Math 63: Real Analysis **Winter 2024**

PSET 1 — 01/10/2024

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Problems

Problem 1.

Let A, B, C be subsets of a set S. Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in (B \cap C)$.

  1. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$.

  2. If $x \in (B \cap C)$, then x is in **both** B and C.

  Therefore, $x \in (A \cup B)$ and $x \in (A \cup C)$.

  Therefore, $x \in (A \cup B) \cap (A \cup C)$.

(ii) Let $x \in (A \cup B) \cap (A \cup C)$.

Then $x \in (A \cup B)$ and $x \in (A \cup C)$. Since $x \in (A \cup B)$, either $x \in A$ or $x \in B$. Similarly, since $x \in (A \cup C)$, either $x \in A$ or $x \in C$. Thus, either $x \in A$ or x is in **both** B and C.

  1. If $x \in A$, then $x \in A \cup (B \cap C)$.

  2. If x is in **both** B and C, then $x \in (B \cap C)$, so $x \in A \cup (B \cap C)$.

Problem 2.

If A, B, C are sets, show that

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in (A \cup B)$ and $x \notin (A \cap B)$.

  1. If $x \in A$, then $x \notin B$ since $x \notin (A \cap B)$.

  Therefore, $x \in (A \setminus B)$.

  2. If $x \in B$, then $x \notin A$ since $x \notin (A \cap B)$.

  Therefore, $x \in (B \setminus A)$.

Therefore, $x \in (A \setminus B) \cup (B \setminus A)$.

(ii) Let $x \in (A \setminus B) \cup (B \setminus A)$. Then either $x \in (A \setminus B)$ or $x \in (B \setminus A)$.

  1. If $x \in (A \setminus B)$, then $x \in A$ and $x \notin B$. Therefore, $x \in (A \cup B)$ and $x \notin (A \cap B)$.

  2. If $x \in (B \setminus A)$, then $x \in B$ and $x \notin A$. Therefore, $x \in (A \cup B)$ and $x \notin (A \cap B)$.

Therefore, $x \in (A \cup B) \setminus (A \cap B)$.

Problem 3.

Let $f : X \to Y$ be a function. Let C and D be subsets of Y. Prove that

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

Recall the definition of the inverse image of a set:

Definition 3.1. Let $f : X \to Y$ be a function. Let $C \subseteq Y$. The *inverse image* of C under f is the set

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in f^{-1}(C \cup D)$. Then $f(x) \in (C \cup D)$.

  1. If $f(x) \in C$, then $x \in f^{-1}(C)$.

  2. If $f(x) \in D$, then $x \in f^{-1}(D)$.

  Therefore, $x \in f^{-1}(C) \cup f^{-1}(D)$.

(ii) Let $x \in f^{-1}(C) \cup f^{-1}(D)$. Then either $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$.

  1. If $x \in f^{-1}(C)$, then $f(x) \in C$.

  2. If $x \in f^{-1}(D)$, then $f(x) \in D$.

  Therefore, $f(x) \in (C \cup D)$, so $x \in f^{-1}(C \cup D)$.

This means that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$ and $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$, which can only happen if the two sets are equal.

Problem 4.

Let $A = \{1, 2, \dots, n\}$.

(a) Show that the cardinality of $\mathcal{P}(A)$ is 2^n.

For simplicity, let's use the notation A_n to denote the set $\{1, 2, \dots, n\}$.

Proof. This is a proof by induction on n.

(i) *Base Case*: $n = 0$

We show that $|\mathcal{P}(A_0)| = 2^0 = 1$. Note that $A_0 = \varnothing$, since A_0 contains zero elements. The only possible subset is the entire set itself, so $|\mathcal{P}(A_0)| = 2^0 = 1$.

(ii) *Induction Hypothesis*:

Assume that $|\mathcal{P}(A_n)| = 2^n$, we show that $|\mathcal{P}(A_{n+1})| = 2^{n+1}$.

(iii) *Inductive Step*:

We show that $|\mathcal{P}(A_{n+1})| = 2^{n+1}$.

$$\mathcal{P}(A_{n+1}) = \mathcal{P}(A_n) \cup \{a \cup \{(n+1)\} \mid a \in \mathcal{P}(A_n)\}$$

However, $\mathcal{P}(A_n)$ and $\{a \cup \{(n+1)\} \mid a \in \mathcal{P}(A_n)\}$ are disjoint!

This is because each element in $\{a \cup \{(n+1)\} \mid a \in \mathcal{P}(A_n)\}$ contains the element $(n+1)$,

but no element in $\mathcal{P}(A_n)$ contains the element $(n+1)$.

$$\begin{aligned}
\therefore\ |\mathcal{P}(A_{n+1})| &= |\mathcal{P}(A_n)| + |\{a \cup \{(n+1)\} \mid a \in \mathcal{P}(A_n)\}| \\
&= |\mathcal{P}(A_n)| + |\mathcal{P}(A_n)| \\
&= 2 \times |\mathcal{P}(A_n)| \\
&= 2 \cdot 2^n \\
&= 2^{n+1}
\end{aligned}$$

□

(b) How many functions are there from A to A? Explain.

A function sends each point in the domain to exactly one point in the codomain. Since there are n points in the domain and n points in the codomain, there are n choices for each point to go to, so **there are n^n functions from A to A.** However, most of these functions will be neither injective nor surjective!

(c) How many onto (surjective) functions are there from A to A? Explain.

A function is surjective if every point in the codomain is hit by some point in the domain. In this case, the codomain and the domain are the same set, so surjectivity also implies injectivity. Thus; there are n possible points for the first point to go to, then $n-1$ possible points for the second point to go to (since it can no longer map to the same point as the first point), and $n-2$ possible points for the third point to go to, and so on. **In total, there are $n!$ surjective functions from A to A.**

Problem 5.

Give an example of a countably infinite collection of infinite sets $A_1, A_2, A_3, \ldots$ with $A_j \cap A_k$ being inifinite for all j and k such that $\bigcap_{j=1}^{\infty} A_j$ is nonempty and finite.

Let $A = \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$ be the set of all nonnegative integers. For each $i \in \mathbb{N}$, define $A_i = \{a \in A \mid i \text{ divides } a\}$. That is, the A_i is the set of all nonnegative integers that are divisible by i. Then:

$$\begin{aligned} A_1 &= \{0, 1, 2, 3, \ldots\} = A \\ A_2 &= \{0, 2, 4, 6, \ldots\} \\ A_3 &= \{0, 3, 6, 9, \ldots\} \\ A_4 &= \{0, 4, 8, 12, \ldots\} \\ A_5 &= \{0, 5, 10, 15, \ldots\} \\ &\vdots \end{aligned}$$

(i) First, we'll show that $A_j \cap A_k$ is infinite for all j and k. For any arbitrary $j, k \in \mathbb{N}$;

$$(A_j \cap A_k) = \{0, jk, 2 \cdot jk, 3 \cdot jk, \ldots\} = \{n \cdot jk \mid n \in \mathbb{Z}_{\geq 0}\}.$$

Since $\mathbb{Z}_{\geq 0}$ is infinite, $(A_j \cap A_k)$ is also infinite.

(ii) We'll then show that $\bigcap_{j=1}^{\infty} A_j$ is nonempty yet finite.

1. $\bigcap_{j=1}^{\infty} A_j$ is nonempty. Since $0 \in A_j$ for all $j \in \mathbb{N}$, then $0 \in \bigcap_{j=1}^{\infty} A_j$.

2. $\bigcap_{j=1}^{\infty} A_j$ is finite. Precisely, $\bigcap_{j=1}^{\infty} A_j = \{0\}$. Suppose, for contradiction, that $\bigcap_{j=1}^{\infty} A_j$ contains a non-zero element x. This implies that x is in every A_j, thus x is divisible by every positive integer j. Take $y = 2x$, then clearly $\frac{x}{y} = \frac{x}{2x} = \frac{1}{2}$, which is not an integer, so x is not divisible by y, contradicting the assumption that x is divisible by every positive integer.

Problem 6.

Prove that a function is invertible if and only if it is a bijection.

Definition 6.1. A function $f : A \to B$ is *invertible* if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

For the sake of contradiction, suppose that $f : A \to B$ is invertible but not a bijection. Let $g : B \to A$ be its inverse.

(i) Suppose f is not an injection. Then there exists some $a_1, a_2 \in A$, $a_1 \neq a_2$, and some $b \in B$ such that $f(a_1) = f(a_2) = b$. Since g is the inverse of f, $g(f(x)) = x$ for all $x \in A$, which implies that $g(b) = a_1$ and $g(b) = a_2$. However, a function cannot map an element to two different elements, so either f is an injection or it must not be invertible.

(ii) Suppose f is not a surjection. Then there exists some $b \in B$ such that $f(a) \neq b$ for all $a \in A$. However, g, the inverse of f, acts from B to A, so it must assign some element $a' \in A$ to b. This implies that $g(b) = a'$ for some $a' \in A$. But g was taken to be the inverse of f, so it must map $f(a') \mapsto a'$, where $f(a') \neq b$. Therefore g is not an injection (and, by the previous result, not invertible). This contradicts the assumption that $g = f^{-1}$ as that would imply that g is also invertible (with f being its inverse).

Problem 7.

Let $\mathbb{F}$ be a field. Prove that for any $a \in \mathbb{F}$ there exists a unique $b \in \mathbb{F}$ such that $a + b = 0$.

Definition 7.1. A *field* is a set $\mathbb{F}$ with two binary operations $+$ and $\times$ such that:

- $(\mathbb{F}, +)$ is an abelian group with identity 0.
- $(\mathbb{F} \setminus \{0\}, \times)$ is an abelian group with identity 1.
- $\times$ distributes over $+$.

Useful results in a field $\mathbb{F}$:

- $0 \times a = a \times 0 = 0$ for all $a \in \mathbb{F}$.

(i) First, we show that for any element $a \in \mathbb{F}$, there exists some element $b \in \mathbb{F}$ such that $a + b = 0$. Let $a \in \mathbb{F}$ be arbitrary. Then:

$$\begin{aligned} a \times 0 &= 0 \\ a \times (1 + (-1)) &= 0 \\ a \times 1 + a \times (-1) &= 0 \\ a + (-a) &= 0 \end{aligned}$$

(ii) Next, we show that this element b is unique by demonstrating that if any two elements $b_1, b_2 \in \mathbb{F}$ satisfy $a + b_1 = 0$ and $a + b_2 = 0$, then $b_1 = b_2$. Let $b_1, b_2 \in \mathbb{F}$ be selected as above. Then:

$$\begin{aligned} b_1 &= b_1 + 0 \\ &= b_1 + (a + b_2) \\ &= (b_1 + a) + b_2 \\ &= 0 + b_2 \\ &= b_2 \end{aligned}$$