Math 71: Algebra **Fall 2022**

PSET 2 — September 27, 2022

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Abstract Algebra*** by **David S. Dummit & Richard M. Foote**.
(b) ***Algebra*** by **Jacob K. Goldhaber & Gertrude Ehrlich**

Problems

1. (DF 0.1.7) Let $f\colon A \to B$ be a surjective map of sets. For $y \in B$, let

$$f^{-1}(y) := \{x \in A : f(x) = y\}$$

be the *preimage* or *fiber* of f over y. (The map f is bijective if and only if $f^{-1}(y) = \{x\}$ consists of a single element $x \in A$, in which case we can define f^{-1} as a function, removing the set brackets. But we always have fibers.) Define a relation by $a \sim b$ if $f(a) = f(b)$. Show that this relation is an equivalence relation whose equivalence classes are the fibers of f.

2. (sorta-not-really DF 0.3.15(b))

(a) For $a = 69$ and $n = 372$, determine the greatest common divisor $g := \gcd(a, n)$, the least common multiple $\mathbf{lcm}\ ((,\ a), b)$, and write $g = ax + by$ with $x, y\ in\mathbb{Z}$. Is $\overline{a} \in (\mathbb{Z}/n\mathbb{Z})^{\times}$? If so, what is $\overline{a}^{-1}$?

(b) Taking $n = 89$, what is the order of $\overline{2}$ in $(\mathbb{Z}/n\mathbb{Z})^{\times}$?

(c) How many elements are there in $(\mathbb{Z}/360\mathbb{Z})^{\times}$?

3. (DF 1.3.1, sorta 1.3.7)

(a) Let σ be the permutation

$$1 \mapsto 3,\ 2 \mapsto 4,\ 3 \mapsto 5,\ 4 \mapsto 2,\ 5 \mapsto 1$$

and τ be the permutation

$$1 \mapsto 5,\ 2 \mapsto 3,\ 3 \mapsto 2,\ 4 \mapsto 4,\ 5 \mapsto 1.$$

Find the cycle decompositions of each of the following: $\sigma, \tau, \sigma^2,\ sigma^{-1}, \sigma\tau, \tau\sigma, \tau^2\sigma$. Do σ and τ commute?

(b) Write out the cycle decomposition of each element of order 2 in the symmetric group S_4. How many such elements are there of each cycle type?

(c) How many elements are in the set $\{\sigma \in S_5 : \sigma(2) = 5\}$?

4. (some of DF 1.6.6)

(a) Let $\mathbb{R}^\times = \mathbb{R} \smallsetminus \{0\}$ be the set of nonzero real numbers. Then $\mathbb{R}^\times$ is a group under multiplication. Define a second binary operation on $\mathbb{R}^\times$ by $x * y = xy/2$ for $x, y \in \mathbb{R}^\times$. Show that $(\mathbb{R}^\times, *)$ is a group, and find an isomorphism $\phi\colon (\mathbb{R}^\times, \cdot) \xrightarrow{\sim} (\mathbb{R}^\times, *)$. *[Hint: if it helps, write $G = \mathbb{R}^\times$ in the second case with the nonstandard operation.]*

(b) Prove that the groups $\mathbb{Z}$ (under $+$) is not isomorphic to $\mathbb{Q}$ (under $+$). *[Remark: there is a bijection from $\mathbb{Z}$ to $\mathbb{Q}$ that is not a homomorphism, and a homomorphism that is not a bijection!]*

5. Let $\phi\colon G \to H$ be a bijective homomorphism, with inverse $\phi^{-1}\colon H \to G$. Show that ϕ^{-1} is also a homomorphism.