Math 63: Real Analysis **Winter 2024**

PSET 1 — 01/09/2024

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Problems

Problem 1.

Let A, B, C be subsets of a set S. Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in (B \cap C)$.

  1. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$.

  2. If $x \in (B \cap C)$, then x is in **both** B and C.

  Therefore, $x \in (A \cup B)$ and $x \in (A \cup C)$.

  Therefore, $x \in (A \cup B) \cap (A \cup C)$.

(ii) Let $x \in (A \cup B) \cap (A \cup C)$.

Then $x \in (A \cup B)$ and $x \in (A \cup C)$. Since $x \in (A \cup B)$, either $x \in A$ or $x \in B$. Similarly, since $x \in (A \cup C)$, either $x \in A$ or $x \in C$. Thus, either $x \in A$ or x is in **both** B and C.

  1. If $x \in A$, then $x \in A \cup (B \cap C)$.

  2. If x is in **both** B and C, then $x \in (B \cap C)$, so $x \in A \cup (B \cap C)$.

Problem 2.

If A, B, C are sets, show that

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in (A \cup B)$ and $x \notin (A \cap B)$.

   1. If $x \in A$, then $x \notin B$ since $x \notin (A \cap B)$.
   Therefore, $x \in (A \setminus B)$.

   2. If $x \in B$, then $x \notin A$ since $x \notin (A \cap B)$.
   Therefore, $x \in (B \setminus A)$.

   Therefore, $x \in (A \setminus B) \cup (B \setminus A)$.

(ii) Let $x \in (A \setminus B) \cup (B \setminus A)$. Then either $x \in (A \setminus B)$ or $x \in (B \setminus A)$.

   1. If $x \in (A \setminus B)$, then $x \in A$ and $x \notin B$. Therefore, $x \in (A \cup B)$ and $x \notin (A \cap B)$.

   2. If $x \in (B \setminus A)$, then $x \in B$ and $x \notin A$. Therefore, $x \in (A \cup B)$ and $x \notin (A \cap B)$.

   Therefore, $x \in (A \cup B) \setminus (A \cap B)$.

Problem 3.

Let $f : X \to Y$ be a function. Let C and D be subsets of Y. Prove that

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

Recall the definition of the inverse image of a set:

Definition 3.1. Let $f : X \to Y$ be a function. Let $C \subseteq Y$. The *inverse image* of C under f is the set

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in f^{-1}(C \cup D)$. Then $f(x) \in (C \cup D)$.

  1. If $f(x) \in C$, then $x \in f^{-1}(C)$.

  2. If $f(x) \in D$, then $x \in f^{-1}(D)$.

  Therefore, $x \in f^{-1}(C) \cup f^{-1}(D)$.

(ii) Let $x \in f^{-1}(C) \cup f^{-1}(D)$. Then either $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$.

  1. If $x \in f^{-1}(C)$, then $f(x) \in C$.

  2. If $x \in f^{-1}(D)$, then $f(x) \in D$.

  Therefore, $f(x) \in (C \cup D)$, so $x \in f^{-1}(C \cup D)$.

This means that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$ and $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$, which can only happen if the two sets are equal.

Problem 4.

Let $A = \{1, 2, ..., n\}$.

(a) Show that the cardinality of $\mathcal{P}(A)$ is 2^n.

(b) How many functions are there from A to A? Explain.

(c) How many onto (surjective) functions are there from A to A? Explain.

Problem 5.

Give an example of a countably infinite collection of infinite sets $A_1, A_2, A_3, \ldots$ with $A_j \cap A_k$ being inifinite for all j and k such that $\bigcap_{j=1}^{\infty} A_j$ is nonempty and finite.

Let $A = \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$ be the set of all nonnegative integers. For each $i \in \mathbb{N}$, define $A_i = \{a \in A \mid i \text{ divides } a\}$. That is, the A_i is the set of all nonnegative integers that are divisible by i. Then:

$$\begin{aligned}
A_1 &= \{0, 1, 2, 3, \ldots\} = A \\
A_2 &= \{0, 2, 4, 6, \ldots\} \\
A_3 &= \{0, 3, 6, 9, \ldots\} \\
A_4 &= \{0, 4, 8, 12, \ldots\} \\
A_5 &= \{0, 5, 10, 15, \ldots\} \\
&\vdots
\end{aligned}$$

(i) First, we'll show that $A_j \cap A_k$ is infinite for all j and k. For any arbitrary $j, k \in \mathbb{N}$;

$$(A_j \cap A_k) = \{0, jk, 2 \cdot jk, 3 \cdot jk, \ldots\} = \{n \cdot jk \mid n \in \mathbb{Z}_{\geq 0}\}.$$

Since $\mathbb{Z}_{\geq 0}$ is infinite, $(A_j \cap A_k)$ is also infinite.

(ii) We'll then show that $\bigcap_{j=1}^{\infty} A_j$ is nonempty yet finite.

1. $\bigcap_{j=1}^{\infty} A_j$ is nonempty. Since $0 \in A_j$ for all $j \in \mathbb{N}$, then $0 \in \bigcap_{j=1}^{\infty} A_j$.

2. $\bigcap_{j=1}^{\infty} A_j$ is finite. Precisely, $\bigcap_{j=1}^{\infty} A_j = \{0\}$. Suppose, for contradiction, that $\bigcap_{j=1}^{\infty} A_j$ contains a non-zero element x. This implies that x is in every A_j, thus x is divisible by every positive integer j. Take $y = 2x$, then clearly $\frac{x}{y} = \frac{x}{2x} = \frac{1}{2}$, which is not an integer, so x is not divisible by y, contradicting the assumption that x is divisible by every positive integer.

Problem 6.

Prove that a function is invertible if and only if it is a bijection.

Definition 6.1. A function $f : A \to B$ is *invertible* if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

For the sake of contradiction, suppose that $f : A \to B$ is invertible but not a bijection. Let $g : B \to A$ be its inverse.

(i) Suppose f is not an injection. Then there exists some $a_1, a_2 \in A$, $a_1 \neq a_2$, and some $b \in B$ such that $f(a_1) = f(a_2) = b$. Since g is the inverse of f, $g(f(x)) = x$ for all $x \in A$, which implies that $g(b) = a_1$ and $g(b) = a_2$. However, a function cannot map an element to two different elements, so either f is an injection or it must not be invertible.

(ii) Suppose f is not a surjection. Then there exists some $b \in B$ such that $f(a) \neq b$ for all $a \in A$. However, g, the inverse of f, acts from B to A, so it must assign some element $a' \in A$ to b. This implies that $g(b) = a'$ for some $a' \in A$. But g was taken to be the inverse of f, so it must map $f(a') \mapsto a'$, where $f(a') \neq b$. Therefore g is not an injection (and, by the previous result, not invertible). This contradicts the assumption that $g = f^{-1}$ as that would imply that g is also invertible (with f being its inverse).

Problem 7.

Let $\mathbb{F}$ be a field. Prove that for any $a \in \mathbb{F}$ there exists a unique $b \in \mathbb{F}$ such that $a + b = 0$.

Definition 7.1. A *field* is a set $\mathbb{F}$ with two binary operations $+$ and $\times$ such that:

- $(\mathbb{F}, +)$ is an abelian group with identity 0.
- $(\mathbb{F} \setminus \{0\}, \times)$ is an abelian group with identity 1.
- $\times$ distributes over $+$.

Useful results in a field $\mathbb{F}$:

- $0 \times a = a \times 0 = 0$ for all $a \in \mathbb{F}$.

(i) First, we show that for any element $a \in \mathbb{F}$, there exists some element $b \in \mathbb{F}$ such that $a + b = 0$. Let $a \in \mathbb{F}$ be arbitrary. Then:

$$
\begin{aligned}
a \times 0 &= 0 \\
a \times (1 + (-1)) &= 0 \\
a \times 1 + a \times (-1) &= 0 \\
a + (-a) &= 0
\end{aligned}
$$

(ii) Next, we show that this element b is unique by demonstrating that if any two elements $b_1, b_2 \in \mathbb{F}$ satisfy $a+b_1 = 0$ and $a + b_2 = 0$, then $b_1 = b_2$. Let $b_1, b_2 \in \mathbb{F}$ be selected as above. Then:

$$
\begin{aligned}
b_1 &= b_1 + 0 \\
&= b_1 + (a + b_2) \\
&= (b_1 + a) + b_2 \\
&= 0 + b_2 \\
&= b_2
\end{aligned}
$$