

Collaborative Filtering

Mathieu Lagrange



Based on slides by Lester Mackey and Aleksandr Simma



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Outline

- ① Problem Formulation
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- ② Preliminaries
 - Naive Bayes
 - KNN
- ③ Classification/Regression
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- ④ Low Dimensional Matrix Factorization
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What is Collaborative Filtering?

Group of users



Group of items



What is Collaborative Filtering?

Group of users



Group of items



- ⌘ Observe some user-item preferences
- ⌘ Predict new preferences:

Does Bob like strawberries???

Collaborative Filtering in the Wild...

Amazon.com recommends products based on purchase history

Amazon.com Recommended for You

Hello, LESTER We have recommendations for you (Not LESTER?)

LESTER's Amazon.com Today's Deals Gifts & Wish Lists Gift Cards

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LESTER's Amazon.com Your Browsing History Recommended For You Rate These Items Improve Your Recommendations Your Profile Your Communities Learn More

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Today's Recommendations For You

Here's a daily sample of items recommended for you. Click here to [see all recommendations](#).

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FQNews

Obama Nobel Peace Prize: Obama wins, and partisan fighting continues

Chicago Tribune - [Mark Z. Barabak](#), [Geraldine Baum](#) - 45 minutes ago

President Barack Obama's winning of the Nobel Peace Prize brought nothing of the sort at home, as political combatants were quick to assume their usual battles: Democrats largely hailed the ...

Video: Did Obama Deserve Nobel Prize? CBS

If Obama can get one, you can, too Detroit Free Press

[New York Times](#) - [Philadelphia Inquirer](#) - [Fort Worth Star Telegram](#) - [Wikipedia: 2009 Nobel Peace Prize](#)
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Chippewa Herald

US Senate panel votes to extend security law

Reuters - [Thomas Farrago](#), [Anthony Roadie](#) - Oct 8, 2009

WASHINGTON (Reuters) - A Senate Judiciary Committee, drawing criticism from both liberals and conservatives, voted on Thursday to extend expiring provisions of a post-September 11 law designed to protect the United States from another attack.

AP Interview: White House expands climate campaign The Associated Press

US Senate Panel Unlikely To Debate CO2 Bill Before Nov Wall Street Journal

[New York Times](#) - [Houston Chronicle](#) - [Politico](#) - [Red, Green, and Blue](#)
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Portfolio.com

Barnes & Noble May Sell Its Own E-reader

PC World - [Harry McCracken](#) - Oct 9, 2009

Is bookstore behemoth Barnes & Noble about to enter the e-book fray with its own Android-powered device? I like these rumors: The Wall Street Journal is reporting that bookstore behemoth Barnes & Noble will soon start selling its own e-reader device, ...

Barnes & Noble's E-Reader Gets Real Wired News

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Birmingham Star

Dow Ends Week at Highest Level in a Year

Washington Post - 3 hours ago

US stocks gained last week, pushing the Dow Jones industrial average to its highest close in a year, as Alcoa unexpectedly reported a profit and economic data signaled the US recession is ending.

Dun of IBM, Intel Propels Dow's Run Wall Street Journal

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✂ Google News recommends new articles based on click and search history

✂ Millions of users, millions of articles

Das et al., 2007

Collaborative Filtering in the Wild...

Netflix predicts other “Movies You’ll ♥” based on past numeric ratings (1-5 stars)

The screenshot shows the Netflix homepage with a red header. The navigation bar includes links for Browse DVDs, Browse Instant, Your Queue, Movies You'll ♥, Friends & Community, and DVD Sale \$5.99. A search bar is on the right. Below the navigation bar, there are tabs for Home, Genres, New Releases, Netflix Top 100, Critics' Picks, and Award Winners. The main content area features a section titled "Because you enjoyed:" with links to Chinatown, Vertigo, and Dr. Strangelove. Below this, it says "We think you'll enjoy:" and recommends "The Last Laugh" with an "Add" button. A movie poster for "The Last Laugh" is shown with a 4-star rating and a "Not Interested" link. To the right, the "YOUR RECENT ACTIVITY" section lists three items: "We shipped" (04/14), "We received" (04/14), and "We received" (03/26). Below that, the "SUGGESTIONS FOR YOU" section says "You have new suggestions in Movies You'll ♥".

- ⌘ Recommendations drive 60% of Netflix’s DVD rentals
- ⌘ Mostly smaller, independent movies (Thompson 2008)

Collaborative Filtering in the Wild...



⌘ **Netflix Prize:**
Beat Netflix
recommender system,
using Netflix data →

Win \$1 million

⌘ **Data:**
480,000 users
18,000 movies
100 million observed
ratings = only 1.1% of
ratings observed

“The Netflix Prize seeks to substantially improve the accuracy of predictions about how much someone is going to love a movie based on their movie preferences.”

What is Collaborative Filtering?

Insight: Personal preferences are correlated

- ⌘ If Jack loves A and B, and Jill loves A, B, and C, then Jack is more likely to love C

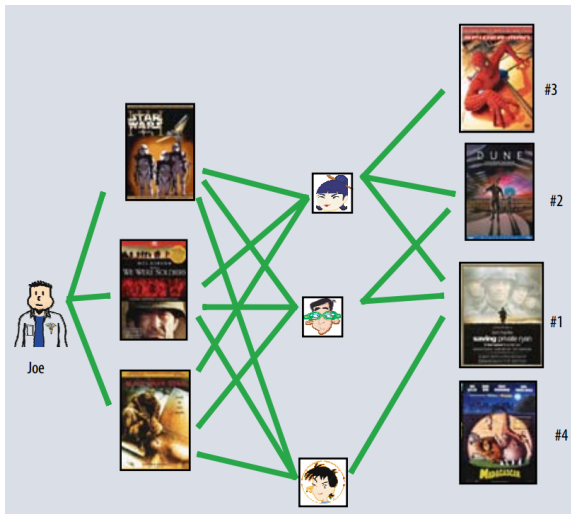
Collaborative Filtering Task

- ⌘ Discover patterns in observed preference behavior (e.g. purchase history, item ratings, click counts) across community of users
- ⌘ Predict new preferences based on those patterns

Does not rely on item or user attributes (e.g. demographic info, author, genre)

- ⌘ Content-based filtering: complementary approach

What is Collaborative Filtering?



What is Collaborative Filtering?

Given:

- ⊞ Users $u \in \{1, \dots, U\}$
- ⊞ Items $i \in \{1, \dots, M\}$
- ⊞ Training set \mathcal{T} with observed, real-valued preferences r_{ui} for some user-item pairs (u, i)
 - ⊞ r_{ui} = e.g. purchase indicator, item rating, click count ...

Goal: Predict unobserved preferences

- ⊞ Test set Q with pairs (u, i) not in \mathcal{T}

View as matrix completion problem

- ⊞ Fill in unknown entries of sparse preference matrix

$$\mathbf{R} = \underbrace{\begin{bmatrix} ? & ? & 1 & \dots & 4 \\ 3 & ? & ? & \dots & ? \\ ? & 5 & ? & \dots & 5 \end{bmatrix}}_{M \text{ items}} \Bigg\} U \text{ users}$$

What is Collaborative Filtering?

Measuring success

- Interested in error on unseen test set Q , not on training set
- For each (u, i) let r_{ui} = true preference, \hat{r}_{ui} = predicted preference

- Root Mean Square Error**

$$\text{RMSE} = \sqrt{\frac{1}{|Q|} \sum_{(u,i) \in Q} (r_{ui} - \hat{r}_{ui})^2}$$

- Mean Absolute Error

$$\text{MAE} = \frac{1}{|Q|} \sum_{(u,i) \in Q} |r_{ui} - \hat{r}_{ui}|$$

- Ranking-based objectives

- e.g. What fraction of true top-10 preferences are in predicted top 10?

Centering Your Data

⌘ What?

- ⌘ Remove bias term from each rating before applying CF methods: $\tilde{r}_{ui} = r_{ui} - b_{ui}$

⌘ Why?

- ⌘ Some users give systematically higher ratings
- ⌘ Some items receive systematically higher ratings
- ⌘ Many interesting patterns are in variation around these systematic biases
- ⌘ Some methods assume mean-centered data
 - ⌘ Recall PCA required mean centering to measure variance around the mean

Centering Your Data

⌘ What?

- ⌘ Remove bias term from each rating before applying CF methods: $\tilde{r}_{ui} = r_{ui} - b_{ui}$

⌘ How?

⌘ Global mean rating

- ⌘ $b_{ui} = \mu := \frac{1}{|\mathcal{T}|} \sum_{(u,i) \in \mathcal{T}} r_{ui}$

⌘ Item's mean rating

- ⌘ $b_{ui} = b_i := \frac{1}{|R(i)|} \sum_{u \in R(i)} r_{ui}$

- ⌘ $R(i)$ is the set of users who rated item i

⌘ User's mean rating

- ⌘ $b_{ui} = b_u := \frac{1}{|R(u)|} \sum_{i \in R(u)} r_{ui}$

- ⌘ $R(u)$ is the set of items rated by user u

⌘ Item's mean rating + user's mean deviation from item mean

- ⌘ $b_{ui} = b_i + \frac{1}{|R(u)|} \sum_{i \in R(u)} (r_{ui} - b_i)$

Shrinkage

⌘ What?

- ⌘ Interpolating between an estimate computed from data and a fixed, predetermined value

⌘ Why?

- ⌘ Common task in CF: Compute estimate (e.g. a mean rating) for each user/item
- ⌘ Not all estimates are equally reliable
- ⌘ Some users have orders of magnitude more ratings than others
- ⌘ Estimates based on fewer datapoints tend to be noisier

		A	B	C	D	E	F	User mean
R =	<i>Alice</i>	2	5	5	4	3	5	4
	<i>Bob</i>	2	?	?	?	?	?	2
	<i>Craig</i>	3	3	4	3	?	4	3.4

- ⌘ Hard to trust mean based on one rating

Shrinkage

- ⌘ What?
 - ⌘ Interpolating between an estimate computed from data and a fixed, predetermined value
- ⌘ How?
 - ⌘ e.g. Shrunk User Mean:

$$\tilde{b}_u = \frac{\alpha}{\alpha + |R(u)|} * \mu + \frac{|R(u)|}{\alpha + |R(u)|} * b_u$$

- ⌘ μ is the global mean, α controls degree of shrinkage
- ⌘ When user has many ratings, $\tilde{b}_u \approx$ user's mean rating
- ⌘ When user has few ratings, $\tilde{b}_u \approx$ global mean rating

		A	B	C	D	E	F	User mean	Shrunk mean
R =	Alice	2	5	5	4	3	5	4	3.94
	Bob	2	?	?	?	?	?	2	2.79
	Craig	3	3	4	3	?	4	3.4	3.43

Global mean $\mu = 3.58$, $\alpha = 1$

Classification/Regression for CF

Interpretation: CF is a set of M classification/regression problems, one for each item

- ⌘ Consider a fixed item i
- ⌘ Treat each user as incomplete vector of user's ratings for all items except i : $\vec{r}_u = (3, ?, ?, 4, ?, 5, ?, 1, 3)$
- ⌘ Class of each user w.r.t. item i is the user's rating for item i (e.g. 1, 2, 3, 4, or 5)
- ⌘ Predicting rating $r_{ui} \equiv$ Classifying user vector \vec{r}_u

Classification/Regression for CF

Approach:

- ☞ Choose your favorite classifier/regression algorithm
- ☞ Train separate predictor for each item
- ☞ To predict r_{ui} for user u and item i , apply item i 's predictor to vector of user u 's incomplete ratings vector

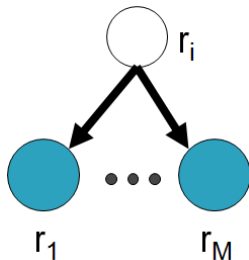
Pros:

- ☞ Reduces CF to a well-known, well-studied problem
- ☞ Many good prediction algorithms available

Cons:

- ☞ Predictor must handle missing data (unobserved ratings)
- ☞ Training M independent predictors can be expensive
- ☞ Approach may not take advantage of problem structure
 - ☞ Item-specific subproblems are often related

Naive Bayes Classifier



- ⌘ Treat distinct rating values as classes
- ⌘ Consider classification for item i
- ⌘ Main assumption
 - ⌘ For any items $j \neq k \neq i$, r_j and r_k are conditionally independent given r_i
 - ⌘ When we know rating r_{ui} all of a user's other ratings are independent
- ⌘ Parameters to estimate
 - ⌘ Prior class probabilities: $P(r_i = v)$
 - ⌘ Likelihood: $P(r_j = w | r_i = v)$

Naive Bayes Classifier

Train classifier with all users who have rated item i

⊞ Use counts to estimate prior and likelihood

$$P(r_i = v) = \frac{\sum_{u=1}^U \mathbf{1}(r_{ui} = v)}{\sum_{w=1}^V \sum_{i=1}^U \mathbf{1}(r_{ui} = w)}$$

$$P(r_j = w | r_i = v) = \frac{\sum_{u=1}^U \mathbf{1}(r_{ui} = v, r_{uj} = w)}{\sum_{z=1}^V \sum_{u=1}^U \mathbf{1}(r_{ui} = v, r_{uj} = z)}$$

⊞ Complexity

⊞ $O(\sum_{u=1}^U |R(u)|^2)$ time and $O(M^2 V^2)$ space for all items

Predict rating for (u, i) using posterior

$$P(r_{ui} = v | r_{u1}, \dots, r_{uM}) = \frac{P(r_{ui} = v) \prod_{j \neq i} P(r_{uj} | r_{ui} = v)}{\sum_{w=1}^V P(r_{ui} = w) \prod_{j \neq i} P(r_{uj} | r_{ui} = w)}$$

Naive Bayes Summary

Pros:

- ⌘ Easy to implement
- ⌘ Off-the-shelf implementations readily available

Cons:

- ⌘ Large space requirements when storing parameters for all M predictors
- ⌘ Makes strong independence assumptions
- ⌘ Parameter estimates will be noisy for items with few ratings

⌘ E.g. $P(r_j = w | r_i = v) = 0$ if no user rated both i and j

Addressing cons:

- ⌘ Tie together parameter learning in each item's predictor
- ⌘ Shrinkage/smoothing is an example of this

K Nearest Neighbor Methods

Most widely used class of CF methods

- ⌘ Flavors: **Item-based** and User-based
- ⌘ Represent each item as incomplete vector of user ratings:
 $\vec{r}_i = (3, ?, ?, 4, ?, 5, ?, 1, 3)$
- ⌘ To predict new rating r_{ui} for query user u and item i :
 - 1 Compute similarity between i and every other item
 - 2 Find K items rated by u most similar to i
 - 3 Predict weighted average of similar items' ratings
- ⌘ Intuition: Users rate similar items similarly.

KNN: Computing Similarities

How to measure similarity between items?

⊞ Cosine similarity

$$S(\vec{r}_i, \vec{r}_j) = \frac{\langle \vec{r}_i, \vec{r}_j \rangle}{\|\vec{r}_i\| \|\vec{r}_j\|}$$

⊞ Pearson correlation coefficient

$$S(\vec{r}_i, \vec{r}_j) = \frac{\langle \vec{r}_i - \text{mean}(\vec{r}_i), \vec{r}_j - \text{mean}(\vec{r}_j) \rangle}{\|\vec{r}_i - \text{mean}(\vec{r}_i)\| \|\vec{r}_j - \text{mean}(\vec{r}_j)\|}$$

⊞ Inverse Euclidean distance

$$S(\vec{r}_i, \vec{r}_j) = \frac{1}{\|\vec{r}_i - \vec{r}_j\|}$$

Problem: These measures assume complete vectors

Solution: Compute over subset of users rated by both items

Complexity: $O(\sum U, |R(U)|^2)$ time

KNN: Choosing K neighbors

How to choose K nearest neighbors?

- ✚ Select K items with largest similarity score to query item i

Problem: Not all items were rated by query user u

Solution: Choose K most similar items rated by u

Complexity: $O(\min(KM, M \log M))$

Herlocker et al., 1999

KNN: Forming Weighted Predictions

Predicted rating for query user u and item i

- ⌘ $N(i; u)$ is the *neighborhood* of item i for user u
 - ⌘ i.e. the K most similar items rated by u
- ⌘ $\hat{r}_{ui} = b_{ui} + \sum_{N(i;u)} w_{ij}(r_{uj} - b_{uj})$

How to choose weights for each neighbor?

- ⌘ Equal weights: $w_{ij} = \frac{1}{|N(i;u)|}$
- ⌘ Similarity weights: $w_{ij} = \frac{S(i,j)}{\sum_{j \in N(i;u)} S(i,j)}$ (Herlocker et al., 1999)
- ⌘ Learn optimal weights for each user (Bell and Koren, 2007)
- ⌘ Learn optimal global weights (Koren, 2008)

Complexity: $O(K)$

KNN: User Optimized Weights

Intuition: For a given query user u and item i , choose weights that best predict other known ratings of item i using only $N(i; u)$:

$$\min_{\mathbf{w}_i} \sum_{s \in R(i), s \neq u} \left(r_{si} - \sum_{j \in N(i; u)} w_{ij} r_{sj} \right)^2$$

With no missing ratings, this is a linear regression problem:

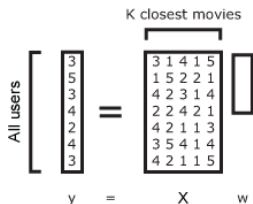
K closest movies

$$\begin{array}{c} \text{All users} \end{array} \begin{bmatrix} 3 \\ 5 \\ 3 \\ 4 \\ 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 \\ 1 & 5 & 2 & 2 & 1 \\ 4 & 2 & 3 & 1 & 4 \\ 2 & 2 & 4 & 2 & 1 \\ 4 & 2 & 1 & 1 & 3 \\ 3 & 5 & 4 & 1 & 4 \\ 4 & 2 & 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$y = Xw$

KNN: User Optimized Weights

- ⌘ Optimal solution: $w = A^{-1}b$ for $A = X^T X, b = X^T y$
- ⌘ **Problem:** X contains missing entries
 - ⌘ Not all items in $N(i; u)$ were rated by all users
- ⌘ **Solution:** Approximate A and b



Bell and Koren, 2007

$$\hat{A}_{jk} = \frac{\sum_{s \in R(j) \cap R(k)} r_{sj} r_{sk}}{|R(j) \cap R(k)|}$$

$$\hat{b}_k = \frac{\sum_{s \in R(i) \cap R(k)} r_{si} r_{sk}}{|R(i) \cap R(k)|}$$

$$\hat{w} = \hat{A}^{-1} \hat{b}$$

- ⌘ Estimates based on users who rated each pair of items

KNN: User Optimized Weights

Benefits

- ⌘ Weights optimized for the task of rating prediction
 - ⌘ Not just borrowed from the neighborhood selection phase
- ⌘ Weights not constrained to sum to 1
 - ⌘ Important if all nearest neighbors are dissimilar
- ⌘ Weights derived simultaneously
 - ⌘ Accounts for correlations among neighbors
- ⌘ Outperforms KNN with similarity or equal weights
- ⌘ Can compute entries of \hat{A} and \hat{b} offline in parallel

Drawbacks

- ⌘ Must solve additional $K \times K$ system of linear equations per query

KNN: Globally Optimized Weights

Consider the following KNN prediction rule for query (u, i) :

$$\hat{r}_{ui} = b_{ui} + |N(i; u)|^{-\frac{1}{2}} \sum_{j \in N(i; u)} w_{ij} (r_{uj} - b_{uj})$$

Could learn a single set of KNN weights w_{ij} , shared by all users, that minimize regularized MSE:

$$E = \frac{1}{|\mathcal{T}|} \sum_{(u, i) \in \mathcal{T}} \frac{1}{2} (\hat{r}_{ui} - r_{ui})^2 + \lambda \sum_{i=1}^M \sum_{j=1}^M \frac{1}{2} w_{ij}^2 = \frac{1}{|\mathcal{T}|} \sum_{(u, i) \in \mathcal{T}} E_{ui}$$

Optimize objective using **stochastic gradient descent**:

↪ For each example $(u, i) \in \mathcal{T}$, update $w_{ij} \forall j \in N(i; u)$

$$\begin{aligned} w_{ij}^{t+1} &= w_{ij}^t - \gamma \frac{\partial}{\partial w_{ij}} E_{ui} \\ &= w_{ij}^t - \gamma (|N(i; u)|^{-\frac{1}{2}} (\hat{r}_{ui} - r_{ui}) (r_{uj} - b_{uj}) + \lambda w_{ij}^t) \end{aligned}$$

KNN: Globally Optimized Weights

Benefits

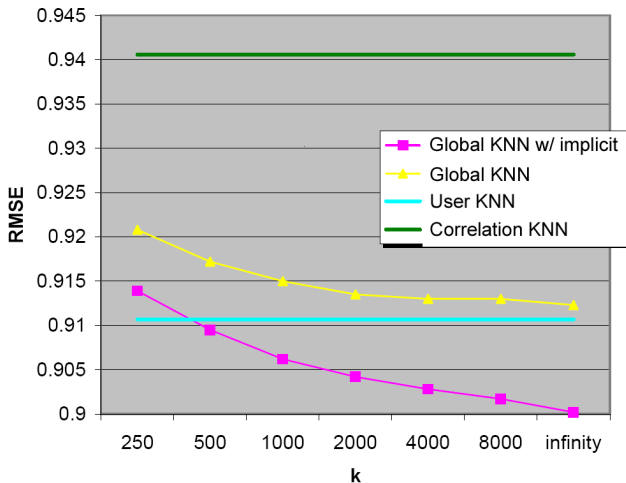
- ⌘ Weights optimized for the task of rating prediction
 - ⌘ Not just borrowed from the neighborhood selection phase
- ⌘ Weights not constrained to sum to 1
 - ⌘ Important if all nearest neighbors are dissimilar
- ⌘ Weights derived simultaneously
 - ⌘ Accounts for correlations among neighbors
- ⌘ Outperforms KNN with similarity or equal weights

Drawbacks

- ⌘ Must solve global optimization problem at training time
- ⌘ Must store $O(M^2)$ weights in memory

KNN: Summary

Comparison of KNN weighting schemes on Netflix quiz data



KNN: Summary

Pros

- ⌘ Intuitive interpretation
- ⌘ When weights not learned...
 - ⌘ Easy to implement
 - ⌘ Zero training time
- ⌘ Learning prediction weights can greatly improve accuracy for little overhead in space and time

Cons

- ⌘ When weights not learned...
 - ⌘ Need to store all item (or user) vectors in memory
 - ⌘ May redundantly recompute similarity scores at test time
 - ⌘ Similarity/equal weights not always suitable for prediction
- ⌘ When weights learned...
 - ⌘ Need to store $O(M^2)$ or $O(U^2)$ parameters
 - ⌘ Must update stored parameters when new ratings occur

Low Dimensional Matrix Factorization

Matrix Completion

- ⌘ Filling in the unknown ratings in a sparse $U \times M$ matrix R

$$\mathbf{R} = \begin{bmatrix} ? & ? & 1 & \dots & 4 \\ 3 & ? & ? & \dots & ? \\ ? & 5 & ? & \dots & 5 \end{bmatrix}$$

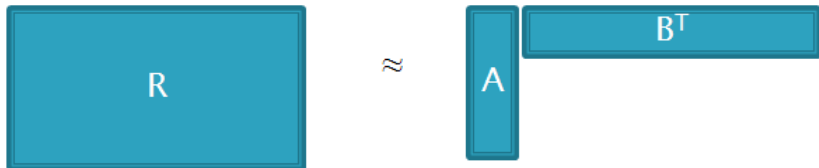
Low dimensional matrix factorization

- ⌘ Model R as a product of two lower dimensional matrices



- ⌘ A is $U \times K$ “user factor” matrix, $K \ll U, M$
- ⌘ B is $M \times K$, “item factor” matrix
- ⌘ Learning A and B allows us to reconstruct all of R

Low Dimensional Matrix Factorization



Interpretation: Rows of A and B are low dimensional feature vectors a_u and b_i for each user u and item i

Motivation: Dimensionality reduction

- ⌘ Compact representation: only need to learn and store $UK + MK$ parameters
- ⌘ Matrices can often be adequately represented by low rank factorizations

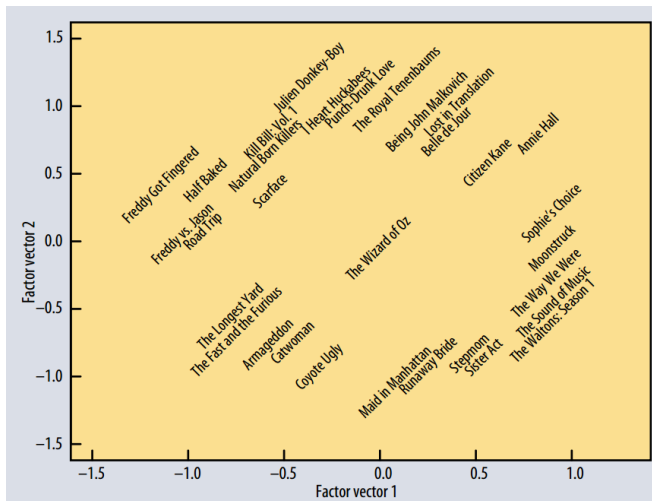
Low Dimensional Matrix Factorization



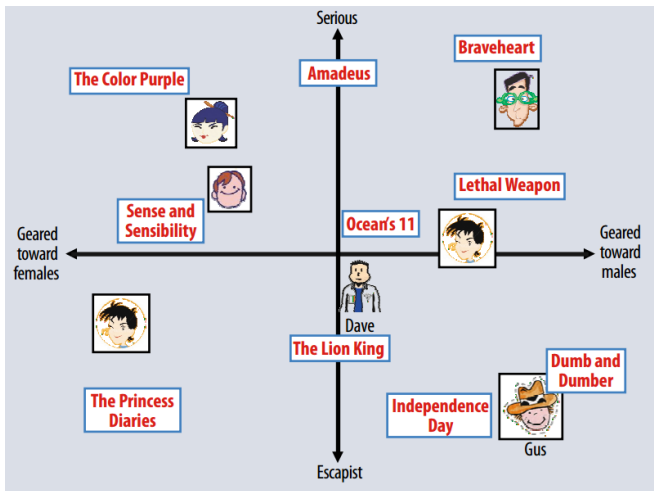
Very general framework that encapsulates many ML methods

- ⌘ Singular value decomposition
- ⌘ Clustering
 - ⌘ A can represent cluster centers
 - ⌘ B probabilities of belonging to each cluster
- ⌘ Factor Analysis/Probabilistic PCA

Projecting movies



Projecting users and movies



Singular Value Decomposition

Squared error objective for MF

$$\operatorname{argmin}_{A,B} \|R - AB^T\|_2^2 = \operatorname{argmin}_{A,B} \sum_{u=1}^U \sum_{i=1}^M (r_{ui} - \langle a_u, b_i \rangle)^2$$

⊞ Reasonable objective since RMSE is our error metric

When all of R is observed, this problem is solved by singular value decomposition (SVD)

⊞ **SVD:** $R = H\Sigma V^T$

⊞ H is $U \times U$ with $H^T H = I_{U \times U}$

⊞ V is $M \times M$ with $V^T V = I_{M \times M}$

⊞ Σ is $U \times M$ and diagonal

⊞ **Solution:** Take first K pairs of singular vectors

⊞ Let $A = H_{U \times K} \Sigma_{K \times K}$ and $B = V_{M \times K}$

SVD with Missing Values

Weighted SE objective

$$\operatorname{argmin}_{A,B} \sum_{u=1}^U \sum_{i=1}^M W_{ui} (r_{ui} - \langle a_u, b_i \rangle)^2$$

Binary weights

- ⊞ $W_{ui} = 1$ if r_{ui} observed, $W_{ui} = 0$ otherwise
- ⊞ Only penalize errors on known ratings

How to optimize?

- ⊞ Straightforward singular value decomposition no longer applies
- ⊞ Local minima exist \Rightarrow algorithm initialization is important

SVD with Missing Values

Insight: Chicken and egg problem

- ⌘ If we knew the missing values in R , could apply SVD
- ⌘ If we could apply SVD, we could find the missing values in R
- ⌘ Idea: Fill in unknown entries with best guess; apply SVD; repeat

Expectation-Maximization (EM) algorithm

- ⌘ Alternate until convergence:
 - 1 E step: $X = W * R + (1 - W) * \hat{R}$
(* represents entrywise product)
 - 2 M step: $[H, \Sigma, V] = SVD(X), \hat{R} = H_{U \times K} \Sigma_{K \times K} V_{M \times K}^T$

Complexity: $O(UM)$ space and $O(UMK)$ time per EM iteration

- ⌘ What if UM or UMK is very large?
 - ⌘ $UM = 8.5$ billion for Netflix Prize dataset
- ⌘ Complete ratings matrix may not even fit into memory!

SVD with Missing Values

Regularized weighted SE objective

$$\operatorname{argmin}_{A,B} \sum_{u=1}^U \sum_{i=1}^M W_{ui} (r_{ui} - \langle a_u, b_i \rangle)^2 + \lambda \left(\sum_{u=1}^U \|a_u\|^2 + \sum_{i=1}^M \|b_i\|^2 \right)$$

Equivalent form

$$\operatorname{argmin}_{A,B} \sum_{(u,i) \in \mathcal{T}} (r_{ui} - \langle a_u, b_i \rangle)^2 + \lambda \left(\sum_{u=1}^U \|a_u\|^2 + \sum_{i=1}^M \|b_i\|^2 \right)$$

Motivation

- ⊖ Counters *overfitting* by implicitly restricting optimization space
 - ⊖ Shrinks entries of A and B toward 0
- ⊖ Can improve *generalization error*, performance on unseen test data

SVD with Missing Values

Insight: If we knew B , could solve for each row of A via ridge regression and vice-versa

- ↪ Alternate between optimizing A and optimizing B with the other matrix held fixed

Alternating least squares (ALS) algorithm

- ↪ Alternate until convergence:
 - 1 For each user u , update
$$a_u \leftarrow (\sum_{i \in R(u)} b_i b_i^T + \lambda I)^{-1} \sum_{i \in R(u)} r_{ui} b_i$$
 - 2 For each item i , update
$$b_i \leftarrow (\sum_{u \in R(i)} a_u a_u^T + \lambda I)^{-1} \sum_{u \in R(i)} r_{ui} a_u$$

Complexity: $O(UK + MK)$ space, $O(UK^3 + MK^3)$ time per iteration

- ↪ Note: updates for vectors a_u can all be performed in parallel (same for b_i)
- ↪ No need to store completed ratings matrix

SVD with Missing Values

Insight: Use standard gradient descent

$$\epsilon \quad \nabla_{a_u} E = \lambda a_u + \sum_{i \in R(u)} b_i (\langle a_u, b_i \rangle - r_{ui})$$

$$\epsilon \quad \nabla_{b_i} E = \lambda b_i + \sum_{u \in R(i)} a_u (\langle a_u, b_i \rangle - r_{ui})$$

Gradient descent algorithm

ϵ Repeat until convergence:

① For each user u , update

$$a_u \leftarrow a_u - \gamma (\lambda a_u + \sum_{i \in R(u)} b_i (\langle a_u, b_i \rangle - r_{ui}))$$

② For each item i , update

$$b_i \leftarrow b_i - \gamma (\lambda b_i + \sum_{u \in R(i)} a_u (\langle a_u, b_i \rangle - r_{ui}))$$

ϵ Can update all a_u in parallel (same for b_i)

Complexity: $O(UK + MK)$ space, $O(NK)$ time per iteration

ϵ No need to store completed ratings matrix

ϵ No K^3 overhead from solving linear regressions

SVD with Missing Values

Insight: Update parameter after each observed rating

$$\Leftarrow \nabla_{a_u} E_{ui} = \lambda a_u + b_i(\langle a_u, b_i \rangle - r_{ui})$$

$$\Leftarrow \nabla_{b_i} E_{ui} = \lambda b_i + a_u(\langle a_u, b_i \rangle - r_{ui})$$

Stochastic gradient descent algorithm

\Leftarrow Repeat until convergence:

① For each $(u, i) \in \mathcal{T}$

① Calculate error: $e_{ui} \leftarrow (\langle a_u, b_i \rangle - r_{ui})$

② Update $a_u \leftarrow a_u - \gamma(\lambda a_u + b_i e_{ui})$

③ Update $b_i \leftarrow b_i - \gamma(\lambda b_i + a_u e_{ui})$

Complexity: $O(UK + MK)$ space, $O(NK)$ time per pass through training set

\Leftarrow No need to store completed ratings matrix

\Leftarrow No K^3 overhead from solving linear regressions

Constrained MF as Clustering

Insight: Soft clustering of items is MF

- ⌵ Row b_i represents item i 's fractional belonging to each cluster
- ⌵ Columns of A are cluster centers
- ⌵ Yields greater interpretability

Constrained weighted SE objective

$$\operatorname{argmin}_{A,B} \sum_{u=1}^U \sum_{i=1}^M W_{ui} (r_{ui} - \langle a_u, b_i \rangle)^2 \text{ s.t. } \forall i \quad b_i \geq 0, \sum_{k=1}^K b_{ik} = 1$$

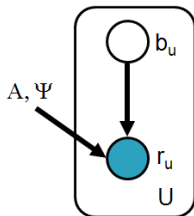
- ⌵ Wu and Li (2008) penalize constraints in the objective and optimize via stochastic gradient descent

Takeaway: Can add your favorite constraints and optimize with standard techniques

Factor Analysis

Motivation

- ⌘ Explain data variability in terms of latent *factors*
- ⌘ Provide model for how data is generated



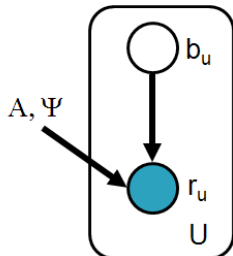
The Model

- ⌘ For each user, r_u = partially observed ratings vector in \mathbb{R}^M
- ⌘ For each user, b_u = latent factor vector in \mathbb{R}^K
- ⌘ A is an $M \times K$ matrix of parameters (*factor loading matrix*)
- ⌘ Ψ is an $M \times M$ covariance matrix
 - ⌘ Probabilistic PCA: Special case when $\Psi = \sigma^2 I$
- ⌘ To generate ratings for user u :
 - 1 Draw $b_u \sim \mathcal{N}(0, I_{K \times K})$
 - 2 Draw $r_u \sim \mathcal{N}(Ab_u, \Psi)$

Factor Analysis

Parameter Learning

- ⌘ Only need to learn A and Ψ
- ⌘ b_u are variables to be integrated out
- ⌘ Typically use EM algorithm (Canny, 2002)
 - ⌘ Can be very slow for large datasets
- ⌘ Alternative: Stochastic gradient descent on negative log likelihood (Lawrence and Urtasun, 2009)



Low Dimensional MF: Summary

Pros

- ⌘ Data reduction: only need to store $UK + MK$ parameters at test time
 - ⌘ $MK + M^2$ needed for Factor Analysis
- ⌘ Gradient descent and ALS procedures are easy to implement and scale well to large datasets
- ⌘ Empirically yields high accuracy in CF tasks
- ⌘ Matrix factors could be used as inputs into other learning algorithms (e.g. classifiers)

Cons

- ⌘ Missing data MF objectives plagued by many local minima
- ⌘ Initialization is important
- ⌘ EM approaches tend to be slow for large datasets

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