Math 63: Real Analysis **Winter 2024**

PSET 6 — 02/14/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Let $f(x) = \dfrac{1+x}{1+x^3}$.

(i) Find a largest subset $U \subseteq \mathbb{R}$ where f is well-defined. Is f continuous on U?

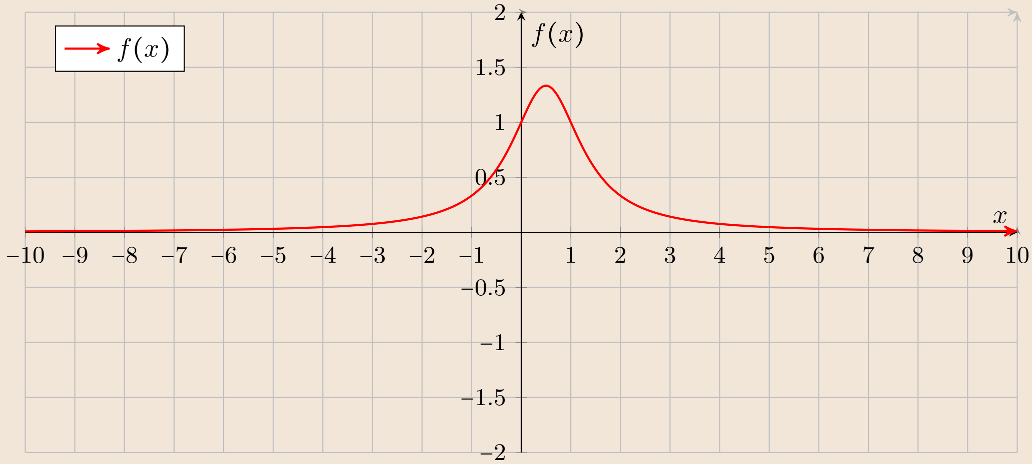
Since f is a rational function where the numerator and denominator consist of polynomial functions, it is well-defined where the denominator is nonzero. Thus, $U = \mathbb{R} \setminus \{-1\}$.

Since U is open and f is a rational function, f is continuous on U (We proved this result in class.).

(ii) Let U be as in part (i). Let g be a function such that $g(x) = f(x)$ if $x \in U$.

Is there a way to define g on $\mathbb{R} \setminus U$ to obtain a continuous function g on $\mathbb{R}$?

Graph for $f(x) = \frac{1+x}{1+x^3}$



Note: value at $x = -1$ is not defined, which is not demonstrated in this plot.

For g to be continuous on $\mathbb{R}$, we need to define g at -1 such that $\lim_{x\to-1} g(x) = g(-1)$. Since g has an indeterminate form at -1, we use L'Hôpital's rule to find the limit:

$$\begin{aligned}\lim_{x\to-1} \frac{1+x}{1+x^3} &= \lim_{x\to-1} \left[\frac{\mathrm{d}}{\mathrm{d}x}(1+x) \Big/ \frac{\mathrm{d}}{\mathrm{d}x}(1+x^3)\right] \\ &= \lim_{x\to-1} \frac{1}{3x^2} \\ &= \frac{1}{3}\end{aligned}$$

Thus, we can define

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \setminus \{-1\} \\ \frac{1}{3} & \text{if } x = -1 \end{cases}$$

to obtain a continuous function on $\mathbb{R}$.

Problem 2.

Determine the points where the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous.

Claim 2.1. *The function f is continuous at all points $(x,y) \neq (0,0)$.*

Proof. Note that the function f is a composition of two functions; $f = h \circ g$, where

$$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto x^2 + y^2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} h : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \frac{1}{t}. \end{aligned}$$

Since $U := \left\{p \in \mathbb{R}^2 \mid p \neq (0,0)\right\}$ is open, both g is continuous over U, and, in turn, f is continuous over U (composition of continuous functions is continuous). $\square$

Claim 2.2. *The function f is not continuous at $(0,0)$.*

Proof. f is continuous at $(0,0)$ if and only if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

However;

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty,$$

so the limit does not exist. However, $f(0,0) = 0$ so f is not continuous at $(0,0)$. $\square$

Problem 3.

Let $f : [a, b] \to \mathbb{R}$ with $a < b$ be continuous. Show that functions

$$m(x) = \inf \{f(y) \mid a \le y \le x\} \quad \text{and} \quad M(x) = \sup \{f(y) \mid a \le y \le x\}$$

are continuous on $[a, b]$.

We will show that m is continuous on $[a, b]$. The proof for M is analogous.

Claim 3.1. *The function m is continuous on $[a, b]$.*

Proof. Let $x_0 \in [a, b]$. We need to show that $\lim_{x \to x_0} m(x) = m(x_0)$.

First, note that:

1. $m(x) \le f(x)$ for all $x \in [a, b]$, by definition of m, so m is bounded above by f.
2. $m(x)$ is monotonically decreasing on $[a, b]$. This also follows from the definition of m, since it is the infimum of all values of f at points less than or equal to x.

Since f is continuous on $[a, b]$, we have $\lim_{x \to x_0} f(x) = f(x_0)$. Thus, for any $x_0 \in [a, b]$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. For any such ε and δ, pick x so that $|x - x_0| < \delta$ (and, consequently, $|f(x) - f(x_0)| < \varepsilon$). We claim that $|m(x) - m(x_0)| < \varepsilon$.

Since f is continuous, by definition of m as an infimum of all previous values of f, if $x < x_0$ and $m(x) \ne m(x_0)$ then $m(x) = f(y)$ for some $y \in [x, x_0]$. Since all points in that range are within δ of x_0, we have $|f(y) - f(x_0)| < \varepsilon$. Furthermore, since $m(x)$ is monotone decreasing, $m(x_0) \le m(x)$ when $x < x_0$, so $m(x_0) = f(z)$ for some $z \in [x, x_0]$ if $m(x) \ne m(x_0)$. Thus, either $m(x) = m(x_0)$ and $|m(x) - m(x_0)| = 0$ or

$$|m(x) - m(x_0)| = |f(y) - f(z)| < \varepsilon \text{ for some } x_0 - \delta < x < y, z < x_0.$$

Since this works for any ε and any x_0, we have $|m(x) - m(x_0)| < \varepsilon$. $\square$

Claim 3.2. *The function M is continuous on $[a, b]$.*

Proof. The proof is analogous to the proof for m. Since $M(x) \geq f(x)$ for all $x \in [a, b]$, by definition of M, M is bounded below by f. Also, $M(x)$ is monotonically increasing on $[a, b]$. This also follows from the definition of M, since it is the supremum of all values of f at points less than or equal to x. Since f is continuous on $[a, b]$, we have $\lim_{x \to x_0} f(x) = f(x_0)$. Thus, for any $x_0 \in [a, b]$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. For any such ε and δ, pick x so that $|x - x_0| < \delta$ (and, consequently, $|f(x) - f(x_0)| < \varepsilon$). We claim that $|M(x) - M(x_0)| < \varepsilon$.

Since f is continuous, by definition of M as an supremum of all previous values of f, if $x < x_0$ and $M(x) \neq M(x_0)$ then $M(x) = f(y)$ for some $y \in [x, x_0]$. Since all points in that range are within δ of x_0, we have $|f(y) - f(x_0)| < \varepsilon$. Furthermore, since $M(x)$ is monotone increasing, $M(x_0) \geq M(x)$ when $x < x0$, so $M(x_0) = f(z)$ for some $z \in [x, x_0]$ if $M(x) \neq M(x_0)$. Thus, either $M(x) = M(x_0)$ and $|M(x) - M(x_0)| = 0$ or

$$|M(x) - M(x_0)| = |f(y) - f(z)| < \varepsilon \text{ for some } x_0 - \delta < x < y, z < x_0.$$

Since this works for any ε and any x_0, we have $|M(x) - M(x_0)| < \varepsilon$. □

Problem 4.

Show if each of this functions is uniformly continuous on $\mathbb{R}$ or not.

(i) $f(x) = x^2$.

Claim 4.1. *The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.*

Proof. Suppose for the sake of contradiction that $f(x) = x^2$ is uniformly continuous on $\mathbb{R}$.

Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Consider $\varepsilon = 1$ and suppose there exists such a δ. For any $x_1 \in \mathbb{R}$, pick $x_2 = x_1 + \frac{\delta}{2}$. Then,

$$|f(x_1) - f(x_2)| = \left|(x_1 + \frac{\delta}{2})^2 - x_1^2\right| = \left|\delta x_1 + \frac{\delta^2}{4}\right| < \varepsilon = 1$$

However, since $\mathbb{R}$ is unbounded, we can pick x_1 such that $x_1 > \frac{1}{\delta}$, so that $\delta x_1 > 1$. Since $\frac{\delta^2}{4} > 0$ for all $\delta > 0$, we have $\left|\delta x_1 + \frac{\delta^2}{4}\right| > 1$, contradicting the assumption that $\left|\delta x_1 + \frac{\delta^2}{4}\right| < 1$. Hence, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. □

(ii) $f(x) = \sqrt{|x|}$.

Claim 4.2. *The function $f(x) = \sqrt{|x|}$ is uniformly continuous on $\mathbb{R}$.*

Proof. We will show that f is uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$. We need to find a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. We claim that $\delta = \varepsilon^2$ works. Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta = \varepsilon^2$. Then,

$$\begin{aligned}
|f(x) - f(y)| &= \left|\sqrt{|x|} - \sqrt{|y|}\right| \\
&\leq \sqrt{|x| - |y|} \\
&\leq \sqrt{|x - y|} \\
&< \sqrt{\delta} = \varepsilon && \text{since } |x - y| < \delta
\end{aligned}$$

Thus, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Hence, f is uniformly continuous on $\mathbb{R}$. □

Problem 5.

Let (E, d_E) be a compact metric space, and let $f, f_1, f_2, f_3, \ldots : E \to \mathbb{R}$ be continuous real-values functions on E, with $\lim\limits_{n\to\infty} f_n = f$. Prove that if $f_1(p) \leq f_2(p) \leq f_3(p) \leq \cdots$ for all $p \in E$ then the sequence $f_1, f_2, f_3, \ldots$ converges uniformly.

We will show that the sequence $f_1, f_2, f_3, \ldots$ converges uniformly to f. Let $\varepsilon > 0$. We need to find an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $p \in E$, $|f_n(p) - f(p)| < \varepsilon$. Since $f_n \to f$, for every $p \in E$, there exists an $N_p \in \mathbb{N}$ such that for all $n \geq N_p$, $|f_n(p) - f(p)| < \frac{\varepsilon}{3}$. Since E is compact, there exists a finite subcover $\{U_1, U_2, \ldots, U_k\}$ of E. Let $N = \mathbf{max}\{N_{p_1}, N_{p_2}, \ldots, N_{p_k}\}$. Then, for all $n \geq N$ and all $p \in E$, $|f_n(p) - f(p)| < \varepsilon$. Thus, the sequence $f_1, f_2, f_3, \ldots$ converges uniformly to f.

Problem 6.

Let (X, d_X) and (Y, d_Y) be metric spaces. Assume (Y, d_Y) is complete. Show that a sequence of functions $f_n : X \to Y$ converges uniformly on X if and only if it is uniformly Cauchy on X.

We will show that a sequence of functions $f_n : X \to Y$ converges uniformly on X if and only if it is uniformly Cauchy on X.

Claim 6.1. *The sequence of functions $f_n : X \to Y$ converges uniformly on X if and only if it is uniformly Cauchy on X.*

Proof. We will first show that if the sequence of functions $f_n : X \to Y$ converges uniformly on X, then it is uniformly Cauchy on X. We will then show that if the sequence of functions $f_n : X \to Y$ is uniformly Cauchy on X, then it converges uniformly on X.

1. $(\Longrightarrow)$ Suppose the sequence of functions $f_n : X \to Y$ converges uniformly on X. Then, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ and all $x \in X$, $|f_n(x) - f_m(x)| < \varepsilon$. Thus, for all $n, m \geq N$,
$$\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon.$$
Therefore, the sequence of functions $f_n : X \to Y$ is uniformly Cauchy on X.

2. $(\Longleftarrow)$ Suppose the sequence of functions $f_n : X \to Y$ is uniformly Cauchy on X. Then, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$,
$$\sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon.$$
Since (Y, d_Y) is complete, the sequence of functions $f_n : X \to Y$ converges pointwise to some function $f : X \to Y$. We need to show that the sequence converges uniformly on X. Meaning, for $\varepsilon > 0$. We need to find an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$.

   1. Since the sequence of functions $f_n : X \to Y$ is uniformly Cauchy on X, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\sup_{x \in X} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$.
   2. Furthermore, since the sequence of functions $f_n : X \to Y$ converges pointwise to $f : X \to Y$, there exists an $N' \in \mathbb{N}$ such that for all $n \geq N'$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in X$.
   3. Let $N'' = \max\{N, N'\}$. Then, for all $n \geq N''$ and all $x \in X$,
$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_{N''}(x)| + |f_{N''}(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, the sequence of functions $f_n : X \to Y$ converges uniformly on X. □

Problem 7.

Let (X, d_X) be a metric space. A function $g : X \to \mathbb{R}$ is bounded on X if $\exists M$ such that $|g(x)| \leq M$ for all $x \in X$. Suppose that $f_n : X \to \mathbb{R}$ is bounded on X for each $n \in \mathbb{N}$. Show that if a sequence of of f_n converges uniformly to a function $f : X \to \mathbb{R}$ then f is bounded on X.

We will show that if a sequence of of f_n converges uniformly to a function $f : X \to \mathbb{R}$ then f is bounded on X.

Claim 7.1. *If a sequence of of f_n converges uniformly to a function $f : X \to \mathbb{R}$ then f is bounded on X.*

Proof. Since $f_n \to f$ uniformly, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. Since f_n is bounded on X for each $n \in \mathbb{N}$, there exists an M_n such that $|f_n(x)| \leq M_n$ for all $x \in X$. Let $M = \mathbf{max}\{M_1, M_2, \ldots, M_N\}$. Then, for all $x \in X$,

$$|f(x)| \leq \underbrace{|f(x) - f_N(x)|}_{<\varepsilon} + \underbrace{|f_N(x)|}_{<M} < \varepsilon + M$$

Since ε is arbitrary, we have $|f(x)| \leq M$ for all $x \in X$. Thus, f is bounded on X. $\square$