Math 63: Real Analysis	**Winter 2024**

PSET 4 — 01/31/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

If $a_1, a_2, a_3, \ldots$ is a bounded sequence of real numbers, define

$$\limsup_{n\to\infty} a_n := \sup\{x \in \mathbb{R} \mid a_n > x \text{ for infinitely many } n \in \mathbb{N}\}$$

$$\liminf_{n\to\infty} a_n := \inf\{x \in \mathbb{R} \mid a_n < x \text{ for infinitely many } n \in \mathbb{N}\}$$

Prove that $\liminf\limits_{n\to\infty} a_n \le \limsup\limits_{n\to\infty} a_n$ with the equality holding if and only if the sequence converges.

Let $A = a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers as defined above.

Claim 1.1. $\liminf\limits_{n\to\infty} a_n \le \limsup\limits_{n\to\infty} a_n$

Proof. Let $L = \liminf\limits_{n\to\infty} a_n$ and $U = \limsup\limits_{n\to\infty} a_n$. By the definitions of $\liminf$ and $\limsup$, for any $\varepsilon > 0$,

$$a_n \ge L - \varepsilon \quad \text{for infinitely many } n$$

$$a_n \le U + \varepsilon \quad \text{for infinitely many } n$$

Thus, for infinitely many n, we have

$$L - \varepsilon \le a_n \le U + \varepsilon.$$

By making ε arbitrarily small, we have $L \le a_n \le U$ which implies $L \le U$.

Remark 1.2. Why can we make ε arbitrarily small? This argument stakes on the fact that L is the *infimum* of the set of all numbers such that r such that $a_n < r$ for infinitely many n, which means that for any other $L' < L$, the condition does not hold (or else L would not be the infimum). For any of the infinitely many n, $\neg(a_n < L') \implies (a_n \ge L')$. The same argument holds for U by replacing L' with $U' > U$ and reversing the inequality.

∎

Claim 1.3. *If the sequence converges, then* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof. Suppose the sequence $A = a_1, a_2, a_3, \ldots$ converges to some point a.

Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ for all $n \geq N$, so that $a - \varepsilon < a_n < a + \varepsilon$ for infinitely many $n \geq N$. Therefore, $\liminf_{n\to\infty} a_n > a - \varepsilon$ and $\limsup_{n\to\infty} a_n < a + \varepsilon$. So we have:

$$a - \varepsilon < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < a + \varepsilon$$

for any $\varepsilon > 0$.

Given that $\{a_n\}_1^\infty$ converges, we can make ε arbitrarily small, so that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$. $\square$

Problem 2.

Let $x_n = \left(1 + \dfrac{1}{n}\right)^n$ for all $n \in \mathbb{N}$.

Remark 2.1. The Euler number e can be defined as $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$.

(i) Using induction, show that for all $x > -1$ and $n \in \mathbb{N}$, we have

$$(1 + x)^n \geq 1 + nx.$$

We shall prove the claim by induction on n.

Proof. Let $x > -1$ and $n \in \mathbb{N}$.

(a) Base case: $n = 1$.

$$\left[(1 + x)^n\right]_{n=1} = (1 + x)^1 = 1 + x = 1 + 1 \cdot x = 1 + nx$$

Since $1 + nx \geq 1 + nx$ (since any number is greater than or equal to itself), the inequality holds for $n = 1$.

(b) Inductive step: Assume $(1 + x)^n \geq 1 + nx$. We show that the invariant holds for $n + 1$.

$$\begin{aligned} (1 + x)^{n+1} &= (1 + x)^n (1 + x) \\ &\geq (1 + nx)(1 + x) = 1 + x + nx + nx^2 \\ &\geq 1 + x + nx \quad = 1 + (n + 1)x \end{aligned}$$

Therefore, $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$. $\square$

(ii) Using the previous item, show that $\frac{x_{n+1}}{x_n} \geq 1$ so x_n is monotonically increasing.

To show this, we need to show that $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

$$
\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} \\
&= \frac{\left(1+\frac{1}{n+1}\right)^{n}}{\left(1+\frac{1}{n}\right)^n} \cdot \left(1+\frac{1}{n+1}\right) \\
&= \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^n \cdot \left(1+\frac{1}{n+1}\right) \\
&= \left(\frac{1+\frac{1}{n+1}}{\frac{n+1}{n}}\right)^n \cdot \left(1+\frac{1}{n+1}\right) \\
&= \left(\frac{n+\frac{n}{n+1}}{n+1}\right)^n \cdot \left(1+\frac{1}{n+1}\right) \\
&= \left(\frac{n}{n+1}+\frac{n}{(n+1)^2}\right)^n \cdot \left(1+\frac{1}{n+1}\right) \\
&= \left(1-\frac{1}{n+1}+\frac{n}{(n+1)^2}\right)^n \cdot \left(1+\frac{1}{n+1}\right) \\
&= \left(1-\frac{-(n+1)+n}{(n+1)^2}\right)^n \cdot \left(1+\frac{1}{n+1}\right) \\
&= \left(1-\frac{1}{(n+1)^2}\right)^n \cdot \left(1+\frac{1}{n+1}\right) \\
&\geq \left(1-\frac{n}{(n+1)^2}\right) \cdot \left(1+\frac{n}{n+1}\right) && \text{(by the previous proof)} \\
&> \left(1-\frac{n+1}{(n+1)^2}\right) \cdot \left(1+\frac{n}{n+1}\right) && \text{(subtracting a bigger term)} \\
&= \left(1-\frac{1}{n+1}\right) \cdot \left(1+\frac{n}{n+1}\right) \\
&= 1-\frac{1}{n+1}+\frac{1}{n}-\frac{1}{n(n+1)} \\
&= \frac{n(n+1)-n+(n+1)-1}{n(n+1)} \\
&= \frac{n(n+1)}{n(n+1)} \\
&= 1
\end{aligned}
$$

(iii) Show that x_n is bounded using the binomial formula

$$(a+b)^n = \frac{n!}{k!(n-k)!}a^{n-k}b^k = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

Fix a and b to arbitrary real numbers. We shall prove this by induction on n.

(a) Base case: $n = 1$

$$\begin{aligned}
(a+b)^n &= a+b \\
&= \binom{1}{0}a^1 \cdot b^0 + \binom{1}{1}a^0 b^1 \\
&= \binom{1}{0}a^{1-0}b^0 + \binom{1}{1}a^{1-1}b^1 = \sum_{k=0}^{1}\binom{n}{k}a^{n-k}b^k
\end{aligned}$$

(b) Inductive step: Assume the invariant holds for $(a+b)^n$. We shall show that it holds for $(a+b)^{(n+1)}$.

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n\,(a+b) \\
&= \left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\right)(a+b) \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \cdot a + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \cdot b \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1} \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k + a^{n-k}b^{k+1} \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k + \binom{n}{k-1}a^{n+1-k}b^k \qquad \text{(Grouping together equal powers)} \\
&= \sum_{k=0}^{n}\left(\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}\right)a^{n+1-k}b^k \\
&= \sum_{k=0}^{n}\left(\frac{(n+1-k)\,n! + kn!}{k!\,(n+1-k)!}\right)a^{n+1-k}b^k \\
&= \sum_{k=0}^{n}\left(\frac{(n+1)\,n!}{k!\,(n+1-k)!}\right)a^{n+1-k}b^k \\
&= \sum_{k=0}^{n}\left(\frac{(n+1)!}{k!\,(n+1-k)!}\right)a^{n+1-k}b^k \\
&= \sum_{k=0}^{n}\binom{n+1}{k}a^{n+1-k}b^k \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k
\end{aligned}$$

Since a and b were arbitrary, this holds for all a and b, including $a = 1$ and $b = \frac{1}{n}$, thus x_n is bounded by the binomial formula since equality satisfies both *less than or equal to* and *greater than or equal to*.

(iv) Show that $\{x_n\}$ is convergent.

As seen in (ii) and (iii), $\{x_n\}$ is monotonically increasing and is bounded above.

Claim 2.2. *$\{x_n\}$ is convergent.*

Proof. Suppose $\{x_n\}$ is not convergent. Since $\{x_n\}$ is monotonically increasing, this implies that it must not be bounded above and $\lim_{n\to\infty} x_n = \infty$, which contradicts the known fact (by remark 2.1) that $\lim_{n\to\infty} x_n = e \neq \infty$. Therefore, the sequence must be convergent. $\square$

Problem 3.

Show that a complete subspace of a metric space is a closed subset.

Definition 3.1. A metric space X is complete if every Cauchy sequence in X has a limit point in X.

Definition 3.2. A subset S of a metric space X is closed if S contains all its limit points.

Claim 3.3. *Every convergent sequence is Cauchy.*

Proof. Let X be a metric space and $P := \{p_n\}_1^\infty \subseteq X$ be a convergent sequence in X. Let $p \in X$ be the limit of P. For any $\varepsilon > 0$, since P converges to p, there exists $N \in \mathbb{N}$ such that $d(p_n, p) < \frac{\varepsilon}{2}$ for all $n \geq N$. Therefore,

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, P is Cauchy. □

Claim 3.4. *A complete subspace of a metric space is a closed subset.*

Proof. Let X be a metric space and $Y \subseteq X$ be a complete subspace of X.
Let $P := \{p_n\}_1^\infty \subseteq Y$ be an arbitrary convergent sequence in Y (i.e. a sequence that has a limit). To show that Y is closed, we need to show that any such P has a limit point in Y.

By claim 3.3, the convergence of P implies that P is Cauchy. Since Y is complete by definition, and P is Cauchy, P has a limit point $p \in Y$.

Since P was arbitrary, Y contains all its limit points, whenever limit points exist, so Y is closed. □

Problem 4.

Definition 4.1. A set S is compact if every open cover of S has a finite subcover.

Definition 4.2. An open cover of a set S is a collection of open sets $\{U_a\}_{a\in A}$ such that $S \subseteq \bigcup_{a\in A} U_a$.

Let $A := \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subset \mathbb{R}$.

(i) Show that A is not compact directly using the definition.

Claim 4.3. *A has no finite subcover.*

Proof. For each $n \in \mathbb{N}$, $a_n = \frac{1}{n} \in A$, set

$$r_n = \frac{1}{2} \cdot \mathbf{min}\left\{\frac{1}{n-1} - \frac{1}{n}, \frac{1}{n} - \frac{1}{n+1}\right\}$$

such that $r_n \leq \frac{1}{2}d(a_{n-1}, a_n)$ and $r_n \leq \frac{1}{2}d(a_n, a_{n+1})$. Define the open ball

$$B_n := B_{r_n}(a_n) = (a_n - r_n, a_n + r_n)\,.$$

Then each B_n is a non-empty open set containing *only* the single element a_n. Furthermore, any two distinct B_n and B_m are disjoint. Therefore, $\{B_n\}_{n\in\mathbb{N}}$ is an infinite open cover of A, (since there are infinitely many B_n), and each a_n is contained in exactly one B_n, so there is no finite subcover. □

(ii) Show that $A \cup \{0\}$ is compact directly using the definition.

Claim 4.4. *$A \cup \{0\}$ has a finite subcover.*

Proof. Let $S := A \cup \{0\}$. First, note that A is bounded below by 0 and bounded above by 1. Let $\{U_a\}_{a\in A}$ be an open cover of S. Since $0 \in S$, there exists $a_0 \in A$ such that $0 \in U_{a_0}$. U_{a_0} is an open set, so there exists some $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U_{a_0}$. However, we know that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and by the order properties of $\mathbb{R}$, $0 < \frac{1}{N} < \varepsilon \implies 0 < \frac{1}{n} \geq \frac{1}{N} < \varepsilon$ for all $n \geq N$.
Therefore, $B_\varepsilon(0)$ contains infinitely many points, and only the points $\frac{1}{k}$ for $k < N$ are not contained in $B_\varepsilon(0)$. Therefore, $\{B_\varepsilon(0)\} \cup \{U_k \mid k < N\}$ is a finite subcover of S. □

Problem 5.

Let (X, d) be a metric space and $S \subset X$. Show directly that if S is sequentially compact then S is limit-point compact without using the theorem we proved in class.

Definition 5.1. A metric space X is sequentially compact if every sequence in X has a convergent subsequence converging to a point in X.

Definition 5.2. A subset S of a metric space X is limit-point compact if every infinite subset of S has a limit point in S.

Claim 5.3. *If a subsequence of a sequence converges to a point in S, then the sequence also converges to that point.*

Proof. Let X be a metric space and $S \subseteq X$. Let $P := \{p_n\}_1^\infty \subseteq S$ be a sequence in S, and let $Q := \{p_{n_k}\}_1^\infty \subseteq P$ be a subsequence of P. Suppose Q converges to $q \in S$. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(q_n, q) < \varepsilon$ for all $n \geq N$. Since $Q \subseteq P$, this implies that $d(p_k, q) < \varepsilon$ for infinitely many distinct $k \in \mathbb{N}$, hence $d(p_k, q) > \varepsilon$ for finite $k \in \mathbb{N}$. Pick $K = \mathbf{max}\,\{k \in \mathbb{N} \mid d(p_k, q) > \varepsilon\}$. Then, for all $n \geq K$, $d(p_n, q) < \varepsilon$, so P also converges to q. □

Claim 5.4. *If S is sequentially compact then S is limit-point compact.*

Proof. For any infinite subset of S, we can construct a sequence P in S by picking any arbitrary element p_1 in the subset, then picking any arbitrary element p_2 in the subset that is not p_1, and so on. Since the subset is infinite, we can always pick an element that is not in the sequence so far.

Suppose S is sequentially compact. Then, by definition of sequential compactness, every such sequence P in S has a convergent subsequence Q converging to some point $q \in S$. By claim 5.3, P also converges to q, so P itself has a limit point in S.
Therefore, S is limit-point compact. □

Problem 6.

Prove that every bounded sequence of real numbers has a convergent subsequence (This statement is known as the *Bolzano-Weierstrass Theorem*).

[Hint]: Construct a Cauchy subsequence from the given sequence by constructing a sequence of nested intervals whose length converges to 0 *and each interval has infinitely many elements from the original sequence.*

Let $P \coloneqq \{p_n\}_1^\infty \subseteq \mathbb{R}$ be a bounded sequence of real numbers. Then P is contained in some interval $[a, b]$. For an interval S, let $\mathcal{L}(S)$ denote the length of the interval. We shall construct a sequence of nested intervals as follows:

1. Let $I_0 \coloneqq [a, b]$. Note that $\mathcal{L}(I_0) = |a - b|$.
2. Let I_1 be whichever of $\left[a, \dfrac{a+b}{2}\right]$ and $\left[\dfrac{a+b}{2}, b\right]$ contains infinitely many elements of P at least one of them must contain infinitely many elements of P, since their union is I_0, which contains infinitely many elements of P. If both contain infinitely many elements of P, pick either one. Note that $\mathcal{L}(I_1) = \dfrac{|a-b|}{2}$ and $I_1 \subseteq I_0$.
3. Construct I_2 to be whichever half of I_1 contains infinitely many elements of P such that $\mathcal{L}(I_2) = \dfrac{|a-b|}{2^2}$ and $I_2 \subseteq I_1$.
4. For each $n \in \mathbb{N}$, recursively construct I_n to be whichever half of I_{n-1} contains infinitely many elements of P so that $\mathcal{L}(I_n) = \dfrac{|a-b|}{2^n}$ and $I_n \subseteq I_{n-1} \subseteq \ldots \subseteq I_0$.

Next, for each $n \in \mathbb{N}$, pick any element q_n from the sequence P such that $q_n \in I_n$.

Claim 6.1. $\{q_n\}$ *is Cauchy.*

Proof. For any $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} \mathcal{L}(I_n) = 0$, there exists $N \in \mathbb{N}$ such that $\mathcal{L}(I_N) < \varepsilon$. Note that for any $n, m > N$, $q_m \in I_N$ and $q_n \in I_N$. Therefore,

$$d(q_m, q_n) \leq \mathcal{L}(I_N) < \varepsilon$$

$\square$

Thus, $\{q_n\}_1^\infty$ is Cauchy, and $\{q_n\}_1^\infty \subseteq [a, b]$, so $\{q_n\}_1^\infty$ is bounded. By the completeness of $\mathbb{R}$, $\{q_n\}_1^\infty$ converges.