Math 71: Algebra **Fall 2022**

PSET 3 — October 4, 2022

Prof. Voight ***Student: Amittai Siavava***

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Abstract Algebra*** by **David S. Dummit & Richard M. Foote**.

(b) ***Algebra*** by **Jacob K. Goldhaber & Gertrude Ehrlich**

Problems

1. (sorta DF 1.7.18) Let G be a group, let X be a set with an action by $G \circlearrowright X$.

(a) Prove that the relation $x \sim y$ if $x = g \cdot y$ for some $g \in G$ defines an equivalence relation on X. The set of equivalence classes are called the emphorbits of X under G.

(b) Show that the multiplicative group $G = \mathbb{R}^\times$ acts on the xy-plane $X = R^2$ by $r \cdot (x, y) = (rx, y)$. What are the orbits of G acting on X? Compute the stabilizers of G on the points $(1, 1)$ and $(0, 0)$.

2. (sorta DF 1.7.18) Let F be a field, let $G = \mathrm{GL}_2(F)$, and let

$$H := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G : a, d \in F^\times,\ b \in F \right\}.$$

(a) Show that H is a subgroup of G, and show H is nonabelian whenever $\#F > 2$. What happens when $\#F = 2$ (so $F \simeq \mathbb{Z}/2\mathbb{Z}$)?

(b) Show that the map

$$\begin{aligned} \phi \colon H &\to F^\times \\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} &\mapsto a \end{aligned}$$

is a surjective group homomorphism that is not an isomorphism.

3. (DF 1.3.1, sorta 1.3.7)
Let G be a group and let $A \subseteq G$ be a subset. For $g \in G$, write

$$gAg^{-1} := \{gag^{-1} : a \in A\}.$$

The *normalizer* of A in G is defined to be:

$$N_G(A) := \{g \in G : gAg^{-1} = A\}.$$

(a) Let $g \in G$. Show that the map

$$\begin{aligned} \phi_g \colon G &\to G \\ x &\mapsto gxg^{-1} \end{aligned}$$

is an isomorphism of groups. (We call an isomorphism from a group to itself an emphautomorphism.)

(b) Show that $N_G(A)$ is a subgroup of G that contains the centralizer $C_G(A)$. *[Hint: if $gAg^{-1} = A$ then $h(gAg^{-1})h^{-1} = hAh^{-1}$, we just took two equal sets and conjugated their elements by h.]*

(c) Let $G = Q_8$ and let $A = \{\pm i\}$. Compute $C_G(A)$ and $N_G(A)$.

(d) Show that if $H \leq G$ is a subgroup, then $H \leq N_G(H)$. *[Hint: use (a), with $G = H$.]*

4. (DF 2.1.8)
Let $H, K \leq G$ be subgroups of a group G. Prove that the union $H \cup K$ is a subgroup if and only if $H \supseteq K$ or $K \subseteq H$. *[Hint: if there exists $h \in H$ with $h \notin K$, show that $K \subseteq H$ by consider hk for $k \in K$.]*

5. (DF 2.3.10)

(a) Let $G = \langle a \rangle$ be a cyclic group of order $n \in \mathbb{Z}_{\geq 1}$. For $k \in \mathbb{Z}$, show that a^k has order n/g where $g = \mathbf{gcd}\ ((,\ k), n)$. *[Hint: what is $\#\langle a^k \rangle$?]*

(b) What is the order of $\overline{30}$ in $\mathbb{Z}/54\mathbb{Z}$? Write out all of the elements in $\langle \overline{30} \rangle$ and their orders.

(c) For which values of $n \in \{8, 9, 10, 11, 12\}$ is $(\mathbb{Z}/n\mathbb{Z})^\times$ a cyclic group?