Math 63: Real Analysis	**Winter 2024**

PSET 5 — 02/0-7/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Are the following functions $f : \mathbb{R} \to \mathbb{R}$ continuous? Justify your answer.

(i) $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$

f is continuous.

Proof. Let $x_0 \in \mathbb{R}$.

(a) Case 1: $x_0 < 0$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 0 = 0 = f(x_0)$$

(b) Case 2: $x_0 \geq 0$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0 = f(x_0)$$

Therefore, f is continuous at x_0 for all $x_0 \in \mathbb{R}$. $\square$

(ii) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime.} \end{cases}$

f is not continuous.

Problem 2.

Let E, E' be metric spaces, $f : E \to E'$ be a function, and suppose that S_1 and S_2 are subsets of E such that $S_1 \cup S_2 = E$. Show that if the restrictions of f to S_1 and S_2 are continuous, then f is continuous.

Claim 2.1. *If the restrictions of f to S_1 and S_2 are continuous, then f is continuous.*

Proof. Let $x_0 \in E$.

(a) Case 1: $x_0 \in S_1$. Since $f|_{S_1}$ is continuous, $\mathbf{lim}_{x \to x_0} f(x) = f(x_0)$.

(b) Case 2: $x_0 \in S_2$. Since $f|_{S_2}$ is continuous, $\mathbf{lim}_{x \to x_0} f(x) = f(x_0)$.

Therefore, f is continuous at x_0 for all $x_0 \in E$. □

Problem 3.

Let U, V be open intervals in $\mathbb{R}$, and let $f : U \to V$ be a function that is strictly increasing (i.e. if $x, y \in U$ and $x < y$, then $f(x) < f(y)$) and onto. Prove that f and f^{-1} are continuous.

Claim 3.1. *f is continuous.*

Proof. Let $x_0 \in U$. Let $\varepsilon > 0$. Since f is onto, there exists $y_0 \in V$ such that $f(x_0) = y_0$. Since f is strictly increasing, there exists $\delta > 0$ such that $f(x_0 - \delta) < y_0 - \varepsilon$ and $f(x_0 + \delta) > y_0 + \varepsilon$. Therefore, $f(B_\delta(x_0)) \subseteq B_\varepsilon(y_0)$. Therefore, f is continuous at x_0 for all $x_0 \in U$. □

Claim 3.2. *f^{-1} is continuous.*

Proof. Let $y_0 \in V$. Let $\varepsilon > 0$. Since f is onto, there exists $x_0 \in U$ such that $f(x_0) = y_0$. Since f is strictly increasing, there exists $\delta > 0$ such that $f(x_0 - \delta) < y_0 - \varepsilon$ and $f(x_0 + \delta) > y_0 + \varepsilon$. Therefore, $f(B_\delta(x_0)) \subseteq B_\varepsilon(y_0)$. Therefore, f^{-1} is continuous at y_0 for all $y_0 \in V$. □

Problem 4.

Let (E, d) and (E', d') be metric spaces, $f : E \to E'$ be a function, and let $p \in E$. Define the oscillation of f at p to be

$$\textbf{inf}\,\{a \in \mathbb{R} \mid \exists \text{ open ball in } B_r(p) \in E \text{ such that } \forall x, y \in B_r(p), d'(f(x), f(y)) \leq a\,\}$$

if the set is nonempty, and $+\infty$ otherwise. Prove that f is continuous at p if and only if the oscillation of f at p is 0, and that for any real number ε, the set of points of E at which the oscillation of f is at least ε is closed.

Claim 4.1. *f is continuous at p if and only if the oscillation of f at p is 0.*

Proof.

(a) ($\Rightarrow$) Suppose f is continuous at p. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $x \in E$ with $d(x, p) < \delta$, $d'(f(x), f(p)) < \varepsilon$. Therefore, the oscillation of f at p is at most ε. Since $\varepsilon > 0$ was arbitrary, the oscillation of f at p is 0.

(b) ($\Leftarrow$) Suppose the oscillation of f at p is 0. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $x \in E$ with $d(x, p) < \delta$, $d'(f(x), f(p)) < \varepsilon$. Therefore, f is continuous at p.

□

Claim 4.2. *For any real number ε, the set of points of E at which the oscillation of f is at least ε is closed.*

Proof. Let $A := \{p \in E \mid \text{oscillation of } f \text{ at } p \text{ is at least } \varepsilon\}$. We shall show that A is closed. Let $p \in E \setminus A$. Then the oscillation of f at p is less than ε. Let $a \in \mathbb{R}$ such that $a < \varepsilon$. Then there exists an open ball $B_r(p)$ in E such that for all $x, y \in B_r(p)$, $d'(f(x), f(y)) \leq a$. Therefore, $B_r(p) \subseteq E \setminus A$. Therefore, $E \setminus A$ is open, so A is closed. □

Problem 5.

Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that if f is one-to-one then $f([a, b])$ is either $[f(a), f(b)]$ or $[f(b), f(a)]$ (whichever expression makes sense).

Claim 5.1. *If f is one-to-one then $f([a, b])$ is either $[f(a), f(b)]$ or $[f(b), f(a)]$.*

Proof. Let $y_0 \in f([a, b])$. Then there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$. Let $y_1 \in f([a, b])$. Then there exists $x_1 \in [a, b]$ such that $f(x_1) = y_1$. Since f is one-to-one, $x_0 \neq x_1$ implies $y_0 \neq y_1$. Therefore, $f([a, b])$ is an interval. Since f is continuous, $f([a, b])$ is connected. Therefore, $f([a, b])$ is an interval. Since $f(a)$ and $f(b)$ are in $f([a, b])$, $f([a, b])$ is either $[f(a), f(b)]$ or $[f(b), f(a)]$. $\square$

Problem 6.

Let (X, d) be a metric space and $a \subset X$. The closure of A is the set

$$\overline{A} := \bigcap \{E \subset X \mid E \text{ is closed and } A \subset E\}.$$

Assume A is a connected set. Is $\overline{A}$ connected? Prove or find a counterexample.

$\overline{A}$ is connected.

Proof. Let $x_0, y_0 \in \overline{A}$. Let $E \subset X$ be closed and $A \subset E$. Since E is closed, E^c is open. Since $x_0, y_0 \in \overline{A}$, $x_0, y_0 \in E$. Since A is connected, there exists a continuous function $f : A \to \{0, 1\}$ such that $f(x_0) = 0$ and $f(y_0) = 1$. Since f is continuous, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are closed. Since $f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = A$ and $A \subset E$, $f^{-1}(\{0\}) \cup f^{-1}(\{1\}) \subset E$. Since $f^{-1}(\{0\}) \neq \varnothing$ and $f^{-1}(\{1\}) \neq \varnothing$, $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) \neq \varnothing$. Therefore, $f^{-1}(\{0\}) \cap f^{-1}(\{1\})$ is connected. Since $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) \subset A$, $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) \subset \overline{A}$. Therefore, $\overline{A}$ is connected. $\square$