Math 71: Algebra	**Fall 2022**

PSET 8 — 2022-11-11

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Abstract Algebra*** by **David S. Dummit & Richard M. Foote**.

(b) ***Algebra*** by **Jacob K. Goldhaber & Gertrude Ehrlich**

Problems

1. (DF 7.3.34)

Let I, J be ideals of a ring R.

(a) Define the *sum* of I and J to be

$$I + J = \{a + b : a \in I, b \in J\}.$$

Prove that $I + J$ is the smallest ideal of R containing both I and J.

[Hint: Show that $I + J$ is an ideal, that $I, J \subset I + J$, and that if N is an ideal containing both I and J then $I + J \subset N$.]

(a) $I + J$ is an ideal.

Consider arbitrary $i_1, i_2 \in I, j_1, j_2 \in J, r \in R$. Let $k_1 = i_1 + j_1$ and $k_2 = i_2 + j_2$. Then $k_1, k_2 \in I + J$, and:

$$ir, ri \in I\, \forall r \in R \quad \text{(since } I \text{ is an ideal)} \tag{0.1}$$

$$jr, rj \in J\, \forall r \in R \quad \text{(since } J \text{ is an ideal)} \tag{0.2}$$

$$k_1 + k_2 = (i_1 + j_1) + (i_2 + j_2) = i_3 + j_3 \in I + J \quad (I, J \text{ are closed under } +) \tag{0.3}$$

$$k_1 r = (i_1 + j_1)r = i_1 r + j_1 r = i_4 + j_4 \in I + J \quad \text{(by 0.1 and 0.2)} \tag{0.4}$$

$$r k_1 = r(i_1 + j_1) = r i_1 + r j_1 = i_5 + j_5 \in I + J \quad \text{(by 0.1 and 0.2)} \tag{0.5}$$

Therefore, $I+J$ is closed under addition with other elements in $I+J$ and closed under multiplication by any element of R. This means that $I + J$ is an ideal.

(b) $I, J \subset I + J$.

Let the element $0 \in R$ defined to be the additive identity. Then, every ideal in R contains 0 (since $0r = r0 = 0$ for all $r \in R$). Therefore, $0 \in I$ and $0 \in J$. Then, $I + \{0\} = I \in I + J$, and $\{0\} + J = J \in I + J$.

(c) If N is an ideal containing both I and J, then $I + J \subset N$.

We know that ideals **must be closed under addition**. Suppose N is an ideal of R, $I \in N$, and $J \in N$. Then, it follows that $i + j \in N$ for every $i \in I$ and every $j \in J$, even when $i + j \notin I \cup J$. Particularly, the set $\{i + j : i \in I, j \in J\} = I + J \subseteq N$.

(b) Define the *product* of I and J to be

$$IJ = \{x_1y_1 + \cdots + x_ny_n : x_i \in I, y_i \in J\}$$

to be finite sums of products of elements from I and J. Prove that IJ is an ideal contained in $I \cap J$.

(a) $I \cap J$ is an ideal. Let's define $I \cap J$ to be the set of all elements $x \in R$ such that $x \in I$ and $x \in J$. Take any elements $y, u \in I \cap J$, then $x + y \in I$ and $x + y \in J$. Consequently, $x + y \in I \cap J$ and $I \cap J$ is closed under addition.

Similarly, take $x, y \in I \cap J$ and $r \in R$. Then $rx \in I$ and $rx \in J$, so $rx \in I \cap J$.

Likewise, $xr \in I$ and $xr \in J$, so $xr \in I \cap J$. Therefore, $I \cap J$ is closed under multiplication by any element of R. This makes $I \cap J$ an ideal.

(b) $IJ = \{x_1y_1 + \cdots + x_ny_n : x_i \in I, y_i \in J\} \subseteq I \cap J$

Let $x_1 \in I \subseteq R$ and $y_1 \in J \subseteq R$. Then the product $x_1y_1 \in I$ even when $x_1 \notin J$ since I is an ideal (meaning $ir, ri \in I$ for all $r \in R$). Likewise, $x_1y_1 \in J$ since J is also an ideal. Therefore, $x_1y_1 \in I \cap J$ for all $x_1 \in I, y_1 \in J$. Consequently, IJ is a set of finite sums of elements in $I \cap J$, and, since $I \cap J$ is an ideal that is closed under addition, $IJ \subseteq I \cap J$.

(c) Give an example where $IJ \neq I \cap J$.

Let $R = \mathbb{Z}$. Consider the ideals $I = 2\mathbb{Z}$ and $J = 4\mathbb{Z}$. Then:

(a) $J \subset I$.

(b) Consequently, $I \cap J = J$.

(c) However, $IJ = 8\mathbb{Z} \neq J$.

(d) Prove that if R is commutative and if $I + J = R$ then $IJ = I \cap J$.

[Hint: since $I + J = R$, we have $s + t = 1$ with $s \in I$ and $t \in J$.]

Since $I + J = R$, we know that $1 \in I + J$. Therefore, for every $s \in I$, there exists some $t \in J$ such that $s + t = 1$. Likewise, for every $t \in J$, there exists some $s \in I$ such that $s + t = 1$.
We saw in (a) above that $I, J \in I + J$. Consider the set $I \cdot (I + J) = I^2 + IJ$. Since I is an ideal, $I^2 \in I$

2.

Let R be a commutative ring, and consider the ring $\mathrm{M}_n(R)$.

(a) Let $I \subset R$ be an ideal. Show that

$$M_n(I) := \{A = (a_{ij})_{i,j} \in \mathrm{M}_n(R) : a_{ij} \in I \text{ for all } i, j\}$$

is an ideal of $M_n(R)$.

Let $A \in M_n(I)$ and $M \in M_n(R)$. Then

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

having all a_{ii} where $0 < i \leq n$ as elements in I. Therefore:

$$AM = \begin{pmatrix} \sum_{k=1}^n a_{1k}m_{k1} & \sum_{k=1}^n a_{1k}m_{k2} & \cdots & \sum_{k=1}^n a_{1k}m_{kn} \\ \sum_{k=1}^n a_{2k}m_{k1} & \sum_{k=1}^n a_{2k}m_{k2} & \cdots & \sum_{k=1}^n a_{2k}m_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{nk}m_{k1} & \sum_{k=1}^n a_{nk}m_{k2} & \cdots & \sum_{k=1}^n a_{nk}m_{kn} \end{pmatrix} \tag{0.6}$$

$$MA = \begin{pmatrix} \sum_{k=1}^n m_{1k}a_{k1} & \sum_{k=1}^n m_{1k}a_{k2} & \cdots & \sum_{k=1}^n m_{1k}a_{kn} \\ \sum_{k=1}^n m_{2k}a_{k1} & \sum_{k=1}^n m_{2k}a_{k2} & \cdots & \sum_{k=1}^n m_{2k}a_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^n m_{nk}a_{k1} & \sum_{k=1}^n m_{nk}a_{k2} & \cdots & \sum_{k=1}^n m_{nk}a_{kn} \end{pmatrix} \tag{0.7}$$

Notice that the matrices AM and MA have elements which are summations of products or elements a_i and m_i where $a_i \in I$ and $m_i \in R$. Since I is an ideal, we have that all such $a_i m_i \in I$. Furthermore, since ideals are closed under addition, each summation term is in I. Therefore, $AM, MA \in M_n(I)$.

(b) Let

$$J := \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} : x, y \in R \right\}.$$

Show that J is a left ideal of $M_2(R)$ but not a right ideal.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Then:

$$JM = \begin{pmatrix} xc & xd \\ yc & yd \end{pmatrix} \notin J$$

$$MJ = \begin{pmatrix} 0 & ax + bx \\ 0 & cx + dy \end{pmatrix} \in J$$

Since $xc = 0$ implies that either $c = 0$ or $x = 0$, JM is clearly not in J. Therefore, J is not a right ideal of $M_2(R)$.

However, MJ is in J since the its first column is all zeros. Therefore, J is a left ideal of $M_2(R)$.

(c) Prove that every (two-sided) ideal $J \subseteq \mathrm{M}_n(R)$ is equal to $\mathrm{M}_n(I)$ for some (two-sided) ideal I of R.

[Hint: Use previous homework to show that the subset $I \subseteq R$ formed by all $(1,1)$ entries of all matrices in J is itself an ideal in R.]

Let

$$A = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \in J \subseteq M_n(R)$$

$$B = \begin{pmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{pmatrix} \in M_n(R)$$

Then $AB \in J$ and $BA \in J$ since J is an ideal. Then:

$$AB = \begin{pmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \cdots & \sum_{i=1}^n a_{1i}b_{in} \\ \sum_{i=1}^n a_{2i}b_{i1} & \sum_{i=1}^n a_{2i}b_{i2} & \cdots & \sum_{i=1}^n a_{2i}b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{ni}b_{i1} & \sum_{i=1}^n a_{ni}b_{i2} & \cdots & \sum_{i=1}^n a_{ni}b_{in} \end{pmatrix} \in J$$

$$BA = \begin{pmatrix} \sum_{i=1}^n b_{1i}a_{i1} & \sum_{i=1}^n b_{1i}a_{i2} & \cdots & \sum_{i=1}^n b_{1i}a_{in} \\ \sum_{i=1}^n b_{2i}a_{i1} & \sum_{i=1}^n b_{2i}a_{i2} & \cdots & \sum_{i=1}^n b_{2i}a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n b_{ni}a_{i1} & \sum_{i=1}^n b_{ni}a_{i2} & \cdots & \sum_{i=1}^n b_{ni}a_{in} \end{pmatrix} \in J$$

This suggests that every sum $\sum a_{ij}b_{ji}$ is in the set generating members of matrices in J. Likewise, every corresponding sum $\sum b_{ji}a_{ij}$ is an element of the set corresponding members of matrices in J. Therefore, the set must be closed under addition, and left multiplication, and right multiplication, making the set an ideal in R.

3. List all ideals of $\mathbb{Z}/12\mathbb{Z}$, and label each indicating if it is prime or maximal (or both or neither).

(a) $(1) = \mathbb{Z}/12\mathbb{Z}$ is not prime, not maximal since it is the entire $\mathbb{Z}/12\mathbb{Z}$.

(b) $(2) = \{0, 2, 4, 6, 8, 10\}$ is prime and maximal.

(c) $(3) = \{0, 3, 6, 9\}$ is prime and maximal.

(d) $(4) = \{0, 4, 8\}$ is not prime, not maximal.

(e) $(6) = \{0, 6\}$ is not prime, not maximal.

(f) $(0) = \{0\}$ is not prime, not maximal.

4. (DF 7.3.11)

Let R be the set of all *continuous* real-valued functions on the closed interval $[0, 1]$.

(a) Show that R is a commutative ring by adding and multiplying values:

for $f, g \in R$, define $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x)g(x)$ for all $x \in [0, 1]$.

[Hint: no need to belabor this, but quickly check what needs to be checked.

How is this different from the problem in the short exam?]

Given a function $f \in R$, note that $f : [0, 1] \to \mathbb{R}$. Suppose $f(x) = x' \in \mathbb{R}$, $g(x) = x'' \in \mathbb{R}$, then:

(a) $(f + g)(x) = f(x) + g(x) = x' + x'' \in \mathbb{R}$, so $(f + g) : [0, 1] \to \mathbb{R}$. so $f + g \in R$.

Since addition is commutative in $\mathbb{R}$, then $x' + x'' = x'' + x'$ implies $f + g = g + f$.

(b) $(f \cdot g)(x) = f(x)g(x) = x'x'' \in \mathbb{R}$, so $(f \cdot g) : [0, 1] \to \mathbb{R}$. Therefore, $f \cdot g \in R$.

(c) Since multiplication is commutative in $\mathbb{R}$, we have that $x'x'' = x''x'$, therefore $f \cdot g = g \cdot f$.

Therefore, R is a ring by (a) and (b), and a commutative ring extension with (c).

(b) Let $a \in \mathbb{R}$. Show that the evaluation map

$$\begin{aligned} \mathrm{ev}_a \colon R &\to \mathbb{R} \\ f &\mapsto f(a) \end{aligned}$$

is a ring homomorphism. What is the kernel, and what does the FHT tell you? Conclude that the kernel is a maximal ideal.

Let $f, g \in R$. Then:

$$\mathrm{ev}_a(f + g) = f(a) + g(a) = \mathrm{ev}_a(f) + \mathrm{ev}_a(g)$$

$$\mathrm{ev}_a(f \cdot g) = f(a)g(a) = \mathrm{ev}_a(f) \cdot \mathrm{ev}_a(g)$$

Therefore, ev_a is a ring homomorphism.

The kernel of the homomorphism ev_a is the set of all functions f such that $f(a) = 0$:

$$\ker(\mathrm{ev}_a) = \{f \in R : f(a) = 0\}$$

The FHT tells us that $R/\ker(\mathrm{ev}_a) \simeq \mathrm{ev}_a[R]$. That is, the image of ev_a is isomorphic to the quotient ring $R/\ker(\mathrm{ev}_a)$. But in this case, the image of ev_a is the set of all real numbers. Therefore, $\ker(\mathrm{ev}_a)$ is a maximal ideal.

(c) Prove that the map $\phi := R \to \mathbb{R}$ defined by $\phi(f) = \int_0^1 f(t)\,dt$ is a homomorphism of additive groups but not a ring homomorphism.

Let $f, g \in R$. Then:

$$\begin{aligned}\phi(f+g) &= \int_0^1 (f+g)(t)\,dt \\ &= \int_0^1 f(t) + g(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt \\ &= \phi(f) + \phi(g)\end{aligned}$$

However:

$$\begin{aligned}\phi(f \cdot g) &= \int_0^1 (f \cdot g)(t)\,dt \\ &= \int_0^1 f(t)g(t)\,dt \\ &= \left[f(t)\int g(t)\,dt - \int g(t)\left(\frac{d}{dt}f(t)\right)\,dt\right]_0^1 \\ &\neq \int_0^1 f(t)\,dt \int_0^1 g(t)\,dt = \phi(f)\phi(g)\end{aligned}$$

Therefore, ϕ is a homomorphism of additive groups but not a homomorphism of multiplicative groups, hence not a ring homomorphism.

5. Let R be a commutative ring, and let $I := (x) \subseteq R[x]$. Show that I is a prime ideal if and only if R is a domain.

[Hint: consider evaluation.]

We can prove this in 2 steps:

(a) $R[x]$ is a domain if and only if R is a domain.

Suppose R is a domain but $R[x]$ is not a domain. Then, there exists $f, g \in R[x]$ such that $f, g \neq 0$ and $fg = 0$. Let $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{n} b_i x^i$. Then, $fg = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i b_j x^{i+j} = 0$. This implies that $a_i b_j = 0$ for some i, j having $a_i \neq 0, b_i \neq 0$, which contradicts the fact that R is a domain. Therefore, $R[x]$ must be a domain if R is a domain.

In the other direction, suppose $R[x]$ is a domain but R is not a domain. This implies that R has some elements $a, b \neq 0$ such that $ab = 0$. By inclusion of R in $R[x]$ as the constant polynomials, $R \subset R[x]$. Therefore, $a \in R[x]$ and $b \in R[x]$. Since $R[x]$ is a domain, we have that $ab \neq 0$ unless $a = 0$ or $b = 0$. This contradicts the fact that $a, b \neq 0$ as picked from R. Therefore, it must be that R is a domain if $R[x]$ is a domain.

(b) Let's show that if I is a prime ideal then $R[x]$ is a domain. Let I be a prime ideal of $R[x]$, and $R[x]/I$ be the quotient ring. Let $i = a + I, j = b + I$ be nonzero elements in $R[x]/I$. Then $a \notin I$ and $b \notin I$. Consider their product, $ij = (ab) + I$. Since $ab \notin I$ (because $a \notin I$ and $b \notin I$), then the product $ij = (ab) + I$ is nonzero in $R[x]/I$. Therefore, the product of any two nonzero elements in $R[x]/I$ is nonzero, and $R[x]/I$ is a domain. By extension, $R[x]$ is a domain.

In the other direction, if $R[x]$ is a domain then $R[x]/I$ is a domain, and $ij = (ab) + I = 0$ implies either $i = a + I = 0$ or $j = b + I = 0$, which implies $a \in I$ or $b \in I$ whenever $ab \in I$.