

PSET 6 — 02/14/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

- (a) *Introduction to Analysis* by Maxwell Rosenlicht

Problem 1.

Let $f(x) = \frac{1+x}{1+x^3}$.

- (i) Find a largest subset $U \subseteq \mathbb{R}$ where f is well-defined. Is f continuous on U ?

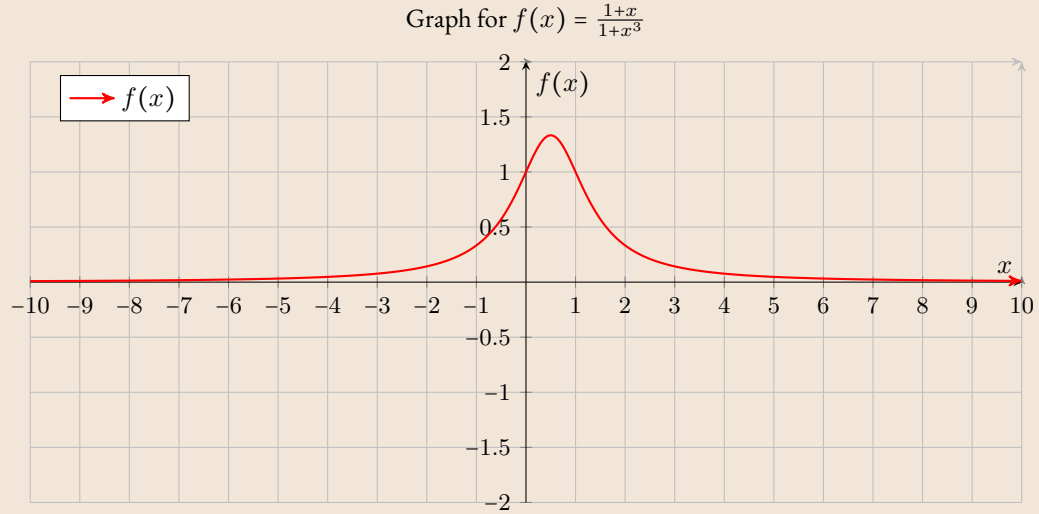
Since f is a rational function, it is well-defined where the denominator is nonzero.

Thus, $U = \mathbb{R} \setminus \{-1\}$.

Since U is open and f is a rational function, f is continuous on U (This was proven in class.).

(ii) Let U be as in part (i). Let g be a function such that $g(x) = f(x)$ if $x \in U$.

Is there a way to define g on $\mathbb{R} \setminus U$ to obtain a continuous function g on \mathbb{R} ?



Note: value at $x = -1$ is not defined, which is not demonstrated in this plot.

For g to be continuous on \mathbb{R} , we need to define g at -1 such that $\lim_{x \rightarrow -1} g(x) = g(-1)$. Since g has an indeterminate form at -1 , we use L'Hôpital's rule to find the limit:

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{1+x}{1+x^3} &= \lim_{x \rightarrow -1} \left[\frac{\frac{d}{dx}(1+x)}{\frac{d}{dx}(1+x^3)} \right] \\ &= \lim_{x \rightarrow -1} \frac{1}{3x^2} \\ &= \frac{1}{3} \end{aligned}$$

Thus, we can define

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \setminus \{-1\} \\ \frac{1}{3} & \text{if } x = -1 \end{cases}$$

to obtain a continuous function on \mathbb{R} .

Problem 2.

Determine the points where the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{1}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous.

Claim 2.1. *The function f is continuous at all points $(x, y) \neq (0, 0)$.*

Proof. Note that the function f is a composition of two functions; $f = h \circ g$, where

$$\begin{array}{ccc} g : \mathbb{R}^2 \rightarrow \mathbb{R} & & h : \mathbb{R} \rightarrow \mathbb{R} \\ (x, y) \mapsto x^2 + y^2 & \text{and} & t \mapsto \frac{1}{t}. \end{array}$$

Since g and h are both continuous over $U := \{p \in \mathbb{R}^2 \mid p \neq (0, 0)\}$, Their is composition is also continuous over U . \square

Claim 2.2. *The function f is not continuous at $(0, 0)$.*

Proof. f is continuous at $(0, 0)$ if and only if

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0).$$

However;

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{1}{x^2 + y^2} = \infty,$$

so the limit does not exist. However, $f(0, 0) = 0$ so f is not continuous at $(0, 0)$. \square

Problem 3.

Let $f : [a, b] \rightarrow \mathbb{R}$ with $a < b$ be continuous. Show that functions

$$m(x) = \inf \{f(y) \mid a \leq y \leq x\} \quad \text{and} \quad M(x) = \sup \{f(y) \mid a \leq y \leq x\}$$

are continuous on $[a, b]$.

We will show that m is continuous on $[a, b]$. The proof for M is analogous.

Claim 3.1. *The function m is continuous on $[a, b]$.*

Proof. Let $x_0 \in [a, b]$. We need to show that $\lim_{x \rightarrow x_0} m(x) = m(x_0)$.

First, note that by definition of m , $m(x) \leq f(x)$ for all $x \in [a, b]$. Thus, m is bounded above by f . Since f is continuous on $[a, b]$, we have $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

For any such ε and δ , pick x so that $|x - x_0| < \delta$. We claim that $|m(x) - m(x_0)| < \varepsilon$. To show this, we shall demonstrate that $m(x_0) - \varepsilon < m(x) < m(x_0) + \varepsilon$.

1. Proof that $m(x_0) - \varepsilon < m(x)$:

$$\begin{aligned} m(x_0) - \varepsilon &\leq f(x_0) - \varepsilon && \text{since } m(x_0) \leq f(x_0) \\ &< f(x) && \text{since } |f(x) - f(x_0)| < \varepsilon \\ &\leq \inf \{f(y) \mid a \leq y \leq x\} \\ &= m(x) \end{aligned}$$

2. Proof that $m(x) < m(x_0) + \varepsilon$:

$$\begin{aligned} m(x) &\leq f(x) \\ &< f(x_0) + \varepsilon \\ &\leq \inf \{f(y) \mid a \leq y \leq x_0\} + \varepsilon \\ &= m(x_0) + \varepsilon \end{aligned}$$

□

Problem 4.

Show if each of this functions is uniformly continuous on \mathbb{R} or not.

(i) $f(x) = x^2$.

Claim 4.1. *The function $f(x) = x^2$ is not uniformly continuous on \mathbb{R} .*

Proof. Suppose for the sake of contradiction that $f(x) = x^2$ is uniformly continuous on \mathbb{R} .

Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Consider $\varepsilon = 1$ and suppose there exists such a δ . For any $x_1 \in \mathbb{R}$, pick $x_2 = x_1 + \frac{\delta}{2}$. Then,

$$\begin{aligned} |f(x_1) - f(x_2)| &< \varepsilon \\ \implies \left| x_1^2 - \left(x_1 + \frac{\delta}{2}\right)^2 \right| &< 1 \\ \implies \left| x_1^2 - x_1^2 - \delta x_1 - \frac{\delta^2}{4} \right| &< 1 \\ \implies \left| \delta x_1 + \frac{\delta^2}{4} \right| &< 1 \end{aligned}$$

However, since \mathbb{R} is unbounded, we can pick x_1 such that $x_1 > \frac{1}{\delta}$, so that $\delta x_1 > 1$. Since $\frac{\delta^2}{4} > 0$ for all $\delta > 0$, we have $\left| \delta x_1 + \frac{\delta^2}{4} \right| > 1$, contradicting the assumption that $\left| \delta x_1 + \frac{\delta^2}{4} \right| < 1$. Hence, $f(x) = x^2$ is not uniformly continuous on \mathbb{R} . \square

(ii) $f(x) = \sqrt{|x|}$.

Claim 4.2. *The function $f(x) = \sqrt{|x|}$ is uniformly continuous on \mathbb{R} .*

Proof. We will show that f is uniformly continuous on \mathbb{R} . Let $\varepsilon > 0$. We need to find a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. We claim that $\delta = \varepsilon^2$ works. Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta = \varepsilon^2$. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left| \sqrt{|x|} - \sqrt{|y|} \right| \\ &\leq \sqrt{||x| - |y||} \\ &\leq \sqrt{|x - y|} \\ &< \sqrt{\delta} = \varepsilon \quad \text{since } |x - y| < \delta \end{aligned}$$

Thus, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Hence, f is uniformly continuous on \mathbb{R} . \square

Problem 5.

Let (E, d_E) be a compact metric space, and let $f, f_1, f_2, f_3, \dots : E \rightarrow \mathbb{R}$ be continuous real-valued functions on E , with $\lim_{n \rightarrow \infty} f_n = f$. Prove that if $f_1(p) \leq f_2(p) \leq f_3(p) \leq \dots$ for all $p \in E$ then the sequence f_1, f_2, f_3, \dots converges uniformly.

We will show that the sequence f_1, f_2, f_3, \dots converges uniformly to f . Let $\varepsilon > 0$. We need to find an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $p \in E$, $|f_n(p) - f(p)| < \varepsilon$.

Claim 5.1. *There exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $p \in E$, $|f_n(p) - f(p)| < \varepsilon$.*

Proof. Since $f_n \rightarrow f$, for every $p \in E$, there exists an $N_p \in \mathbb{N}$ such that for all $n \geq N_p$, $|f_n(p) - f(p)| < \frac{\varepsilon}{3}$. Since E is compact, there exists a finite subcover $\{U_1, U_2, \dots, U_k\}$ of E . Let $N = \max\{N_{p_1}, N_{p_2}, \dots, N_{p_k}\}$. Then, for all $n \geq N$ and all $p \in E$, $|f_n(p) - f(p)| < \varepsilon$. □

Thus, the sequence f_1, f_2, f_3, \dots converges uniformly to f .

Problem 6.

Let (X, d_X) and (Y, d_Y) be metric spaces. Assume (Y, d_Y) is complete. Show that a sequence of functions $f_n : X \rightarrow Y$ converges uniformly on X if and only if it is uniformly Cauchy on X .

We will show that a sequence of functions $f_n : X \rightarrow Y$ converges uniformly on X if and only if it is uniformly Cauchy on X .

Claim 6.1. *The sequence of functions $f_n : X \rightarrow Y$ converges uniformly on X if and only if it is uniformly Cauchy on X .*

Proof. We will first show that if the sequence of functions $f_n : X \rightarrow Y$ converges uniformly on X , then it is uniformly Cauchy on X . We will then show that if the sequence of functions $f_n : X \rightarrow Y$ is uniformly Cauchy on X , then it converges uniformly on X .

1. (\implies) Suppose the sequence of functions $f_n : X \rightarrow Y$ converges uniformly on X . Then, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ and all $x \in X$, $|f_n(x) - f_m(x)| < \varepsilon$. Thus, for all $n, m \geq N$,

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon.$$

Therefore, the sequence of functions $f_n : X \rightarrow Y$ is uniformly Cauchy on X .

2. (\impliedby) Suppose the sequence of functions $f_n : X \rightarrow Y$ is uniformly Cauchy on X . Then, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon.$$

Since (Y, d_Y) is complete, the sequence of functions $f_n : X \rightarrow Y$ converges pointwise to some function $f : X \rightarrow Y$. We need to show that the sequence converges uniformly on X . Meaning, for $\varepsilon > 0$. We need to find an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$.

1. Since the sequence of functions $f_n : X \rightarrow Y$ is uniformly Cauchy on X , there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\sup_{x \in X} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$.
2. Furthermore, since the sequence of functions $f_n : X \rightarrow Y$ converges pointwise to $f : X \rightarrow Y$, there exists an $N' \in \mathbb{N}$ such that for all $n \geq N'$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in X$.
3. Let $N'' = \max\{N, N'\}$. Then, for all $n \geq N''$ and all $x \in X$,

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_{N''}(x)| + |f_{N''}(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, the sequence of functions $f_n : X \rightarrow Y$ converges uniformly on X . □

Problem 7.

Let (X, d_X) be a metric space. A function $g : X \rightarrow \mathbb{R}$ is bounded on X if $\exists M$ such that $|g(x)| \leq M$ for all $x \in X$. Suppose that $f_n : X \rightarrow \mathbb{R}$ is bounded on X for each $n \in \mathbb{N}$. Show that if a sequence of f_n converges uniformly to a function $f : X \rightarrow \mathbb{R}$ then f is bounded on X .

We will show that if a sequence of f_n converges uniformly to a function $f : X \rightarrow \mathbb{R}$ then f is bounded on X .

Claim 7.1. *If a sequence of f_n converges uniformly to a function $f : X \rightarrow \mathbb{R}$ then f is bounded on X .*

Proof. Since $f_n \rightarrow f$ uniformly, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. Since f_n is bounded on X for each $n \in \mathbb{N}$, there exists an M_n such that $|f_n(x)| \leq M_n$ for all $x \in X$. Let $M = \max \{M_1, M_2, \dots, M_N\}$. Then, for all $x \in X$, $|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < \varepsilon + M$. Since ε is arbitrary, we have $|f(x)| \leq M$ for all $x \in X$. Thus, f is bounded on X . \square