Math 63: Real Analysis **Winter 2024**

PSET 8 — 02/28/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Prove that $\int_0^1 f(x)\,\mathrm{d}x = 0$ if $f(\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$ and $f(x) = 0$ for all other x.

Claim 1.1. *f is Rieman integrable on $[0, 1]$.*

Proof. Since $f = 0$ at all points $x \in [0, 1] \setminus \{1/n : n \in \mathbb{N}\}$, f is continuous at all such x. Therefore, we can consider the points of the form $1/n, n \in \mathbb{N}$ as discontinuities of f. However, since $\mathbb{N}$ has measure zero (since it is countable), $\{1/n : n \in \mathbb{N}\}$ also has measure zero. By the Lebesgue criterion for Riemann integrability, f is Riemann integrable, so $\int_0^1 f(x)\,\mathrm{d}x$ exists. □

Claim 1.2. $\int_0^1 f(x)\,dx = 0.$

Proof. Given a partition P_n of $[0, 1]$ into n subintervals,

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \le \int_0^1 f(x)\,\mathrm{d}x \le \sum_{i=1}^{n} M_i \Delta x_i = U(f, P),$$

where $m_i = \textbf{inf}\,\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i = \textbf{sup}\,\{f(x) : x \in [x_{i-1}, x_i]\}$. Furthermore;

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0 \tag{1.3}$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i. \tag{1.4}$$

Since the set of points where f is nonzero has measure zero, as we make the partitions finer and finer, $M_i \Delta x_i$ will either be zero or approach 0 for all i, so $U(f, P)$ will also approach 0. On the other hand, $L(f, P)$ will always be 0. Thus, we can make $U(f, P) - L(f, P)$ arbitrarily small, so $\int_0^1 f(x)\,\mathrm{d}x = 0$. □

Problem 2.

Prove that if f is a continuous real-valued function on the interval $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $f(x) > 0$ for some $x \in [a, b]$, then $\int_a^b f(x)\,\mathrm{d}x > 0$.

Claim 2.1. $\int_a^b f(x)\,dx > 0$.

Proof. First, note that f is Riemann integrable on $[a, b]$ since it is continuous on $[a, b]$ and $[a, b]$ is a closed interval. We are given that $f(x) > 0$ for some $x \in [a, b]$. Since f is continuous over $[a, b]$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x_1, x_2 \in [a, b]$, if $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \varepsilon$. Let $y = \mathbf{inf}\,\{f(x) \mid x \in [a, b]\}$, with ξ as the corresponding value for x. Since $f > 0$ at some point over $[a, b]$, $y > 0$. Pick $\varepsilon = y/2$, and pick δ as above. Then, for all $x \in [\xi - \delta, \xi + \delta] \cap [a, b]$, $f(x) > y - \varepsilon = y/2 > 0$. Therefore,

$$\text{Therefore, } \int_{\mathbf{max}(a,\xi-\varepsilon)}^{\mathbf{min}(b,\xi+\varepsilon)} f(x)\,\mathrm{d}x > 0.$$

Since $f(x) \geq 0$ for all $x \in [a, b]$, and $[a, b]$ is a superset of $[\mathbf{max}(a, \xi - \varepsilon), \mathbf{min}(b, \xi + \varepsilon)]$, it follows that

$$\int_a^b f(x)\,\mathrm{d}x \geq \int_{\mathbf{max}(a,\xi-\varepsilon)}^{\mathbf{min}(b,\xi+\varepsilon)} f(x)\,\mathrm{d}x > 0.$$

Therefore, $\int_a^b f(x)\,\mathrm{d}x > 0$. $\square$

Problem 3.

Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (i.e. not rational).} \\ \frac{1}{q} & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0. \end{cases}$$

Show that $\int_0^1 f(x)\,\mathrm{d}x$ exists and is equal to 0.

Hint: Use the Lebesgue criterion for integrability. In particular, you need to determine at what points f is continuous.

Claim 3.1. *$\int_0^1 f(x)\,dx$ exists.*

Proof. We are given that f is only nonzero for rational numbers of the form p/q, where $p, q \in \mathbb{N}$ and p, q are coprime. Since the rational numbers are countable, $\mathbb{Q}$ has Lebesgue measure zero, meaning that the set of points where f is nonzero, which is a subset of $\mathbb{Q}$, also has measure zero. By the Lebesgue criterion, f is Riemann integrable on $[0, 1]$ since the set of points where f is discontinuous has measure zero, so $\int_0^1 f(x)\,\mathrm{d}x$ exists. □

Claim 3.2. *$\int_0^1 f(x)\,dx = 0$.*

Proof. Given a partition P_n of $[0, 1]$ into n subintervals,

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \le \int_0^1 f(x)\,\mathrm{d}x \le \sum_{i=1}^{n} M_i \Delta x_i = U(f, P),$$

where $m_i = \mathbf{inf}\,\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i = \mathbf{sup}\,\{f(x) : x \in [x_{i-1}, x_i]\}$. Furthermore;

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0 \tag{3.3}$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i. \tag{3.4}$$

Since the set of points where f is nonzero has measure zero, as we make the partitions finer and finer, $M_i \Delta x_i$ will approach 0 for all i, so $U(f, P)$ will also approach 0 (since only a smaller subset will have nonzero M_i). On the other hand, $L(f, P)$ will always be 0. For any $\varepsilon > 0$, take P_ε to be a partition such that $U(f, P_\varepsilon) < \varepsilon$, then $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$. Therefore, $\int_0^1 f(x)\,\mathrm{d}x = 0$. □

Problem 4.

Prove that if the real-valued function f on the interval $[a, b]$ is integrable on $[a, b]$, then so is $|f|$, and

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x.$$

Claim 4.1. *Suppose f is integrable on $[a, b]$, then so is $|f|$.*

Proof. Since f is integrable on $[a, b]$, for any $\varepsilon > 0$, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Since $|a| - |b| \le |a - b|$[a] for all $a, b \in \mathbb{R}$, we have $U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P) < \varepsilon$, so $|f|$ is also integrable on $[a, b]$. □

Claim 4.2. *Given that both f and $|f|$ are integrable on $[a, b]$, then $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.*

Proof. Since f is integrable on $[a, b]$, for any $\varepsilon > 0$, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon,$$

and

$$L(f, P) \le R(f, P) \le U(f, P),$$

with $R(f, P) = \sum_1^n f(c_i)$ for some $c_i \in [a_{i-1}, a_i]$. As $\|P\| \to 0$, $\varepsilon \to 0$, so the Riemann sum $R(f, P)$ converges to the integral;

$$\lim_{\|P\|\to 0} \sum_{i=1}^n f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x \qquad \text{and} \qquad \lim_{\|P\|\to 0} \sum_{i=1}^n |f(c_i)|\,\Delta x_i = \int_a^b |f(x)|\,\mathrm{d}x.$$

Thus, our original claim is equivalent to:

$$\begin{aligned}
\left|\int_a^b f(x)\,\mathrm{d}x\right| &= \left|\lim_{\|P\|\to 0} \sum_{i=1}^n f(c_i)\Delta x_i\right| && \text{(for some } c_i \in [x_{i-1}, x_i]) \\
&\le \lim_{\|P\|\to 0} \sum_{i=1}^n |f(c_i)|\,\Delta x_i && \text{(triangle inequality)} \\
&= \int_a^b |f(x)|\,\mathrm{d}x,
\end{aligned}$$

□

[a] This can be shown using Cauchy-Schwarz inequality:

$$\begin{aligned}
|a| = |(a - b) + b| &\le |a - b| + |b| && \text{(triangle inequality)} \\
|a| - |b| &\le |a - b| && \text{(deduct } |b| \text{ to both sides)}
\end{aligned}$$

Problem 5.

Prove integration by parts. That is, suppose F and G are continuously differentiable functions on $[a, b]$. Then, prove that

$$\int_a^b F(x)G'(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,\mathrm{d}x.$$

Proof. Since F and G are continuously differentiable on $[a, b]$, they are also continuous on $[a, b]$. Thus, by the fundamental theorem of calculus, we have Let $H(x) = F(x)G(x)$, then

$$\begin{aligned} H'(x) &= \frac{d}{dx}[F(x)G(x)] \\ &= F'(x)G(x) + F(x)G'(x) \qquad \text{(by the product rule).} \end{aligned}$$

What happens if we integrate both sides of this equation?

$$\begin{aligned} \int_a^b H'(x)\,\mathrm{d}x &= \int_a^b [F'(x)G(x) + F(x)G'(x)]\,\mathrm{d}x \\ H(b) - H(a) &= \int_a^b F'(x)G(x)\,\mathrm{d}x + \int_a^b F(x)G'(x)\,\mathrm{d}x \end{aligned}$$

Rearranging this gives us

$$H(b) - H(a) - \int_a^b F'(x)G(x)\,\mathrm{d}x = \int_a^b F(x)G'(x)\,\mathrm{d}x$$

Since $H(x) = F(x)G(x)$, we have

$$\int_a^b F(x)G'(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,\mathrm{d}x$$

$\square$

Problem 6.

Let $g, f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on any interval $[a, b] \subset \mathbb{R}$. Is it true that $g \circ f$ is also Riemann integrable on any interval $[a, b] \subset \mathbb{R}$?

<u>Hint:</u> Consider g such that $g(x) = 0$ if $x = 0$ and $g(x) = 1$ if $x \neq 0$, and f as in Problem 3:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (i.e. not rational).} \\ \frac{1}{q} & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0. \end{cases}$$

No, the composition of two Riemann integrable functions is not necessarily Riemann integrable. As a counterexample, consider the functions $g, f : \mathbb{R} \to \mathbb{R}$ as defined anove, and their composition $g \circ f$. Both g and f are Riemann integrable on any interval $[a, b] \subset \mathbb{R}$, as priorly shown. However, Let's look at $g \circ f$:

$$(g \circ f)(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (i.e. not rational).} \\ 1 & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0. \end{cases}$$

Note that $g \circ f$ is not continuous at any point $x \in \mathbb{Q}$, since there exists a rational number between any two distinct irrationals [a] and there exists an irrational number between any two distinct rationals [b]. Consequently, if we take any $0 < \varepsilon < 1$, then there is no value for $\delta > 0$ that satisfies the ε-δ criterion for continuity at any point $x \in \mathbb{R}$ since. Take x_2 to be any number in the interval $(x - \delta, x + \delta)$, then:

1. If $x \in \mathbb{Q}$ and $x_2 \in \mathbb{R} \setminus \mathbb{Q}$, then $(g \circ f)(x) = 1$ and $(g \circ f)(x_2) = 0$, so
$$|(g \circ f)(x) - (g \circ f)(x_2)| = 1 > \varepsilon.$$
2. If $x \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}$, then there exists some irrational number x_3 between x and x_2, then $(g \circ f)(x) = 1$ and $(g \circ f)(x_3) = 0$. Therefore,
$$|(g \circ f)(x) - (g \circ f)(x_3)| = 1 > \varepsilon.$$
3. If $x \in \mathbb{R} \setminus \mathbb{Q}$ and $x_2 \in \mathbb{Q}$, then $(g \circ f)(x) = 0$ and $(g \circ f)(x_2) = 1$, so
$$|(g \circ f)(x) - (g \circ f)(x_2)| = 1 > \varepsilon.$$
4. If $x \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}$, then there exists some irrational number x_3 between x and x_2, then $(g \circ f)(x) = 1$ and $(g \circ f)(x_3) = 0$. Therefore,
$$|(g \circ f)(x) - (g \circ f)(x_3)| = 1 > \varepsilon.$$

Thus, $g \circ f$ is not continuous at any point $x \in \mathbb{R}$, therefore not Riemann integrable on any interval $[a, b] \subset \mathbb{R}$.

[a] **Proof that there exists a rational number between any two distinct irrationals.**

Let $a, b \in \mathbb{R} \setminus \mathbb{Q}$ with $a < b$. Let $c = b - a$. By the properties of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > 1/c$, which implies that $cn > 1$. Since we took $c = b - a$, this implies that $nb - na > 1$. Therefore, there exists some integer N such that $na < N < nb$. Dividing by n, we get $a < N/n < b$. Thus, N/n is a rational number between a and b.

[b] **Proof that there exists an irrational number between any two distinct rationals.**

Let $a, b \in \mathbb{Q}$ with $a < b$. Then $b - a > 0$, $b - a > \frac{b-1}{\sqrt{2}}$, and $\frac{b-a}{\sqrt{2}} \notin \mathbb{Q}$. Therefore $a + \frac{b-a}{\sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$, and it is contained in the interval (a, b).