Math 63: Real Analysis	Winter 2024

PSET 3 — 01/24/2024

Prof. Erchenko — ***Student: Amittai Siavava***

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Let $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$.

(i) Show that $X_1 := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ is open in $(\mathbb{R}^n, d_E)$.

Definition 1.1. A set S is open if all points have an open neighborhood (ball) contained in S.

Definition 1.2. The Euclidean metric d_E is defined as

$$d_E(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Remark 1.3. Note that for any $r \in \mathbb{R}$, $r^2 = |r|^2$. This means that:

$$d_E(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}.$$

Since $|x_i - y_i| \geq 0$ for all $i = 1, 2, \ldots, n$, this means:

$$\sqrt{|x_k - y_k|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \text{ for all } i = 1, 2, \ldots, n, \text{ and } k \in [1, n].$$

$$|x_k - y_k| \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \text{ for all } i = 1, 2, \ldots, n, \text{ and } k \in [1, n]. \tag{1.4}$$

Let $x = \langle x_1, x_2, \ldots, x_n \rangle \in X_1$ be arbitrary.

To show that X_1 is open, we shall show that any such x has an open neighborhood contained in X_1.

Pick $\varepsilon = \mathbf{min} \{|x_1 - b_1|, |x_1 - a_1|, |x_2 - b_2|, |x_2 - a_2|, \ldots, |x_n - b_n|, |x_n - a_n|\}$ such that

$$\varepsilon \leq |x_i - b_i| \text{ and } \varepsilon \leq |x_i - a_i| \text{ for all } i = 1, 2, \ldots, n. \tag{1.5}$$

Let $B_\varepsilon(x)$ be the open ball of radius ε centered at x.

Claim 1.6. $B_\varepsilon(x) \subseteq X_1$.

Proof. Let $y \in B_\varepsilon(x)$ be arbitrary. Since $B_\varepsilon(x)$ is open, $d_E(x, y) < \varepsilon$.

This means $|x_i - y_i| < \varepsilon$ for all $i = 1, 2, \ldots, n$ (by 1.4).

Thus, $|x_i - y_i| < |x_i - b_i|$ and $|x_i - y_i| < |x_i - a_i|$ for all $i = 1, 2, \ldots, n$ (by 1.5).

Thus, $y_i \in (a_i, b_i)$ for all $i = 1, 2, \ldots, n$ (by 1.5).

Thus, $y \in X_1$. □

(ii) Show that $X_2 := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is closed in $(\mathbb{R}^n, d_E)$.

Definition 1.7. A set S is closed if it contains all its limit points.

To show that X_2 is closed, we shall show that any convergent sequence in X_2 converges to a point in X_2.

Let $\{p_i\}_{i=1}^{\infty} \subseteq X_2$ be a convergent sequence in X_2, such that each $p_i = \langle p_{i,1}, p_{i,2}, \ldots, p_{i,n} \rangle \in X_2$.

Furthermore, suppose that $\lim_{i \to \infty} p_i = p$ for some $p = \langle p_1, p_2, \ldots, \_n \rangle \in \mathbb{R}^n$.

Claim 1.8. $p \in X_2$.

Proof. Let $i \in \mathbb{N}$ be arbitrary. Since $p_i \in X_2$, $p_{i,k} \in [a_k, b_k]$ for all $k = 1, 2, \ldots, n$.

Note that $[a_k, b_k]$ is closed for all $k = 1, 2, \ldots, n$ (since $[a_k, b_k]$ contains its boundary points).

Since $p_i \to p$, $p_{i,k} \to p_k$ for all $k = 1, 2, \ldots, n$.

Thus, $p_k \in [a_k, b_k]$ for all $k = 1, 2, \ldots, n$.

Thus, $p \in X_2$. □

Problem 2.

Prove that any bounded open subset of $\mathbb{R}$ is the union of disjoint open intervals.

Let $E \subseteq \mathbb{R}$ be a bounded open subset of $\mathbb{R}$. Since E is bounded *and* open, there exists some $a, b \in \mathbb{R}$ such that $a < x < b$ for all $x \in E$. Thus, $E \subseteq (a, b)$.

For each element $x \in E$, let $I_x = (a_x, b_x)$ be the largest *continuous* open interval around x that is contained entirely in E.

Claim 2.1. *For any two such intervals I_x and I_y, either $I_x = I_y$ or $I_x \cap I_y = \varnothing$.*

Proof. Since we pick I_x and I_y to be the largest open intervals around x and y respectively that are contained in E, if $I_x \cap I_y \neq \varnothing$, then there exists some element z in both I_x and I_y. If I_z is the maximal interval around z, then every element in I_x is in I_z and every element in I_y is in I_z. Likewise, every element in I_z is in both I_x and I_y (since x is in both I_x and I_y). Therefore, $I_x = I_y = I_z$. □

Claim 2.2. $E = \bigcup\limits_{x \in E} I_x$.

Proof. We shall show that the two sets are equal using double containment.

1. $E \supseteq \bigcup_{x \in E} I_x$:
   Let $a \in \bigcup_{x \in E} I_x$ be arbitrary, with I_a as the largest open interval around a that is contained in E, By definition of I_a, every point in I_a is contained in E, so $a \in E$.
2. $E \subseteq \bigcup\limits_{x \in E} I_x$:
   Let $a \in E$ be arbitrary, with I_a being the largest open interval around x that is contained in E. By definition of I_a, $a \in I_a$, so $a \in \bigcup\limits_{x \in E} I_x$.

□

Claim 2.3. *Any bounded open subset of $\mathbb{R}$ is the union of disjoint open intervals.*

Proof. By connecting claims 2.1 and 2.2, we have that $E = \bigcup\limits_{x \in E} I_x$, where each I_x is a disjoint open interval. □

I attempted an alternative proof on the next page. I would love some feedback on whether it is correct or not, but if you only have to grade one then please grade the first one.

I wasn't sure if the previous technique is correct, so I attempted an alternative construction.

Claim 2.4. *Any bounded open subset of $\mathbb{R}$ is the union of disjoint open intervals.*

Proof. Let $E \subseteq \mathbb{R}$ be a bounded open subset of $\mathbb{R}$. Since E is bounded *and* open, there exists some $a, b \in \mathbb{R}$ such that $a < x < b$ for all $x \in E$. Thus, $E \subseteq (a, b)$.

(i) If $E = \varnothing$, then it is the union of disjoint open intervals since the empty set is trivially open.

(ii) If $E = (a, b)$, then it is the union of the disjoint open intervals since (a, b) is open.

(iii) If $E \neq (a, b)$, then there exists some $c \in (a, b)$ such that $c \notin E$. Write $E = E_1 \cup E_2$ with $E_1 \subseteq (a, c)$ and $E_2 \subseteq (c, b)$. E_1 and E_2 are disjoint since $a < c < b$, and $c \notin E_1, c \notin E_2$.

- If $E_1 = (a, c)$ and $E_2 = (c, b)$, then we are done since (a, c) and (c, b) are both open.
- If $E_1 \neq (a, c)$ or $E_2 \neq (c, b)$, repeat the process of splitting E_1 and/or E_2 into smaller open intervals until equality.

We can then write $E = E_i \cup E_j \cup \dots$ for some open intervals $E_i, E_j, \dots$ contained in (a, b).

□

Problem 3.

Prove that if the points of a convergent sequence of points in a metric space are reordered, the new sequence converges to the same limit.

Let $\langle p \rangle = \{p_i\}_1^\infty$ be a convergent sequence in a metric space (X, d) such that $\lim_{i\to\infty} p_i = p$.
Let $\langle q \rangle = \{q_i\}_1^\infty$ be a *re-ordering* of $\langle p \rangle$ such that for all $i \in \mathbb{N}$, $q_i = p_{f(i)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$. We will show that $\lim_{i\to\infty} q_i = p$.

Claim 3.1. $\lim_{i\to\infty} q_i = p$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\langle p \rangle$ converges to p, there exists some $N \in \mathbb{N}$ such that $d(p_i, p) < \varepsilon$ for all $i \geq N$. This means that for any such ε, there are only finitely many p_i such that $d(p_i, p) \geq \varepsilon$, i.e. the set

$$\{i \in \mathbb{N} \mid d(p_i, p) \geq \varepsilon\}$$

is finite.

Since $q_i = p_k$ for some $k \in \mathbb{N}$ for all $i \in \mathbb{N}$ and the reordering is a bijection, the set

$$\{i \in \mathbb{N} \mid d(q_i, p) \geq \varepsilon\}$$

is also finite.

For any ε, pick

$$M := \mathbf{sup}\,\{i \in \mathbb{N} \mid d(q_i, p) \geq \varepsilon\}\,.$$

Then we have $d(q_i, p) < \varepsilon$ for all $i \geq M + 1$, meaning $\lim_{i\to\infty} q_i = p$. $\square$

Problem 4.

Prove that if $\lim_{n\to\infty} p_n = p$ in a given metric space (X, d), then the set of points $S = \{p, p_1, p_2, \dots\}$ is closed.

Definition 4.1. A set S is closed if every convergent sequence in S converges to a point in S.

Let $\{p_i\}_1^\infty$ be a convergent sequence in a metric space (X, d) and $\lim_{i\to\infty} p_i = p \in X$ as outlined in the problem, and let $S = \{p, p_1, p_2, \dots\}$.

To show that S is closed, we shall show that any convergent sequence in S converges to a point in S. In fact, we'll make the stronger claim that any such sequence converges to p.

Claim 4.2. *If $Q = \{q_i\}_1^\infty$ is a convergent sequence in S, then $\lim_{i\to\infty} q_i = p$.*

Proof. Since Q converges, there exists some point q (in X, but not necessarily in S) such that $\lim_{i\to\infty} q_i = q$.

Since $\{q_i\}_1^\infty \subseteq S$, $q_i \in S$ for all $i \in \mathbb{N}$. This means that for all $i \in \mathbb{N}$, either $q_i = p$ or $q_i = p_k$ for some $k \in \mathbb{N}$.

However, since the sequence $P = \{p_i\}_1^\infty$ converges to p, for every $\varepsilon > 0$, there are only finitely many p_i such that $d(p_i, p) \geq \varepsilon$, hence there are only finitely many q_i such that $d(q_i, p) \geq \varepsilon$ (since each q_i is either equivalent to p or equivalent to $= p_k$ for some $k \in \mathbb{N}$) Thus, for any ε, pick M to be the largest such i in the set of all i such that $d(q_i, p) \geq \varepsilon$. Then we have $d(q_i, p) < \varepsilon$ for all $i \geq M + 1$, meaning $\lim_{i\to\infty} q_i = p$.

Since $p \in S$, S is closed. $\square$

Problem 5.

Let $a_n = \frac{n}{n+1}$ for $n \in \mathbb{N}$. Show, using the definition of a limit, that $\lim\limits_{n\to\infty} a_n = 1$.

Definition 5.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in a metric space (X, d).

We say that $\lim\limits_{n\to\infty} a_n = a$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ for all $n \geq N$.

Remark 5.2. For any $n \in \mathbb{N}$;

1. $N + 1 > 0$

2. $0 < N < N + 1 \implies \frac{1}{N} > \frac{1}{N+1} > 0.$

We shall show that $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$ in three steps, outlined in the following claims.

Claim 5.3. *The sequence is bounded above by* 1.

Proof. Let $n \in \mathbb{N}$ be arbitrary.

$$\begin{aligned} a_n &= \frac{n}{n+1} \\ &= \frac{n+1-1}{n+1} \\ &= 1 - \frac{1}{n+1} < 1 - 0 = 1 \qquad \text{since } \frac{1}{n+1} > \frac{1}{n} > 0 \text{ (by 2.)} \end{aligned}$$

□

Claim 5.4. *The sequence is monotonically increasing, i.e.* $a_n < a_{n+1}$ *for all* $n \in \mathbb{N}$.

Proof.

$$\begin{aligned} a_n &= \frac{n}{n+1} \\ a_n &= \frac{n+1-1}{n+1} \\ a_n &= 1 - \frac{1}{n+1} < 1 - \frac{1}{n+2} \qquad \text{since } \frac{1}{n+1} > \frac{1}{n+2} > 0 \text{ (by 2.)} \\ a_n &\qquad\qquad < \frac{(n+2)-1}{n+2} \\ a_n &\qquad\qquad < \frac{n+1}{n+2} \\ a_n &\qquad\qquad < a_{n+1} \end{aligned}$$

□

Claim 5.5. *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(a_n, 1) < \varepsilon$ for all $n \geq N$.*

Proof. Pick N to be the smallest integer greater than $\frac{1}{\varepsilon}$, so that $\frac{1}{N} < \varepsilon$.

By definition of the series, $a_N = \frac{N}{N+1}$. We'll show that $d(a_N, 1) < \varepsilon$.

$$\begin{aligned} a_N &= \frac{N}{N+1} \\ &= \frac{N+1-1}{N+1} \\ &= 1 - \frac{1}{N+1} > 1 - \frac{1}{N} > 1 - \varepsilon \qquad \text{since } \frac{1}{N+1} < \frac{1}{N} \text{ (by 2.) and } \frac{1}{N} < \varepsilon. \end{aligned}$$

Therefore, $d(a_N, 1) = |a_N - 1| < |1 - \varepsilon - 1| = \varepsilon$. □

Tying together claims 5.3, 5.4, and 5.5, we have shown that the sequence is monotonically increasing, is bounded above by 1, and for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(a_N, 1) < \varepsilon$. But these three properties mean that for all $n \geq N$, $d(a_n, 1) < \varepsilon$ (since the sequence will never decrease below a_N and never increase above 1).

Therefore, $\lim_{n \to \infty} a_n = 1$.

Problem 6.

Consider the sequence $A \coloneqq \{a_n\}_{n=1}^{\infty}$ such that $a_1 \geq a_2 \geq a_3 \geq \dots$ (i.e., it is a monotonically decreasing sequence). Assume that there exists $m > 0$ such that $a_n > m$ for all n. Show that A converges in $\mathbb{R}$.

Definition 6.1. A sequence $\{p_n\}_{n=1}^{\infty}$ converges to a point p if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(p_n, a) < \varepsilon$ for all $n \geq N$.

Remark 6.2. Since $a_n > m$ for all n, and A is monotonically decreasing,

1. $a_n - m > 0$ for all n.

2. $a_n - a_{n+1} \geq 0$ for all n.

3. (**1.**) and (**2.**) imply that $d(a_n, m) \geq d(a_{n+1}, m)$ for all n.

Let S be the set of all points in $a_n \in A$ for each $n \in \mathbb{N}$. Then S has the following properties:

(i) S is nonempty since $a_1 \in S$.

(ii) S is bounded below by m.

For our purposes, let $M \coloneqq \textbf{inf}\, S$.

Claim 6.3. *A converges to M.*

Proof. Since $M = \textbf{inf}\, S$, for all $a_n \in S$, we have $a_n \geq M$.

For any $\varepsilon > 0$, note that $M + \varepsilon$ is not a lower bound for S (since we picked M to be the biggest such lower bound). This implies that there exists some $a_N \in S$ such that $a_N < M + \varepsilon$ (or equivalently, $d(a_N, M) < \varepsilon$). But, since A is monotonically decreasing, this implies that $a_n < M + \varepsilon$ for all $n > M$, meaning $d(a_n, M) < \varepsilon$, for all $n > N$.
By definition of convergence, this means that the sequence $A = \{a_i\}_1^{\infty}$ converges to M. □