Math 63: Real Analysis **Winter 2024**

PSET 6 — 02/14/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Let $f(x) = \dfrac{1+x}{1+x^3}$.

(i) Find a largest subset $U \subseteq \mathbb{R}$ where f is well-defined. Is f continuous on U?

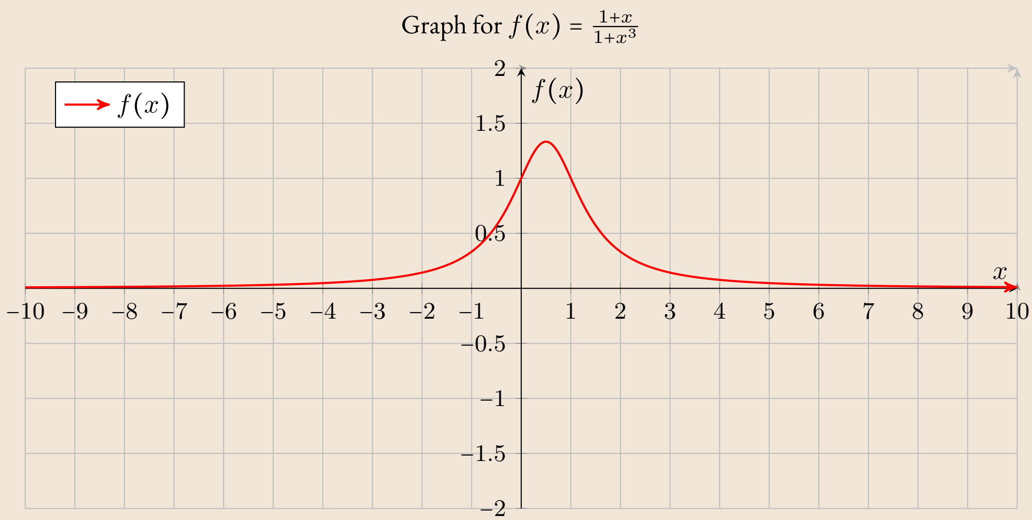
Since f is a rational function, it is well-defined where the denominator is nonzero.

Thus, $U = \mathbb{R} \setminus \{-1\}$.

Since U is open and f is a rational function, f is continuous on U (This was proven in class.).

(ii) Let U be as in part (i). Let g be a function such that $g(x) = f(x)$ if $x \in U$.

Is there a way to define g on $\mathbb{R} \setminus U$ to obtain a continuous function g on $\mathbb{R}$?



Note: value at $x = -1$ is not defined, which is not demonstrated in this plot.

For g to be continuous on $\mathbb{R}$, we need to define g at -1 such that $\lim_{x\to -1} g(x) = g(-1)$. Since g has an indeterminate form at -1, we use L'Hôpital's rule to find the limit:

$$\begin{aligned}
\lim_{x\to -1} \frac{1+x}{1+x^3} &= \lim_{x\to -1} \left[\frac{\mathrm{d}}{\mathrm{d}x}(1+x) \bigg/ \frac{\mathrm{d}}{\mathrm{d}x}(1+x^3) \right] \\
&= \lim_{x\to -1} \frac{1}{3x^2} \\
&= \frac{1}{3}
\end{aligned}$$

Thus, we can define

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \setminus \{-1\} \\ \frac{1}{3} & \text{if } x = -1 \end{cases}$$

to obtain a continuous function on $\mathbb{R}$.

Problem 2.

Determine the points where the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous.

Claim 2.1. *The function f is continuous at all points $(x,y) \neq (0,0)$.*

Proof. Note that the function f is a composition of two functions; $f = h \circ g$, where

$$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto x^2 + y^2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} h : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \frac{1}{t}. \end{aligned}$$

Since g and h are both continuous over $U := \{p \in \mathbb{R}^2 \mid p \neq (0,0)\}$, Their is composition is also continuous over U. □

Claim 2.2. *The function f is not continuous at $(0,0)$.*

Proof. f is continuous at $(0,0)$ if and only if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

However;

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty,$$

so the limit does not exist. However, $f(0,0) = 0$ so f is not continuous at $(0,0)$. □

Problem 3.

Let $f : [a, b] \to \mathbb{R}$ with $a < b$ be continuous. Show that functions

$$m(x) = \inf \{f(y) \mid a \le y \le x\} \quad \text{and} \quad M(x) = \sup \{f(y) \mid a \le y \le x\}$$

are continuous on $[a, b]$.

We will show that m is continuous on $[a, b]$. The proof for M is analogous.

Claim 3.1. *The function m is continuous on $[a, b]$.*

Proof. Let $x_0 \in [a, b]$. We need to show that $\lim_{x \to x_0} m(x) = m(x_0)$.

First, note that by definition of m, $m(x) \le f(x)$ for all $x \in [a, b]$. Thus, m is bounded above by f. Since f is continuous on $[a, b]$, we have $\lim_{x \to x_0} f(x) = f(x_0)$.

Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

For any such ε and δ, pick x so that $|x - x_0| < \delta$. We claim that $|m(x) - m(x_0)| < \varepsilon$. To show this, we shall demonstrate that $m(x_0) - \varepsilon < m(x) < m(x_0) + \varepsilon$.

1. Proof that $m(x_0) - \varepsilon < m(x)$:

$$\begin{aligned} m(x_0) - \varepsilon &\le f(x_0) - \varepsilon && \text{since } m(x_0) \le f(x_0) \\ &< f(x) && \text{since } |f(x) - f(x_0)| < \varepsilon \\ &\le \inf \{f(y) \mid a \le y \le x\} \\ &= m(x) \end{aligned}$$

2. Proof that $m(x) < m(x_0) + \varepsilon$:

$$\begin{aligned} m(x) &\le f(x) \\ &< f(x_0) + \varepsilon \\ &\le \inf \{f(y) \mid a \le y \le x_0\} + \varepsilon \\ &= m(x_0) + \varepsilon \end{aligned}$$

$\square$

Problem 4.

Show if each of this functions is uniformly continuous on $\mathbb{R}$ or not.

(i) $f(x) = x^2$.

(ii) $f(x) = \sqrt{|x|}$.

Problem 5.

Let (E, d_E) be a compact metric space, and let $f, f_1, f_2, f_3, \ldots : E \to \mathbb{R}$ be continuous real-values functions on E, with $\lim\limits_{n\to\infty} f_n = f$. Prove that if $f_1(p) \leq f_2(p) \leq f_3(p) \leq \cdots$ for all $p \in E$ then the sequence $f_1, f_2, f_3, \ldots$ converges uniformly.

Problem 6.

Let (X, d_X) and (Y, d_Y) be metric spaces. Assume (Y, d_Y) is complete. Show that a sequence of functions $f_n : X \to Y$ converges uniformly on X if and only if it is uniformly Cauchy on X.

Problem 7.

Let (X, d_X) be a metric space. A function $g : X \to \mathbb{R}$ is bounded on X if $\exists M$ such that $|g(x)| \leq M$ for all $x \in X$.

Suppose that $f_n : X \to \mathbb{R}$ is bounded on X for each $n \in \mathbb{N}$. Show that if a sequence of of f_n converges uniformly to a function $f : X \to \mathbb{R}$ then f is bounded on X.