Model Predictive Control
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Problem n.1 (15 points)

Given the discrete-time system S(A,B,C,D) described below:

$$\begin{cases} x_1(k+1) = -x_1(k) + x_2(k) \\ x_2(k+1) = x_1(k) + u(k) \end{cases} \qquad y(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Where:

$$r(k) = \begin{bmatrix} 1(k) \\ 1(k) \end{bmatrix} \qquad 1(k) = 1 \quad \forall k \geq 0$$

And the cost function is:

$$V(k) = \sum_{i=H_W}^{H_P} [r(k+i) - x(k+i)]^T \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} [r(k+i) - x(k+i)]$$

$$H_P = 2 \qquad H_w = 1 \qquad H_c = 2$$

Questions:

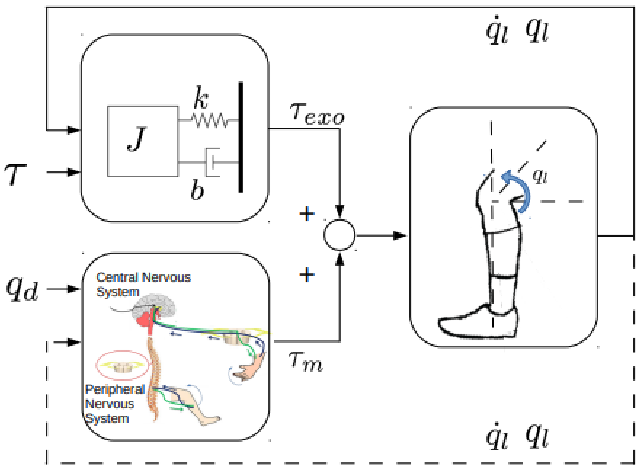
- **A.** Compute the cost function $V(k)$.
- **B.** Compute $u(k)$
- **C.** Compute $u(k+1)$
- **D.** Considering the computed u(k), what can be said about the stability of the closed-loop system?
- **E.** By considering the cost function defined below, compute $V_1(k), u(k), u(k+1)$

$$V_1(k) = \sum_{i=H_W}^{H_P} [r(k+i) - x(k+i)]^T \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} [r(k+i) - x(k+i)] + \sum_{i=0}^{H_C-1} 2[u(k+i)]^2$$

Table n.1 (see Problem n.2)	
Name	**Value [unit]**
q_d	[rad]
q_l	[rad]
q_m	[rad]
τ	[Nm]
τ_m	[Nm]
k	0.80 [Nm/rad]
b	$2.0 * 10^{-2}$ [Nm/rad/s]
m	1.2 [kg]
l	0.40 [m]
J_m	$0.5 * 10^{-3}$ $[kg\ m^2]$
k_P	40 [N m/ rad]
k_d	15 [N m/ rad/s]

Problem n.2 (15 points)

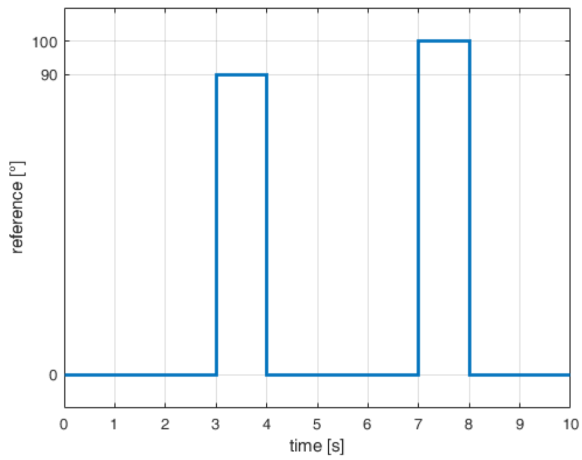
The following image shows a conceptual representation of the dynamic system consisting of a leg with an exoskeleton applied to increase the torque available at the knee joint. In particular, it is considered that the joint is implemented by the pair of muscle groups, quadriceps femoris and biceps femoris, and by the actuator of the exoskeleton. Muscle actuation is overall modeled by a moment acting on the joint controlled by the nervous system (central and peripheral). The artificial actuation is instead modeled as a torque-controlled motor with viscoelastic transmission.



The equations that govern the dynamics of the system in the figure are shown below:

$$x = [x_1\ x_2\ x_3\ x_4]^T = [q_l\ q_m\ \dot{q}_l\ \dot{q}_m]^T \quad u = \tau \quad y = q_l$$

$$\dot{x_1} = x_3$$
$$\dot{x_2} = x_4$$
$$\dot{x_3} = \frac{-mglcos(x_1) - k(x_1 - x_2) - b(x_3 - x_4) + \tau_m}{ml^2}$$
$$\dot{x_4} = \frac{\tau - k(x_2 - x_1) + b(x_3 - x_4)}{J_m}$$



where q_l represents the angular position of the thigh with respect to the calf, q_m the position of the motor, k the stiffness of the motor transmission, b the damping of the motor transmission, m the mass of the thigh, l the distance of the center of gravity of the thigh from the joint, J_m the inertia of the motor, τ_m the resulting moment exerted by the muscle groups.

According to the Servo-Hypothesis, in the execution of positioning, it is possible to consider that τ_m is controlled by the nervous system to carry out an action similar to that of a controller in feedback on the desired position. In this case, consider the following proportional, derivative law to approximate the action of the muscles, where q_d represents the desired position of the thigh $\tau_m = k_p(q_d - x_1) - k_d x_3$. Saturate τ_m so that $-2 \leq \tau_m \leq 2$.

The goal of this problem, is to design a nonlinear MPC controller that follows the reference depicted in figure.

Questions:

A. Calculate and specify an analytical Jacobian for the output function and the state function.
B. Consider $T_S = 0.1s$. The desired settling time is 1s. Initialize the MPC controller with the proper parameters. Validate the MPC Object. Run a simulation lasting $10s$ and comment the results.
C. We want to meet the following requirements: No overshoots, Position error at steady state: $\pm 2°$. Tune the controller to meet the requirements. Explain all the design choices and comment on the results obtained.
D. We are told that for safety reasons the speed of the joint cannot exceed 80°/s. Constrains the controller accordingly. Run a simulation lasting $10s$ and comment the results. Did the constraints come into effect?
E. Analyze the computational time required for each move update. Can you control the system in real time?
F. Plot the evolution of the cost over time and comment it.

IMPORTANT: To get a full score it is necessary to comment on the design choices.

1

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$\begin{cases} x_1(k+1) = -x_1(k) + x_2(k) \\ x_2(k+1) = x_1(k) + u(k) \end{cases} \rightarrow x(k+1) = A\,x(k) + B\,u(k)$$

allora $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ e $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y(k) = x(k)$

il riferimento é $r(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$H_p = 2, \; H_w = 1, \; H_c = 2$

la funzione costo é:

$$V(k) = \sum_{i=H_w}^{H_p} [r(k+i) - x(k+i)]^T \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} [r(k+i) - x(k+i)]$$

Predizione dello stato (passaggio preliminare)

$$x(k+1) = \begin{bmatrix} -x_1(k) + x_2(k) \\ x_1(k) + u(k) \end{bmatrix}$$

$$x(k+2) = A\,x(k+1) + B\,u(k+1) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -x_1(k) + x_2(k) \\ x_1(k) + u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k+1) = \begin{bmatrix} \overbrace{x_1(k) - x_2(k) + x_1(k)}^{2x_1(k)} + u(k) \\ -x_1(k) + x_2(k) + u(k+1) \end{bmatrix}$$

A

Definendo l'errore: $e(k+i) = r(k+i) - x(k+i)$

Il termine di costo generico é: $e(k+i)^T \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} e(k+i) = -2\, e_1(k+i)^2 + 3\, e_2(k+i)^2$

Quindi:

$$V(k) = -2\big(1 + x_1(k) - x_2(k)\big)^2 + 3\big(1 - x_1(k) - u(k)\big)^2 - 2\big(1 - 2x_1(k) + x_2(k) - u(k)\big)^2 +$$
$$+ 3\big(1 + x_1(k) - x_2(k) - u(k+1)\big)^2$$

B

$$\frac{\partial V(k)}{\partial u(k)} = 0 \rightarrow \quad 6\big(1 - x_1(k) - u(k)\big)(-1) - 4\big(1 - 2x_1(k) + x_2(k) - u(k)\big)(-1) = 0$$

$$\rightarrow \quad -6 + 6x_1(k) + 6u(k) + 4 - 8x_1(k) + 4x_2(k) - 4u(k) = 0$$

$$\rightarrow \quad -2 - 2x_1(k) + 2u(k) + 4x_2(k) = 0 \rightarrow u(k) = \frac{2x_1(k) - 4x_2(k) + 2}{2}$$

$$\rightarrow \quad u(k) = x_1(k) - 2x_2(k) + 1$$

C

$$\frac{\partial V(k)}{\partial u(k+1)} = 0 \rightarrow \quad 6\big(1 + x_1(k) - x_2(k) - u(k+1)\big)(-1) = 0$$

$$\rightarrow \quad -6 - 6x_1(k) + 6x_2(k) + 6u(k+1) = 0$$

$$\rightarrow \quad u(k+1) = x_1(k) - x_2(k) + 1$$

D

Per calcolare la stabilità del sistema in anello chiuso sostituiamo $u(k)$ trovata al punto B nell'equazione di stato; troviamo:

$$\begin{cases} x_1(k+1) = -x_1(k) + x_2(k) \\ x_2(k+1) = 2x_1(k) - 2x_2(k) + 1 \end{cases}$$

Il sistema omogeneo Closed-Loop é definito dalla matrice $A_{CL} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

Calcoliamo gli autovalori:

$$|A_{CL} - \lambda I| = \begin{vmatrix} -1-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = 0 \rightarrow (-1-\lambda)(-2-\lambda) - 2 = 0 \rightarrow \cancel{2} + 3\lambda + \lambda^2 - \cancel{2} = 0 \rightarrow \lambda(\lambda+3) = 0$$

$\rightarrow \lambda_1 = 0 \; ; \; \lambda_2 = -3$

In un sistema Tempo Discreto per avere stabilitá bisogna avere tutti gli autovalori a modulo strettamente minore di 1 (tutti i poli all'interno della circonferenza di raggio unitario).
In questo caso quindi il sistema in anello chiuso é INSTABILE

E

Calcoliamo una nuova funzione costo che aggiunge una penalitá sul quadrato dell'azione di controllo ($H_c = 2$).

$$V_1(k) = V(k) + \sum_{i=0}^{H_c - 1} 2\left[u(k+i)\right]^2 = V(k) + 2u^2(k) + 2u^2(k+1)$$

$$\frac{\partial V_1(k)}{\partial u(k)} = 0 \rightarrow \quad -2 - 2x_1(k) + 2u(k) + 4x_2(k) + 4u(k) = 0 \rightarrow$$

$$\rightarrow \quad 6u(k) = 2x_1(k) - 4x_2(k) + 2$$

$$\rightarrow u(k) = \frac{1}{3}x_1(k) - \frac{2}{3}x_2(k) + \frac{1}{3}$$

$$\frac{\partial V_1(k)}{\partial u(k+1)} = 0 \rightarrow \quad -6 - 6x_1(k) + 6x_2(k) + 6u(k+1) + 4u(k+1) = 0$$

$$\rightarrow \quad 10u(k+1) = 6x_1(k) - 6x_2(k) + 6$$

$$\rightarrow \quad u(k+1) = \frac{3}{5}x_1(k) - \frac{3}{5}x_2(k) + \frac{3}{5}$$