Problem n.1 (15 points)

Given the discrete-time system S(A,B,C,D) described below:

$$\begin{cases} x_1(k+1) = -x_1(k) + x_2(k) \\ x_2(k+1) = x_1(k) + u(k) \end{cases} \qquad y(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Where:

$$r(k) = \begin{bmatrix} 1(k) \\ 1(k) \end{bmatrix} \qquad 1(k) = 1 \quad \forall k \geq 0$$

And the cost function is:

$$V(k) = \sum_{i=H_W}^{H_P} [r(k+i) - x(k+i)]^T \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} [r(k+i) - x(k+i)]$$

$$H_P = 2 \qquad H_w = 1 \qquad H_c = 1$$

Questions:

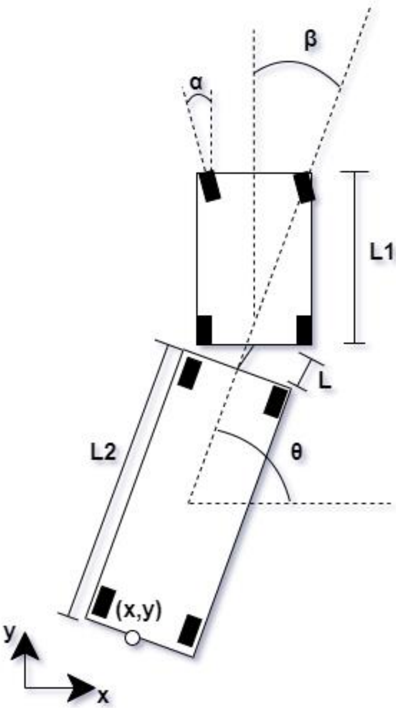
- **A.** Compute the cost function $V(k)$.
- **B.** Compute $u(k)$
- **C.** Compute $u(k+1)$
- **D.** Considering the computed u(k), what can be said about the stability of the closed-loop system?
- **E.** By considering the cost function defined below, compute $V_1(k), u(k), u(k+1)$

$$V_1(k) = \sum_{i=H_W}^{H_P} [r(k+i) - x(k+i)]^T \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} [r(k+i) - x(k+i)] + \sum_{i=0}^{H_C-1} 2[u(k+i)]^2$$

Table n.1 (see Problem n.2)		
Name	**Description**	**Value [unit]**
x	Center of the trailer rear axle on the x axis	[m]
y	Center of the trailer rear axle on the y axis	[m]
θ	Trailer orientation, global angle, 0 = east	[rad]
β	Truck orientation with respect to trailer, 0 = aligned	[rad]
α	Truck steering angle	[rad]
v	Truck longitudinal velocity	[m/s]
L	Hitch length	1 [m]
L_1	Truck length	6 [m]
L_2	Trailer length	10 [m]
x_0	Initial state	[0;0;0;0]
y_{ref}	Reference	[1;-25;π/2;0]

Problem n.2 (15 points)

The following image depicts a truck and trailer system. The nonlinear model below describe its kinematics.



$$\dot{\boldsymbol{x}} = [x_P \quad y_P \quad \theta \quad \beta]^T \qquad \boldsymbol{y} = \boldsymbol{x}$$

$$\begin{cases} \dot{x_P} = v \cdot cos\beta \left(1 + \dfrac{L}{L_1} tan\beta \cdot tan\alpha\right) \cdot \cos\theta \\ \dot{y_P} = v \cdot cos\beta \left(1 + \dfrac{L}{L_1} tan\beta \cdot tan\alpha\right) \cdot \sin\theta \\ \dot{\theta} = v \cdot \left(\dfrac{sin\beta}{L_2} - \dfrac{L}{L_1 L_2}\right) \cdot cos\beta \cdot tan\alpha \\ \dot{\beta} = v \cdot \left(\dfrac{tan\alpha}{L_1} - \dfrac{sin\beta}{L_2} + \dfrac{L}{L_1 L_2} cos\beta \cdot tan\alpha\right) \end{cases}$$

The goal of this problem, is to design a nonlinear MPC controller that finds an optimal route to automatically park a truck with a single trailer from its initial position to its target position.

Table n.1 lists the vehicle parameters.

Questions:

A. Calculate and specify an analytical Jacobian for the output function and the state function.
B. Consider $T_S = 0.5s, H_P = 12, H_C = 5$. Initialize the MPC controller with the proper parameters. Validate the MPC Object. Run a simulation lasting $10s$ and comment the results.
C. The steering angle of the truck must remain in the range $\pm 45°$. The maximum forward speed is $10\ m/s$ and the maximum reverse speed is $10\ m/s$. The angle between truck and trailer cannot go beyond $\pm 90°$ due to mechanics limitations. Constrains the controller accordingly. Run a simulation lasting $10s$ and comment the results. Did the constraints come into effect?
D. We want to meet the following requirements: Position error at steady state: $\pm 1m$, angular error at steady state (on β): $\pm 1°$, maneuver time:$< 6s$, minimize parking speed. Tune the controller to meet the requirements. Explain all the design choices and comment on the results obtained.
E. Analyze the computational time required for each move update. Can you control the system in real time?
F. Plot the evolution of the cost over time and comment it.

SUGGESTION: Analyze the data using four figures:

- fig. 1 showing the subplots of the inputs.
- fig. 2 showing the subplots of the states.
- fig. 3 showing the subplots of the outputs.
- fig. 4 showing on the xy plane of the path traveled by the truck.

Add titles to charts and anything else that makes them easier to understand.

IMPORTANT: To get a full score it is necessary to comment on the design choices.