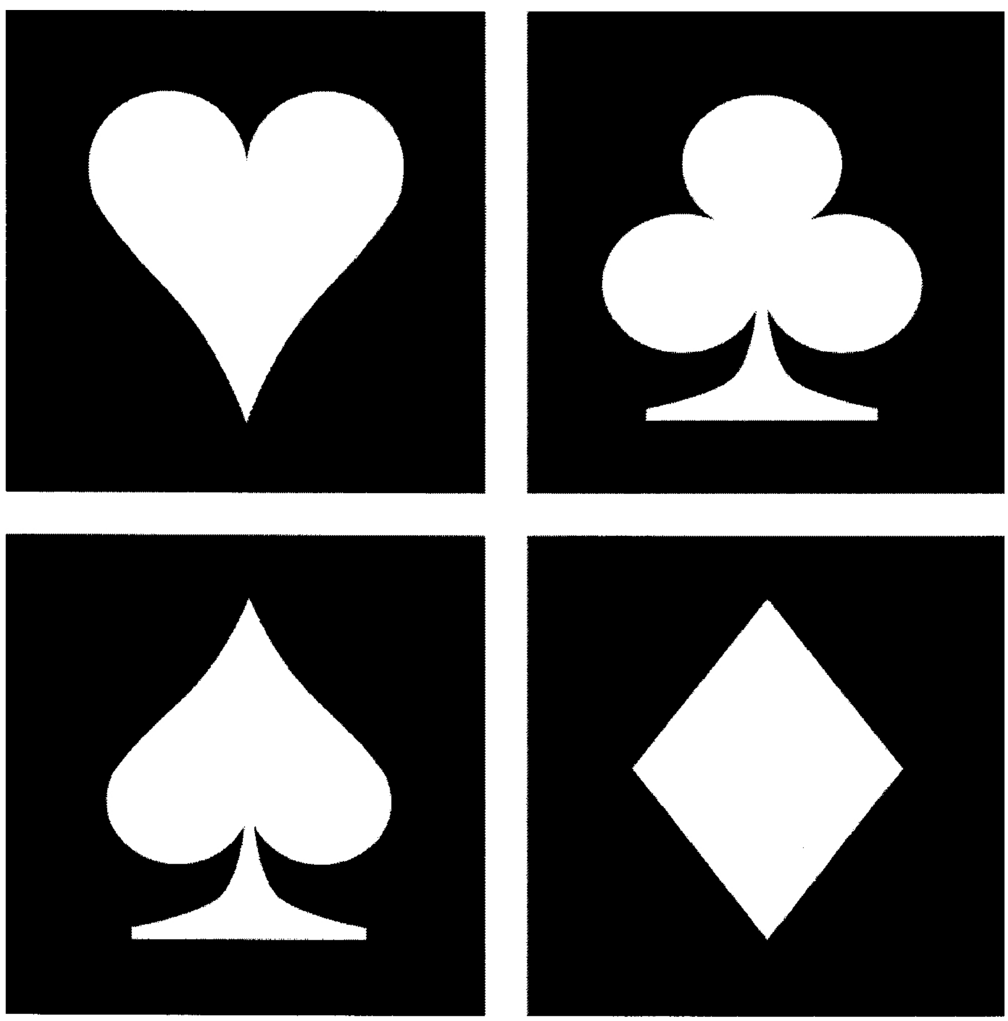
Cayley’s Theorem

3 november 2022



Problems

1. (DF 7.1.1, 7.1.15)

(a) Show that $(-1)^2 = 1$ in any ring R.

(b) A ring R is called *Boolean* if $a^2 = a$ for all $a \in R$. Show that every Boolean ring is commutative. *[Hint: Not every nonzero element of R is a unit.]*

2. (DF 7.1.7) The center of a ring R is

$$Z(R) := \{z \in R : zr = rz \text{ for all } r \in R\}.$$

(a) For a ring R, show that $Z(R) \subseteq R$ is a subring (in particular, containing 1).

(b) Show that the center of a division ring is a field.

3. (sorta DF 7.1.24) Let $D \in \mathbb{Z}$ be a nonsquare.

(a) Suppose that $D \equiv 1 \pmod 4$. Show that

$$\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right] = \left\{a + b\frac{1+\sqrt{D}}{2} : a, b \in \mathbb{Z}\right\} \subset \mathbb{Q}(\sqrt{D})$$

is a subring of $\mathbb{Q}(\sqrt{D})$. What happens when $D \not\equiv 1 \pmod 4$?

(b) Let $\mathcal{O} = Z[\omega]$ where $\omega = \sqrt{D}$ or $(1+\sqrt{D})/2$, according as $D \equiv 2, 3 \pmod 4$ or $D \equiv 1 \pmod 4$. Show for $D = 3, 5, 6, 7$ that $\mathcal{O}^\times$ is infinite. *[Hint: see Example, §7.1, pages 229–230.]*

4.

(a) Let R be a commutative ring. Show that $R[x]$ is never a field (even if R is a field).

(b) How many polynomials of degree d are there in $(\mathbb{Z}/n\mathbb{Z})[x]$?

(c) Show that $(\mathbb{Z}/8\mathbb{Z})[x]^\times > (\mathbb{Z}/8\mathbb{Z})^\times$, i.e., there is a unit which is not a scalar unit.

5. (DF 7.2.6–7.2.7)

Let R be a commutative ring and let $n \in \mathbb{Z}_{\geq 1}$. Let $A = (a_{ij})_{i,j} \in \mathrm{M}_n(R)$ be an $n \times n$-matrix whose (i, j)-entry is $a_{ij} \in R$. Let $E_{ij} \in \mathrm{M}_n(R)$ be the matrix whose (i, j) entry is 1 with all other entries zero. For example,

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{M}_3(R)$$

(a) Prove that $E_{ij}A$ is the $n \times n$-matrix whose ith row is equal to the jth row of A, with all other rows zero:

$$E_{ij}A = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ a_{j1} & \cdots & a_{jn} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

(in the ith row)

(b) Prove that AE_{ij} is the $n \times n$-matrix whose jth column equals the ith column of A, with all other columns zero:

$$AE_{ij} = \begin{pmatrix} 0 & \cdots & 0 & a_{1i} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{ni} & 0 & \cdots & 0 \end{pmatrix}$$

(in the jth column)

(c) Prove that $E_{pq}AE_{rs}$ is the matrix whose (p, s)-entry is a_{qr}, with all other entries zero.

(d) Prove that the center of $\mathrm{M}_n(R)$ is the set (subring!) of scalar matrices (i.e., diagonal matrices with the same entry down the diagonal).

[Hint: if you get lost in the indices, do the cases $n = 2$ and maybe $n = 3$ first.]