Math 71: Algebra **Fall 2022**

PSET 5 — 2022-11-11

Prof. Voight ***Student: Amittai Siavava***

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Abstract Algebra*** by **David S. Dummit & Richard M. Foote**.

(b) ***Algebra*** by **Jacob K. Goldhaber & Gertrude Ehrlich**

Problems

1. In this problem, we prove the main ingredients in the fourth isomorphism theorem. Throughout, let G be a group, let $N \trianglelefteq G$ be a normal subgroup. Let $H \leq G$ be a subgroup.

(a) Define $H/N := \{hN : h \in H\}$. Show that $H/N \leq G/N$ is a subgroup. (So given a subgroup of G, we can make a subgroup of G/N.)

(a) Since both H and N are subgroups of G, they both contain the identity element of G, ϵ. So the set $H/N = \{hn\colon h \in H,\ n \in N\}$ contains ϵ, particularly when $h = \epsilon$ and $n = \epsilon$.

(b) Let $h_1n_1 \in H/N$ and $h_2n_2 \in H/N$. Consider the product, $h_1n_1 \cdot h_2n_2$. Since elements of normal subgroups commute:

$$\begin{aligned} h_1n_1h_2n_2 &= (h_1h_2)(n_1n_2) && (n_1h_2 = h_2n_1) \\ &= h_3n_3 \in H/N && (h_3 = h_1h_2 \in H,\ n_3 = n_1n_2 \in N) \end{aligned}$$

So H/N is closed under multiplication.

(c) Let $h_1n_1 \in H/N$. Consider the inverse, $(h_1n_1)^{-1}$. Since elements of normal subgroups commute:

$$\begin{aligned} (h_1n_1)^{-1} &= n_1^{-1}h_1^{-1} \\ &= h_1^{-1}n_1^{-1} && (n_1^{-1}h_1^{-1} = h_1^{-1}n_1^{-1} \text{ since } n_1^{-1} \in N) \\ &= h_2n_2 \in H/N && (h_2 = h_1^{-1} \in H,\ n_2 = n_1^{-1} \in N) \end{aligned}$$

So H/N is closed under inversion.

(b) Show that $(HN)/N = H/N$. (So we get the same subgroups of G/N by taking those subgroups $H \leq G$ containing N.)

> Consider the map $\phi\colon H \to (HN)/N$ defined by $\phi(h) := hnN$, $n \in N$. Take the instance $h \in \ker(G)$, then $hn = \epsilon$, and $hnN = N$. Thus, we see that one of the cosets of HN/H is N. Using this coset, we can define the image of ϕ as a product of H and N — that is, H/N.

(c) If $\phi\colon G \to G'$ is a group homomorphism and $H' \leq G'$ is a subgroup, show that $\phi^{-1}(H')$ is a subgroup of G. Apply this to the map $\pi\colon G \to G/N$ to conclude that if $H' \leq G/N$ is a subgroup, then $\pi^{-1}(H') \leq G$ is a subgroup of G containing N. (So we can go backwards.) *[Hint: recall that if $f\colon A \to B$ is a map and $Y \subseteq B$ is a subset, then the preimage is $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$. The use of this symbol does not imply that f has an inverse.]*

> (a) Let ϵ' be the identity element of G'. Then $\phi(\epsilon) = \epsilon' \in H'$. So $\phi^{-1}(\{\epsilon'\}) = \{\epsilon\}$. Therefore, $\phi^{-1}(H')$ contains the identity element of G.
>
> (b) For $h_1, h_2 \in H'$, let $g_1 = \phi^{-1}(h_1)$ and $g_2 \in \phi^{-1}(h_2)$. Then $\phi(g_1) \in H'$ and $\phi(g_2) \in H'$. So $\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \in H'$. By definition of the isomorphism mapping elements in G to elements in H',
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> Let $g \in \ker(\phi)$, then $\phi(g) = \epsilon$. So $\phi^{-1}(\epsilon) = \{g \in G : \phi(g) = \epsilon\} = \{g \in G : g = \epsilon\} = \{\epsilon\}$. Thus, $\phi^{-1}(H')$ is closed under the identity element. Let $g_1 \in \phi^{-1}(H')$ and $g_2 \in \phi^{-1}(H')$. Then $\phi(g_1) \in H'$ and $\phi(g_2) \in H'$. So $\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \in H'$. Thus, $\phi^{-1}(H')$ is closed under multiplication. Let $g \in \phi^{-1}(H')$. Then $\phi(g) \in H'$. So $\phi(g^{-1}) = (\phi(g))^{-1} \in H'$. Thus, $\phi^{-1}(H')$ is closed under inversion.

(d) Show that if $H \trianglelefteq G$ is normal, then $H/N \trianglelefteq G/N$ is normal.

2. (DF 3.5.3)

(a) Prove that S_n is generated by the set$\{(1\ 2), (2\ 3), \ldots, (n\ -\ 1\ n)\}$. *[Hint: Consider conjugates, e.g.* $(2\,3)(1\,2)(2\,3)^{-1}$*.]*

First, we note that every permutation can be written as a product of transpositions. Consider the trivial cases of $n = 1$. Then S_1 is generated by the single identity element. For S_2, we have the two elements $(1\,2)$ and $(2\,1)$, which are both transpositions generated by $(1\,2)$. For each $n \geq 3$, supposing we have already shown that $S_{(n-1)}$ is generated by the set $\{(1\,2), (2\,3), \ldots, (n-2\ n-1)\}$, we can generate any element $(i\ \ n)$ by conjugation:

$$(i\ \ (n-1)) \cdot ((n-1)\ \ n) \cdot ((n-1)\ \ i) = (i\ \ n)$$

(b) Show that every element in A_n for $n\ \geq\ 3$ can be written as the product of (not necessarily disjoint) 3-cycles.

The alternating group A_n is generated by all cycles that can be written as an even product of transpositions. For any $a, b, c \in \{1, 2, \ldots, n\}$, The cycle $(a\,b\,c)$ can be written as two transpositions of a:

$$(a\,b\,c) = (a\,b)(b\,c)$$

Essentially, every 3-cycle is a product of two transpositions. Since every elements in A_n can be written as a product of **even** transpositions, every element in A_n can also be written as a product of the 3-cycles constituting contiguous pairs of those transpositions.

3. (DF 4.1.4)

Let S_3 act on the set Ω of ordered pairs: $\{[i,j] : 1 \leq i,j \leq 3\}$ by $\sigma([i,j]) = [\sigma(i), \sigma(j)]$. Let's write square brackets around these ordered pairs so there is no chance we will get confused between them and permutations (we should have been OK since there are commas involved).

(a) For each $\sigma \in S_3$ find the cycle decomposition of σ under this action (i.e., find its cycle decomposition when σ is considered as an element of S_9—first fix a labelling of these nine ordered pairs). Is the action faithful?

(b) Find the orbits of S_3 on Ω. Is the action transitive?

(c) For each orbit $\mathcal{O}$ of S_3 acting on these nine points, pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3. What does this have to do with the orbit-stabilizer lemma (DF, Proposition 2, §4.1, p. 114)?

4. (DF 3.4.2, 4.3.2)

(a) Give a composition series for each of the groups D_8, Q_8, and $\mathbb{Z}/8\mathbb{Z}$.

(a) Composition series for $D_8 = \langle r, s : r^4 = s^2 = 1, rs = sr^{-1} \rangle$:

$$\begin{aligned} 1 \quad &< \quad \langle s \rangle \cong \langle r^2 \rangle \cong \langle sr^n, n \in \mathbb{Z}^+ \rangle \cong C_2 \\ &< \quad \langle r \rangle \cong C_4, \{1, r^2, s, sr^2\} \cong \{1, r^2, sr, sr^3\} \cong V_4 \quad < \quad D_8 \end{aligned}$$

Note: subgroups isomorphic to V_4 in D_8 are: $\{1, r^2, s, sr^2\}$ and $\{1, r^2, sr, sr^3\}$.

(b) Composition series for $Q_8 = \langle i, j, k \colon i^2 = j^2 = k^2 = -1, ij = k,\ jk = i,\ ki = j \rangle$

$$1 \quad < \quad \langle -1 \rangle \cong C_2 \quad < \quad \langle i \rangle \cong \langle j \rangle \cong \langle k \rangle \quad < \quad Q_8$$

(c) Composition series for $\mathbb{Z}/8\mathbb{Z}$

$$1 \quad < \quad \mathbb{Z}/2\mathbb{Z} \cong C_2 \quad < \quad \mathbb{Z}/4\mathbb{Z} \cong C_4 \quad < \quad \mathbb{Z}/8\mathbb{Z} \cong C_8$$

(b) List all conjugacy classes in the groups D_8, Q_8, $\mathbb{Z}/8\mathbb{Z}$.

Conjugacy classes are the sets of elements that are conjugate to each other.

(a) D_8 has 5 conjugacy classes:

$$\{1\}, \{r, r^3\}, \{r^2\}, \{s, sr^2\}, \{sr, sr^3\}$$

(b) Q_8 has 5 conjugacy classes:

$$\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$$

(c) $\mathbb{Z}/8\mathbb{Z}$ has 8 conjugacy classes. Because the group is Abelian, each element is its own conjugate.

5. (DF 4.3.25)

Almost by definition, a normal subgroup is a union of conjugacy classes. If $H < G$ is a proper subgroup of a finite group G, then G is not the union of conjugates of H, i.e., $G \neq \bigcup_{g \in G} gHg^{-1}$. (You are not asked to prove this.)

Let $G := \mathrm{GL}_2(\mathbb{C})$, and let $H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C}, ac \neq 0 \right\}$.

(a) Show that H is not normal in G.

Let $g \in G, h \in H$. Suppose H is normal in G, then $ghg^{-1} \in H$.

Take $g = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ and let's define $g^{-1} = \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix}$. Let's take $h = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

Then:

$$\begin{aligned} hg^{-1} &= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix} \\ &= \begin{pmatrix} aw' + by' & ax' + bz' \\ cy' & cz' \end{pmatrix} \\ g(hg^{-1}) &= \begin{pmatrix} w & x \\ y & z \end{pmatrix} \cdot \begin{pmatrix} aw' + by' & ax' + bz' \\ cy' & cz' \end{pmatrix} \\ &= \begin{pmatrix} w(aw' + by') + xcy' & w(ax' + bz') + xcz' \\ zcy' & zcz' \end{pmatrix} \end{aligned}$$

If $ghg^{-1} \in H$, then $zcy' = 0$. That would require that either z, c, y' is 0.

(a) In the case of $z = 0$, then clearly the choice of g doesn't cover the entirety of G.

(b) In the case of $c = 0$, then the matrix h is not in H, since in the definition of H we restricted the product $ac \neq 0$. This causes a contradiction.

(c) In the case $y' = 0$, then w' and z' must non-zero (for the matrix g^{-1} to be invertible), and, in-fact, $g^{-1} \in H$, which implies that $g \in H$. If $g \in H$, and $H < G$, then the choice of g doesn't cover the entirety of G.

Therefore, we can conclude that H is not normal in G because it is not normal to all the elements in G.

(b) Prove that every element of G is conjugate to some element of the subgroup H and deduce that G is the union of conjugates of H. *[Hint: Every polynomial over $\mathbb{C}$ has a root; use this to show that every element of* $\mathrm{GL}_2(\mathbb{C})$ *has an eigenvector.]*

Consider $g = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in G$, and $h = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in H$. Let's define $g^{-1} = \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix}$.
Every matrix $g \in G$ has the characteristic equation: $x^2 - (w + z)x + (wx - yz) = 0$. Since every polynomial over $\mathbb{C}$ has a root, every matrix $g \in G$ has an eigenvector.