Question 1.

Conversion to SEF:

Maximise $-x_1 + 3x_2$

Subject to $x_1 - x_2 + x_3 = 1$

$-x_1 + x_2 + x_4 = 1$

$2x_1 - x_2 + x_5 = 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

In matrix form:

Maximise $z(x) = (-1, 3, 0, 0, 0)x$

Subject to $\begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Insert into tableau:

$$T^1 = \begin{array}{c|ccccc|c} 1 & 1 & -3 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 & 3 \end{array}$$

Basis = {3, 4, 5}

Column of pivot = First negative entry in top row: column 2

Row of pivot = t $= \min\left\{-, \frac{1}{1}, -\right\}$: row 1

Note: Negative values do not restrict t

$$T^2 = \begin{array}{c|ccccc|c} 1 & -2 & 0 & 0 & 3 & 0 & 3 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \end{array}$$

Column of pivot = First negative entry in top row: column 1

Row of pivot t $= \min\left\{\infty, -, \frac{4}{1}\right\}$: row 3

$$T^3 = \begin{array}{c|ccccc|c} 1 & 0 & 0 & 0 & 5 & 2 & 11 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \end{array}$$

T^3 has non-negative top row so optimal solution is $x = (4, 5, 2, 0, 0)^T$, with value 11.

$x_1 = 4, x_2 = 5.$

Question 2.

Insert into tableau:

$$T^1 = \begin{array}{c|cccccc|c} 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & -3 & -1 & 0 & 0 & 1 & 6 \end{array}$$

Basis = {4, 5, 6}

Column of pivot = First negative entry in top row: column 1

Row of pivot $\mathrm{t} = \min\left\{-, \frac{2}{1}, \frac{6}{2}\right\}$: row 2

$$T^2 = \begin{array}{c|cccccc|c} 1 & 0 & -3 & 1 & 0 & 2 & 0 & 4 \\ \hline 0 & 0 & -1 & 1 & 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 & 0 & -2 & 1 & 2 \end{array}$$

Column of pivot = First negative entry in top row: column 2

Row of pivot $\mathrm{t} = \min\{-, -, -\}$

As there is no positive minimum in column 2 upon performing the ratio test, this LP is unbounded. $x'_B = b - tA_k$, where $k = 2$. Therefore $x_2 = t, x_3 = 0, x_5 = 0$.

Certificate of unboundedness:

$$\begin{pmatrix} x_4 \\ x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} - t \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$x(t) = (2 \quad 0 \quad 0 \quad 5 \quad 0 \quad 2)^T + t(1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1)^T$$

As $t \to \infty$, $z((x)t) \to \infty$, the certificates of unboundedness are $\bar{x} = (2 \quad 0 \quad 0 \quad 5 \quad 0 \quad 2)$ and $d = (1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1)$.

Question 3.

Variables:

Let r, g, b and k be the amount of gallons of red, green, blue and black paint produced.

Let c, y and m represent one gallon of cyan, yellow and magenta paint.

Two gallons of red is made from mixing one gallon of yellow and one gallon of magenta.

Therefore $r = \frac{y+m}{2}$. Likewise, $g = \frac{y+c}{2}$, $b = \frac{m+c}{2}$ and $k = \frac{c+m+y}{2}$.

Objective Function:

Maximise income $z = 10r + 15g + 25b + 25k$

Constraints:

Total amount of yellow paint = 11 gallons.

As yellow paint is mixed in to make red, green and black, $\frac{r}{2}+\frac{g}{2}+\frac{k}{3}\leq 11$.

Likewise, $\frac{r}{2}+\frac{b}{2}+\frac{k}{3}\leq 5$ for magenta paint and $\frac{g}{2}+\frac{b}{2}+\frac{k}{3}\leq 10$ for cyan paint.

Amount of red, blue, green and black paint produced cannot be negative. Therefore, $r,g,b,k\geq 0$.

More concisely:

Maximise $\quad z=10x_1+15x_2+25x_3+25x_4$

Subject to $\quad \frac{x_1}{2}+\frac{x_2}{2}+\frac{x_4}{3}\leq 11$

$$\frac{x_1}{2}+\frac{x_3}{2}+\frac{x_4}{3}\leq 5$$

$$\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{3}\leq 10$$

$$x_1,x_2,x_3,x_4\geq 0$$

Optimal Solution:

$x_1=0, x_2=10, x_3=0, x_4=15$ with value 525

Gallons of red paint = 0

Gallons of green paint = 10

Gallons of blue paint = 0

Gallons of black paint = 15

This means that 5 gallons of yellow paint is mixed with 5 gallons of cyan paint to make 10 gallons of green paint. 5 gallons of cyan paint, 5 gallons of magenta paint and 5 gallons of yellow paint are all mixed to create 15 gallons of black paint. This mixing achieves the maximal income of £525.

Question 4.

(a)

Variables:

Let x_i for $i=1,\ldots,6$ be the number of item A, B, C, D, E, F to take.

Objective Function:

Maximise total value of the items $z=60x_1+70x_2+40x_3+70x_4+16x_5+100x_6$

Constraints:

Cannot take more than one of each item: $x_i \leq 1$ for $i = 1, \ldots, 6$.

Maximum weight allowance of all items is 20kg: $6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 \leq 20$

Cannot take negative items: $x_i \geq 0$ for $i = 1, \ldots, 6$.

Each item must be taken as a whole. Therefore $x_i \in \mathbb{Z}$ for $i = 1, \ldots, 6$.

More concisely:

Maximise $z = 60x_1 + 70x_2 + 40x_3 + 70x_4 + 16x_5 + 100x_6$

Subject to $6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 \leq 20$

$x_i \geq 0$ for $i = 1, \ldots, 6$.

$x_i \leq 1$ for $i = 1, \ldots, 6$.

$x_i \in \mathbb{Z}$ for $i = 1, \ldots, 6$.

Optimal Solution:

$x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0, x_5 = 0, x_6 = 1$ with value 210.

To maximise the total value, take items B, C and F. The total value is £210.

(b)

Add constraint $x_3 \leq x_4$.

If $x_4 = 0$, item x_3 cannot be taken.

If $x_4 = 1$, item x_4 can be taken.

Optimal Solution:

$x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 1, x_6 = 1$ with value 186.

To maximise the total value, take items B, E and F. The total value is £186.

(c)

Variables:

Let x_i for $i = 1, \ldots, 6$ be the number of item A, B, C, D, E, F to take.

Let y be the number of additional kgs of weight over 20kg.

Let $b \in \{0, 1\}$ as an indicator to whether the total weight is over 20kg.

Objective Function:

Maximise total value of the items minus the cost paid to the airline:

$$z = 60x_1 + 70x_2 + 40x_3 + 70x_4 + 16x_5 + 100x_6 - 15y$$

Constraints:

As before:

Cannot take more than one of each item: $x_i \leq 1$ for $i = 1, \dots, 6$.

Cannot take negative items: $x_i \geq 0$ for $i = 1, \dots, 6$.

Each item must be taken as a whole. Therefore $x_i \in \mathbb{Z}$ for $i = 1, \dots, 6$.

Moreover, $y, b \in \mathbb{Z}$ as each item is an integer and b is a binary variable.

Hence $b \geq 0$ and $b \leq 1$.

Set $b = 0$ if the total weight is 20kg or below.

Set $b = 1$ if the total weight is over 20kg.

As the maximal weight of all items is 37kg, the following can be deduced:

$$6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 \leq 20 + 17b$$

$$6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 \geq 20b$$

If $b = 0$, The total weight must be between 0 and 20. If $b = 1$, the total weight must be between 20 and 37.

As the number of additional kgs cannot be negative, $y \geq 0$.

Similarly, the maximum number of additional kgs is $37 - 20 = 17$. However, if $b = 0$, $y = 0$ as the total weight is below 20kg. Hence $y \leq 17b$.

Otherwise, if $b = 1$, $y = 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 - 20$. In inequality form this becomes:

$$y \leq 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 - 20 + 20(1 - b)$$

$$y \geq 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 - 20 - 17(1 - b)$$

If $b = 0$:

$$0 \leq 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6$$

$$0 \geq 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 - 37$$

Therefore the new inequalities for $b = 1$ have no effect on $b = 0$

More concisely:

Maximise $\quad z = 60x_1 + 70x_2 + 40x_3 + 70x_4 + 16x_5 + 100x_6 - 15y$

Subject to $\quad 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 \leq 20 + 17b$

$\quad 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 \geq 20b$

$\quad y \leq 17b$

$$y \leq 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 - 20 + 20(1 - b)$$

$$y \geq 6x_1 + 7x_2 + 4x_3 + 9x_4 + 5x_5 + 8x_6 - 20 - 17(1 - b)$$

$x_i, b, y \geq 0$ for $i = 1, \dots, 6$.

$x_i, b \leq 1$ for $i = 1, \dots, 6$.

$x_i, b, y \in \mathbb{Z}$ for $i = 1, \dots, 6$.

Optimal Solution:

$b = 1, x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 1, y = 1$ with value 215.

To maximise the total value, take items A, B and F. The total value of selected items minus the cost paid to the airline is £215.