Problem n.1 (15 points)

Given the discrete-time system S(A,B,C,D) described below:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Where:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(k) = 2\begin{bmatrix} 1(k) \\ 1(k) \\ 1(k) \end{bmatrix} \qquad 1(k) = 1 \quad \forall k > 0$$

Cost function:

$$V(k) = \sum_{i=H_W}^{H_P} [r(k+i) - y(k+i)]^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} [r(k+i) - y(k+i)]$$

$H_P = 2 \qquad H_w = 1 \qquad H_c = 2$

Questions:

- **A.** Compute the cost function V(k)
- **B.** Compute u(k)
- **C.** Compute u(k+1)
- **D.** Considering the computed u(k), what can be said about the stability of the closed-loop system?
- **E.** By considering the cost function defined below, compute $V_1(k), u(k), u(k+1)$

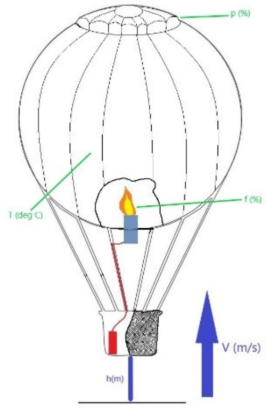
$$V_1(k) = V(k) + \sum_{i=1}^{H_C-1} [u(k+i) - u(k+i-1)]^T \begin{bmatrix} \frac{1}{2} \end{bmatrix} [u(k+i) - u(k+i-1)]$$

Table n.1 (see Problem n.2)

Name	Description	Value [unit]	Name	Description	Value [unit]
u_1	Fuel valve opening (f)	[%]	y_1	Ballon height (h)	[m]
u_2	Vent valve opening (p)	[%]	y_2	Ballon velocity (v)	[m/s]
x_1	Ballon height (h)	[Km]	y_3	Ballon temperature (T)	[°C]
x_2	Ballon velocity (v)	[Km/t_r]	α	Ballon coefficient	5.098
x_3	Ballon temperature (T)	[°K]	β	Heat loss coefficient	0.01683
f_r	Fuel scaling parameter	4870	γ	Atmosphere coefficient	5.257
p_r	Vent scaling parameter	1485	δ	Thermal drop coefficient	0.0255
h_r	Height scaling parameter	1000	μ	Ratio payload weight to total weigth	0.1961
t_r	Time scaling parameter	10.10			
T_r	Temperature scaling parameter	288.2	ω	Drag coefficient	8.544
F_d	Wind gust disturbance (MD)	[Km]			

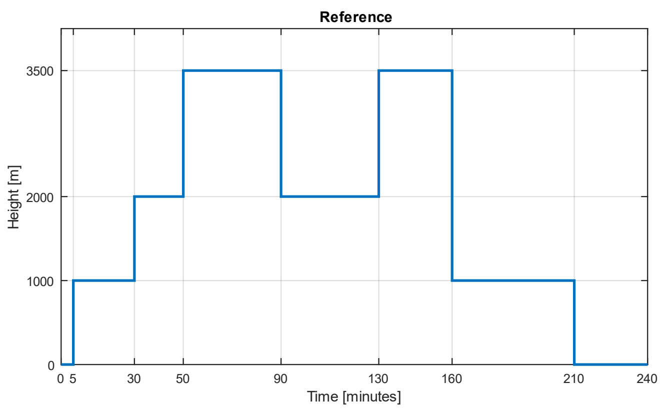
Model Predictive Control
Professor: Prof. Saverio Mascolo
Exam of 22/04/2022

Problem n.2 (15 points)



Hot air balloons (HABs) are based on a very basic scientific principle: lift force is equal to the weight of the air displaced by the envelope (Archimedes' principle). To keep the balloon rising, you need a way to reheat the air. Hot air balloons do this with a fuel burner. HABs also have a cord to open the parachute valve at the top of the envelope. When the pilot pulls the attached cord, some hot air can escape from the envelope, decreasing the inner air temperature. This causes the balloon to slow its ascent or descent. The envelope interior temperature T, changes based on fuel valve position f, vent valve position p, and heat loss to the atmosphere. The dynamics of the system can be described, in a very simplified way, through the following state space equations:

$$\begin{cases} \dot{x}_1 = \alpha\mu((1-\delta x_2)^{\gamma-1})\left(1-\frac{1-\delta x_2}{x_3}\right)-\mu-\omega x_1|x_1| \\ \dot{x}_2 = x_1 + F_d \\ \dot{x}_3 = -\big(x_3-(1-\delta x_1)\big)(\beta+u_2)+u_1 \end{cases} \qquad \begin{cases} y_1 = x_1\frac{h_r}{t_r} \\ y_2 = x_2 h_r \\ y_3 = x_3 T_r - 273.2 \end{cases} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v \\ h \\ T \end{bmatrix}$$



$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f \\ p \end{bmatrix}$$

The controller's goal is to control <u>only</u> the height of the balloon. The customer asks us that the validation of the controller must be performed during a specific maneuver provided (on picture above). The balloon parameters are written in Table n.1.

Questions:

A. The system has a very slow dynamics whose order of magnitude is minutes. The settling time required by the balloon to perform a 1000m elevation is 20 minutes. Considering Ts = 1 and Hc = 2, choose the appropriate Hp value. The initial state of the system is x0 = [1.244 0 0].
B. The fuel valve and the vent valve are proportional valves, therefore their opening is limited in a range (0-100%). The material of which the balloon is made is able to withstand up to the maximum temperature of 150 ° C. The height of the balloon cannot be negative.
C. Calculate and implement the Jacobian for the StateFunction and the OutputFunction.
D. At time t = 80, a gust of wind hits the balloon making it rise by 0.5km. The gust of wind can be modeled as an impulsive disturbance with an amplitude equal to the induced elevation. Carry out a simulation and comment on the results obtained.
E. Analyze the computational time required for each move update. Can you control the system in real time? Is computational time affected by the disturbance of the previous point (question D)?

$\boxed{1}$

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Dove :

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(k) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \forall\, k > 0$$

$$V(k) = \sum_{i=H_w}^{H_p} [r(k+i) - y(k+i)]^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} [r(k+i) - y(k+i)]$$

con $H_w = 1$, $H_p = 2$, $H_c = 2$

$$x(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u(k) = \begin{bmatrix} -u(k) \\ x_1(k) + x_2(k) \\ 3x_1(k) \end{bmatrix}$$

$$y(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k+1) = \begin{bmatrix} -u(k) \\ x_1(k) + x_2(k) \\ 3x_1(k) \end{bmatrix}$$

$$x(k+2) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -u(k) \\ x_1(k) + x_2(k) \\ 3x_1(k) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u(k+1) = \begin{bmatrix} -u(k+1) \\ -u(k) + x_1(k) + x_2(k) \\ -3u(k) \end{bmatrix}$$

$$y(k+2) = \begin{bmatrix} -u(k+1) \\ -u(k) + x_1(k) + x_2(k) \\ -3u(k) \end{bmatrix}$$

(A) Calcolo della funzione Costo

$$V(k) = \left[r(k+1) - y(k+1)\right]^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left[r(k+1) - y(k+1)\right] + \left[r(k+2) - y(k+2)\right]^T \cdot$$

$$\cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left[r(k+2) - y(k+2)\right] =$$

$$= \begin{bmatrix} 2 + u(k) \\ 2 - x_1(k) - x_2(k) \\ 2 - 3x_1(k) \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 + u(k) \\ 2 - x_1(k) - x_2(k) \\ 2 - 3x_1(k) \end{bmatrix} +$$

$$+ \begin{bmatrix} 2 + u(k+1) \\ 2 + u(k) - x_1(k) - x_2(k) \\ 2 + 3u(k) \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 + u(k+1) \\ 2 + u(k) - x_1(k) - x_2(k) \\ 2 + 3u(k) \end{bmatrix} =$$

$$= \begin{bmatrix} 4 + 2u(k) & 2 - x_1(k) - x_2(k) & -2 + 3x_1(k) \end{bmatrix} \begin{bmatrix} 2 + u(k) \\ 2 - x_1(k) - x_2(k) \\ 2 - 3x_1(k) \end{bmatrix} +$$

$$+ \begin{bmatrix} 4 + 2u(k+1) & 2 + u(k) - x_1(k) - x_2(k) & -2 - 3u(k) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 2 + u(k+1) \\ 2 + u(k) - x_1(k) - x_2(k) \\ 2 + 3u(k) \end{bmatrix} = 2\left(2 + u(k)\right)^2 + \left(2 - x_1(k) - x_2(k)\right)^2 -$$

$$- \left(2 - 3x_1(k)\right)^2 + 2\left(2 + u(k+1)\right)^2 + \left(2 + u(k) - x_1(k) - x_2(k)\right)^2 -$$

$$- \left(2 + 3u(k)\right)^2$$

B

$$\frac{\partial V(k)}{\partial u(k)} = 0 \rightarrow 4\big(2+u(k)\big) + 2\big(2+u(k)-x_1(k)-x_2(k)\big) -$$

$$- 2\big(2+3u(k)\big)(3) = 0 \rightarrow$$

$$\rightarrow \cancel{8} + 4u(k) + \cancel{4} + 2u(k) - 2x_1(k) - 2x_2(k)$$

$$- \cancel{12} - 18u(k) = 0 \rightarrow$$

$$\rightarrow u(k) = \frac{-2x_1(k) - 2x_2(k)}{12} = -\frac{1}{6}x_1(k) - \frac{1}{6}x_2(k)$$

C

$$\frac{\partial V(k)}{\partial u(k+1)} = 0 \rightarrow 4\big(2+u(k+1)\big) = 0 \rightarrow 8 + 4u(k+1) = 0$$

$$\rightarrow u(k+1) = -2$$

D

ho sostituito u(k) trovato al punto B

$$A_{CL} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} +\frac{1}{6}x_1(k) + \frac{1}{6}x_2(k) \\ x_1(k) + x_2(k) \\ 3x_1(k) \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A_{CL}| = \begin{vmatrix} \lambda - 1/6 & -1/6 & 0 \\ -1 & \lambda - 1 & 0 \\ -3 & 0 & \lambda \end{vmatrix} = 0 \rightarrow$$

$$\rightarrow \lambda(\lambda - 1/6)(\lambda - 1) - [1/6\, \lambda] = 0 \rightarrow$$

$$\rightarrow \lambda\left(\lambda^2 - 1/6\,\lambda - \lambda + 1/6\right) - 1/6\,\lambda = 0 \rightarrow$$

$$\rightarrow \lambda^3 - \frac{7}{6}\lambda^2 + \cancel{\frac{1}{6}\lambda} - \cancel{\frac{1}{6}\lambda} = 0 \rightarrow \lambda^2\left(\lambda - \frac{7}{6}\right) = 0$$

$$\rightarrow \lambda_{1,2} = 0 \qquad \lambda_3 = \frac{7}{6} > 1$$

il sistema in anello chiuso é INSTABILE (ha un polo esterno alla circonferenza di raggio unitario).

E

$$V_1(k) = V(k) + [u(k+1) - u(k)]^T \left[\frac{1}{2}\right] [u(k+1) - u(k)] =$$

$$= V(k) + \frac{1}{2}\left(u(k+1) - u(k)\right)^2$$

$$\frac{\partial V_1(k)}{\partial u(k)} = 0 \rightarrow 4u(k) + 2u(k) - 2x_1(k) - 2x_2(k) - 18u(k) +$$

$$+ \frac{1}{2} \cdot 2\Big(u(k+1) - u(k)\Big)\big(-1\big) = 0 \rightarrow$$

$$\rightarrow \quad u(k) = \frac{-2x_1(k) - 2x_2(k) - u(k+1)}{11}$$

$$\frac{\partial V_1(k)}{\partial u(k+1)} = 0 \rightarrow 8 + 4u(k+1) + \frac{1}{2} \cdot 2\Big(u(k+1) - u(k)\Big) = 0$$

$$\rightarrow \quad u(k+1) = \frac{u(k)}{5} - \frac{8}{5}$$