Math 63: Real Analysis **Winter 2024**

PSET 3 — 01/24/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

If $a_1, a_2, a_3, \ldots$ is a bounded sequence of real numbers, define

$$\limsup_{n\to\infty} a_n := \sup\{x \in \mathbb{R} \mid a_n > x \text{ for infinitely many } n \in \mathbb{N}\}$$

$$\liminf_{n\to\infty} a_n := \inf\{x \in \mathbb{R} \mid a_n < x \text{ for infinitely many } n \in \mathbb{N}\}$$

Prove that $\liminf\limits_{n\to\infty} a_n \le \limsup\limits_{n\to\infty} a_n$ with the equality holding if and only if the sequence converges.

Problem 2.

Let $x_n = \left(1 + \frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$.

Remark 2.1. The Euler number e can be defined as $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

(i) Using induction, show that for all $x > -1$ and $n \in \mathbb{N}$, we have

$$(1 + x)^n \geq 1 + nx.$$

(ii) Using th eprevious item, show that $\frac{x_{n+1}}{x_n} \geq 1$ so x_n is monotonically increasing.

(iii) Show that x_n is bounded using the binomial formula

$$(a + b)^n = \frac{n!}{k!(n-k)!} a^{n-k} b^k = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

(iv) Show that $\{x_n\}$ is convergent.

Problem 3.

Show that a complete subspace of a metric space is a closed subset.

Definition 3.1. A metric space X is complete if every Cauchy sequence in X converges.

Definition 3.2. A subset S of a metric space X is closed if it contains all its limit points.

Claim 3.3. *A complete subspace of a metric space is a closed subset.*

Proof. Let X be a metric space and $Y \subseteq X$ be a complete subspace of X. Let $S \subseteq Y$ be a Cauchy sequence in Y. Since Y is a subspace of X, S is also a Cauchy sequence in X. Since X is complete, S converges in X. Since S is a Cauchy sequence in Y, S converges in Y. Thus, Y contains all its limit points. Thus, Y is closed. $\square$

Problem 4.

Let $A := \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subset \mathbb{R}$.

(i) Show that A is not compact directly using the definition.

(ii) Show that $A \cup \{0\}$ is compact directly using the definition.

Problem 5.

Let (X, d) be a metric space and $S \subset X$. Show directly that if S is sequentially compact then S is limit-point compact without using the theorem we proved in class.

Problem 6.

Prove that every bounded sequence of real numbers has a convergent subsequence (This statement is known as the *Bolzano-Weierstrass Theorem*).

[Hint]: Construct a Cauchy subsequence from the given sequence by constructing a sequence of nested intervals whose length converges to 0 *and each interval has infinitely many elements from the original sequence.*