

## PSET 3 — 01/31/2024

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## Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

- (a) *Introduction to Analysis* by Maxwell Rosenlicht

## Problem 1.

If  $a_1, a_2, a_3, \dots$  is a bounded sequence of real numbers, define

$$\limsup_{n \rightarrow \infty} a_n := \sup \{x \in \mathbb{R} \mid a_n > x \text{ for infinitely many } n \in \mathbb{N}\}$$

$$\liminf_{n \rightarrow \infty} a_n := \inf \{x \in \mathbb{R} \mid a_n < x \text{ for infinitely many } n \in \mathbb{N}\}$$

Prove that  $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$  with the equality holding if and only if the sequence converges.

Let  $a_1, a_2, a_3, \dots$  be a bounded sequence of real numbers as defined above. Let

$$L := \liminf_{n \rightarrow \infty} a_n, \quad U := \limsup_{n \rightarrow \infty} a_n.$$

For convenience, define

$$S_U := \{x \in \mathbb{R} \mid a_n > x \text{ for infinitely many } n \in \mathbb{N}\}$$

$$S_L := \{x \in \mathbb{R} \mid a_n < x \text{ for infinitely many } n \in \mathbb{N}\}$$

such that  $L = \inf S_L$  and  $U = \sup S_U$ .

We shall first prove that  $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$ , then show that the equality holds if and only if the sequence converges.

**Claim 1.1.**  $u < l$  for all  $l \in S_L$  and  $u \in S_U$ .

*Proof.* Let  $l \in S_L$  and  $u \in S_U$ . By definition of  $S_L$  and  $S_U$ ,  $u < a_n < l$  for infinitely many  $n \in \mathbb{N}$ . Therefore,  $u < l$ .  $\square$

**Claim 1.2.**  $L \leq U$ .

*Proof.*  $L$  is a lower bound of  $S_L$  and  $U$  is an upper bound of  $S_U$ . This means that  $L \leq l$  for all  $l \in S_L$  and  $u \leq U$  for all  $u \in S_U$ . By the previous claim,  $u < l$ , Suppose  $U < L$ . Since  $L$  was a lower bound on  $S_L$ , this would imply that  $U$  is also a lower bound on  $S_L$ , hence  $U \leq l$  for all  $l \in S_L$ .

□

**Problem 2.**

Let  $x_n = \left(1 + \frac{1}{n}\right)^n$  for all  $n \in \mathbb{N}$ .

*Remark 2.1.* The Euler number  $e$  can be defined as  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ .

(i) Using induction, show that for all  $x > -1$  and  $n \in \mathbb{N}$ , we have

$$(1+x)^n \geq 1+nx.$$

We shall prove the claim by induction on  $n$ .

*Proof.* Let  $x > -1$  and  $n \in \mathbb{N}$ .

(a) Base case:  $n = 1$ .

$$(1+x)^n = (1+x)^1 = 1+x = 1+1 \cdot x = 1+nx$$

Since  $1+nx \geq 1+nx$ , the inequality holds for  $n = 1$ .

(b) Inductive step: Suppose  $(1+x)^n \geq 1+nx$ .

$$\begin{aligned} (1+x)^{n+1} &= (1+x)^n (1+x) \\ &\geq (1+nx)(1+x) = 1+x+nx+nx^2 \\ &\geq 1+x+nx = 1+(n+1)x \end{aligned}$$

Therefore,  $(1+x)^n \geq 1+nx$  for all  $n \in \mathbb{N}$ . □

- (ii) Using the previous item, show that  $\frac{x_{n+1}}{x_n} \geq 1$  so  $x_n$  is monotonically increasing.

To show this, we need to show that  $x_{n+1} \geq x_n$  for all  $n \in \mathbb{N}$ .

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} \\
&= \frac{\left(1 + \frac{1}{n+1}\right)^n}{\left(1 + \frac{1}{n}\right)^n} \cdot \left(1 + \frac{1}{n+1}\right) \\
&= \left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^n \cdot \left(1 + \frac{1}{n+1}\right) \\
&= \left(\frac{1 + \frac{1}{n+1}}{\frac{n+1}{n}}\right)^n \cdot \left(1 + \frac{1}{n+1}\right) \\
&= \left(\frac{n + \frac{n}{n+1}}{n+1}\right)^n \cdot \left(1 + \frac{1}{n+1}\right) \\
&= \left(\frac{n}{n+1} + \frac{n}{(n+1)^2}\right)^n \cdot \left(1 + \frac{1}{n+1}\right) \\
&= \left(1 - \frac{1}{n+1} + \frac{n}{(n+1)^2}\right)^n \cdot \left(1 + \frac{1}{n+1}\right) \\
&= \left(1 - \frac{-(n+1)+n}{(n+1)^2}\right)^n \cdot \left(1 + \frac{1}{n+1}\right) \\
&= \left(1 - \frac{1}{(n+1)^2}\right)^n \cdot \left(1 + \frac{1}{n+1}\right) \\
&\geq \left(1 - \frac{n}{(n+1)^2}\right) \cdot \left(1 + \frac{n}{n+1}\right) \quad (\text{by the previous proof}) \\
&> \left(1 - \frac{n+1}{(n+1)^2}\right) \cdot \left(1 + \frac{n}{n+1}\right) \quad (\text{subtracting a bigger term}) \\
&= \left(1 - \frac{1}{n+1}\right) \cdot \left(1 + \frac{n}{n+1}\right) \\
&= 1 - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n(n+1)} \\
&= \frac{n(n+1) - n + (n+1) - 1}{n(n+1)} \\
&= \frac{n(n+1)}{n(n+1)} \\
&= 1
\end{aligned}$$

(iii) Show that  $x_n$  is bounded using the binomial formula

$$(a + b)^n = \frac{n!}{k!(n-k)!} a^{n-k} b^k = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

Fix  $a$  and  $b$  to arbitrary real numbers. We shall prove this by induction on  $n$ .

(a) Base case:  $n = 1$

$$\begin{aligned} (a + b)^n &= a + b \\ &= \binom{1}{0} a^1 \cdot b^0 + \binom{1}{1} a^0 b^1 \\ &= \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1 = \sum_{k=0}^1 \binom{n}{k} a^{n-k} b^k \end{aligned}$$

(b) Inductive step: Assume the invariant holds for  $(a + b)^n$ . We shall show that it holds for  $(a + b)^{(n+1)}$ .

$$\begin{aligned} (a + b)^{n+1} &= (a + b)^n (a + b) \\ &= \left( \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \right) (a + b) \\ &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \cdot a + \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \cdot b \\ &= \sum_{k=0}^n \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^n \binom{n}{k} a^{n-k} b^{k+1} \\ &= \sum_{k=0}^n \binom{n}{k} a^{n+1-k} b^k + a^{n-k} b^{k+1} \\ &= \sum_{k=0}^n \binom{n}{k} a^{n+1-k} b^k + \binom{n}{k-1} a^{n+1-k} b^k \quad (\text{Grouping together equal powers}) \\ &= \sum_{k=0}^n \left( \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \right) a^{n+1-k} b^k \\ &= \sum_{k=0}^n \left( \frac{(n+1-k) n! + k n!}{k! (n+1-k)!} \right) a^{n+1-k} b^k \\ &= \sum_{k=0}^n \left( \frac{(n+1) n!}{k! (n+1-k)!} \right) a^{n+1-k} b^k \\ &= \sum_{k=0}^n \left( \frac{(n+1)!}{k! (n+1-k)!} \right) a^{n+1-k} b^k \\ &= \sum_{k=0}^n \binom{n+1}{k} a^{n+1-k} b^k \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k \end{aligned}$$

Since  $a$  and  $b$  were arbitrary, this holds for all  $a$  and  $b$ , including  $a = 1$  and  $b = \frac{1}{n}$ , thus  $x_n$  is bounded by the binomial formula since equality satisfies both *less than or equal to* and *greater than or equal to*.

(iv) Show that  $\{x_n\}$  is convergent.

As seen in (ii) and (iii),  $\{x_n\}$  is monotonically increasing and is bounded above.

**Claim 2.2.**  $\{x_n\}$  is convergent.

*Proof.* Suppose  $\{x_n\}$  is not convergent. Since  $\{x_n\}$  is monotonically increasing, this implies that it must not be bounded above and  $\lim_{n \rightarrow \infty} x_n = \infty$ , which contradicts the known fact (by remark 2.1) that  $\lim_{n \rightarrow \infty} x_n = e \neq \infty$ . Therefore, the sequence must be convergent.  $\square$

### Problem 3.

Show that a complete subspace of a metric space is a closed subset.

**Definition 3.1.** A metric space  $X$  is complete if every Cauchy sequence in  $X$  has a limit point in  $X$ .

**Definition 3.2.** A subset  $S$  of a metric space  $X$  is closed if  $S$  contains all its limit points.

**Claim 3.3.** *Every convergent sequence is Cauchy.*

*Proof.* Let  $X$  be a metric space and  $P := \{p_n\}_1^\infty \subseteq X$  be a convergent sequence in  $X$ . Let  $p \in X$  be the limit of  $P$ . For any  $\varepsilon > 0$ , since  $P$  converges to  $p$ , there exists  $N \in \mathbb{N}$  such that  $d(p_n, p) < \frac{\varepsilon}{2}$  for all  $n \geq N$ . Therefore,

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore,  $P$  is Cauchy. □

**Claim 3.4.** *A complete subspace of a metric space is a closed subset.*

*Proof.* Let  $X$  be a metric space and  $Y \subseteq X$  be a complete subspace of  $X$ .

Let  $P := \{p_n\}_1^\infty \subseteq Y$  be an arbitrary convergent sequence in  $Y$  (i.e. a sequence that has a limit). To show that  $Y$  is closed, we need to show that any such  $P$  has a limit point in  $Y$ .

By claim 3.3, the convergence of  $P$  implies that  $P$  is Cauchy. Since  $Y$  is complete by definition, and  $P$  is Cauchy,  $P$  has a limit point  $p \in Y$ .

Since  $P$  was arbitrary,  $Y$  contains all its limit points, whenever limit points exist, so  $Y$  is closed. □

**Problem 4.**

**Definition 4.1.** A set  $S$  is compact if every open cover of  $S$  has a finite subcover.

**Definition 4.2.** An open cover of a set  $S$  is a collection of open sets  $\{U_a\}_{a \in A}$  such that  $S \subseteq \bigcup_{a \in A} U_a$ .

Let  $A := \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subset \mathbb{R}$ .

(i) Show that  $A$  is not compact directly using the definition.

**Claim 4.3.**  $A$  has no finite subcover.

*Proof.* For each  $n \in \mathbb{N}$ ,  $a_n = \frac{1}{n} \in A$ , set

$$r_n = \frac{1}{2} \cdot \min \left\{ \frac{1}{n-1} - \frac{1}{n}, \frac{1}{n} - \frac{1}{n+1} \right\}$$

such that  $r_n \leq \frac{1}{2}d(a_{n-1}, a_n)$  and  $r_n \leq \frac{1}{2}d(a_n, a_{n+1})$ . Define the open ball

$$B_n := B_{r_n}(a_n) = (a_n - r_n, a_n + r_n).$$

Then each  $B_n$  is a non-empty open set containing *only* the single element  $a_n$ . Furthermore, any two distinct  $B_n$  and  $B_m$  are disjoint. Therefore,  $\{B_n\}_{n \in \mathbb{N}}$  is an infinite open cover of  $A$ , (since there are infinitely many  $B_n$ ), and each  $a_n$  is contained in exactly one  $B_n$ , so there is no finite subcover.  $\square$

(ii) Show that  $A \cup \{0\}$  is compact directly using the definition.

**Claim 4.4.**  $A \cup \{0\}$  has a finite subcover.

*Proof.* Let  $S := A \cup \{0\}$ . First, note that  $A$  is bounded below by 0 and bounded above by 1. Let  $\{U_a\}_{a \in A}$  be an open cover of  $S$ . Since  $0 \in S$ , there exists  $a_0 \in A$  such that  $0 \in U_{a_0}$ .  $U_{a_0}$  is an open set, so there exists some  $\varepsilon > 0$  such that  $B_\varepsilon(0) \subseteq U_{a_0}$ . However, we know that for all  $\varepsilon > 0$ , there exists some  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \varepsilon$ , and by the order properties of  $\mathbb{R}$ ,  $0 < \frac{1}{N} < \varepsilon \implies 0 < \frac{1}{n} \geq \frac{1}{N} < \varepsilon$  for all  $n \geq N$ .

Therefore,  $B_\varepsilon(0)$  contains infinitely many points, and only the points  $\frac{1}{k}$  for  $k < N$  are not contained in  $B_\varepsilon(0)$ . Therefore,  $\{B_\varepsilon(0)\} \cup \{U_k \mid k < N\}$  is a finite subcover of  $S$ .  $\square$



**Problem 5.**

Let  $(X, d)$  be a metric space and  $S \subset X$ . Show directly that if  $S$  is sequentially compact then  $S$  is limit-point compact without using the theorem we proved in class.

**Problem 6.**

Prove that every bounded sequence of real numbers has a convergent subsequence (This statement is known as the *Bolzano-Weierstrass Theorem*).

*[Hint]: Construct a Cauchy subsequence from the given sequence by constructing a sequence of nested intervals whose length converges to 0 and each interval has infinitely many elements from the original sequence.*