 1.1 (Piecewise Continuously Differentiable)**.** Let $D = [a, b] \subseteq \mathbb{R}$ be a closed interval (this includes the cases when $a = -\infty$ or $b = \infty$, or both). The mapping $x\colon D \to \mathbb{R}^n$ is called n-times *piecewise continuously differentiable* if and only if there is a finite partition (ordered set) $\{a = t_0 < t_1 < \ldots < t_m = b\}$ of D (i.e. $a = t_0 < t_1 < \ldots < t_m = b$) such that x is n-times continuously differentiable on each interval (t_i, t_{i+1}) with *càdlàg* (« continue à droite, limite à gauche ») derivative.

This means that everywhere on D, the function x and each of its derivatives $x^{(k)}$ are right continuous and have left limits.



(a) Admissible piecewise continuous function.

(b) Not allowed!

Figure 1.1: Examples to Definition 1.1.

More precisely, for all $i = 0, \dots, m-1$ and for all $k = 0, \dots, n$ exist the left limits

$$\lim_{\substack{t \nearrow t_{i+1} \\ t \in (t_i, t_{i+1})}} x^{(k)}(t)$$

as well as the right limits

$$\lim_{\substack{t \searrow t_i \\ t \in (t_i, t_{i+1})}} x^{(k)}(t) = x^{(k)}(t_i)$$

which additionally coincide with the value of $x^{(k)}$ at this knot t_i. Hence x can have an isolated point only in the right interval limit b.

In the case $n = 0$, we say x is *piecewise continuous*. For a compact interval $D \subset \mathbb{R}$ (this excludes the cases with $\pm\infty$), we denote by $C^n_{\text{pw}}(D, \mathbb{R}^n)$ the set of *n-times piecewise continuously differentiable functions* on D mapping to $\mathbb{R}^n$, and respectively, by $C^0_{\text{pw}}(D, \mathbb{R}^n)$ the set of *piecewise continuous functions* on D.

The supremum norm $\|\cdot\|_{\sup}$ of the Banach space of continuous functions on the compactum D can be extended to $C^n_{\text{pw}}(D, \mathbb{R}^n)$, since each element consists of a finite number of continuous parts.

In the following, when we talk about *piecewise continuous* and *piecewise continuously differentiable*, we refer to it in the sense of Definition 1.1. Let us note some basic observations which will be used subsequently.

Lemma 1.2. *The composition of a continuous (outer) and a piecewise continuous function (inner) is again piecewise continuous with the same partition.*

Proof. The limits exist, because they commute with the continuous function and exist for the piecewise-continuous function. $\square$

Lemma 1.3. *A piecewise continuous function is (Riemann) integrable.*

Proof. This proof is usually given in every standard analysis book, see for example Theorem 6.10 in [40] or Example 11.16b in [7]. □

The following lemma generalizes the fundamental theorem of calculus to piecewise continuous derivatives.

Lemma 1.4. *Let $F \in C^0([a,b]) \cap C^1_{\mathrm{pw}}([a,b])$ with piecewise derivative f. Then*

$$F(t) - F(a) = \int_a^t f(s)\,\mathrm{d}s$$

for all $t \in [a,b]$.

Proof. On each compact interval $[t_{i-1}, t_i]$ of the partition, f is piecewise continuous and hence integrable (Lemma 1.3).

By precondition is F differentiable on $[t_{i-1}, \zeta]$ with $F' = f$ for all $\zeta \in (t_{i-1}, t_i)$. For that reason, the fundamental theorem of calculus (cf. standard analysis literature, e.g. [7, 40]) yields

$$\int_{t_{i-1}}^{\zeta} f(s)\,\mathrm{d}s = F(\zeta) - F(t_{i-1})$$

and by the continuity of F that

$$\int_{t_{i-1}}^{t_i} f(s)\,\mathrm{d}s = \lim_{\zeta \to t_i} \int_{t_{i-1}}^{\zeta} f(s)\,\mathrm{d}s = \lim_{\zeta \to t_i} F(\zeta) - F(t_{i-1}) = F(t_i) - F(t_{i-1})$$

For any $t \in [a,b]$, there is a $k \in \{1, \dots, m\}$ such that $t \in [t_{k-1}, t_k)$ (in the case $t = b$, set $k = m$), summation over $i = 1, \dots, k$ yields the telescoping series

$$F(t) - F(a) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(s)\,\mathrm{d}s + \int_{t_j}^{t} f(s)\,\mathrm{d}s$$

what is by the additivity of the integral equivalent to

$$F(t) - F(a) = \int_a^t f(s)\,\mathrm{d}s.$$

□

1.2 Definition of DDEs

There are different possibilities to define delay differential equations, depending on what application one has in mind. We restrict to a class adapted to our needs and often found in literature, see for example [39] and which cover a wide range of applications.

Definition 1.5 (Delay Differential Equation)**.** Given a function $f\colon \mathbb{R}\times\mathbb{R}^n\times\ldots\times\mathbb{R}^n \to \mathbb{R}^n$ and a set of time delays $\{\tau_j \;:\; 0 < \tau_1 < \ldots < \tau_k\}$, a functional equation of the form

$$x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k)) \tag{1.1}$$

is called (first order) *delay differential equation* (DDE) with *multiple constant, discrete delays* τ_j. It is said to be *autonomous* if its right hand side f is time independent and *pure*, if the right hand side only depends on $x(t-\tau_j)$ but not on $x(t)$. We define its *maximal* and *minimal delay* as $\tau_{\max} \overset{\text{def}}{=} \tau_k$ and $\tau_{\min} \overset{\text{def}}{=} \tau_1$, respectively.

A DDE can be equipped with an *initial condition* $x_\sigma\colon [\sigma-\tau_{\max}, \sigma] \to \mathbb{R}^n$. It specifies the initial state, i.e. the values of x, on which the right hand side depends at $t = \sigma$. Such a pair is called *initial value problem* (IVP):

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k)) & \text{for } t \geq \sigma \\ x(t) = x_\sigma(t) & \text{for } t \in [\sigma - \tau_{\max}, \sigma] \end{cases} \tag{1.2}$$

Definition 1.6 (Solution of DDE)**.** A function $x\colon [\sigma - \tau_{\max}, \sigma + T] \to \mathbb{R}^n$ is called *(local) solution* of the initial value problem (1.2), if and only if there exists a $T > 0$ such that x obeys the initial condition

$$x(t) = x_\sigma(t) \quad \text{for } t \in [\sigma - \tau_{\max}, \sigma]$$

and x is continuous and piecewise continuously differentiable on $[\sigma, \sigma + T]$, fulfilling

$$x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k))$$

on each (open) interval (t_i, t_{i+1}) of its partition $\{\sigma = t_0 < \ldots < t_m = \sigma + T\}$.

If the function x is a solution for all $T > 0$, it is called *global*.

The piecewise continuity of the derivative means

$$\lim_{s \searrow t_i} x'(s) = f(t_i, x(t_i), x(t_i - \tau_1), \ldots, x(t_i - \tau_k))$$

for the right limits in the knots t_i, $i \in \{0, \ldots, m-1\}$. This is equivalent to the fact that it holds for the *right derivative*

$$x'_+(t) \overset{\text{def}}{=} \lim_{s \searrow t} \frac{x(s) - x(t)}{s - t} = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k))$$

for all $t \in [\sigma, \sigma + T]$.

There can be a T such that left limit does not exist. explosion The left limits $\lim_{s \nearrow t_m} x'(s)$ exists not necessarily in the last knot $t_m = \sigma + T$. If it does, the solution is continuable.

1.3 Method of Steps

If we restrict the IVP (Eq. 1.2) onto an interval $[\sigma, \sigma + T_1]$ with $T_1 \le \tau_{\min}$, then the values of all $x(t - \tau_j)$ are specified by the initial condition and can thus be replaced by $x_\sigma(t - \tau_j)$. We obtain an *initial value problem* for an *ordinary differential equation.* If we can solve this IVP, i.e. if we can find a solution of the ODE on $[\sigma, \sigma + T_1]$, then we can reapply this method by plugging the computed solution into the DDE and solving the resulting ODE on the interval $[\sigma + T_1, \sigma + T_2]$, where again $T_2 \le \tau_{\min}$. As long as one can solve the resulting ODE (for suitable f and x_σ, the existence (and uniqueness) of a solution for the ODE is guaranteed by Picard-Lindelöf's theorem), this step can be iterated.

This method, which allows to convert DDE into a ODE on a certain interval, eliminating the explicit dependence on the past by inserting the initial condition, is know as *method of steps.* See [5] for examples.

1.4 Existence and Uniqueness of Solutions

In this section, we show that under certain conditions, we can guarantee the existence of a solution for the DDE-IVP (Eq. 1.2) and that in general, it cannot have more than one.

Definition 1.7 (Lipschitz Continuity)**.** A function $f\colon \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$ is called *(locally) Lipschitz continuous* (in its j-th argument, referring to t as zeroth argument) if and only if for all $a, b \in \mathbb{R}$ and $M > 0$ there is a $L > 0$, such that

$$\|f(t, x_1, \ldots, x_j, \ldots, x_k) - f(t, x_1, \ldots, y_j, \ldots, x_k)\| \le L\|x_j - y_j\|$$

for all $t \in [a, b]$ and $x_j, y_j \in \mathbb{R}^n$ with $\|x_j\|, \|y_j\|, \le M$.

Lemma 1.8. *Finding a solution of the initial value problem* (1.2) *is equivalent to solving the integral equation*

$$\begin{cases} x(t) = x_\sigma(\sigma) + \int_\sigma^t f(s, x(s), x(s - \tau_1), \ldots, x(s - \tau_k))\,\mathrm{d}s & \text{for } t \ge \sigma \\ x(t) = x_\sigma(t) & \text{for } t \in [\sigma - \tau_{\max}, \sigma] \end{cases}$$

where $f\colon \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and Lipschitz continuous in all but its zeroth argument. The integral is meant to be componentwise, if f is vector-valued.

Proof. Let x be a solution of the IVP. Thus x is (by definition) piecewise continuous on $[\sigma - \tau_{\max}, \sigma]$ and continuous and piecewise continuously differentiable on $[\sigma, \sigma + T]$ with (piecewise) derivative $t \mapsto f(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_k))$. By Lemma 1.4 it follows

$$x(t) = x_\sigma(\sigma) + \int_\sigma^t f(s, x(s), x(s - \tau_1), \ldots, x(s - \tau_k))\,\mathrm{d}s$$

for $t \ge \sigma$, since $x_\sigma(\sigma) = x(\sigma)$.

Conversely, let x be a solution of the integral equation. By the fundamental theorem of calculus, x is continuous on $[\sigma, \sigma + T]$.

From the partition $\{t_0 < \ldots < t_m\}$ of x_σ, we define a partition of $[\sigma, \sigma + T]$ by

$$\mathcal{Z} \stackrel{\text{def}}{=} \{\hat{t}_0 < \ldots < \hat{t}_p\} \stackrel{\text{def}}{=} \{\sigma, \sigma + T\} \cup \bigcup_{j=1}^{k} \bigcup_{\substack{i=1 \\ t_i \geq \sigma - \tau_j}}^{m} \{t_i + \tau_j\} \tag{1.3}$$

Let $t \in (\hat{t}_{l-1}, \hat{t}_l)$ for any $l \in \{1, \ldots, p\}$. If for any $j \in \{1, \ldots, k\}$ and $i \in \{0, \ldots, m\}$ was $t - \tau_j = t_i$, then $t = t_i + \tau_j = \hat{t}_r$ for a $r \in \{1, \ldots, p\}$, which would be a contradiction to the choice of t. Hence $t - \tau_j \neq t_i$ for all $j \in \{1, \ldots, k\}$ and $i \in \{0, \ldots, m\}$, what implies that all $s \mapsto x(s - \tau_j)$ are continuous in t. Thus the composition

$$s \mapsto f(s, x(s), x(s - \tau_1), \ldots, x(s - \tau_k))$$

is continuous in t. The fundamental theorem of calculus states in this case that x is differentiable in t and that $x'(t) = f(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_k))$.

For the right limits if follows by the continuity of f and $\lim_{t \searrow \hat{t}_l} x(t - \tau_j) = x(\hat{t}_l - \tau_j)$, since $t - \tau_j \neq t_i$, that

$$\begin{aligned}\lim_{t \searrow \hat{t}_l} x'(t) &= \lim_{t \searrow \hat{t}_l} f(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_k)) \\ &= f(\hat{t}_l, x(\hat{t}_l), x(\hat{t}_l - \tau_1), \ldots, x(\hat{t}_l - \tau_k))\end{aligned}$$

The left limits

$$\lim_{t \nearrow \hat{t}_l} x'(t) = \lim_{t \nearrow \hat{t}_l} f(t, x(s), x(t - \tau_1), \ldots, x(t - \tau_k))$$

exist for the same reason. Summarily, x is continuous and piecewise continuously differentiable on $[\sigma, \sigma + T]$ with piecewise derivative f and it obviously obeys the initial condition, i.e. $x(t) = x_\sigma(t)$ for all $t \in [\sigma - \tau, \sigma]$. □

The most important result for the considered class of delay differential equations is the following theorem. Its proof is an adaption and extension of the existence theorem (Theorem 3.7) given in [41] and the proof of uniqueness in [34]. It essentially reduces the DDE locally to an ODE by applying the method of steps.

Theorem 1.9 (Existence of a unique solution)**.** *For a continuous function $f\colon \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$, satisfying the Lipschitz condition (Def. 1.7) in all but its zeroth argument, consider the IVP for a delay differential equation*

$$\begin{cases} x' = f(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_k)) & \text{for } t \geq \sigma \\ x(t) = x_\sigma(t - \sigma) & \text{for } t \in [\sigma - \tau_{\max}, \sigma] \end{cases} \tag{1.4}$$

with zero-aligned initial function.

Then, for each initial condition $x_\sigma \in C^1_{\text{pw}}([-\tau_{\max}, 0], \mathbb{R}^n)$ *and start time* $\sigma \in \mathbb{R}$, *there* exists *a* unique local solution *of the IVP on a time interval* $[\sigma - \tau_{\max}, \sigma + T]$. *The duration $T > 0$ depends on the sup-norm and the partition of the initial condition, as well as the right hand side f.*

Proof. Let $\{-\tau_{\max} = t_0 < \ldots < t_m = 0\}$ be the partition of x_σ. As a piecewise continuous function, the initial condition can be bounded on $[-\tau, 0]$ by any $M \geq \|x_\sigma\|_{\sup}$.

Since f is continuous, its sup-norm admits a maximum $K > 0$ on the compact set

$$S \stackrel{\text{def}}{=} [\sigma, \sigma + \tau_{\max}] \times \{x \in \mathbb{R}^n \ : \ \|x\| \leq 2M\}^k$$

Let $L > 0$ be the Lipschitz constant of f for that set with respect to its first argument. We put $T \stackrel{\text{def}}{=} \min\{\sigma + \tau_{\min}, \frac{M}{K}\}$ to restrict f as integrand to S.

We construct a series $(x_{(m)})_{m \in \mathbb{N}_0}$ of piecewise continuous functions, which approximates the solution of the initial value problem. Set

$$x_{(0)}(t) = \begin{cases} x_\sigma(0) & \text{for } t \in [\sigma, \sigma + T] \\ x_\sigma(t - \sigma) & \text{for } t \in [\sigma - \tau_{\max}, \sigma] \end{cases}$$

For $m \in \mathbb{N}_{>0}$ define

$$x_{(m)}(t) = \begin{cases} x_\sigma(0) + \displaystyle\int_\sigma^t f(s, x_{(m-1)}(s), x_{(m-1)}(s - \tau_1), \ldots \\ \qquad\qquad\qquad \ldots, x_{(m-1)}(s - \tau_k))\, \mathrm{d}s & \text{for } t \in [\sigma, \sigma + T] \\ x_\sigma(t - \sigma) & \text{for } t \in [\sigma - \tau_{\max}, \sigma] \end{cases}$$

The integral exists, because the integrand is a composition of a continuous and piecewise continuous function, which is again piecewise continuous (Lemma 1.2) and hence by Lemma 1.3 integrable on $[\sigma, \sigma + T]$.

It holds for all $m > 0$ and $t \in [\sigma - \tau_{\max}, \sigma]$ by the definition of this sequence that

$$\left\|x_{(m)}(t) - x_{(m-1)}(t)\right\| = 0$$

We show by induction over m that for all $t \in [\sigma, \sigma + T]$ it holds

$$\left\|x_{(m)}(t) - x_{(m-1)}(t)\right\| \leq \frac{K}{L} \frac{L^m (t - \sigma)^m}{m!}.$$

Let $t \in [\sigma, \sigma + T]$. Since for all $s \in [\sigma - \tau_{\max}, \sigma + T]$ obviously $\|x_{(0)}(t)\| \leq M$, the statement for $m = 0$ follows from the boundedness of f on S and the triangle inequality for integrals:

$$\|x_{(1)}(t) - x_{(0)}(t)\| = \left\|\int_\sigma^t f(s, x_{(0)}(s), x_{(0)}(s - \tau_1), \ldots, , x_{(0)}(s - \tau_k))\, \mathrm{d}s\right\| \leq K(t - \sigma)$$

In the inductive step for any $m > 0$, we use that $\|x_{(m-1)}(t)\| \leq 2M$ for all $s \in [\sigma - \tau_{\max}, \sigma + T]$ implies

$$\begin{aligned} \left\|x_{(m)}(t)\right\| &\leq \|x_\sigma(0)\| + \int_\sigma^t \left\|f(s, x_{(m-1)}(s), x_{(m-1)}(s - \tau_1), \ldots, x_{(m-1)}(s - \tau_k))\right\| \mathrm{d}s \\ &\leq M + K(t - \sigma) \leq M + KT \\ &\leq 2M \end{aligned} \tag{1.5}$$

using the triangle inequality and the choice of $T \leq \frac{M}{K}$.

Given the second restriction for $T \leq \sigma + \tau_{\min}$ It follows by the Lipschitz property of f (for its first argument) that

$$\begin{aligned}
&\left\|x_{(m+1)}(t) - x_{(m)}(t)\right\| = \\
&\quad = \left\| \int_\sigma^t f(s, x_{(m)}(s), x_{(m)}(s-\tau_1), \ldots, x_{(m)}(s-\tau_k)) \right. \\
&\qquad\qquad \left. - f(s, x_{(m-1)}(s), x_{(m-1)}(s-\tau_1), \ldots, x_{(m-1)}(s-\tau_k))\,\mathrm{d}s \right\| \\
&\quad = \left\| \int_\sigma^t f(s, x_{(m)}(s), x_\sigma(s-\tau_1-\sigma), \ldots, x_\sigma(s-\tau_k-\sigma)) \right. \\
&\qquad\qquad \left. - f(s, x_{(m-1)}(s), x_\sigma(s-\tau_1-\sigma), \ldots, x_\sigma(s-\tau_k-\sigma))\,\mathrm{d}s \right\| \\
&\qquad\qquad\qquad \leq L \int_\sigma^t \left\|x_{(m)}(s) - x_{(m-1)}(s)\right\| \mathrm{d}s \\
&\qquad\qquad\qquad \leq \frac{L^m K}{m!} \int_\sigma^t (s-\sigma)^m \,\mathrm{d}s = \frac{L^m K}{(m+1)!}(t-\sigma)^{m+1}
\end{aligned}$$

The Cauchy criterion for convergent series ([7] 6.13, [40] 3.22) applied to the exponential series states that

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}_0\ \forall m \geq k \geq n_0 :\ \sum_{i=k+1}^{m} \frac{(LT)^i}{i!} < \varepsilon$$

So for any $\varepsilon > 0$ exist $k \in \mathbb{N}_0$ and $m \geq k$, such that

$$\begin{aligned}
\left\|x_{(m)}(t) - x_{(k)}(t)\right\| &\leq \left\|x_{(m)}(t) - x_{(m-1)}(t)\right\| + \left\|x_{(m-1)}(t) - x_{(m-2)}(t)\right\| + \\
&\qquad + \ldots + \left\|x_{(k+1)}(t) - x_{(k)}(t)\right\| \\
&\leq \frac{K}{L}\frac{L^m(t-\sigma)^m}{m!} + \frac{K}{L}\frac{L^{m-1}(t-\sigma)^{m-1}}{(m-1)!} + \\
&\qquad + \ldots + \frac{K}{L}\frac{L^{k+1}(t-\sigma)^{k+1}}{(k+1)!} \\
&\leq \frac{K}{L}\sum_{i=k+1}^{m} \frac{(LT)^i}{i!} < \varepsilon
\end{aligned}$$

for all $t \in [\sigma, \sigma+T]$, i.e. $(x_{(m)})$ is a Cauchy sequence

Since each $x_{(m)}$ is continuous on $[\sigma, \sigma+T]$, this Cauchy sequence admits a limit x in the Banach space $C^0([\sigma, \sigma+T], \mathbb{R}^n)$ with respect to the sup-norm.

Again, we extend x to $[\sigma - \tau_{\max}, \sigma]$ with x_σ, such that $x \in C^0_{\mathrm{pw}}([\sigma - \tau, \sigma + T], \mathbb{R}^n)$.

By the continuity of the sup-norm it follows from (1.5) that

$$\|x\|_{\sup} = \lim_{m\to\infty} \left\|x_{(m)}\right\|_{\sup} \leq 2M$$

can by the Lipschitz property of f

$$\begin{aligned}
&\sup_{t\in[\sigma,\sigma+T]} \|f(s, x_{(m)}(s), x_{(m)}(s-\tau_1), \ldots, x_{(m)}(s-\tau_k)) \\
&\qquad\qquad\qquad\qquad - f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_k))\| \\
= &\sup_{t\in[\sigma,\sigma+T]} \|f(s, x_{(m)}(s), x_\sigma(s-\tau_1-\sigma), \ldots, x_\sigma(s-\tau_k-\sigma)) \\
&\qquad\qquad - f(s, x(s), x_\sigma(s-\tau_1-\sigma), \ldots, x_\sigma(s-\tau_k-\sigma))\| \\
\leq &\sup_{t\in[\sigma,\sigma+T]} \left\|x_{(m)}(t) - x(t)\right\|
\end{aligned}$$

The uniform convergence (convergence in sup-norm) of $x_{(m)} \to x$, implies the uniform convergence

$$f(s, x_{(m)}(s), x_{(m)}(s-\tau_1), \ldots, x_{(m)}(s-\tau_k)) \xrightarrow{m\to\infty} f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_k))$$

and hence we can commute the integral and the limit process in

$$\begin{aligned}
x(t) &= \lim_{m\to\infty} x_{(m+1)} \\
&= x_\sigma(0) + \lim_{m\to\infty} \int_\sigma^t f(s, x_{(m)}(s), x_{(m)}(s-\tau_1), \ldots, x_{(m)}(s-\tau_k)) \, \mathrm{d}s \\
&= x_\sigma(0) + \int_\sigma^t f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_k)) \, \mathrm{d}s
\end{aligned}$$

It follows that x solves the integral equation (1.2), which, by Lemma 1.8, proves the existence of a solution to the DDE which fulfills the initial condition.

It remains to show the uniqueness of a solution. Let x and $\bar{x}$ be two solutions of the DDE on $[\sigma, \sigma+T]$, coinciding on $[\sigma-\tau_{\max}, \sigma]$. By Lemma 1.8 they are equivalent to solutions of the integral equations

$$x(t) = x_\sigma(0) + \int_\sigma^t f(s, x(s), x(s-\tau_1), \ldots, x(s-\tau_k)) \, \mathrm{d}s$$

and

$$\bar{x}(t) = x_\sigma(0) + \int_\sigma^t f(s, \bar{x}(s), \bar{x}(s-\tau_1), \ldots, \bar{x}(s-\tau_k)) \, \mathrm{d}s$$

For $t \in [\sigma, \sigma + T]$, we set

$$\begin{aligned}
\rho(t) \stackrel{\text{def}}{=} \|x(t) - \bar{x}(t)\| &\leq \int_\sigma^t \|f(s, x(s), x(s-\tau)) - f(s, \bar{x}(s), \bar{x}(s-\tau))\| \,\mathrm{d}s \\
&= \int_\sigma^t \|f(s, x(s), x_\sigma(s - \tau_1 - \sigma), \dots, x_\sigma(s - \tau_k - \sigma)) \\
&\qquad\qquad - f(s, \bar{x}(s), x_\sigma(s - \tau_1 - \sigma), \dots, x_\sigma(s - \tau_k - \sigma))\| \,\mathrm{d}s \\
&\leq L \int_\sigma^t \|x(s) - \bar{x}(s)\| \,\mathrm{d}s = L \int_\sigma^t \rho(s) \,\mathrm{d}x(s) \\
&= L \int_\sigma^t \mathrm{e}^{-\alpha s} \rho(s) \mathrm{e}^{\alpha s} \,\mathrm{d}s \leq L \sup_{s \in [\sigma, \sigma+T]} \left(\mathrm{e}^{-\alpha s} \rho(s)\right) \int_\sigma^t \mathrm{e}^{\alpha s} \,\mathrm{d}s \\
&\leq \frac{L}{\alpha} \mathrm{e}^{\alpha t} \sup_{s \in [\sigma, \sigma+T]} \left(\mathrm{e}^{-\alpha s} \rho(s)\right)
\end{aligned}$$

The continuity of x also asserts the continuity of ρ. Choosing $\alpha = 2L$ and multiplying with $\mathrm{e}^{-\alpha t} > 0$ leads to

$$\rho(t)\mathrm{e}^{-2Lt} \leq \frac{1}{2} \sup_{s \in [\sigma, \sigma+T]} \left(\mathrm{e}^{-2Ls} \rho(s)\right)$$

for all $t \in [\sigma, \sigma + T]$

$$0 \leq \sup_{t \in [\sigma, \sigma+T]} \left(\rho(t)\mathrm{e}^{-2Lt}\right) \leq \frac{1}{2} \sup_{s \in [\sigma, \sigma+T]} \left(\mathrm{e}^{-2Ls} \rho(s)\right)$$

That is only possible if $\rho(t) = 0$ for all $t \in [\sigma, \sigma + T]$, which means $x(t) = \bar{x}(t)$. □

Corollary 1.10. *Continuability of solution. Get existence of unique solution on* $[\sigma - \tau, \sigma + S]$ *with* $S > T$.

In the following chapters, we will deal with delay differential equations having a polynomial right-hand side.

Corollary 1.11. *If* f *is a polynomial over* t, $x(t)$ *and* $x(t - \tau_j)$, *then there exists a unique solution to the initial value problem with delay differential equation and piece-wise continuous initial condition* (1.2).

Proof. As a polynomial, f is continuously differentiable and hence locally Lipschitz. The existence of a unique solution follows from Theorem 1.9 and Corollary 1.10. □

[39] If we limit T by $\tau_{\min}$, we can see the solution of a delay differential equation as an operator mapping from functions on $[t - \tau_{\max}, t]$ to functions on $[t, t + \tau_{\min}]$. Then the solution of the initial value problem is the sequence of these functions. The notion of solution for an autonomous DDE as given above can be lifted to be a trajectory γ in the state space

$$\gamma \colon [0, T] \to C^1_{\mathrm{pw}}([\tau, 0], \mathbb{R}^n) \tag{1.6}$$

This notion of solution is a *dynamical systems* point of view which later turns out to be useful.

Other results know from ordinary differential equations can be adapted to delay differential equations, such as continuous (or even differentiable) dependence of the solution on initial data, see [4]. In the following chapters, we will only consider autonomous DDEs, i.e. restrict to the case of initial time $\sigma = 0$.

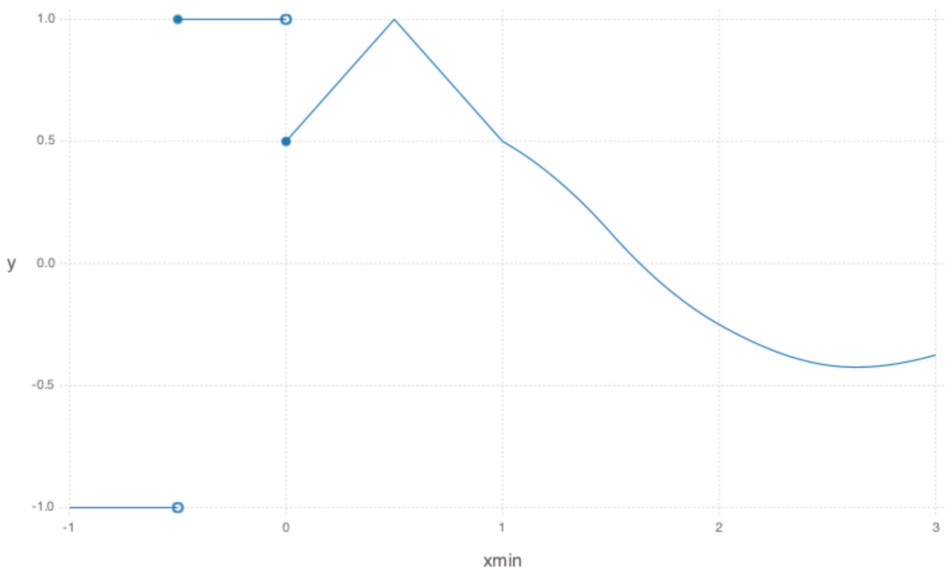
Example 1.12. Delay differential equations can often incorporate a much richer behavior than ordinary differential equations. The basic ordinary IVP

$$\begin{cases} x'(t) = -x(t) \\ x(0) = x_0 \end{cases} \tag{1.7}$$

has the solution $x(t) = x_0 \mathrm{e}^{-t}$. However the similar DDE

$$\begin{cases} x'(t) = -x(t-\tau) & t \geq 0 \\ x(t) = x_0(t) & -\tau \leq t \leq 0 \end{cases} \tag{1.8}$$

has a much richer dynamics, but solution (as series) for $x_0 \equiv 1$, can compute first solutions by method of steps.


1.0
0.5
y
0.0
-0.5
-1.0
-1
0
1
2
3
xmin

2 Delay Differential Dynamic Logic

We extent classical differential dynamic logic ($\mathsf{d}\mathcal{L}$) (see e.g. [30]) with syntax, semantics, axiomatization and proof rules to support reasoning about hybrid dynamical systems with delay.

To that purpose, we allow delay differential equations in hybrid programs, which are then called *delay hybrid program* (dHP).

The definition of *delay differential dynamic logic* ($\mathsf{dd}\mathcal{L}$) provides all operators of first-order logic, as well as modal operators, in order to specify and verify reachability properties about the state of such dHPs.

In the language of $\mathsf{dd}\mathcal{L}$, we can not only model hybrid programs with DDEs in the continuous part, but also with temporal differences in the discrete fragment. For example a controller which approximates a derivative by a difference quotient.

The logic $\mathsf{dd}\mathcal{L}$ is a superset of $\mathsf{d}\mathcal{L}$, i.e. in the absence of any delay, it reduces to classical *differential dynamic logic.*

2.1 Syntax

Terms and formulas in $\mathsf{dd}\mathcal{L}$, as well as dHPs are defined as *words* of finite length, produced by their corresponding grammars in Backus-Naur-form (BNF).

We define by $\mathcal{V}$ be the set of *all variables* and by $\mathcal{V}' \overset{\text{def}}{=} \{x' \mid x \in \mathcal{V}\}$ the corresponding set of *differential symbols.* Let $\mathcal{C} \subset \mathbb{Q}_0^-$ be the set of *constant parameters.* All three sets are supposed to be finite. We denote $\mathcal{V}[\mathcal{C}] \overset{\text{def}}{=} \{x[c] \mid x \in \mathcal{V},\ c \in \mathcal{C}\}$ as the set of *delay variables* and $\mathcal{V}'[\mathcal{C}] \overset{\text{def}}{=} \{x'[c] \mid x' \in \mathcal{V}',\ c \in \mathcal{C}\}$ as the set of *delay differentials.*

We will usually write variables as $x, y, z \in \mathcal{V}$ and their differential symbols as $x', y', z' \in \mathcal{V}'$. *Function symbols* f, g, h and *constant symbols* $a, b \in \mathbb{Q}$ are as in first-order logic (cf. Section **??**).

Moreover, we write θ, η for $\mathsf{dd}\mathcal{L}$ terms, ϕ, ψ for $\mathsf{dd}\mathcal{L}$ formulas and α, β for dHPs. For formulas of first-order logic of real arithmetic ($\mathrm{FOL}_\mathbb{R}$), we use the symbols χ and φ.

Definition 2.1 (s-Terms)**.** The syntax of *terms* of *delay differential dynamic logic* is defined by the following grammar:

$$\begin{aligned}\theta(s), \eta(s) ::= {}& x[s] \mid x'[s] \mid x[c] \mid x'[c] \mid a \mid \\ & f(\theta_1(s), \ldots, \theta_k(s)) \mid \theta(s) + \eta(s) \mid \theta(s) \cdot \eta(s) \mid (\theta(s))'\end{aligned}$$

where $x \in \mathcal{V}, x' \in \mathcal{V}'$ and f is a function symbol of arity k. The symbol $a \in \mathcal{C}$ stands for a constant value in $\mathbb{Q}$. The constant parameters $c \in \mathbb{Q}_0^-$ are not allowed to be positive.

The s-terms listed in the first line are called *atomic*, as opposed to the *composite* s-terms in the second line. S-terms generally depend on the time parameter $s \in \mathbb{R}_0^-$. This is why we write them as $\theta(s)$. If a s-term $\theta(s)$ does neither contain $x[s]$ nor $x'[s]$, we say $s \notin \theta(s)$ and abbreviate its notation to θ. Writing $\theta(b)$ means that all occurences of s in $\theta(s)$ have been replaced with $b \in \mathbb{Q}_0^-$. Moreover, we agree on abbreviating $x[0]$ to x and $x'[0]$ to x'. Note that $s \notin \mathcal{V} \cup \mathcal{V}' \cup \mathcal{C}$. It is a special variable symbol.

The *differential* $(\theta(s))'$ of a term $\theta(s)$ is its syntactic (total) derivation, obtained by standard differentiation rules. Lemma 2.12 shows the validity of these rules and that the result is again a s-term.

Subtraction can be defined using addition and multiplication, division would also be possible, if we can exclude any division by zero. The grammar allows in particular the construction of polynomial forms.

Example 2.2. Let us consider the s-term

$$\theta(s) = x[s] + x[-\tau].$$

Setting $s = -1$ gives the term

$$\theta(-1) = x[-1] + x[-\tau].$$

Delay differential dynamic logic uses hybrid programs with delay differential equations as system model. The grammar defining these *delayed hybrid programs* is the same as for classical HPs (cf. [32]).

Definition 2.3 (Delay Hybrid Programs)**.** The syntax of *delay hybrid programs* (dHPs) is defined by

$$\alpha, \beta ::= x := \theta \mid x' := \theta \mid ?\chi \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid x' = \theta \,\&\, \chi$$

where α, β denote dHPs, x a variable and θ a term (possibly containing x or $x[b]$, but no $x[s]$). The formula χ is of $\mathrm{FOL}_{\mathbb{R}}$, containing only normal variable symbols from $\mathcal{V}$.

Note that the syntax only allows autonomous DDEs, though with multiple constant delays.

Atomic dHPs are given by instantaneous discrete *assignments* $x := \theta(0)$ and *differential assignments* $x' := \theta(0)$, which change the value of the given variable only at the current time instant, not the past, *tests* $?\chi$, which pass only if the current state satisfies first-order formula χ of real arithmetic and abords the program execution if not, as well as evolutions along *delay differential equation* systems $x' = \theta(0) \,\&\, \chi$ of an arbitrary amount of time, but restricted by the evolution domain constraint χ.

Compound dHPs combine atomic programs, and comprise *nondeterministic choices* $\alpha \cup \beta$, running either α or β, *sequential compositions* $\alpha;\beta$, executing β after α and *nondeterministic repetitions* α^*, repeating α any number of times, zero times included.

Observe that ODEs are still expressible by this syntax and that hybrid programs are hence only delayed hybrid programs with zero delay.

The difference between classical HPs (as defined in $\mathsf{d}\mathcal{L}$, cf. [26, 30, 32]) and *delay hybrid programs* is not syntactical, but only given by their semantics.

Definition 2.4 (s-Formulas)**.** The syntax for *formulae* of *delay differential dynamic logic* is defined by the grammar

$$\begin{aligned}\phi(s), \psi(s) ::= \theta(s) = \eta(s) \mid \theta(s) \geq \eta(s) \mid p(\theta_1(s), \dots, \theta_k(s)) \mid \forall[-T)\, \phi(s) \mid \\ \neg\phi(s) \mid \phi(s) \wedge \psi(s) \mid \forall x\, \phi(s) \mid \exists x\, \phi(s) \mid [\alpha]\phi(s) \mid \langle\alpha\rangle\phi(s)\end{aligned}$$

with $\theta(s), \eta(s), \theta_1(s), \dots, \theta_k(s)$ as s-terms, p as predicate symbol, x as variable, and α as dHP.

These formulae combine connectives of propositional logic with first-order quantifiers (which both have standard meaning) and two modalities, describing *necessary* and *possible* properties.

The other comparison operators $<, \leq, >$ and logic connectives $\vee, \rightarrow, \leftrightarrow$ can be defined using $=, \wedge, \neg$ and are hence not explicitely mentioned in the grammar. Analogoulsy is $\exists x\, \phi$ expressible as $\neg\forall x\, \neg\phi$ and the modal formula $[\alpha]\phi$ (ϕ holds in the state after all runs of α) by its dual $\langle\alpha\rangle\phi \equiv \neg[\alpha]\neg\phi$ (there is at least one state reachable by α such that ϕ holds). The quantifiers $\forall$ and $\exists$ quantify over the state space $C^1_{\text{pw}}([-T, 0], \mathbb{R}^n)$.

Like the s-terms defined above, the s-formulae depend on a time parameter $s \in [-T, 0]$. The symbol T is a symbolic constant related to the length of the domain of the state space, which is induced by the occurence of delay symbols. Its value is defined by the static semantics and set by proof rules. The only way to bind the variable s in a formula $\phi(s)$ is by using $\forall[-T)\, \phi(s)$, which quantifies s over the domain of the state space, except for the current time point 0.

We note ϕ to indicate that s is not a free variable of $\phi(s)$ and $\phi(b)$ with $b \in \mathbb{Q}_0^-$ to express that each term $\theta(s)$ in the formula was replaced by its corresponding $\theta(b)$, even if it was bound by a $\forall[-T)$.

Formulas of first-order logic of real arithmetic constitute a subset of $\mathsf{dd}\mathcal{L}$, i.e. every $\text{FOL}_{\mathbb{R}}$ formula is also a formula of delay differential dynamic logic.

Convention 2.5. The frequently appearing fact that $\phi(s)$ is not only supposed to hold for $s \in [-T, 0)$ but also in $s = 0$

$$\forall[-T)\, \phi(s) \wedge \phi(0)$$

can also be written as

$$\forall[-T]\, \phi(s)$$

For convenience, we allow the latter, abbreviated notation, which is implicitely replaced by the former, syntactically correct version.

In order to simplify notation by eliminating parentheses, we agree on the following

Convention 2.6. The operators in $\mathsf{dd}\mathcal{L}$ formulae obey the following binding priorities (from highest to lowest):

- the quantifiers $\forall, \exists$ and the modal operators $[\cdot], \langle\cdot\rangle$ bind strongest

- negation $\neg$ binds stronger than
- conjunction $\wedge$ binds stronger than
- disjunction $\vee$ binds stronger than
- implication $\rightarrow$ binds stronger than
- equivalence $\leftrightarrow$, which binds weakest.

Moreover, when a s-formula does not depend on the quantified parameter s, we can drop the quantifier $\forall[-T)$ in which this formula appears.

Example 2.7. Consider the two well-formed $\mathsf{dd}\mathcal{L}$ formulae:

$$\forall[-T)\,(x + x[s] \geq 0)$$
$$\forall[-T)\,(x + x[-\tau] \geq 0)$$

The quantification over s in the second formula can be dropped, what leads to the equivalent formula

$$x + x[-\tau] \geq 0.$$

2.2 Dynamic Semantics

In this section, we give meaning to the syntax introduced above, by defining its semantics in a compositional way.

Following the remark to the solution of a DDE (cf. Section **??**), we define the *state space* in $\mathsf{dd}\mathcal{L}$ as $C^1_{\text{pw}}([-T,0],\mathbb{R}^n)$, the set of piecewise continuously differentiable functions on $[-T,0]$, as defined in Definition **??**. This means that a variable remembers a limited part of its evolution history, what demands hence an implicit notion of a underlying time.

We denote by $\mathcal{S}$ the *set of states.* A *state* $\nu \in \mathcal{S}$ is a mapping

$$\nu\colon \mathcal{V} \cup \mathcal{V}' \to C^1_{\text{pw}}([-T,0],\mathbb{R}^n) \tag{2.1}$$

which assigns a *history* (function) to each variable and differential symbol.

By $\nu[x \mapsto y]$ we denote the state which is equal to state ν, except for the value of the variable x, which is set to $y \in C^1_{\text{pw}}([-T,0],\mathbb{R}^n)$.

Definition 2.8 (Semantics of s-terms)**.** The *semantics* of a s-term $\theta(s)$ in the state $\nu \in \mathcal{S}$ with respect to the time instant $r \in [-T,0]$ is a value in $\mathbb{R}$ and defined inductively as follows:

1. $[\![x[s]]\!]^I_{\nu,r} = \nu(x)(r)$ for a variable $x \in \mathcal{V}$
2. $[\![x'[s]]\!]^I_{\nu,r} = \nu(x')(r) \stackrel{\text{def}}{=} \lim_{t\searrow r} \frac{\nu(x)(t)-\nu(x)(r)}{t-r}$ (except in $r = 0$)
3. $[\![x[c]]\!]^I_{\nu,r} = \nu(x)(c)$ for a variable $x \in \mathcal{V}$

4. $[\![x'[c]]\!]^I_{\nu,r} = \nu(x')(c) \stackrel{\text{def}}{=} \lim_{t\searrow c} \frac{\nu(x)(t)-\nu(x)(c)}{t-c}$ (except in $c = 0$)

5. $[\![a]\!]^I_{\nu,r} = I(a)$ for a constant $a \in \mathcal{C}$

6. $[\![f(\theta_1(s), \ldots, \theta_k(s))]\!]^I_{\nu,r} = I(f)([\![\theta_1(s)]\!]^I_{\nu,r}, \ldots, [\![\theta_k(s)]\!]^I_{\nu,r})$ for a function symbol f

7. $[\![\theta(s) + \eta(s)]\!]^I_{\nu,r} = [\![\theta(s)]\!]^I_{\nu,r} + [\![\eta(s)]\!]^I_{\nu,r}$

8. $[\![\theta(s) \cdot \eta(s)]\!]^I_{\nu,r} = [\![\theta(s)]\!]^I_{\nu,r} \cdot [\![\eta(s)]\!]^I_{\nu,r}$

9. $$[\![(\theta(s))']\!]^I_{\nu,r} = \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(I(c)) \frac{\partial [\![\theta(s)]\!]^I_{\nu,r}}{\partial x[b]} + \nu(x')(r) \frac{\partial [\![\theta(s)]\!]^I_{\nu,r}}{\partial x[s]}$$

where $c \in \mathbb{Q}_0^-$ is a non-positive rational number.

The meaning of the variable and differential symbols is determined by the state. Additionally, the value of a differential symbol has to coincide with the right derivative of the corresponding variable, except for $r = 0$.

The meaning of the differential of an arbitrary term is the total derivative of its value with respect to the underlying continuous time. As a composition of smooth functions is $[\![\theta(s)]\!]^I_{\nu,r}$ smooth itself and hence these derivatives exist. The sum is finite, since each term only mentions finitely many variables.

In the precondition, no values are associated to the differential symbols. In general, the initial function is only piecewise continuous. Since for later time instances, the values of the differential symbols derive from the DDE, they become (locally) smooth function.

Definition 2.9 (Semantics of s-formulae)**.** The semantics of a $\mathsf{dd}\mathcal{L}$ formula ϕ is the subset of all states $[\![\phi]\!]^I_r \subseteq \mathcal{S}$ in which ϕ is true. This set is given inductively by

1. $[\![\theta(s) = \eta(s)]\!]^I_r = \left\{\nu \in \mathcal{S} \;\middle|\; [\![\theta(s)]\!]^I_{\nu,r} = [\![\eta(s)]\!]^I_{\nu,r}\right\}$

2. $[\![\theta(s) \geq \eta(s)]\!]^I_r = \left\{\nu \in \mathcal{S} \;\middle|\; [\![\theta(s)]\!]^I_{\nu,r} \geq [\![\eta(s)]\!]^I_{\nu,r}\right\}$

3. $[\![p(\theta_1(s), \ldots, \theta_k(s))]\!]^I_r = \left\{\nu \in \mathcal{S} \;\middle|\; \left([\![\theta_1(s)]\!]^I_{\nu,r}, \ldots, [\![\theta_k(s)]\!]^I_{\nu,r}\right) \in I(p)\right\}$

4. $[\![\neg\phi(s)]\!]^I_r = \left([\![\phi(s)]\!]^I_r\right)^{\complement} = \mathcal{S} \setminus [\![\phi(s)]\!]^I_r$

5. $[\![\phi(s) \wedge \psi(s)]\!]^I_r = [\![\phi(s)]\!]^I_r \cap [\![\psi(s)]\!]^I_r$

6. $[\![\forall[-T]\, \phi(s)]\!]^I_r = \left\{\nu \in \mathcal{S} \;\middle|\; \forall \tilde{r} \in [-T, 0) \,:\, \nu \in [\![\phi(s)]\!]^I_{\tilde{r}}\right\}$

7. $[\![\forall x\, \phi(s)]\!]^I_r = \left\{\nu \in \mathcal{S} \;\middle|\; \nu[x \mapsto y] \in [\![\phi(s)]\!]^I_r \text{ for all } y \in C^1_{\text{pw}}([-T, 0], \mathbb{R}^n)\right\}$

8. $[\![\exists x\, \phi(s)]\!]_r^I = \left\{\nu \in \mathcal{S} \,\middle|\, \nu[x \mapsto y] \in [\![\phi(s)]\!]_r^I \text{ for some } y \in C^1_{\mathrm{pw}}([-T,0],\mathbb{R}^n)\right\}$

9. $[\![[\alpha]\phi(s)]\!]_r^I = \left\{\nu \in \mathcal{S} \,\middle|\, \omega \in [\![\phi(s)]\!]_r^I \text{ for all } \omega \text{ such that} (\nu,\omega) \in \rho(\alpha)\right\}$,
   i.e. $= \left\{\nu \in \mathcal{S} \,\middle|\, \forall \omega \in \mathcal{S} :\ (\nu,\omega) \in \rho(\alpha) \Rightarrow \omega \in [\![\phi(s)]\!]_r^I\right\}$

10. $[\![\langle\alpha\rangle\phi(s)]\!]_r^I = \left\{\nu \in \mathcal{S} \,\middle|\, \omega \in [\![\phi(s)]\!]_r^I \text{ for some } \omega \text{ such that} (\nu,\omega) \in \rho(\alpha)\right\}$,
   i.e. $= \left\{\nu \in \mathcal{S} \,\middle|\, \exists \omega \in \mathcal{S} :\ (\nu,\omega) \in \rho(\alpha) \wedge \omega \in [\![\phi(s)]\!]_r^I\right\}$

The fact that formula $\phi(s)$ is true in state ν under the interpretation I at past time instant $r \in [-T,0]$, i.e. $\nu \in [\![\phi(s)]\!]_r^I$ can also be written as $I,\nu,r \models \phi(s)$. A formula $\phi(s)$ is called valid, written as $\models \phi(s)$, if and only if $\phi(s)$ is true in all states, for all $r \in [-T,0]$ and under all interpretations.

As in classic first-order logic, the interpretation of a predicate symbol of arity n is a relation $I(p) \subseteq \mathbb{R}^n$.

Lemma 2.10 (Barcan formula)**.** *The box modality and the quantification over s commute*

$$[\![\forall[-T]\,[\alpha]\phi(s)]\!]_r^I = [\![[\alpha]\,(\forall[-T]\,\phi(s))]\!]_r^I$$

Proof. Since $\forall x :\ (p \Rightarrow q(x)) \equiv p \Rightarrow \forall x :\ q(x)$, it holds

$$\begin{aligned}
&[\![\forall[-T]\,[\alpha]\phi(s)]\!]_r^I = \\
&\quad = \left\{\nu \in \mathcal{S} \,\middle|\, \forall \tilde{r} \in [-T,0) :\ \forall \omega \in \mathcal{S} :\ \Big((\nu,\omega) \in \rho(\alpha) \Rightarrow \omega \in [\![\phi(s)]\!]_{\tilde{r}}^I\Big)\right\} \\
&\quad = \left\{\nu \in \mathcal{S} \,\middle|\, \forall \omega \in \mathcal{S} :\ \forall \tilde{r} \in [-T,0) :\ \Big((\nu,\omega) \in \rho(\alpha) \Rightarrow \omega \in [\![\phi(s)]\!]_{\tilde{r}}^I\Big)\right\} \\
&\quad = \left\{\nu \in \mathcal{S} \,\middle|\, \forall \omega \in \mathcal{S} :\ \Big((\nu,\omega) \in \rho(\alpha) \Rightarrow \forall \tilde{r} \in [-T,0) :\ \omega \in [\![\phi(s)]\!]_{\tilde{r}}^I\Big)\right\} \\
&\quad = \left\{\nu \in \mathcal{S} \,\middle|\, \forall \omega \in \mathcal{S} :\ \Big((\nu,\omega) \in \rho(\alpha) \Rightarrow \omega \in [\![\forall[-T]\,\phi(s)]\!]_r^I\Big)\right\} \\
&\quad = [\![[\alpha]\,(\forall[-T]\,\phi(s))]\!]_r^I
\end{aligned}$$

□

However, the diamond modality does not commute with the s-quantification.

$$[\![\forall[-T]\,\langle\alpha\rangle\phi(s)]\!]_r^I \neq [\![\langle\alpha\rangle\forall[-T]\,\phi(s)]\!]_r^I$$

Definition 2.11 (Transition semantics of dHPs)**.** The interpretation of a dHP is given by a binary *reachability relation* $\rho(\alpha) \subseteq \mathcal{S} \times \mathcal{S}$ between states:

1. $\rho(x := \theta) = \left\{(\nu,\omega)\ :\ \omega = \nu \text{ except } \omega(x) = \left(r \mapsto \begin{cases} [\![\theta(s)]\!]_{\nu,r}^I & r = 0 \\ \nu(x)(r) & r \in [-T,0) \end{cases}\right)\right\}$

2. $\rho\left(x' := \theta\right) = \left\{(\nu, \omega) \ : \ \omega = \nu \text{ except } \omega(x') = \left(r \mapsto \begin{cases} [\![\theta(s)]\!]^{I}_{\nu,r} & r = 0 \\ \nu(x')(r) & r \in [-T, 0) \end{cases}\right)\right\}$

3. $\rho\left(?\chi\right) = \left\{(\nu, \nu) \ : \ \nu \in [\![\chi]\!]^{I}_{r}\right\}$

4. $\rho\left(\alpha \cup \beta\right) = \rho\left(\alpha\right) \cup \rho\left(\beta\right)$

5. $\rho\left(\alpha; \beta\right) = \{(\nu, \omega) \ : \ (\nu, \mu) \in \rho\left(\alpha\right), (\mu, \omega) \in \rho\left(\beta\right)\}$

6. $\rho\left(\alpha^{*}\right) = \bigcup_{n \in \mathbb{N}_0} \rho\left(\alpha^{n}\right)$ with $\alpha^{n+1} \equiv (\alpha^{n}; \alpha)$ and $\alpha^{0} \equiv (?\mathit{true})$

7. $\rho\left(x' = \theta \,\&\, \chi\right) = \{(\nu, \omega) \mid \forall \zeta \in [0, r] \ : \ \gamma(\zeta) \in [\![x' = \theta \wedge \chi]\!]^{I}_{r}$ and $\nu = \gamma(0)$ on $\{x'\}^{\complement}$ and $\omega = \gamma(r)$ for a $\gamma\colon [0, r] \to \mathcal{S}\}$, i.e. there exists a $r \geq 0$ and a trajectory $\gamma\colon [0, r] \to \mathcal{S}$, which fulfills $\gamma(\zeta)(x')(s) \stackrel{\text{def}}{=} \frac{\mathrm{d}\gamma(t)(x)(s)}{\mathrm{d}t}(\zeta) \stackrel{!}{=} [\![\theta]\!]^{I}_{\gamma(\zeta+s),r}$ and satisfies χ for all $s \in [-\min\{\zeta, T\}, 0]$. On $[-T, -\min\{\zeta, T\})$ it holds $\gamma(\zeta)(\cdot)(s) = \nu(\cdot)(s + \zeta)$ for all variables.

The semantics of a delay differential equation is motivated by the definition of a solution for a DDE-IVP (cf. Definition 1.6), following the evolution for a nondeterministic period of time, as long as the evolution domain constraint holds.

initial value $\nu(x')$ may not be compatible with derivative final values coincide

For the *discrete assignment*, we only allow the values at the current time instant to be changed. A functional assignment would essentially allow to rewrite history, which is not permitted.

The jump behavior caused be discrete assignments is the actual reason why we need to consider piecewise continuous evolutions.

Time is implicit and usually not revealed. If it is explicitely needed, a clock variable t can be introduced by $t' = 1$.

As a $\mathrm{FOL}_{\mathbb{R}}$ formula, χ do not contain any delayed variables and thus only depends on the values at the current time instant (and not over the entire interval $[-T, 0)$).

Lemma 2.12 (Derivations)**.** *Standard analysis derivation rules also hold in the semantics of* $\mathsf{dd}\mathcal{L}$ *terms, i.e. the following equations are valid* $\mathsf{dd}\mathcal{L}$ *formulas*

$$(x[s])' = x'[s] \tag{2.2}$$
$$(x[c])' = x'[c] \tag{2.3}$$
$$(a)' = 0 \tag{2.4}$$
$$(f(\theta_1, \ldots, \theta_k))' = \tag{2.5}$$
$$(\theta + \eta)' = (\theta)' + (\eta)' \tag{2.6}$$
$$(\theta \cdot \eta)' = (\theta)' \cdot \eta + \theta \cdot (\eta)' \tag{2.7}$$
$$\tag{2.8}$$

This allows to apply these rules on a syntactic level, what will be done in the form of axioms (see **??***).*

Proof.

$$\begin{aligned}
[\![(x[s])']\!]^I_{\nu,r} &= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![x[s]]\!]^I_{\nu,r}}{\partial x[c]} + \nu(x')(r)\frac{\partial[\![x[s]]\!]^I_{\nu,r}}{\partial x[s]} \\
&= \nu(x')(r)\frac{\partial[\![x[s]]\!]^I_{\nu,r}}{\partial x[s]} = \nu(x')(r) = [\![x'[s]]\!]^I_{\nu,r}
\end{aligned}$$

$$\begin{aligned}
[\![(x[c])']\!]^I_{\nu,r} &= \sum_{x[d]\in\mathcal{V}[\mathcal{C}]} \nu(x')(d)\frac{\partial[\![x[c]]\!]^I_{\nu,r}}{\partial x[d]} + \nu(x')(r)\frac{\partial[\![x[c]]\!]^I_{\nu,r}}{\partial x[s]} \\
&= \nu(x')(c)\frac{\partial[\![x[c]]\!]^I_{\nu,r}}{\partial x[c]} = \nu(x')(c) = [\![x'[b]]\!]^I_{\nu,r}
\end{aligned}$$

$$\begin{aligned}
[\![(a)']\!]^I_{\nu,r} &= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![a]\!]^I_{\nu,r}}{\partial x[c]} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\![(f(\theta_1(s),\ldots,\theta_k(s)))']\!]^I_{\nu,r} &= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![f(\theta_1(s),\ldots,\theta_k(s))]\!]^I_{\nu,r}}{\partial x[c]} \\
&\quad + \nu(x')(r)\frac{\partial[\![f(\theta_1(s),\ldots,\theta_k(s))]\!]^I_{\nu,r}}{\partial x[s]}
\end{aligned}$$

$$\begin{aligned}
[\![(\theta(s)+\eta(s))']\!]^I_{\nu,r} &= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![\theta(s)+\eta(s)]\!]^I_{\nu,r}}{\partial x[c]} + \nu(x')(r)\frac{\partial[\![\theta(s)+\eta(s)]\!]^I_{\nu,r}}{\partial x[s]} \\
&= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![\theta(s)]\!]^I_{\nu,r}+[\![\eta(s)]\!]^I_{\nu,r}}{\partial x[c]} + \nu(x')(r)\frac{\partial[\![\theta(s)]\!]^I_{\nu,r}+[\![\eta(s)]\!]^I_{\nu,r}}{\partial x[s]} \\
&= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![\theta(s)]\!]^I_{\nu,r}}{\partial x[c]} + \nu(x')(r)\frac{\partial[\![\theta(s)]\!]^I_{\nu,r}}{\partial x[s]} \\
&+ \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![\eta(s)]\!]^I_{\nu,r}}{\partial x[c]} + \nu(x')(r)\frac{\partial[\![\eta(s)]\!]^I_{\nu,r}}{\partial x[s]} \\
&= [\![(\theta(s))']\!]^I_{\nu,r} + [\![(\eta(s)']\!]^I_{\nu,r} = [\![(\theta(s))' + (\eta(s))']\!]^I_{\nu,r}
\end{aligned}$$

$$
\begin{aligned}
[\![(\theta(s)\cdot\eta(s))']\!]^I_{\nu,r} &= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![\theta(s)\cdot\eta(s)]\!]^I_{\nu,r}}{\partial x[c]} + \nu(x')(r)\frac{\partial[\![\theta(s)\cdot\eta(s)]\!]^I_{\nu,r}}{\partial x[s]} \\
&= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial\left([\![\theta(s)]\!]^I_{\nu,r}\cdot[\![\eta(s)]\!]^I_{\nu,r}\right)}{\partial x[c]} + \nu(x')(r)\frac{\partial\left([\![\theta(s)]\!]^I_{\nu,r}\cdot[\![\eta(s)]\!]^I_{\nu,r}\right)}{\partial x[s]} \\
&= \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![\theta(s)]\!]^I_{\nu,r}}{\partial x[c]}[\![\eta(s)]\!]^I_{\nu,r} + \nu(x')(r)\frac{\partial[\![\theta(s)]\!]^I_{\nu,r}}{\partial x[s]}[\![\eta(s)]\!]^I_{\nu,r} \\
&+ \sum_{x[c]\in\mathcal{V}[\mathcal{C}]} \nu(x')(c)\frac{\partial[\![\eta(s)]\!]^I_{\nu,r}}{\partial x[c]}[\![\theta(s)]\!]^I_{\nu,r} + \nu(x')(r)\frac{\partial[\![\eta(s)]\!]^I_{\nu,r}}{\partial x[s]}[\![\theta(s)]\!]^I_{\nu,r} \\
&= [\![(\theta(s))']\!]^I_{\nu,r}\cdot[\![\eta(s)]\!]^I_{\nu,r} + [\![\theta(s)]\!]^I_{\nu,r}\cdot[\![(\eta(s)']\!]^I_{\nu,r} \\
&= [\![(\theta(s))'\cdot\eta(s)+\theta(s)\cdot(\eta(s)']\!]^I_{\nu,r}
\end{aligned}
$$

□

Definition 2.13. We define by

$$\mathcal{C}_\theta \stackrel{\text{def}}{=} \{c\in\mathcal{C} \mid \exists x[]\in\mathcal{V}:\ x[c]\in\theta(s)\}$$

the set of constant parameter symbols occuring in the s-term $\theta(s)$.

Note that this set does not contain s, since it is, as a special purpose symbol, not in $\mathcal{C}$.

Definition 2.14 (Sampled trajectory)**.** Since a s-term $\theta(s)$ only comprises a finite number of atomic terms, its valuation can also be seen as a mapping

$$[\![\theta(s)]\!]^I : \mathbb{R}^{|\mathcal{K}|} \to \mathbb{R}$$

from the concrete values for each element of $\mathcal{K} \stackrel{\text{def}}{=} \mathcal{V}[\mathcal{C}_\theta]\cup\mathcal{V}'[\mathcal{C}_\theta]\cup\{x[s],x'[s]\}$ into the reals, if we assign a fixed $r\in[-T,0]$ to s.

This gives rise to the definition of the *sampled trajectory* $\hat{\gamma}^r_\theta : [0,R]\to\mathbb{R}^{|\mathcal{K}|}$ for a fixed $r\in[-T,0]$ and s-term $\theta(s)$ (without loss of generality, considering $\mathcal{V}=\{x\}$)

$$\hat{\gamma}^r_\theta(t) \stackrel{\text{def}}{=} \begin{pmatrix}\gamma(t)(x)(c_1)\\ \vdots \\ \gamma(t)(x)(c_n)\end{pmatrix}$$

The following lemma shows the consistency of the semantics for differentials with the semantics of the evolution of a delay differential equation. This means that along a DDE, the values of differential symbols coincide with the time derivative of the value of the corresponding variable.

Lemma 2.15 (Differential Lemma)**.** *The value of a s-term $\eta(s)$ along a trajectory $\gamma\colon [0,R] \to \mathcal{S}$ satisfying a DDE for any duration $R > 0$, i.e. $I, \gamma \models (x' = \theta \wedge \chi)$, is piecewise continuously differentiable and for all $\zeta \in [0,R]$ and $r \in [-T,0]$ it holds:*

$$[\![(\eta(s))']\!]^I_{\gamma(\zeta),r} = \frac{\mathrm{d}[\![\eta(s)]\!]^I_{\gamma(t),r}}{\mathrm{d}t}(\zeta)$$

As in Definition 1.1, the derivative at a discontinuity point of is to be understood as right derivative.

Proof. Without loss of generality, we restrict in this proof to a single variable x. If $\eta(s)$ depends on more variables, consider the union of their partitions in the initial condition. Let $\{-T = t_0 < \ldots < t_m = 0\}$ be the partition of the initial condition $\gamma(0)(x) \in C^1_{\mathrm{pw}}([-T,0],\mathbb{R}^n)$.

We choose an arbitray but fixed valuation for s from $[-T,0]$, such that the symbol s can be treated as a constant, in the same way as any $c \in \mathbb{Q}_0^-$. Depending on the s-term $\eta(s)$ and the fixed s, we define a partition $\mathcal{Z}^s_\eta = \{\hat{t}_0 < \ldots < \hat{t}_p\}$ of $[0,\infty)$ (which can be limited to $[0,R]$) by

$$\mathcal{Z}^s_\eta \stackrel{\text{def}}{=} \{0\} \cup \bigcup_{i=0}^{m} \bigcup_{\substack{c\in\mathcal{K}\\ t_i \geq c}} \{t_i - c\} \cup \bigcup_{i=0}^{m} \bigcup_{j=1}^{k} \bigcup_{\substack{c\in\mathcal{K}\\ t_i+\tau_j \geq c}} \{t_i + \tau_j - c\}$$

where $\mathcal{K} \stackrel{\text{def}}{=} \mathcal{C}_\eta \cup \{s\}$ is the set of the constants (the interpretations/valuations of the constant symbols) appearing in the term η and $\tau_j \in \mathcal{C}_\theta$ the delays in the right hand side of the DDE. The set $\mathcal{Z}^s_\eta$ is finite and non-empty, since it contains at least the value 0 and R.

We show first that $\gamma(t)(x)(c)$ is piecewise continuously differentiable in t for all $c \in \mathcal{K}$ with partition $\mathcal{Z}^s_\eta$:

Let $c \in \mathcal{K}$ and $\zeta \in (\hat{t}_l, \hat{t}_{l+1})$. Assume that $\zeta + c = t_i$ for some i. This implies $\zeta = t_i - c = \hat{t}_k$ for some k by the definition of the partition. This is not possible by the choice of ζ lying between two consequtive $\hat{t}_l$. We apply the same argumentation to the assumption $\zeta + c = t_i + \tau_j$. These contradictions show that for $\zeta \in (\hat{t}_j, \hat{t}_{j+1})$, it holds that $\zeta + c \neq t_i$ and $\zeta + c \neq t_i + \tau_j$ for all $c \in \mathcal{K}$ and for all $i \in \{0,\ldots,m\}$, $j \in \{1,\ldots,k\}$. We now distinguish two cases:

If $\zeta + c < 0$, it holds by the definition of the DDE semantics (Definition **??**(7)) that $\gamma(\zeta)(x)(c) = \gamma(0)(x)(\zeta + c)$, which is continuously differentiable as initial condition, if $\zeta + c \neq t_i$. Hence it follows

$$\frac{\mathrm{d}\gamma(t)(x)(c)}{\mathrm{d}t}(\zeta) = \frac{\mathrm{d}\gamma(0)(x)(r)}{\mathrm{d}r}(\zeta + c) = \gamma(0)(x')(\zeta + c) = \gamma(\zeta)(x')(c)$$

For the right limit it holds

$$\begin{aligned}\lim_{\zeta\searrow\hat{t}_j} \frac{\mathrm{d}\gamma(t)(x)(c)}{\mathrm{d}t}(\zeta) &= \lim_{\zeta\searrow t_i - c} \frac{\mathrm{d}\gamma(0)(x)(r)}{\mathrm{d}r}(\zeta + c)\\ &= \lim_{\zeta\searrow t_i} \frac{\mathrm{d}\gamma(0)(x)(r)}{\mathrm{d}r}(\zeta) = \gamma(0)(x')(t_i)\end{aligned}$$

And analogously for the existence of the left limit for $\zeta \nearrow \hat{t}_{j+1}$

If $\zeta + c \geq 0$, then $\gamma(\zeta)(x)(r) = \gamma(\zeta + c)(x)(0)$ is differentiable in ζ with

$$\gamma(\zeta)(x')(r) = \frac{\mathrm{d}\gamma(t)(x)(r)}{\mathrm{d}t}(\zeta)$$

by the semantics of the DDEs, if $\zeta + c \neq t_i + \tau_j$.

Let $\hat{\gamma}^s_\eta$ be the η-sampled trajectory for the considered delay differential equation and the fixed s. It follows with the above results

$$\begin{aligned}
\frac{\mathrm{d}[\![\eta]\!]^I_{\hat{\gamma}^s_\eta(\zeta)}}{\mathrm{d}t} &= \Big([\![\eta]\!]^I \circ \hat{\gamma}^s_\eta(\zeta)\Big)' = \nabla[\![\eta]\!]^I(\hat{\gamma}^s_\eta(\zeta)) \cdot \frac{\mathrm{d}\hat{\gamma}^s_\eta}{\mathrm{d}t}(\zeta) \\
&= \sum_{x[c] \in \mathcal{V}[\mathcal{C}_\eta]} \frac{\mathrm{d}\gamma(t)(x)(c)}{\mathrm{d}t}(\zeta) \frac{\partial [\![\eta]\!]^I_{\hat{\gamma}^s_\eta(\zeta),s}}{\partial(x[c])} \\
&= \sum_{x[c] \in \mathcal{V}[\mathcal{C}]} \gamma(\zeta)(x')(c) \frac{\partial [\![\eta]\!]^I_{\hat{\gamma}^s_\eta(\zeta),s}}{\partial(x[c])} \\
&= [\![(\eta')]\!]^I_{\hat{\gamma}^s_\eta(\zeta)}
\end{aligned}$$

where each sum only consits of finitely many summands. Moreover, it holds for the right limits

$$\lim_{\zeta \searrow \hat{t}_j} \frac{\mathrm{d}[\![\eta]\!]^I_{\hat{\gamma}^s_\eta(\zeta),s}}{\mathrm{d}t} = [\![(\eta')]\!]^I_{\hat{\gamma}^s_\eta(\hat{t}_j)}$$

and the left limits for $\zeta \nearrow \hat{t}_{j+1}$ exist. □

Example 2.16. As an example for the construction of the partition in Proof 2.2, consider the s-term $\eta(s) \equiv x + x[-3.5] + x[s]$ together with the DDE $x' = x[-4]$. Let

$$\mathcal{Z} = \{-4, -3.25, -2, -1.2, 0\}$$

be the partition of some initial condition. Choosing $r = -1.8$ for s, we obtain

$$\mathcal{Z}^r_\eta = \{0, 0.25, 0.6, 0.75, 1.5, 1.8, 2, 2.3, 2.55, 2.8, 3.5, 3.8, 4\}$$

when we restrict the evolution to $[0, 4]$. Figure 2.1 depicts an example for the piecewise continuous differentiability of the term's evolution, given some initial condition.

Lemma 2.17 (Differential assignment)**.** *Let $\gamma\colon [0, R] \to \mathcal{S}$ be a trajectory satisfying a DDE for any duration $R \geq 0$, i.e. $I, \gamma \models (x' = \theta \wedge \chi)$. Then it holds:*

$$I, \gamma, r \models \phi(s) \leftrightarrow \gamma(\zeta) \in [\![[x' := \theta]\phi(s)]\!]^I_r$$

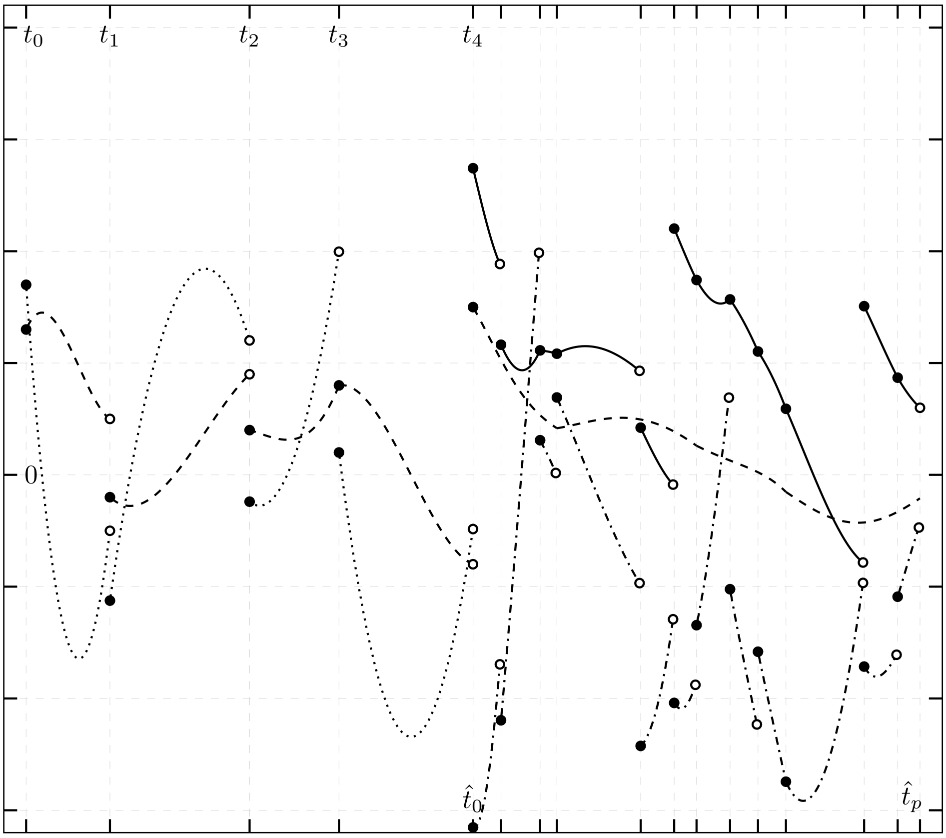


Figure 2.1: Plot for Example 2.16, initial condition and solution of DDE (dashed), derivatives (dotted), value of term (solid) and the derivative of the term (dash-dotted).

Proof. Let $\zeta \in [0, R]$. It is $\gamma(\zeta) \in [\![x'[0] = \theta]\!]^I_r$ and $\gamma(\zeta) \in [\![\chi]\!]^I_r$, which means $\gamma(\zeta)(x')(0) = [\![\theta]\!]^I_{\gamma(\zeta),r}$, since θ is independent of s. By Definition 2.11(1) of the assignment's semantics, this implies $(\gamma(\zeta), \omega) \in \rho\,(x' := \theta)$ if and only if $\omega = \gamma(\zeta)$. Finally, this implies the equivalence

$$\begin{aligned}\gamma(\zeta) \in [\![\phi(s)]\!]^I_r &\leftrightarrow \forall \omega \in \mathcal{S} :\ \Big((\gamma(\zeta), \omega) \in \rho\,(x' := \theta) \to \omega \in [\![\phi(s)]\!]^I_r\Big) \\ &\leftrightarrow \gamma(\zeta) \in [\![[x' := \theta]\phi(s)]\!]^I_r\end{aligned}$$

□

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