

PSET 5 — 2022-11-05

Prof. Voight

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

- (a) *Abstract Algebra* by David S. Dummit & Richard M. Foote.
- (b) *Algebra* by Jacob K. Goldhaber & Gertrude Ehrlich

Problems

1. In this problem, we prove the main ingredients in the fourth isomorphism theorem. Throughout, let G be a group, let $N \trianglelefteq G$ be a normal subgroup. Let $H \leq G$ be a subgroup.

- (a) Define $H/N := \{hN : h \in H\}$. Show that $H/N \leq G/N$ is a subgroup. (So given a subgroup of G , we can make a subgroup of G/N .)

- (a) Since both H and N are subgroups of G , they both contain the identity element of G , ϵ . So the set $H/N = \{hn : h \in H, n \in N\}$ contains ϵ , particularly when $h = \epsilon$ and $n = \epsilon$.
- (b) Let $h_1n_1 \in H/N$ and $h_2n_2 \in H/N$. Consider the product, $h_1n_1 \cdot h_2n_2$. Since elements of normal subgroups commute:

$$\begin{aligned} h_1n_1h_2n_2 &= (h_1h_2)(n_1n_2) & (n_1h_2 &= h_2n_1) \\ &= h_3n_3 \in H/N & (h_3 &= h_1h_2 \in H, n_3 = n_1n_2 \in N) \end{aligned}$$

So H/N is closed under multiplication.

- (c) Let $h_1n_1 \in H/N$. Consider the inverse, $(h_1n_1)^{-1}$. Since elements of normal subgroups commute:

$$\begin{aligned} (h_1n_1)^{-1} &= n_1^{-1}h_1^{-1} \\ &= h_1^{-1}n_1^{-1} & (n_1^{-1}h_1^{-1} &= h_1^{-1}n_1^{-1} \text{ since } n_1^{-1} \in N) \\ &= h_2n_2 \in H/N & (h_2 &= h_1^{-1} \in H, n_2 = n_1^{-1} \in N) \end{aligned}$$

So H/N is closed under inversion.

- (b) Show that $(HN)/N = H/N$. (So we get the same subgroups of G/N by taking those subgroups $H \leq G$ containing N .)

Consider the map $\phi: H \rightarrow (HN)/N$ defined by $\phi(h) := hnN$, $n \in N$. Take the instance $h \in \ker(\phi)$, then $hn = e$, and $hnN = N$. Thus, we see that one of the cosets of HN/N is N . Using this coset, we can define the image of ϕ as a product of H and N — that is, H/N .

- (c) If $\phi: G \rightarrow G'$ is a group homomorphism and $H' \leq G'$ is a subgroup, show that $\phi^{-1}(H')$ is a subgroup of G . Apply this to the map $\pi: G \rightarrow G/N$ to conclude that if $H' \leq G/N$ is a subgroup, then $\pi^{-1}(H') \leq G$ is a subgroup of G containing N . (So we can go backwards.) [Hint: recall that if $f: A \rightarrow B$ is a map and $Y \subseteq B$ is a subset, then the preimage is $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$. The use of this symbol does not imply that f has an inverse.]

(a) Let e' be the identity element of G' . Then $\phi(e) = e' \in H'$. So $\phi^{-1}(\{e'\}) = \{e\}$. Therefore, $\phi^{-1}(H')$ contains the identity element of G .

(b) For $h_1, h_2 \in H'$, let $g_1 = \phi^{-1}(h_1)$ and $g_2 = \phi^{-1}(h_2)$. Then $\phi(g_1) \in H'$ and $\phi(g_2) \in H'$. So $\phi(g_1g_2) = \phi(g_1)\phi(g_2) \in H'$. By definition of the isomorphism mapping elements in G to elements in H' ,

Let $g \in \ker(\phi)$, then $\phi(g) = e$. So $\phi^{-1}(e) = \{g \in G : \phi(g) = e\} = \{g \in G : g = e\} = \{e\}$. Thus, $\phi^{-1}(H')$ is closed under the identity element. Let $g_1 \in \phi^{-1}(H')$ and $g_2 \in \phi^{-1}(H')$. Then $\phi(g_1) \in H'$ and $\phi(g_2) \in H'$. So $\phi(g_1g_2) = \phi(g_1)\phi(g_2) \in H'$. Thus, $\phi^{-1}(H')$ is closed under multiplication. Let $g \in \phi^{-1}(H')$. Then $\phi(g) \in H'$. So $\phi(g^{-1}) = (\phi(g))^{-1} \in H'$. Thus, $\phi^{-1}(H')$ is closed under inversion.

- (d) Show that if $H \trianglelefteq G$ is normal, then $H/N \trianglelefteq G/N$ is normal.

2. (DF 3.5.3)

- (a) Prove that S_n is generated by the set $\{(1\ 2), (2\ 3), \dots, (n-1\ n)\}$. [Hint: Consider conjugates, e.g. $(2\ 3)(1\ 2)(2\ 3)^{-1}$.]

First, we note that every permutation can be written as a product of transpositions. Consider the trivial cases of $n = 1$. Then S_1 is generated by the single identity element. For S_2 , we have the two elements $(1\ 2)$ and $(2\ 1)$, which are both transpositions generated by $(1\ 2)$. For each $n \geq 3$, supposing we have already shown that $S_{(n-1)}$ is generated by the set $\{(1\ 2), (2\ 3), \dots, (n-2\ n-1)\}$, we can generate any element $(i\ n)$ by conjugation:

$$(i\ (n-1)) \cdot ((n-1)\ n) \cdot ((n-1)\ i) = (i\ n)$$

- (b) Show that every element in A_n for $n \geq 3$ can be written as the product of (not necessarily disjoint) 3-cycles.

The alternating group A_n is generated by all cycles that can be written as an even product of transpositions. For any $a, b, c \in \{1, 2, \dots, n\}$, The cycle $(a\ b\ c)$ can be written as two transpositions of a :

$$(a\ b\ c) = (a\ b)(b\ c)$$

Essentially, every 3-cycle is a product of two transpositions. Since every elements in A_n can be written as a product of **even** transpositions, every element in A_n can also be written as a product of the 3-cycles constituting contiguous pairs of those transpositions.

3. (DF 4.1.4)

Let S_3 act on the set Ω of ordered pairs: $\{[i, j] : 1 \leq i, j \leq 3\}$ by $\sigma([i, j]) = [\sigma(i), \sigma(j)]$. Let's write square brackets around these ordered pairs so there is no chance we will get confused between them and permutations (we should have been OK since there are commas involved).

- (a) For each $\sigma \in S_3$ find the cycle decomposition of σ under this action (i.e., find its cycle decomposition when σ is considered as an element of S_9 —first fix a labelling of these nine ordered pairs). Is the action faithful?
- (b) Find the orbits of S_3 on Ω . Is the action transitive?
- (c) For each orbit \mathcal{O} of S_3 acting on these nine points, pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3 . What does this have to do with the orbit-stabilizer lemma (DF, Proposition 2, §4.1, p. 114)?

4. (DF 3.4.2, 4.3.2)

- (a) Give a composition series for each of the groups
- D_8
- ,
- Q_8
- , and
- $\mathbb{Z}/8\mathbb{Z}$
- .

(a) Composition series for $D_8 = \langle r, s : r^4 = s^2 = 1, rs = sr^{-1} \rangle$:

$$1 < \langle s \rangle \cong \langle r^2 \rangle \cong \langle sr^n, n \in \mathbb{Z}^+ \rangle \cong C_2 \\ < \langle r \rangle \cong C_4, \{1, r^2, s, sr^2\} \cong \{1, r^2, sr, sr^3\} \cong V_4 < D_8$$

Note: subgroups isomorphic to V_4 in D_8 are: $\{1, r^2, s, sr^2\}$ and $\{1, r^2, sr, sr^3\}$.(b) Composition series for $Q_8 = \langle i, j, k : i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j \rangle$

$$1 < \langle -1 \rangle \cong C_2 < \langle i \rangle \cong \langle j \rangle \cong \langle k \rangle < Q_8$$

(c) Composition series for $\mathbb{Z}/8\mathbb{Z}$

$$1 < \mathbb{Z}/2\mathbb{Z} \cong C_2 < \mathbb{Z}/4\mathbb{Z} \cong C_4 < \mathbb{Z}/8\mathbb{Z} \cong C_8$$

- (b) List all conjugacy classes in the groups
- D_8
- ,
- Q_8
- ,
- $\mathbb{Z}/8\mathbb{Z}$
- .

Conjugacy classes are the sets of elements that are conjugate to each other.

(a) D_8 has 5 conjugacy classes:

$$\{1\}, \{r, r^3\}, \{r^2\}, \{s, sr^2\}, \{sr, sr^3\}$$

(b) Q_8 has 5 conjugacy classes:

$$\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$$

(c) $\mathbb{Z}/8\mathbb{Z}$ has 8 conjugacy classes. Because the group is Abelian, each element is its own conjugate.

5. (DF 4.3.25)

Almost by definition, a normal subgroup is a union of conjugacy classes. If $H < G$ is a proper subgroup of a finite group G , then G is not the union of conjugates of H , i.e., $G \neq \bigcup_{g \in G} gHg^{-1}$. (You are not asked to prove this.)

Let $G := \text{GL}_2(\mathbb{C})$, and let $H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C}, ac \neq 0 \right\}$.

- (a) Show that H is not normal in G .

Let $g \in G, h \in H$. Suppose H is normal in G , then $ghg^{-1} \in H$.

Take $g = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ and let's define $g^{-1} = \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix}$. Let's take $h = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

Then:

$$\begin{aligned} hg^{-1} &= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix} \\ &= \begin{pmatrix} aw' + by' & ax' + bz' \\ cy' & cz' \end{pmatrix} \\ g(hg^{-1}) &= \begin{pmatrix} w & x \\ y & z \end{pmatrix} \cdot \begin{pmatrix} aw' + by' & ax' + bz' \\ cy' & cz' \end{pmatrix} \\ &= \begin{pmatrix} w(aw' + by') + xcy' & w(ax' + bz') + xcz' \\ zcy' & zcz' \end{pmatrix} \end{aligned}$$

If $ghg^{-1} \in H$, then $zcy' = 0$. That would require that either z, c, y' is 0.

- (a) In the case of $z = 0$, then clearly the choice of g doesn't cover the entirety of G .
- (b) In the case of $c = 0$, then the matrix h is not in H , since in the definition of H we restricted the product $ac \neq 0$. This causes a contradiction.
- (c) In the case $y' = 0$, then w' and z' must non-zero (for the matrix g^{-1} to be invertible), and, in-fact, $g^{-1} \in H$, which implies that $g \in H$. If $g \in H$, and $H < G$, then the choice of g doesn't cover the entirety of G .

Therefore, we can conclude that H is not normal in G because it is not normal to all the elements in G .

- (b) Prove that every element of G is conjugate to some element of the subgroup H and deduce that G is the union of conjugates of H . [Hint: Every polynomial over \mathbb{C} has a root; use this to show that every element of $\text{GL}_2(\mathbb{C})$ has an eigenvector.]

Consider $g = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in G$, and $h = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in H$. Let's define $g^{-1} = \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix}$.

Every matrix $g \in G$ has the characteristic equation: $x^2 - (w + z)x + (wx - yz) = 0$. Since every polynomial over \mathbb{C} has a root, every matrix $g \in G$ has an eigenvector.