Math 63: Real Analysis	Winter 2024

PSET 5 — 02/07/2024

Prof. Erchenko | ***Student: Amittai Siavava***

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Are the following functions $f : \mathbb{R} \to \mathbb{R}$ continuous? Justify your answer.

(i) $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$

f is continuous.

Proof. Let $x_0 \in \mathbb{R}$.

(a) Case 1: $x_0 < 0$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 0 = 0 = f(x_0)$$

(b) Case 2: $x_0 \geq 0$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0 = f(x_0)$$

Therefore, f is continuous at x_0 for all $x_0 \in \mathbb{R}$. □

(ii) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime.} \end{cases}$

f is not continuous.

Proof. Let $x_0 \in \mathbb{R}$.

(a) Case 1: $x_0 \in \mathbb{R} \setminus \mathbb{Q}$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 0 = 0 = f(x_0)$$

Therefore, f is continuous at the irrational numbers.

(b) Case 2: $x_0 \in \mathbb{Q}$. Let $x_0 = \frac{p}{q}$ with p, q coprime. Let $\varepsilon := \frac{1}{2q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $d(x, x_0) < \varepsilon$. Then $f(x) = 0$ and $d(f(x), f(x_0)) = \frac{1}{q} > \varepsilon$. Therefore, f is not continuous at x_0.

Therefore, f is not continuous in $\mathbb{R}$, since it is not continuous at rational numbers. □

Problem 2.

Let E, E' be metric spaces, $f : E \to E'$ be a function, and suppose that S_1 and S_2 are subsets of E such that $S_1 \cup S_2 = E$. Show that if the restrictions of f to S_1 and S_2 are continuous, then f is continuous.

Claim 2.1. *If the restrictions of f to S_1 and S_2 are continuous, then f is continuous.*

Proof. Let $x_0 \in E$.

(a) Case 1: $x_0 \in S_1$. Since $f\big|_{S_1}$ is continuous, $\lim_{x \to x_0} f(x) = f(x_0)$.

(b) Case 2: $x_0 \in S_2$. Since $f\big|_{S_2}$ is continuous, $\lim_{x \to x_0} f(x) = f(x_0)$.

Therefore, f is continuous at x_0 for all $x_0 \in E$. $\square$

Problem 3.

Let U, V be open intervals in $\mathbb{R}$, and let $f : U \to V$ be a function that is strictly increasing (i.e. if $x, y \in U$ and $x < y$, then $f(x) < f(y)$) and onto. Prove that f and f^{-1} are continuous.

Claim 3.1. *f is continuous.*

Proof. We need to show that for every $x \in U$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $u \in U$, if $d(u, x) < \delta$ then $d(f(u), f(x)) < \varepsilon$.

For any $x \in U$, let $x' \in V$ such that $x' = f(x)$.
For any $\varepsilon > 0$, pick $a', b' \in V$ such that $x' - \varepsilon = a' < x' < b' = x' + \varepsilon$.
Since f is onto, there exists $a, b \in U$ such that $f(a) = a'$ and $f(b) = b'$. Furthermore, since f is strictly increasing, $a' < x' < b'$ implies $a < x < b$. Set $\delta = \mathbf{min}\{x - a, b - x\}$. Then, for any $u \in U$, if $d(u, x) < \delta$, then $a < u < b$, meaning $a' < f(u) < b'$ (since f is strictly increasing). This means $d(f(u), x') < \varepsilon$, since (a', b') is the open interval or radius ε around x'.
Thus, f is continuous. □

Claim 3.2. *f^{-1} is continuous.*

Proof. We need to show that for every $y \in V$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $v \in V$, if $d(v, y) < \delta$ then $d(f^{-1}(v), f^{-1}(y)) < \varepsilon$.

For any $y' \in V$, let $y \in U$ such that $y = f^{-1}(y')$ — (or, $y' = f(y)$).
For any $\varepsilon > 0$, pick $a, b \in U$ such that $y - \varepsilon = a < y < b = y + \varepsilon$.
Let $a', b' \in V$ such that $f(a) = a'$ and $f(b) = b'$. Since f is strictly increasing, $a < y < b$ implies $a' < y' < b'$. Set $\delta = \mathbf{min}\{y' - a', b' - y'\}$. Then, for any $v \in V$, let $u \in U$ such that $v = f(u)$ and $u = f^{-1}(v)$. Then, if $d(v, y') < \delta$, then $a' < v' < b'$, meaning $a < f^{-1}(v') < b$ (since f is strictly increasing). This means $d(f^{-1}(v), y') < \varepsilon$, since (a, b) is the open interval or radius ε around y'.
Thus, f^{-1} is continuous. □

Problem 4.

Let (E, d) and (E', d') be metric spaces, $f : E \to E'$ be a function, and let $p \in E$. Define the oscillation of f at p to be

$$\textbf{inf}\,\{a \in \mathbb{R} \mid \exists \text{ open ball } B_r(p) \in E \text{ such that } \forall x, y \in B_r(p), d'(f(x), f(y)) \leq a\,\}$$

if the set is nonempty, and $+\infty$ otherwise. Prove that f is continuous at p if and only if the oscillation of f at p is 0, and that for any real number ε, the set of points of E at which the oscillation of f is at least ε is closed.

Claim 4.1. *f is continuous at p if and only if the oscillation of f at p is 0.*

Proof. First, we shall show that if f is continuous at p, then the oscillation of f at p is 0. Then, we shall show that if the oscillation of f at p is 0, then f is continuous at p.

1. ($\Rightarrow$) Suppose f is continuous at p. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $x \in E$ with $d(x, p) < \delta$, $d'(f(x), f(p)) < \varepsilon$. This means that for all $x, y \in B_\delta(p)$, $d'(f(x), f(y)) < \varepsilon$. Knowing that f is contunyous at p, we can pick ε such that the corresponding δ is arbitrarily small, so the oscillation of f at p is 0.
2. ($\Leftarrow$) Suppose the oscillation of f at p is 0. This means that for all $\varepsilon > 0$, there exists an open ball $B_r(p)$ in E such that for all $x, y \in B_r(p)$, $d'(f(x), f(y)) \leq \varepsilon$. Set $\delta = r$, and we have that for all $x \in E$ with $d(x, p) < \delta$, $d'(f(x), f(p)) < \varepsilon$, which is the definition of continuity. Therefore, f is continuous at p.

□

Claim 4.2. *For any real number ε, the set of points of E at which the oscillation of f is at least ε is closed.*

Proof. Let $A = \{p \in E \mid \text{oscillation of } f \text{ at } p \geq \varepsilon\}$. We will show that A is closed by showing that every limit point of A is in A. Let $p \in E$ be a limit point of A and suppose, for contradiction, that $p \notin A$. Then, the oscillation of f at p is less than ε (by the definition of A). This means that there exists an open ball $B_r(p)$ in E such that for all $x, y \in B_r(p)$, $d'(f(x), f(y)) < \varepsilon$. But since p is a limit point of A, there exists $q \in A$ such that $q \in B_r(p)$. Then, the oscillation of f at q is at least ε (by the definition of A, since $q \in A$). This means that there exists an open ball $B_s(q)$ in E such that for all $x, y \in B_s(q)$, $d'(f(x), f(y)) \geq \varepsilon$. Pick a point $a \in B_s(q)$ such that $a \in B_r(p)$, then we have:

1. $d(f(a), f(q)) \geq \varepsilon$ (since $q \in A$).
2. $d(f(a), f(q)) \geq \varepsilon$ (since $a, q \in B_r(p)$).

Clearly, both of these statements cannot be true at the same time. This is a contradiction, so our assumption that $p \notin A$ must be false. Therefore, $p \in A$, so A is closed. □

Problem 5.

Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that if f is one-to-one then $f([a, b])$ is either $[f(a), f(b)]$ or $[f(b), f(a)]$ (whichever expression makes sense).

Claim 5.1. *If f is one-to-one then $f([a, b])$ is either $[f(a), f(b)]$ or $[f(b), f(a)]$.*

Proof. We will show that $f([a, b])$ is an interval.

Let $x_0, x_1 \in [a, b]$ with $x_0 < x_1$, and let $y_0 = f(x_0)$ and $y_1 = f(x_1)$.

1. Since f is continuous, by the intermediate value theorem, for any $y \in [\mathbf{min}\{y_0, y_1\}, \mathbf{max}\{y_0, y_1\}]$ there exists some $x \in [x_0, x_1]$ such that $f(x) = y$. Thus, f takes on every value in $[\mathbf{min}\{y_0, y_1\}, \mathbf{max}\{y_0, y_1\}]$.
2. Since f is injective, $x_a \neq x_b$ implies $y_a \neq y_b$. This means that f does not repeat any values over the interval $[x_0, x_1]$, so $[\mathbf{min}\{y_0, y_1\}, \mathbf{max}\{y_0, y_1\}]$ is also an interval.

By setting $x_0 = a$ and $x_1 = b$, we have $f([a, b]) = [\mathbf{min}\{f(a), f(b)\}, \mathbf{max}\{f(a), f(b)\}]$. □

Problem 6.

Let (X, d) be a metric space and $A \subset X$. The closure of A is the set

$$\overline{A} := \bigcap \{E \subset X \mid E \text{ is closed and } A \subset E\}.$$

Assume A is a connected set. Is $\overline{A}$ connected? Prove or find a counterexample.

$\overline{A}$ is connected.

Proof. Suppose, for contradiction, that $\overline{A}$ is not connected. Then, there exist nonempty disjoint sets $U, V \subset X$ such that:

1. $\overline{A} = U \cup V$
2. $\overline{A} \cap U \neq \varnothing$
3. $\overline{A} \cap V \neq \varnothing$
4. $U \cap V = \varnothing$

By the definition of $\overline{A}$, $A \subseteq \overline{A}$ (since $\overline{A}$ is a union of closed sets containing A). Then, by **1.**, $A \subseteq U \cup V$.
But since A is connected, A must be fully contained in either U or V since U and V are disjoint(by **4.**).

1. If $A \subseteq U$, then for any $a \in A$, $a \in U$ and $a \notin V$.
   But, by **3.**, $\overline{A} \cap V \neq \varnothing$, so there must be some element $v \in V$ such that $v \in \overline{A}$. This implies that v is contained in every closed set containing A, so v is contained in U, which is a contradiction of the fact that $U \cap V = \varnothing$.
2. If $A \subseteq V$, then for any $a \in A$, $a \in V$ and $a \notin U$.
   But, by **2.**, $\overline{A} \cap U \neq \varnothing$, so there must be some element $u \in U$ such that $u \in \overline{A}$. This implies that u is contained in every closed set containing A, so u is contained in V, which is a contradiction of the fact that $U \cap V = \varnothing$.

Therefore, $\overline{A}$ is connected.

$\square$