Model Predictive Control
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Exam of 06/06/2022

Problem n.1 (15 points)

Given the discrete-time system S(A,B,C,D) described below:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Where:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(k) = \begin{bmatrix} 1(k) \\ 1(k) \\ 1(k) \end{bmatrix} \qquad 1(k) = 1 \quad \forall k > 0$$

Cost function:

$$V(k) = \sum_{i=H_W}^{H_P} [r(k+i) - y(k+i)]^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} [r(k+i) - y(k+i)]$$

$$H_P = 2 \qquad H_w = 1 \qquad H_c = 2$$

Questions:

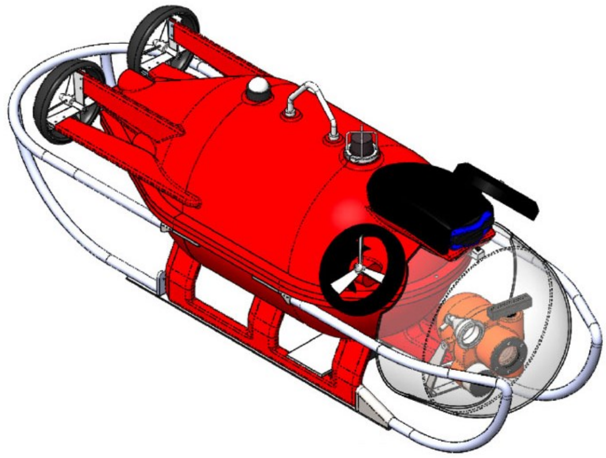
A. Compute the cost function V(k)
B. Compute u(k)
C. Compute u(k+1)
D. Considering the computed u(k), what can be said about the stability of the closed-loop system?
E. By considering the cost function defined below, compute $V_1(k), u(k), u(k+1)$

$$V_1(k) = V(k) + \sum_{i=1}^{H_C - 1} 2[u(k+i)]^2$$

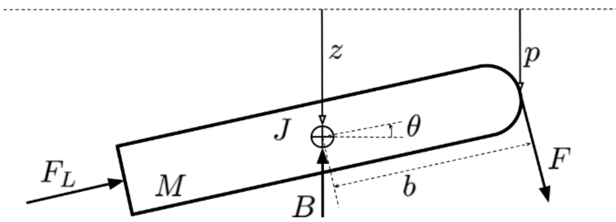
Table n.1 (see Problem n.2)

Name	Description	Value [unit]	Name	Description	Value [unit]
u_1	Rear propeller force F	[N]	y_1	Posizion of the bow (p)	[m]
u_2	Directional propeller force F_L	[N]	y_2	Depth of the ROV	[m]
x_1	Depth of the ROV (z)	[m]	y_3	Pitch angle	[rad]
x_2	Pitch angle (θ)	[rad]	y_4	Descent speed of the ROV	[m/s]
x_3	Descent speed of the ROV ($\dot{z}$)	[m/s]	y_5	Angular speed on pitch axis	[rad/s]
x_4	Angular speed on pitch axis ($\dot{\theta}$)	[rad/s]	K_r	System parameter	168 [Nms]
B	Buoyancy (MD)	[N]	K_θ	System parameter	143 [Nm]
b	Height scaling parameter	1.2 [m]	J	Moment of inertia	560 [Nms^2]
K_z	System parameter	133 [Ns/m]			
M	Mass of the ROV	580 [kg]			

Problem n.2 (15 points)



The figure shows a remotely controlled submarine (ROV) and its model in the pitch plane. The system has mass M and moment of inertia J (with respect to the pitch axis) and is subject to three forces: F, F_L, provided by the onboard engines, and the buoyancy B (which is a measured disturbance). The z indicates the position of the center of mass of the ROV, with p the position of the bow of the ROV and with θ the pitch angle. The sign conventions adopted are visible in the figure. The dynamics of the system can be described, in a very simplified way, through the following equations:



$$\begin{cases} \dot{x_1} = x_3 \\ \dot{x_2} = x_4 \\ \dot{x_3} = \frac{u_1}{M}\cos(x_2) - \frac{K_z}{M}\cos(x_2)\,x_3 - \frac{u_2}{M}\sin(x_2) - \frac{B}{M} \\ \dot{x_4} = -\frac{K_r}{J}x_4 - \frac{K_\theta}{J}\sin(x_2) - \frac{b}{J}u_1 \end{cases}$$

$$\begin{cases} y_1 = x_1 - b\,sin(x_2) \\ y_2 = x_1 \\ y_3 = x_2 \\ y_4 = x_3 \\ y_5 = x_4 \end{cases} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z \\ \theta \\ \dot{z} \\ \dot{\theta} \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F \\ F_L \end{bmatrix} \qquad y_{ref} = [1 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

The controller's goal is to control <u>only</u> the position of the bow and the pitch angle of the ROV. The ROV parameters are written in Table n.1 in page 1.

Questions:

A. The system has a slow dynamics whose order of magnitude is seconds. The settling time required by the ROV to perform a 1m dive is 40 minutes. Considering Ts = 1s and Hc = 2, choose the appropriate Hp value. The initial state of the system is x0 = [0 0 0 0 0]. Carry out a simulation and <u>comment on the results obtained</u>.
B. The rear propeller motor can deliver a force in the range of $\pm$10N, the directional propeller instead in the range of $\pm$2N. The depth of the ROV cannot be negative. Constrain the system accordingly.
C. Calculate and implement the Jacobian for the StateFunction and the OutputFunction.
D. From time t_1 = 20s to time t_2=30s, the ROV is inside a sea current which exerts a force of on the hull. The sea current can be modeled as a sinusoidal disturbance with the expression: $MD(t) = 5\sin(10t)$. Carry out a simulation and <u>comment on the results obtained</u>. Can the MPC controller reject the disturbance?
E. Analyze the computational time required for each move update. Can you control the system in real time? Is computational time affected by the disturbance of the previous point (question D)?
F. Suppose we are in the presence of a depth sensor measurement noise of the type $n(t) = 100\sin(10t)$. Comment out the measurable noise section in the code and add this noise to the feedback. Carry out a simulation and <u>comment on the results obtained</u>.

1

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

dove: $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Funzione costo:

$$V(k) = \sum_{i=H_w}^{H_p} [r(k+i) - y(k+i)]^T Q [r(k+i) - y(k+i)]$$

con orizzonte di predizione $H_p = 2$; ($H_w = 1$), ed orizzonte di controllo $H_c = 2$.

INIZIO ORIZZONTE DI PREDIZIONE

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad r(k) = \begin{bmatrix} 1(k) \\ 1(k) \\ 1(k) \end{bmatrix} \qquad \text{dove } 1(k) = 1 \quad \forall k > 0$$

Calcolo:

$$x(k+1) = Ax(k) + Bu(k) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) = \begin{bmatrix} u(k) \\ x_1(k) - x_2(k) \\ 3x_1(k) \end{bmatrix}$$

$$y(k+1) = Cx(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u(k) \\ x_1(k) - x_2(k) \\ 3x_1(k) \end{bmatrix} = \begin{bmatrix} u(k) \\ 2x_1(k) - 2x_2(k) \\ 9x_1(k) \end{bmatrix}$$

per $i = 2$:

$$x(k+2) = Ax(k+1) + Bu(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(k) \\ x_1(k) - x_2(k) \\ 3x_1(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k+1) = \begin{bmatrix} u(k+1) \\ u(k) - x_1(k) + x_2(k) \\ 3u(k) \end{bmatrix}$$

$$y(k+2) = Cx(k+2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u(k+1) \\ u(k) - x_1(k) + x_2(k) \\ 3u(k) \end{bmatrix} = \begin{bmatrix} u(k+1) \\ 2u(k) - 2x_1(k) + 2x_2(k) \\ 9u(k) \end{bmatrix}$$

A calcolo della funzione costo

$$V(k) = \left[r(k+1) - y(k+1)\right]^T Q \left[r(k+1) - y(k+1)\right] + \left[r(k+2) - y(k+2)\right]^T Q \left[r(k+2) - y(k+2)\right] =$$

$$= \begin{bmatrix} 1 - u(k) \\ 1 - 2x_1(k) + 2x_2(k) \\ 1 - 9x_1(k) \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 - u(k) \\ 1 - 2x_1(k) + 2x_2(k) \\ 1 - 9x_1(k) \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 - u(k+1) \\ 1 - 2u(k) + 2x_1(k) - 2x_2(k) \\ 1 - 9u(k) \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 - u(k+1) \\ 1 - 2u(k) + 2x_1(k) - 2x_2(k) \\ 1 - 9u(k) \end{bmatrix} =$$

$$= \begin{bmatrix} 2 - 2u(k) & 1 - 2x_1(k) + 2x_2(k) & -1 + 9x_1(k) \end{bmatrix} \begin{bmatrix} 1 - u(k) \\ 1 - 2x_1(k) + 2x_2(k) \\ 1 - 9x_1(k) \end{bmatrix} +$$

$$+ \begin{bmatrix} 2 - 2u(k+1) & 1 - 2u(k) + 2x_1(k) - 2x_2(k) & -1 + 9u(k) \end{bmatrix} \begin{bmatrix} 1 - u(k+1) \\ 1 - 2u(k) + 2x_1(k) - 2x_2(k) \\ 1 - 9u(k) \end{bmatrix} =$$

$$= 2\left(1 - u(k)\right)^2 + \left(1 - 2x_1(k) + 2x_2(k)\right)^2 - \left(1 - 9x_1(k)\right)^2 +$$

$$+ 2\left(1 - u(k+1)\right)^2 + \left(1 - 2u(k) + 2x_1(k) - 2x_2(k)\right)^2 - \left(1 - 9u(k)\right)^2$$

METODO ALTERNATIVO (veloce):

$$e(k+i) = r(k+i) - y(k+i)$$

dato che $Q = \mathrm{diag}(2, 1, -1) \Rightarrow e(k+i)^T Q\, e(k+i) = 2 e_1(k+i)^2 + e_2(k+i)^2 - e_3(k+i)^2$

calcolare $V_1 = e(k+1) = \begin{bmatrix} e_1(k+1) \\ e_2(k+1) \\ e_3(k+1) \end{bmatrix}$ e $V_2 = e(k+2)$

per ottenere $V(k) = V_1 + V_2$

(B) calcolo dell'azione di controllo ottimale $u(k)$

Bisogna minimizzare $V(k)$, ponendo la derivata parziale rispetto a $u(k)$ uguale a zero:

$$\frac{\partial V(k)}{\partial u(k)} = 0 \rightarrow 2\cdot 2\left(1-u(k)\right)\cdot(-1) + 0 + 0 + 0 + 2\left(1 - 2u(k) + 2x_1(k) - 2x_2(k)\right)(-2) -$$

$$- 2\left(1 - 9u(k)\right)(-9) = 0 \rightarrow$$

$$\rightarrow -4 + 4u(k) - 4 + 8u(k) - 8x_1(k) + 8x_2(k) + 18 - 162u(k) = 0$$

$$\rightarrow -150u(k) - 8x_1(k) + 8x_2(k) + 10 = 0$$

dividendo num e den per 2

$$\rightarrow u(k) = \frac{10 - 8x_1(k) + 8x_2(k)}{150} = \frac{5 - 4x_1(k) + 4x_2(k)}{75}$$

(C) calcolo dell'azione di controllo ottimale $u(k+1)$

Ragionamento analogo a prima.

$$\frac{\partial V(k)}{\partial u(k+1)} = 0 \rightarrow 0+0+0+2\cdot 2\left(1-u(k+1)\right)(-1) + 0 + 0 = 0 \rightarrow 4u(k+1) - 4 = 0 \rightarrow$$

$$\rightarrow u(k+1) = 1$$

(D) Stabilità del sistema in anello chiuso (sostituisco $u(k)$ con quello trovato in (B))

$$A_{CL} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -\frac{4}{75}x_1(k) + \frac{4}{75}x_2(k) \\ x_1(k) - x_2(k) \\ 3x_1(k) \end{bmatrix} \doteq \begin{bmatrix} -\frac{4}{75} & \frac{4}{75} & 0 \\ 1 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A_{CL}| = \begin{vmatrix} \lambda + \frac{4}{75} & -\frac{4}{75} & 0 \\ -1 & \lambda + 1 & 0 \\ -3 & 0 & \lambda \end{vmatrix} = 0 \rightarrow$$

$$\rightarrow (\lambda^2 + \lambda)\left(\lambda + \frac{4}{75}\right) - \frac{4}{75}\lambda = 0 \rightarrow$$

$$\rightarrow \lambda^3 + \lambda^2 + \frac{4}{75}\lambda^2 + \cancel{\frac{4}{75}\lambda} - \cancel{\frac{4}{75}\lambda} = 0 \rightarrow$$

$$\rightarrow \lambda^2\left(\lambda + \frac{79}{75}\right) = 0 \rightarrow \lambda_1 = \lambda_2 = 0$$

$$\lambda_3 = -\frac{79}{75} > 1$$

INSTABILE

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E

$$V_1(k) = V(k) + 2(u(k+1))^2$$

il calcolo di $u(k)$ non varia, mentre cambia:

$$\frac{\partial V_1(k)}{\partial u(k+1)} = 0 \rightarrow 4u(k+1) - 4 + 4u(k+1) = 0 \rightarrow u(k+1) = \frac{1}{2}$$