46 Derivatives

A *tangent line* to a circle is a line that meets the circle at exactly one point. A *secant line* to a circle is a line that meets the circle at exactly two points.

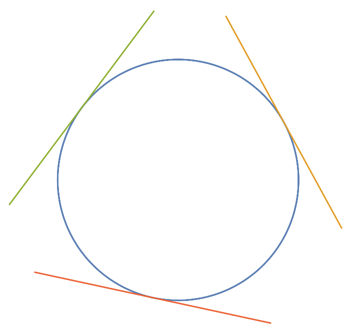

Figure 1: Tangent Line to a Circle

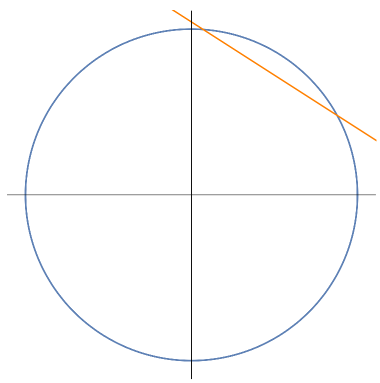

Figure 2: Secant Line to a Circle

We can generalize the concept of a tangent and secant line to an arbitrary curve.

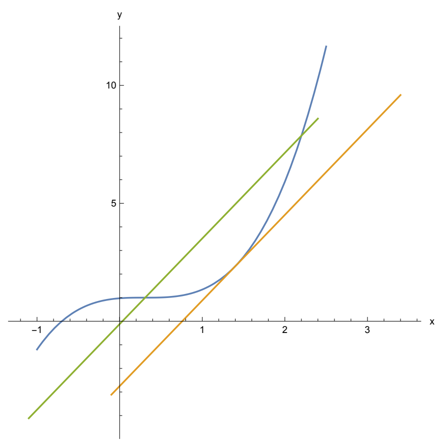

Figure 3: Tagent and Secant line to a curve

Given an arbitrary curve we cannot simply say "a line that meets the curve at exactly 1 point" in order to be a tangent.

Let the curve C in Figure 4 be the graph of the function $y = f(x)$, and let $P(a,\ f(a))$ be a point on the curve. Let further $Q(a+h, f(a+h))$ be another point on the curve. A line through PQ is called a *secant*. Now, keep P fixed and let Q move along the curve approaching P. The secant line will rotate about P. The limiting straight line is called the *tangent line* to the curve $y = f(x)$ at the point $P(a,\ f(a))$.

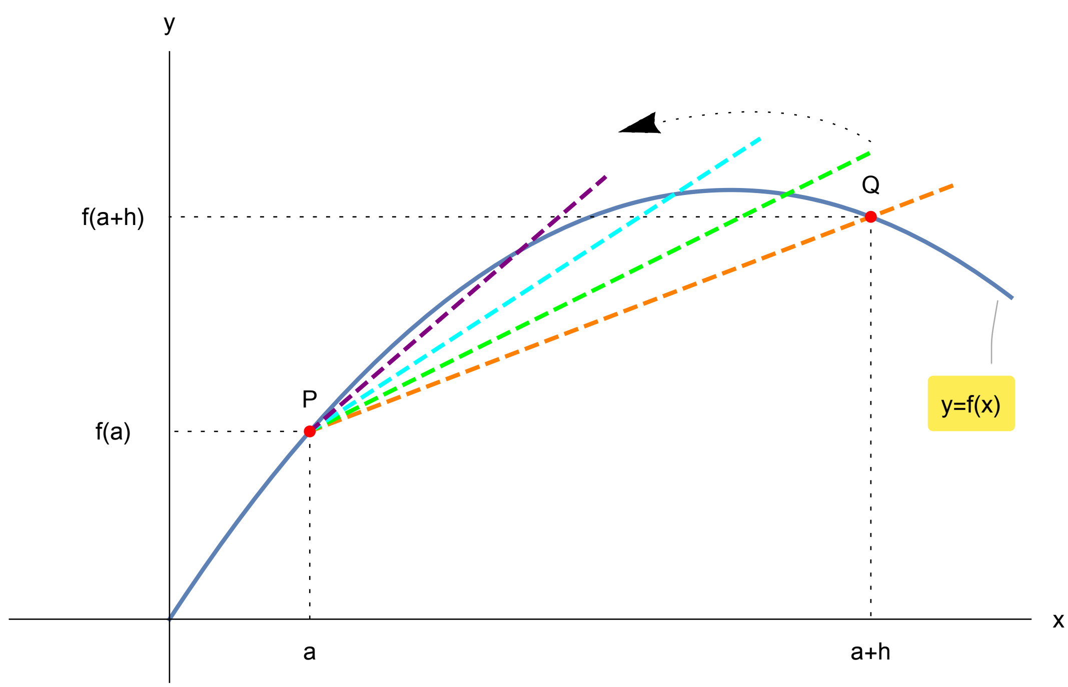

Figure 4: Tangent line as a limit of secant lines

The slope of the secant line PQ is given by

$$m_{PQ} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

Definition: The slope of the tangent line t to the curve $y = f(x)$ at the point $(a,\ f(a))$ is given by

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.

Remark: If $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, then the equation of the tangent line to the curve $y = f(x)$ at the point $(a,\ f(a))$ is given by

$$y - f(a) = m_{tan}(x - a).$$

Example 1: Find the equation of the tangent line to $f(x) = x^2 + x$ at $(1, 2)$.

Solution:

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 + (1+h) - 2}{h} = \cdots = \lim_{h\to 0} \frac{h^2 + 3h}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(h+3)}{\cancel{h}} = 3 \end{aligned}$$

So, the tangent line to $y = x^2 + x$ at the point $(1, 2)$ is

$$y - 2 = 3(x - 1),$$

or equivalently $y = 3x - 1$.

Example 2: Show that

$$f(x) = |x| = \begin{cases} x & , \ x \geq 0 \\ -x & , \ x < 0 \end{cases}$$

does not have a tangent line at $(0, 0)$.

Solution: The slope of the tangent line to $f(x) = |x|$ at the point $(0, 0)$ is given by

$$m_{tan} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

if this limit exists. We claim that this limit doesn't exist and we will prove the claim by showing that

$$\lim_{h\to 0^+} \frac{|h|}{h} \neq \lim_{h\to 0^-} \frac{|h|}{h}.$$

We have

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{\cancel{h}}{\cancel{h}} = \lim_{h\to 0^+} 1 = 1$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-\cancel{h}}{\cancel{h}} = \lim_{h\to 0^-} (-1) = -1$$

which proves the claim.

Definition: Given a function $y = f(x)$, the *derivative* of $f(x)$ with respect to (w.r.t.) x is denotes by $f'(x)$ (pronounced f prime of x) and it is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided that this limit exists.

<u>Remark</u>:

- Given $y = f(x)$, $f'(x)$ is also a function and $Dom(f') \subseteq Dom(f)$
- Derivatives are also denoted by y', $\frac{dy}{dx}$, D_x
- At the point $a \in Dom(f)$ where $f'(x)$ exists, $f'(a) = m_{tan}$ is the slope of the tangent line to $y = f(x)$ at $x = a$.

Example 1: Compute $f'(x)$ for $f(x) = x^2$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(h+2x)}{\cancel{h}}$$
$$= \lim_{h\to 0}(h+2x) = 2x.$$

<u>**Remark**</u>: In general,

$$\boxed{\text{if } f(x) = x^n \text{, then } \ f'(x) = nx^{n-1}, \ \ n \in \mathbb{Q}} \tag{1}$$

Example 2: Compute $f'(x)$ for $f(x) = \frac{1}{x}$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)}$$
$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \qquad \text{for } x \neq 0$$

Example 3: Compute $f'(x)$ using formula (1), if $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

Solution: Plugging $n = \frac{1}{2}$ into formula (1) we get

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

We see that $Dom(f') = (0, \infty) \subsetneq Dom(f) = [0, \infty)$.

Example 4: Compute $f'(x)$ if $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Solution: If $x < 0$, $f(x) = -x$ and we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = -1$$

If $x > 0$, $f(x) = x$ and we similarly have $f'(x) = 1$.
If $x = 0$, $f'(x) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist. So,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Example 5: Compute $f'(x)$ if $f(x) = e^x$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{(x+h)} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$$

Remark: If $f(x) = a^x$, then $f'(x) = a^x \cdot \ln a$

Example 6: Let $f(x) = \sqrt[3]{x}$.

a) Compute $f'(x)$.

b) Does $f'(0)$ exist? Why or why not?

Solution: a) $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Using formula (1) for $n = \frac{1}{3}$ we have

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}, \quad \text{for} \quad x \neq 0.$$

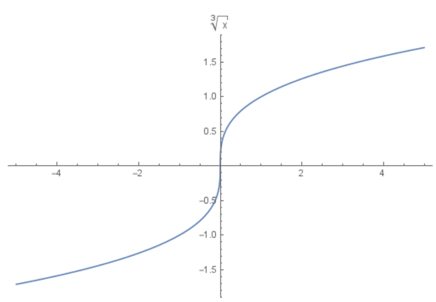

Figure 5: The graph of $y = \sqrt[3]{x}$

b) At $x = 0$, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{\frac{2}{3}}}.$$

$\lim_{h\to 0} \frac{1}{h^{\frac{2}{3}}}$ does not exist! We have

$$\lim_{h\to 0^+} \frac{1}{h^{\frac{2}{3}}} = \lim_{h\to 0^-} \frac{1}{h^{\frac{2}{3}}} = +\infty.$$

So, $f'(0)$ does not exist and therefore the function $f(x) = \sqrt[3]{x}$ does not have a tangent line at $x = 0$.

Definition: We say that a function $y = f(x)$ is *differentiable* at $x = a$ if $f'(a)$ exists, i.e., $f(x)$ has a derivative at a.

Example 7: For $f(x) = |x|$ we have:

$f(x) = |x|$ is continuous on $\mathbb{R}$.
$f(x) = |x|$ is differentiable for all $x \neq 0$.
So, $f(x) = |x|$ is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R}$.

In general

$$\text{CONTINUITY} \not\Longrightarrow \text{DIFFERENTIABILITY}$$
$$\text{DIFFERENTIABILITY} \Longrightarrow \text{CONTINUITY}$$

Remark: Usually a continuous function fails to have a derivative at a point a if a graph makes an abrupt change of direction at $(a,\ f(a))$. We call such a point a *corner*.

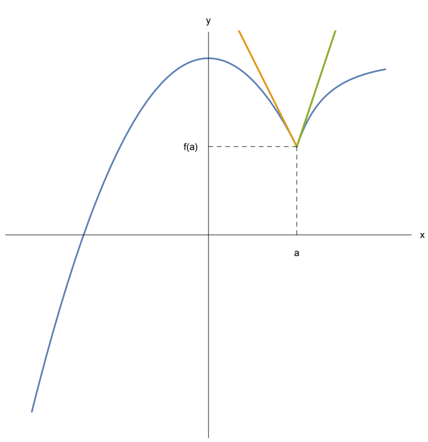

Figure 6: No derivative at $x = a$

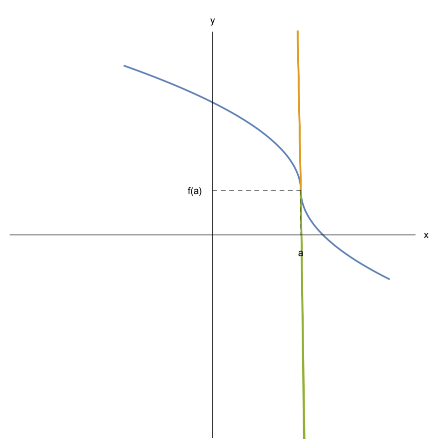

Figure 7: Undefined slope of the tangent at $x = a$

Example 8: Given

$$f(x) = \begin{cases} x^2 + ax + b & , \ x \leq 0 \\ x - 1 & , \ x > 0 \end{cases}$$

find the values of a and b such that $f(x)$ is continuous and has a derivative at $x = 0$.

Solution: For $x \neq 0$, $f(x)$ is a polynomial and hence continuous.

For $x = 0$, we want $\lim_{x \to 0} f(x) = f(0) = b$.

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x - 1) = -1$,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + ax + b) = b$.

Therefore, we must have $b = -1$.

<u>Conclusion</u>: f is continuous on $\mathbb{R}$ iff $b = -1$.

For $x \neq 0$, $\quad f'(x) = \begin{cases} 2x + a & , \ x < 0 \\ 1 & , \ x > 0 \end{cases}$

For $x = 0$,

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2 + ah - 1 + 1}{h} = \lim_{h \to 0^-} (h + a) = a.$$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 1 + 1}{h} = \lim_{h \to 0^+} 1 = 1.$$

So, f has a derivative at $x = 0$ iff $a = 1$ and $b = -1$.