Math 63: Real Analysis **Winter 2024**

PSET 2 — 01/17/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Let A be a finite set. Show that A cannot be an ordered field.

To show this, we will show that any ordered field must have infinitely many elements.

Let F be an ordered field, with 0 and 1 as the additive and multiplicative identities, and $0 < 1$.

Recall the following properties of ordered fields:

1. $0 < 1$
2. $0 < a < b \implies 0 < a^{-1} < b^{-1}$
3. $0 < a < b \implies 0 < a + c < b + c$
4. $0 < a < b \wedge 0 < c \implies 0 < ac < bc$

Since $0 < 1$, adding 1 to both sides implies $1 < 1 + 1$. Repeating this process infinitely many times, we get:

$$0 < 1$$
$$1 < 1 + 1 \qquad (= 2)$$
$$1 + 1 < 1 + 1 + 1 \qquad (= 3)$$
$$1 + 1 + 1 < 1 + 1 + 1 + 1 \qquad (= 4)$$
$$\vdots$$

For any element $a \in F$, the element $a + 1$ must be in F (since fields are closed under addition) and it must be greater than a. Similarly, no addition of two elements a, b, with $0 < a$ and $0 < b$, can ever equal 0. This can only happen if F is infinite.

Problem 2.

Let $F = \{0, 1, 2\}$. Prove that there is exactly one way to define addition and multiplication so that F is a field if 0 is the additive identity and 1 is the multiplicative identity.

Multiplication

First, note that $0 \cdot a = a \cdot 0 = 0$ for all $a \in F$.

This is because multiplication distributes over addition, so $0 \cdot a = (1 + (-1)) \cdot a = a + (-a) = 0$.

Next, since 1 is the multiplicative identity, $1 \cdot a = a \cdot 1 = a$ for all $a \in F$.

There are three possible ways to define $2 \cdot 2$:

1. If $2 \cdot 2 = 0$, then multiplying both sides by 2^{-1} implies that $2 = 0$. This is a contradiction.
2. If $2 \cdot 2 = 2$, then multiplying both sides by 2^{-1} implies that $2 = 1$. This is a contradiction.
3. Therefore, multiplication is only well-defined if $2 \cdot 2 = 1$.

Since 0 is the additive identity, $0 + a = a + 0 = a$ for all $a \in F$.

Similarly, $1 \cdot a = a \cdot 1 = a$ for all $a \in F$.

Addition

There are three potential ways to define addition:

1. If $1 + 1 = 0$, this implies $1 = -1$.

   For addition to be well-defined, we must also have $2 + 2 = 0$ since 2 must also have an additive inverse.

   How do we define $1 + 2$?

   (i) If $1 + 2 = 0$, then $1 = -2$, suggesting that 1 has two additive inverses since $1 + 1 = 1 + 2 = 0$. But each element must have a unique inverse, so this is a contradiction.

   (ii) If $1 + 2 = 1$, then adding -1 to both sides implies that $2 = 0$, a contradiction.

   (iii) If $1 + 2 = 2$, then adding -2 to both sides implies that $1 = 0$, a contradiction.
2. If $1 + 1 = 2$. then $1 + 2 = 2 + 1 = 0$ (since both 1 and 2 must have additive inverses).

   This is not consistent with multiplication as defined in **3.** above, since:

   (i) $2 \cdot (1 + 2) = 2 \cdot 0 = 0$.

   (ii) $2 \cdot (1 + 2) = 2 \cdot 1 + 2 \cdot 2 = 2 + 1 = 0$
3. $1 + 1 = 1$ implies that $1 = 0$ (by adding -1 to both sides). This is a contradiction, since $1 \neq 0$.

Thus, there is only one way to define addition and multiplication such that F is a field:

$$
\begin{aligned}
0 + 0 &= 0 \\
0 + 1 = 1 + 0 &= 1 \\
0 + 2 = 2 + 0 &= 2 \\
1 + 1 &= 2 \\
1 + 2 = 2 + 1 &= 0 \\
2 + 2 &= 1
\end{aligned}
$$

$$
\begin{aligned}
0 \cdot 0 &= 0 \\
0 \cdot 1 = 1 \cdot 0 &= 0 \\
0 \cdot 2 = 2 \cdot 0 &= 0 \\
1 \cdot 1 &= 1 \\
1 \cdot 2 = 2 \cdot 1 &= 2 \\
2 \cdot 2 &= 1
\end{aligned}
$$

Problem 3.

If S_1 and S_2 are nonempty subsets of $\mathbb{R}$ that are bounded above, prove that

$$\mathbf{sup}\{x + y \mid x \in S_1, y \in S_2\} = \mathbf{sup}\, S_1 + \mathbf{sup}\, S_2.$$

We shall prove this in two steps, first showing that $\mathbf{sup}\{x + y \mid x \in S_1, y \in S_2\} \leq \mathbf{sup}\, S_1 + \mathbf{sup}\, S_2$, and then showing that $\mathbf{sup}\{x + y \mid x \in S_1, y \in S_2\} \geq \mathbf{sup}\, S_1 + \mathbf{sup}\, S_2$.

(i) Let $s_1 \in S_1$ and $s_2 \in S_2$ be arbitrary.
Since $s_1 + s_2 \in \{x + y \mid x \in S_1, y \in S_2\}$, $s_1 + s_2 \leq \mathbf{sup}\{x + y \mid x \in S_1, y \in S_2\}$.
This holds for any selection of $s_1 \in S_1$ and any selection of $s_2 \in S_2$,
so $\mathbf{sup}\, S_1 + \mathbf{sup}\, S_2 \leq \mathbf{sup}\{x + y \mid x \in S_1, y \in S_2\}$.

(ii) Let $s \in \{x + y \mid x \in S_1, y \in S_2\}$ be arbitrary.
Then, by definition, $s = x + y$ for some $x \in S_1$ and $y \in S_2$.
Since $x \leq \mathbf{sup}\, S_1$ and $y \leq \mathbf{sup}\, S_2$; $s = x + y \leq \mathbf{sup}\, S_1 + \mathbf{sup}\, S_2$.
This holds for all elements $s \in \{x + y \mid x \in S_1, y \in S_2\}$,
so $\mathbf{sup}\{x + y \mid x \in S_1, y \in S_2\} \leq \mathbf{sup}\, S_1 + \mathbf{sup}\, S_2$.

Problem 4.

Let $S := \{a_k \mid k \in \mathbb{N}\} \cup \{b_k \mid k \in \mathbb{N}\}$, ordered such that $a_k < b_j$ for all k and j, $a_k < a_m$ whenever $k < m$, and $b_k < b_m$ whenever $k < m$.

(a) Show that S is an ordered set.

S is an ordered set if there exists a relation $<$ on S such that:

1. For all $a, b \in S$, exactly one of $a < b$, $a = b$, or $b < a$ holds.
2. If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$, then $a + c < b + c$.

Let $<$ be as defined in the problem statement. We will show that $<$ satisfies the above properties.

Let $x, y, z \in S$ be arbitrary.

(i) If $x \neq y$, then:
   1. If $x = a_k$ and $y = a_m$ with $k < m$ then $x < y$. Otherwise, $y < x$.
   2. If $x = a_k$ and $y = b_j$, then $x < y$.
   3. If $x = b_k$ and $y = b_j$ with $k < j$, then $x < y$.

(ii) If $x < y$ and $y < z$, then:
   1. If $x = a_k$ and $y = a_m$ with $k < m$, then either $z = a_n, n > m$ or $z = b_j$ for some j. Therefore, $x < z$.
   2. If $x = a_k$ and $y = b_j$ for some j, then $z = b_n$ for some $n > j$. Therefore, $x < z$.
   3. If $x = b_j$ and $y = b_k$ for some $j, k, j < k$m then $z = b_n$ for some $n > j$. Therefore, $x < z$.

(b) Show that every subset of S is bounded above and below.

From the definition of S, $a_k < b_j$ for all k and j. If S' is any subset of S, then S' is bounded above by the biggest $b_j \in S'$ and bounded below by the smallest $a_k \in S'$.

(c) Find a bounded subset of S that has no least upper bound.

The set

$$B = \{b_j \mid j \in \mathbb{N}\} \subset S$$

is bounded below (by any $a_k \in S$) and above (by the largest $b_j \in B$), but does not have a least upper bound.

As a proof by contradiction, supposing $\mathbf{l.u.b.}\ B = \sigma$. First note that B contains elements of the form $b_j, j \in \mathbb{N}$. Therefore, σ must be some element b_x for some $x \in \mathbb{N}$, since the ordering rules defined that all a_k are ordered below any b_j. Now, the element b_{x+1} is also in B, and $b_x < b_{x+1}$ by the ordering rules, contradicting that we took b_x to be the least upper bound of B.

Problem 5.

Let $n \in \mathbb{N}$. Show that $(\mathbb{R}^n, d_1)$ is a metric space where

$$d_1(p,q) := \sum_1^n |p_i - q_i|$$

for all $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ in $\mathbb{R}^n$.

(R^n, d_1) is a metric space if d_1 satisfies the following properties:

1. $d_1(p,q) \geq 0$ for all $p, q \in \mathbb{R}^n$. and $d_1(p,q) = 0$ if and only if $p = q$.
2. $d_1(p,q) = d_1(q,p)$ for all $p, q \in \mathbb{R}^n$.
3. $d_1(p,r) \leq d_1(p,q) + d_1(q,r)$ for all $p, q, r \in \mathbb{R}^n$.

Claim 5.1. $d_1(p,q) \geq 0$ *for all* $p, q \in \mathbb{R}^n$.

Proof. Let $p, q \in \mathbb{R}^n$ be arbitrary. Then, since $|x + y| \geq 0$ for all $x, y \in \mathbb{R}$, $|p_i - q_i| \geq 0$ for all $i \in \{1, \dots, n\}$. Therefore,

$$d_1(p,q) = \sum_1^n |p_i - q_i| \geq 0.$$

$\square$

Claim 5.2. $d_1(p,q) = 0$ *if and only if* $p = q$.

Proof. Let $p, q \in \mathbb{R}^n$ be arbitrary.

$(\Longrightarrow)$: Suppose $d_1(p,q) = 0$. Then, since $|p_i - q_i| \geq 0$ for all $i \in \{1, \dots, n\}$,

$$d_1(p,q) = \sum_1^n |p_i - q_i| = 0$$

implies that $|p_i - q_i| = 0$ for all $i \in \{1, \dots, n\}$. This implies that $p_i = q_i$ for all $i \in \{1, \dots, n\}$, and therefore $p = q$.

$(\Longleftarrow)$: Suppose $p = q$. This means that $p_i = q_i$ for all $i \in \{1, \dots, n\}$, meaning $|p_i - q_i| = 0$, so

$$\sum_1^n |p_i - q_i| = d_1(p,q) = 0.$$

$\square$

Claim 5.3. $d_1(p,q) = d_1(q,p)$ *for all* $p, q \in \mathbb{R}^n$.

Proof. Let $p, q \in \mathbb{R}^n$ be arbitrary. Since $|x + y| = |y + x|$ for all $x, y \in \mathbb{R}$, $|p_i - q_i| = |q_i - p_i|$ for all $i \in \{1, \dots, n\}$. Therefore,

$$d_1(p,q) = \sum_1^n |p_i - q_i| = \sum_1^n |q_i - p_i| = d_1(q,p).$$

$\square$

Claim 5.4. $d_1(p,r) \le d_1(p,q) + d_1(q,r)$ *for all* $p,q,r \in \mathbb{R}^n$.

Proof. Let $p,q,r \in \mathbb{R}^n$ be arbitrary, then:

$$\begin{aligned} d_1(p,r) &= \sum_1^n |p_i - r_i| \\ &= \sum_1^n |p_i - q_i + q_i - r_i| \le \sum_1^n |p_i - q_i| + |q_i - q_i| = \sum_1^n |p_i - q_i| + \sum_1^n |q_i - q_i| \\ &= d_1(p,q) + d_1(q,r) \end{aligned}$$

□

Since all the properties hold, $(\mathbb{R}^n, d_1)$ is a metric space.

Problem 6.

Show that the subset of $(\mathbb{R}^2, d_E)$ given by

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2\}$$

is open.

Definition 6.1. A set is open if each point in the set has an open ball contained in the set.

Let $p = (p_1, p_2) \in S$ be arbitrary. We shall show that any such point has some open ball surrounding it that is contained in S.

By definition of S, $p_1 > p_2$. Pick

$$r = \frac{p_1 - p_2}{\sqrt{2}}$$

such that

$$p_1 - \frac{r}{\sqrt{2}} = p_2 + \frac{r}{\sqrt{2}}.$$

Let $\mathfrak{B} = \mathbf{B}_r(p) \subset S$.

Claim 6.2. *$\mathfrak{B}$ is open.*

Proof. To show this, we show that $\mathfrak{B}$ does not contain its boundary.
Take the point $q = (q_1, q_2)$ with $q_1 = p_1 - \dfrac{r}{\sqrt{2}}$ and $q_2 = p_2 + \dfrac{r}{\sqrt{2}}$:

1. q is on the boundary of $\mathfrak{B}$. Or, more precisely, $d_E(p, q) = r$.

$$\begin{aligned} d_E(p,q) &= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \\ &= \sqrt{\left(p_1 - \left(p_1 - \frac{r}{\sqrt{2}}\right)\right)^2 + \left(p_2 - \left(p_2 + \frac{r}{\sqrt{2}}\right)\right)^2} \\ &= \sqrt{\frac{r^2}{2} + \frac{r^2}{2}} \\ &= \sqrt{r^2} \\ &= r \end{aligned}$$

2. q is not in $\mathfrak{B}$. We picked q such that $q_1 = p_1 - \frac{r}{\sqrt{2}} = p_2 + \frac{r}{\sqrt{2}} = q_2$. Since $q_1 = q_2$, q is not in A, and therefore not in $\mathfrak{B}$.

Therefore, $\mathfrak{B}$ is open.

$\square$

Claim 6.3. $\mathfrak{B}$ *is contained in* S.

Proof. Above, we picked q such that $q_1 = p_1 - \frac{r}{\sqrt{2}}$ and $q_2 = p_2 + \frac{r}{\sqrt{2}}$ such that $q_1 = q_2$. This point q lies on the line $x = y$, and is the only such point on the circle of radius r, centered at p, that lies on the line $x = y$. If a point in $\mathfrak{B}$ lies outside S, it must be on or above the line $x = y$. But that is impossible, as that would imply that the line $x = y$ intersects the circle more than once.

This proof sounds very sketchy. I could visualize it, but I wasn't sure how to articulate it better. □

Problem 7.

Let (X, d) be a metric space. Let $A \subset X$. Show that A is open if and only if it is equal to the union of a collection of open balls.

Definition 7.1. A set $A \subset X$ is open if each point in A has an open ball contained in A.

Let A be an open set. Then, by definition, each point in A has some open ball contained in A. Let B be the union of all such open balls.

Claim 7.2. *$A \subseteq B$.*

Proof. Let $a \in A$ be arbitrary. Since A is an open set, a has some open ball $\mathbf{B}_r(a)$ contained in A. This ball must contain a itself, and since B is the union of all such open balls, B must also contain a. Therefore, $a \in B$. $\square$

Claim 7.3. *$A \supseteq B$.*

Proof. Let $b \in B$ be arbitrary. Since B is the union of a collection of open balls contained in A, the point b must be in some open ball contained in A. Therefore, b itself must also be in A, so $b \in A$. $\square$

Since $A \subseteq B$ and $A \supseteq B$, $A = B$.