Math 71: Algebra **Fall 2022**

PSET 1 — 2022-11-11

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Abstract Algebra*** by **David S. Dummit & Richard M. Foote**.

(b) ***Algebra*** by **Jacob K. Goldhaber & Gertrude Ehrlich**

Problems

1. Let $f\colon A \to B$ and $g\colon B \to C$ be maps. Suppose that $g \circ f\colon A \to C$ is injective.

(a) Show that f is injective.

By definition of composition, $(g \circ f)(x) = g(f(x))$.

By definition of injectivity, $g \circ f$ is injective if and only if $(g \circ f)(a) = (g \circ f)(a')$ implies $a = a'$.

Suppose f is not injective.

Then there exists $a, a' \in A, b \in B$ such that $f(a) = f(a') = b \ \wedge \ a \neq a'$.

Then:

$$(g \circ f)(a) = g(f(a)) = g(b) = g(f(a')) = (g \circ f)(a')$$

Thus, $(g \circ f)(a) = (g \circ f)(a')$ for some $a, a' \in A$ such that $a \neq a'$, implying that $g \circ f$ is not injective.

This contradicts the fact that $g \circ f$ is injective.

Therefore, it must hold that $f(a) = f(a') \implies a = a'$ and f is injective.

(b) Is g necessarily injective? Give a proof or a counterexample.

g is necessarily injective when restricted to the codomain of f.

For instance, consider $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) = x + 1$ and $g(x) = x^2$.

We can note that:

(a) f is injective, since $f(x) = x + 1 = y + 1 = f(y) \implies x = y$.

(b) g is injective *as defined* because its domain is limited to $\mathbb{R}^+$, the codomain of f. However, if the domain of g is extended to include all of $\mathbb{R}$, then g would no longer be injective since $b^2 = (-b)^2$ for all $b \in \mathbb{R}$.

2. (sorta DF 1.1.1, 1.1.8) Determine which of the following are groups. Justify your answer.

(a) The set $G = \mathbb{R} \setminus \{0\}$ under the binary operation $*$ defined by $a * b = a/b$ for $a, b \in G$.

G is not a group.

(a) G is not closed under the operation $*$, since $a * b = a/b$ gives elements in $\mathbb{Q}$ for any $a, b \in \mathbb{R}$ such that $b \nmid a$.

(b) G is not associative, since $a * (b * c) = a/(b/c) = ac/b \neq a/bc = (a/b)/c = (a * b) * c$

(c) G does not have an identity element, since:

$$e * a = e/a = a \implies e = a^2$$

$$a * e = a/e = a \implies e = 1$$

$$e/a = a/e = a \quad \implies \quad e = a^2 \wedge e = 1 \quad \implies \quad a = \pm 1.$$

There is no unique identity unique identity that leaves **all** elements in G invariant.

(d) However, **if we invented the identity to be** 1, then each element in G would be it's own inverse since:

$$\forall a \in G: \qquad a/a^{-1} = a^{-1}/a = 1 \quad \implies \quad a = a^{-1}$$

(b) The set $G = \mathbb{R}$ under the binary operation $*$ defined by $a * b = a + b + ab$ for $a, b \in G$.

G is not a group.

(a) G has a unique identity $e = 0$.

(b) G is closed under $*$.

(c) $*$ is not an associative operation on G.

$$\begin{aligned} a*(b*c) &= a*(b+c+bc) = a*b+a*c+a*bc \\ &= a+b+ab+a+c+ac+a+bc+abc \\ &= 3a+b+c+ab+ac+bc+abc \end{aligned}$$

$$\begin{aligned} (a*b)*c &= (a+b+ab)*c = a*c+b*c+ab*c \\ &= a+c+ac+b+c+bc+ab+c+abc \\ &= a+b+3c+ab+ac+bc+abc \end{aligned}$$

As we can see, $a*(b*c) \neq (a*b)*c$.

(c) The set $G = \{z \in \mathbb{C} : z^n = 1 \text{ for some } n \in \mathbb{Z}_{>0}\}$ under multiplication. *[Hint: be sure to check multiplication is a binary operation on G in the first place!]*

G is a group.

(a) G has a unique identity $e = 1 + 0i = 1$, since 1 leaves all elements in the group invariant after multiplication.

(b) G is closed under multiplication.

$\forall z_1, z_2 \in G, \exists n_1, n_2 \in \mathbb{Z}_{>0}$:

$$z_1 * z_2 = z_1 z_2$$

We need to find an exponent n such that $(z_1 z_2)^n = 1$.

Taking $n = n_1 n_2$ gives:

$$\begin{aligned}(z_1 z_2)^{n_1 n_2} &= z_1^{n_1 n_2} z_2^{n_1 n_2} \\ &= (z_1^{n_1})^{n_2} (z_2^{n_2})^{n_1} \\ &= 1^{n_2} 1^{n_1} \\ &= 1\end{aligned}$$

Therefore, it holds that multiplication of 2 elements in G gives an element in G.

(c) $*$ is associative following from associativity of $\mathbb{C}$.

(d) G has inverses. By definition, if $z \in G$ then $z^n = 1$ for some $n \in \mathbb{Z}_{>0}$.

The multiplicative inverse of z is $1/z$, and $z^n = 1$ implies $(1/z)^n = 1$.

3. (DF 1.1.20) Let G be a group and let $x \in G$. Show that x and x^{-1} have the same order.

Let n be the order of x such that $x^n = e$.

Furthermore, let x^{-1} be the inverse of x, such that $x * x^{-1} = e$.

Using algebraic substitution, we can show that:

$$\begin{aligned} x^n &= e \\ x^n &= x * x^{-1} && \text{(an element multiplied by its inverse)} \\ x^n &= (x * x^{-1})^n && \text{(multiplying } e \text{ by itself } n \text{ times.)} \\ x^n &= x^n * x^{-n} \\ e &= e * x^{-n} && \text{(since we already know that } x^n = e\text{)} \\ e &= x^{-n} = (x^{-1})^n \end{aligned}$$

We see that $(x^{-1})^n = e$. By definition of order, x^{-1} has order n.

4. (sorta DF 1.2.2–1.2.5) Let $D_{2n} = \{1, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}$ be the dihedral group of order $2n$ with presentation $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1}\ rangle$.

(a) Write out the multiplication (Cayley) table for D_6.

	1	r	r^2	s	sr	sr^2
1	1	r	r^2	s	sr	sr^2
r	r	r^2	1	sr	sr^2	s
r^2	r^2	1	r	sr^2	s	sr
s	s	sr^2	sr	1	r^2	r
sr	sr	s	sr^2	r	1	r^2
sr^2	sr^2	sr	s	r^2	r	1

(b) Show that if $x \in D_{2n}$ is a power of r (including $x = r^0 = 1$!), then $rx = xr$ and x has order at most n.

Suppose $x = r^k$ for some $k \in \mathbb{Z}$. Then $rx = rr^k = r^{k+1} = r^k r = xr$.

Furthermore, we know that $r^n = 1$ in D_{2n}.

Therefore:

$$x = r^k$$

$$x^n = \left(r^k\right)^n$$

$$x^n = r^{kn} \qquad \text{(multiplication of exponents for nested exponentiation)}$$

$$x^n = (r^n)^k \qquad \text{(rearranging the powers)}$$

$$\because\ r^n = 1$$

$$\therefore x^n = 1^k = 1$$

Therefore, for any element $x \in D_{2n}$ such that $x = r^k$ for some $k \in \mathbb{Z}$, we see that $x^n = 1$ and x has order at most n.

The order can be lower, if $k > 1$ **and** $k \mid n$

(c) Otherwise, if $x \in \{s, sr, \ldots, sr^{n-1}\}$ (not a power of r), then show that $rx = xr^{-1}$ and x has order 2. *[Hint: first prove by induction that $r^m s = sr^{-m}$ for all $m \geq 1$.]*

We aim to prove that $r^m s = sr^{-m}$ for all $m \geq 1$.

Base case: We aim to show that $r^m s = sr^{-m}$ for $m = 1$.

$$rs = sr$$

The proof is trivial. Since s is a flip, it translates a rotation done in any direction (counter-clockwise, in this case) *before the flip* into equivalent rotations done in the opposite direction (clockwise) *after the flip*.

Induction hypothesis: incrementally, we aim to show that $r^m s = sr^{-m}$ if $r^{m-1}s = sr^{-(m-1)}$. Let $m = i + 1$, assuming it has been proven that $r^i s = sr^{-i}$.

$$r^i s = sr^{-i}$$

$$r(r^i s) = r(sr^{-i})$$

$$r^{(i+1)}s = rsr^{-i} \qquad \text{(Simplifying the left side of the equation)}$$

$$r^{(i+1)}s = (rs)r^{-i} \qquad \text{(Grouping the right side of the equation)}$$

$$r^{(i+1)}s = (sr^{-1})r^{-i}$$

$$r^{(i+1)}s = sr^{-i-1} \qquad \text{(Simplifying the left side of the equation)}$$

$$r^{(i+1)}s = sr^{-(i+1)}$$

We now aim to show that if $x = r^m s$ for some $m \in \mathbb{Z}$ then x has order 2, i.e. $x^2 = 1$.

$$x^2 = (r^m s)^2$$

$$x^2 = (r^m s)(sr^{-m}) \qquad \text{(Substituting the equality)}$$

$$x^2 = r^m s^2 r^{-m} \qquad \text{(Expanding the right side of the equation)}$$

$$x^2 = r^m r^{-m} \qquad \text{(Simplifying } s^2 = 1\text{)}$$

$$x^2 = r^{m-m} = r^0 = 1$$

(d) For a group G under $*$, we say that a is *central* if $a * x = x * a$ for all $x \in G$. Show that if $n = 2k$ is even that $z = r^k$ is an element of order 2 which is central.

(a) $z = r^k$ is an element of order 2

Given $z = r^k$, then $z^2 = \left(r^k\right)^2 = r^{2k}$.

Furthermore, we know that $n = 2k$, which means the Dihedral group is D_{4k}. It then follows from the Group axioms that $r^{2k} = 1$ in D_{4k}.

Thus, $(z^k)^2 = 1$ and z has order 2.

(b) z is central

We aim to show that $z * x = x * z$ for all $x \in D_{4k}$.

Given $z = r^k = r^{n/2}$:

$$
\begin{aligned}
z * x &= r^{(n/2)} * x \\
(zx)^2 &= (r^{n/2}x)^2 && \text{(Squaring both sides)} \\
z^2x^2 &= r^n x^2 \\
z^2x^2 &= x^2 \\
zx &= x
\end{aligned}
$$

$$
\begin{aligned}
x * z &= xr^{(n/2)} \\
(xz)^2 &= \left(xr^{n/2}\right)^2 && \text{(Squaring both sides)} \\
xz^2 &= x^2r^n \\
xz^2 &= x^2 \\
xz &= x
\end{aligned}
$$

Thus, we see that $zx = xz = x$ for all $x \in D_{4k}$.

5. (DF 1.2.10) Let G be the group of rigid motions in $\mathbb{R}^3$ of a cube. Show that G is a nonabelian group of order 24. *[Hint: Find the number of positions to which an adjacent pair of vertices can be sent; alternatively, find the number of places to which a given face may be sent and, once a face is fixed, the number of positions to which a vertex on that face may be sent.]*

(a) $G = 24$

There are 6 faces on a cube, and each face has 4 edges.

If we visualize a person placing the cube on a table, there are 6 possible faces that can face the person, and, for each face, there are 4 orientations (or 4 possible edges that could be touching the table). This make for $6 * 4 = 24$ possible positions for the cube.

(b) G is nonabelian G has order 24. For G to be prime, then $|G|$ must either be prime, a square of a prime, or a product of two primes $p, q \in \mathbb{Z}, p < q$such that $p \nmid (q - 1)$.

However, $|G| = 24 = 2^3 \cdot 3$ and does not satisfy any of those properties.