Delay Differential Logic for Hybrid Systems with Delay

Stage de Recherche à CMU

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Time-delay Systems

- $\mathsf{d}\mathcal{L}$ for hybrid (dynamical) systems with ODE
- CPS: connection between physical world and cyber part may be delayed
- introduce **delay differential dynamic logic** ($\mathsf{dd}\mathcal{L}$)
- a first-order modal logic for time-delay systems

Example

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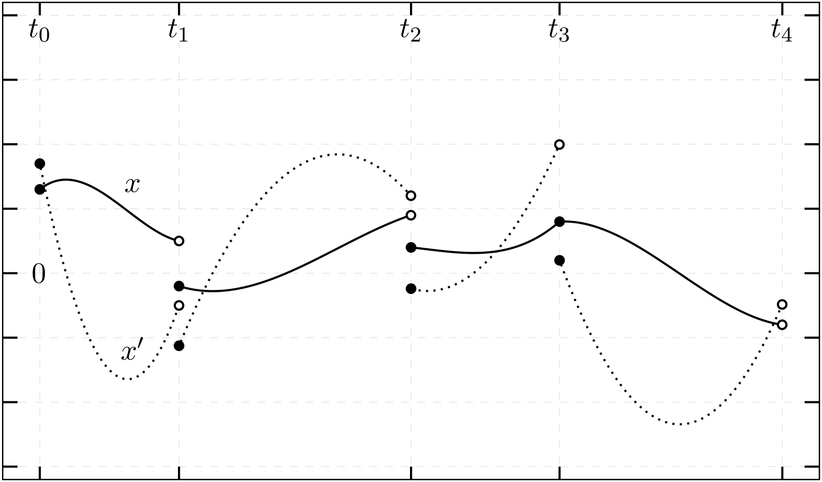
Piecewise



Piecewise

Definition (Piecewise Continuously Differentiable)

Given a finite partition $\{a = t_0 < t_1 < \ldots < t_p = b\}$ of $[a, b]$,

$$x\colon [a, b] \to \mathbb{R}^n$$

is m-times **piecewise continuously differentiable** iff

1. x is m-times continuously differentiable on each (t_i, t_{i+1})
2. $\lim\limits_{\substack{t \nearrow t_{i+1} \\ t \in (t_i, t_{i+1})}} x^{(k)}(t)$ exist
3. $\lim\limits_{\substack{t \searrow t_i \\ t \in (t_i, t_{i+1})}} x^{(k)}(t) = x^{(k)}(t_i)$

for all $k = 0, \ldots, m$.

Delay Differential Equations

Definition (DDE)

For $\{\tau_j \in \mathbb{R} \mid 0 < \tau_1 < \ldots < \tau_k\}$ and $f \colon \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$

$$x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k))$$

is a **delay differential equation** with multiple constant delays. Let $\tau_{\max} \stackrel{\text{def}}{=} \tau_k$ **maximal** and $\tau_{\min} \stackrel{\text{def}}{=} \tau_1$ **minimal delay**.

Definition (IVP)

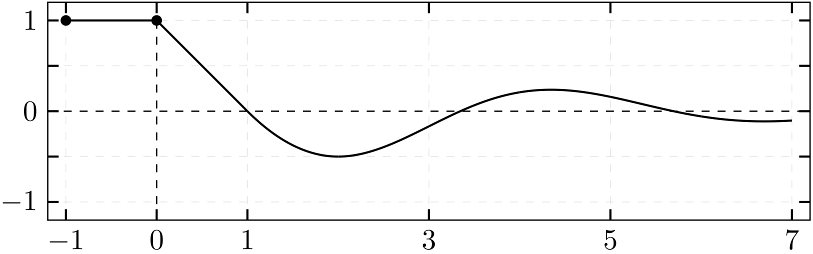
For an **initial condition** $x_\sigma \colon [\sigma - \tau_{\max}, \sigma] \to \mathbb{R}^n$, solving

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k)) & \text{for } t \geq \sigma \\ x(t) = x_\sigma(t) & \text{for } t \in [\sigma - \tau_{\max}, \sigma] \end{cases}$$

is the **initial value problem**.

Example DDE

$$\begin{cases} x'(t) = -x(t-1) & t \geq 0 \\ x(t) = 1 & t \in [-1, 0] \end{cases}$$



Solutions

Theorem (Existence of a unique solution)

Let $f\colon \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}^n$ continuous and Lipschitz in its first argument and x_σ ***piecewise continuous****,*
then there ***exists*** *a* ***unique local solution*** *of the IVP on a time interval $[\sigma - \tau_{\max}, \sigma + T]$.*

dd$\mathcal{L}$ Syntax

Notation for **hybrid programs** with DDEs:

Definition (dHPs)

Delay hybrid programs are defined by

$$\alpha, \beta ::= x := \theta \mid x' := \theta \mid ?\phi \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid x' = \theta \,\&\, \chi$$

with dd$\mathcal{L}$ term θ, dd$\mathcal{L}$ formula ϕ, $\mathrm{FOL}_{\mathbb{R}}$ formula χ.

dd$\mathcal{L}$ Syntax

Definition (s-Terms)

S-terms are defined by

$$
\begin{array}{rll}
\theta(s), \eta(s) ::= & a & \text{constants} \\
& \mid x[s] \mid x'[s] & \text{delay range symbols} \\
& \mid x[c] \mid x'[c] & \text{const. delay symbols} \\
& \mid f(\theta_1(s), \ldots, \theta_k(s)) & \text{functions} \\
& \mid \theta(s) + \eta(s) & \text{addition} \\
& \mid \theta(s) \cdot \eta(s) & \text{multiplication} \\
& \mid (\theta(s))' & \text{differentials}
\end{array}
$$

with $c \in \mathbb{Q}_0^-$ constant parameter, s past parameter, variable $x \in \mathcal{V}$, differential symbol $x' \in \mathcal{V}'$.

If $x[s] \notin \theta(s)$ and $x'[s] \notin \theta(s)$, write θ.

dd$\mathcal{L}$ Syntax

Definition (s-Formulas)

S-formulas are defined by

$$\begin{aligned} \phi(s), \psi(s) ::= {} & \forall[-T)\, \phi(s) && \text{state domain} \\ & \mid \theta(s) = \eta(s) \mid \theta(s) \geq \eta(s) && \text{comparisons} \\ & \mid p(\theta_1(s), \ldots, \theta_k(s)) && \text{predicates} \\ & \mid \neg\phi(s) \mid \phi(s) \land \psi(s) && \text{propositional logic} \\ & \mid \forall x\, \phi(s) \mid \exists x\, \phi(s) && \text{quantifiers} \\ & \mid [\alpha]\phi(s) \mid \langle\alpha\rangle\phi(s) && \text{modalities} \end{aligned}$$

with s-terms $\theta(s), \eta(s)$. $T \geq 0$ is defined by static semantics.

The only way to **bind** s is by $\forall[-T)$. Write ϕ if s is **not free**.
$\phi(r)$ substitutes s by $r \in \mathbb{R}_0^-$, even if in scope of $\forall[-T)$.

Semantics

- state space $C^1_{\mathrm{pw}}([-T,0],\mathbb{R}^n)$
- valuation depends on assignment $r \in [-T,0]$ to s
- Example

Proof Calculus

Hilbert style calculus with proof rules:

$$(\mathrm{G}) \quad \frac{\phi(s)}{[\alpha]\phi(s)}$$

$$(\mathrm{MP}) \quad \frac{\phi(s) \to \psi(s) \quad \phi(s)}{\psi(s)}$$

$$(\forall) \quad \frac{\phi(s)}{\forall x\, \phi(s)}$$

Axiomatization

$\langle\cdot\rangle$	$\langle\alpha\rangle\phi(s) \leftrightarrow \neg[\alpha]\neg\phi(s)$
$[\cup]$	$[\alpha \cup \beta]\phi(s) \leftrightarrow [\alpha]\phi(s) \wedge [\beta]\phi(s)$
$[;]$	$[\alpha;\beta]\phi(s) \leftrightarrow [\alpha][\beta]\phi(s)$
$[*]$	$[\alpha^*]\phi(s) \leftrightarrow \phi(s) \wedge [\alpha][\alpha^*]\phi(s)$
K	$[\alpha]\big(\phi(s) \to \psi(s)\big) \to \big([\alpha]\phi(s) \to [\alpha]\psi(s)\big)$
I	$[\alpha^*]\big(\phi(s) \to [\alpha]\phi(s)\big) \to \big(\phi(s) \to [\alpha^*]\phi(s)\big)$
B	$\forall x\,[\alpha]\phi(s) \to [\alpha]\forall x\,\phi(s) \qquad (x \notin \alpha)$
V	$\phi(s) \to [\alpha]\phi(s) \qquad (\mathrm{FV}(\phi) \cap \mathrm{BV}(\alpha) = \emptyset)$

Axiomatization

Assignment only changes present value, not past!

$$[:=] \quad [x := \theta]\phi(s, x[0]) \leftrightarrow \phi(s, \theta)$$

Test condition over entire state possible:

$$[?] \quad [?\psi]\phi(s) \leftrightarrow \big(\psi \to \phi(s)\big)$$

Axiomatization

$$\text{c}' \quad (a)' = 0$$

$$x[\cdot]' \quad x'[c] = (x[c])', \quad x'[s] = (x[s])'$$

$$+' \quad (\theta(s) + \eta(s))' = (\theta(s))' + (\eta(s))'$$

$$\cdot' \quad (\theta(s) \cdot \eta(s))' = (\theta(s))' \cdot \eta(s) + \theta(s) \cdot (\eta(s))'$$

$$\text{DW} \quad [x' = \theta \,\&\, \chi]\chi$$

$$\text{DC} \quad \big([x' = \theta \,\&\, \chi]\phi(s) \leftrightarrow [x' = \theta \,\&\, \chi \land \varphi]\phi(s)\big) \leftarrow [x' = \theta \,\&\, \chi]\varphi$$

$$\text{DE} \quad [x' = \theta \,\&\, \chi]\phi(s, x, x') \leftrightarrow [x' = \theta \,\&\, \chi][x' := \theta]\phi(s, x, x')$$

$$\text{DI} \quad [x' = \theta \,\&\, \chi]\varphi \leftarrow \big(\chi \to \varphi \land [x' = \theta \,\&\, \chi](\varphi)'\big)$$

Delay Differential Weakening

$$\forall[-T)\,\phi(s) \land \phi(0) \leftrightarrow \forall[-T]\,\phi(s)$$

$$\begin{aligned}
\text{DDW} \quad & \big(\psi \to [x' = \theta \,\&\, \chi]\forall[-T]\,\phi(s)\big) \\
& \quad \leftarrow \big((\psi \to \forall[-T)\,\phi(s)) \land \forall x\,(\chi \to \phi(0))\big)
\end{aligned}$$

$x[c] \notin \phi(s)$

Delay Differential Induction

derived axiom

$$\text{DDI}\ \frac{\psi \to \forall[-T)\,\phi(s) \quad \forall x\,\big(\chi \wedge \varphi \to \phi(0)\big) \quad \psi \wedge \chi \to \varphi \quad \psi \to [x' = \theta \,\&\, \chi](\varphi)'}{\psi \to [x' = \theta \,\&\, \chi]\forall[-T]\,\phi(s)}$$

$x[c] \notin \phi(s)$ $\mathrm{FOL}_{\mathbb{R}}$ formula φ

Axiom of Steps

method of steps for DDEs reduce DDE to ODE

$$[\rightarrowtail] \quad [x' = \theta \,\&\, \chi]\phi(s)$$
$$\leftrightarrow [x' := \theta; (t := 0; t' = 1, x' = \theta \,\&\, \chi \wedge 0 \leq t \leq \tau_{\min})^*]\phi(s)$$

Example

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Outlook

- implementation in KeYmaera X
- complete reduction to $\mathsf{d}\mathcal{L}$ via $[\rightarrowtail]$-axiom
- How to find differntial invariants?
- Examples!
- More general DDEs...