Math 63: Real Analysis **Winter 2024**

PSET 6 — 02/14/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Let $f(x) = \dfrac{1+x}{1+x^3}$.

(i) Find a largest subset $U \subseteq \mathbb{R}$ where f is well-defined. Is f continuous on U?

(ii) Let U be as in part (i). Let g be a function such that $g(x) = f(x)$ if $x \in U$.

Is there a way to define g on $\mathbb{R} \setminus U$ to obtain a continuous function g on $\mathbb{R}$?

Problem 2.

Determine the points where the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Problem 3.

Let $f : [a, b] \to \mathbb{R}$ with $a < b$ be continuous. Show that functions

$$m(x) = \mathbf{inf}\,\{f(y) \mid a \le y \le x\} \quad \text{and} \quad M(x) = \mathbf{sup}\,\{f(y) \mid a \le y \le x\}$$

are continuous on $[a, b]$.

Problem 4.

Show if each of this functions is uniformly continuous on $\mathbb{R}$ or not.

(i) $f(x) = x^2$.

(ii) $f(x) = \sqrt{|x|}$.

Problem 5.

Let (E, d_E) be a compact metric space, and let $f, f_1, f_2, f_3, \ldots : E \to \mathbb{R}$ be continuous real-values functions on E, with $\lim_{n \to \infty} f_n = f$. Prove that if $f_1(p) \leq f_2(p) \leq f_3(p) \leq \cdots$ for all $p \in E$ then the sequence $f_1, f_2, f_3, \ldots$ converges uniformly.

Problem 6.

Let (X, d_X) and (Y, d_Y) be metric spaces. Assume (Y, d_Y) is complete. Show that a sequence of functions $f_n : X \to Y$ converges uniformly on X if and only if it is uniformly Cauchy on X.

Problem 7.

Let (X, d_X) be a metric space. A function $g : X \to \mathbb{R}$ is bounded on X if $\exists M$ such that $|g(x)| \leq M$ for all $x \in X$.

Suppose that $f_n : X \to \mathbb{R}$ is bounded on X for each $n \in \mathbb{N}$. Show that if a sequence of of f_n converges uniformly to a function $f : X \to \mathbb{R}$ then f is bounded on X.