Model Predictive Control
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Problem n.1 (15 points)

Given the discrete-time system S(A,B,C,D) described below:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Where:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad r(k) = 3\begin{bmatrix} 1(k) \\ 1(k) \\ 1(k) \end{bmatrix} \quad 1(k) = 1 \quad \forall k \geq 0$$

And the cost function is:

$$V(k) = \sum_{i=H_W}^{H_P} [r(k+i) - x(k+i)]^T \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [r(k+i) - x(k+i)]$$

$$H_P = 2 \qquad H_w = 1 \qquad H_c = 2$$

Questions:

- **A.** Compute the cost function $V(k)$.
- **B.** Compute $u(k)$
- **C.** Compute $u(k+1)$
- **D.** Considering the computed u(k), what can be said about the stability of the closed-loop system?
- **E.** By considering the cost function defined below, compute $V_1(k), u(k), u(k+1)$

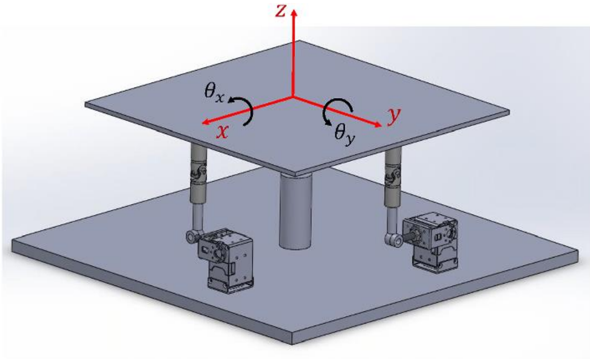
$$V_1(k) = \sum_{i=H_W}^{H_P} [r(k+i) - x(k+i)]^T \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [r(k+i) - x(k+i)]$$
$$+ \sum_{i=1}^{H_C - 1} [u(k+i) - u(k+i-1)]^T \begin{bmatrix} 1 \\ 4 \end{bmatrix} [u(k+i) - u(k+i-1)]$$

Name	Description	Value [unit]
m	Mass of the ball	0.11 [Kg]
J	Mass moment of inertia of the ball	1.76e-5 [Kg m^2]
R	Radius of the ball	0.02 [m]
B	Constant	$m/(m + J/R^2)$
p_x	Position of the ball in the x-axis	[m]
θ_x	Angle of the plate in the x-axis	[rad]
$\dot{p_x}$	Velocity of the ball in the x-axis	[m/s]
$\dot{\theta}_x$	Angular velocity of the plate in the x-axis	[rad/s]
p_y	Position of the ball in the y-axis	[m]
θ_y	Angle of the plate in the y-axis	[rad]
$\dot{p_y}$	Velocity of the ball in the y-axis	[m/s]
$\dot{\theta}_y$	Angular velocity of the plate in the y-axis	[rad/s]
u_x	Angular acceleration of the plate from x-axis	[rad/s^2]
u_y	Angular acceleration of the plate from y-axis	[rad/s^2]

Problem n.2 (15 points)

The following image depicts a ball and plate system. The nonlinear model below describe its kinematics.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \\ \dot{x_7} \\ \dot{x_8} \end{bmatrix} = \begin{bmatrix} x_2 \\ B(x_1x_4^2 + x_4x_5x_8 - g\,sin\,x_3) \\ x_4 \\ 0 \\ x_6 \\ B(x_5x_8^2 + x_1x_4x_8 - g\sin x_7) \\ x_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$



$$\begin{aligned} \boldsymbol{x} &= [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8]^T \\ &= [p_x \quad \dot{p_x} \quad \theta_x \quad \dot{\theta_x} \quad p_y \quad \dot{p_y} \quad \theta_y \quad \dot{\theta_y}]^T \end{aligned}$$

$$\boldsymbol{y} = \begin{bmatrix} x_1 \\ x_5 \end{bmatrix}$$

The ball and plate is a system, where a metal ball stays on a rigid square plate with each side length of 1m. The slope of the plate can be manipulated by two perpendicularly installed step motors, so that the tilting of the plate will make the ball moving. Our goal is to follow the trajectory depicted in the picture on the right.

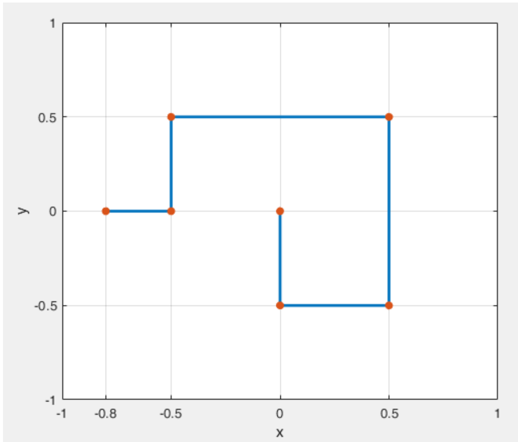


Table n.1 lists the system parameters.

Questions:

A. Calculate and specify an analytical Jacobian for the output function and the state function.
B. Consider $T_S = 0.2s$. Run an open loop simulation to observe the dynamics and determine the appropriate values of H_P, H_C. Initialize the MPC controller with the proper parameters. Validate the MPC Object. Run a simulation lasting $50s$ and comment the results.
C. The plate where the ball moves has each side length of 2m. The angular acceleration of the plate on both axis is limited in $\pm 1rad/s^2$.Constrains the controller accordingly. Run a simulation and comment the results. Did the constraints come into effect?
D. We want to meet the following requirements: Maximum overshoot $\leq 5\%$. Tune the controller to meet the requirements. Explain all the design choices and comment on the results obtained.
E. Analyze the computational time required for each move update. Can you control the system in real time?
F. Plot the evolution of the cost over time and comment it.
G. **BONUS (5 points):** Consider $v = \sqrt{\dot{p_x} + \dot{p_y}}$ the velocity of the ball. Constrain $|v| \leq 4mm/s$.

SUGGESTION: Analyze the data using three figures:

- fig. 1 showing the subplots of the inputs.
- fig. 2 showing the subplots of the outputs.
- fig. 3 showing the path traveled by the ball on the xy plane.

Add titles to charts and anything else that makes them readable.

IMPORTANT: To get a full score it is necessary to comment on the design choices.