Math 63: Real Analysis	**Winter 2024**

PSET 7 — 02/21/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Let $a, b \in \mathbb{R}$, $a < b$, and let f, g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b). Prove that there exists a number $c \in (a, b)$ such that

$$f'(c)\,(g(b) - g(a)) = g'(c)\,(f(b) - f(a))$$

Hint: Consider the function

$$F(x) = (f(x) - f(a))\,(g(b) - g(a)) - (g(x) - g(a))\,(f(b) - f(a))$$

Problem 2.

Use Problem 1 (Cauchy Mean Value Theorem) to prove L'Hôpital's Rule:

Let $U = (a, b) \subset \mathbb{R}$, and let f and g be differentiable real-valued functions on U, with g and g' nowhere zero on U. Suppose that $\lim_{x \to a} f(x) = \lim_{x \to a+} g(x) = 0.$. Then,

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$$

if the limit exists.

Problem 3.

Use Taylor's Theorem to prove the "binomial theorem" for $n \in \mathbb{N}$:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \cdots + x^n.$$

Problem 4.

Let f, g, f_n be real-valued functions on $[a, b] \subset \mathbb{R}$ for all $n \in \mathbb{N}$. Assume $f_n \in C^1([a, b])$ for all $n \in \mathbb{N}$. Suppose that $f_n \to f$ pointwise and $f_n' \to g$ uniformly as $n \to \infty$. Show that $f \in C^1([a, b])$. $f' = g$, and $f_n' \to f'$ uniformly as $n \to \infty$.

Problem 5.

Compute $\int_0^1 x \, \mathrm{d}x$ directly from the definition of the Riemann integral.

Hint: Consider the partition $0 = x_0 < x_1 < \ldots < x_n = 1$ where $x_i = \frac{i}{n}$ for $i = 0, 1, \ldots, n$.