Math 63: Real Analysis **Winter 2024**

PSET 7 — 02/21/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Let $a, b \in \mathbb{R}$, $a < b$, and let f, g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b). Prove that there exists a point $c \in (a, b)$ such that

$$f'(c)\,(g(b) - g(a)) = g'(c)\,(f(b) - f(a))$$

Hint: Consider the function

$$F(x) = (f(x) - f(a))\,(g(b) - g(a)) - (g(x) - g(a))\,(f(b) - f(a))$$

Consider the function $F(x)$ as suggested in the hint. We have $F(a) = F(b) = 0$ because;

1. At $x = a$, the terms $f(x) - f(a)$ and $g(x) - g(a)$ become zero, so

$$F(a) = (f(x) - f(a))\,(g(b) - g(a)) \quad - \quad (g(x) - g(a))\,(f(b) - f(a)) \quad = \quad 0.$$

2. At $x = b$, $(f(x) - f(a)) = (f(b) - f(a))$ and $(g(x) - g(a)) = (g(b) - g(a))$, so

$$F(b) = (f(b) - f(a))(g(b) - g(a)) \quad - \quad (g(b) - g(a))(f(b) - f(a)) \quad = \quad 0.$$

Now, we can apply Rolle's Theorem, which states that if a function is continuous on $[a, b]$ and differentiable on (a, b), and $F(a) = F(b)$, then there exists some $\sigma \in (a, b)$ such that $F'(\sigma) = 0$. Differentiating $F(x)$, we get

$$F'(x) = f'(x)\,(g(b) - g(a)) - g'(x)\,(f(b) - f(a))$$

Setting $F'(\sigma) = 0$ gives $f'(\sigma)\,(g(b) - g(a)) - g'(\sigma)\,(f(b) - f(a)) = 0$, which implies that

$$f'(\sigma)\,(g(b) - g(a)) = g'(\sigma)\,(f(b) - f(a))\,.$$

Therefore, there exists a point $c := \sigma \in (a, b)$ such that

$$f'(c)\,(g(b) - g(a)) = g'(c)\,(f(b) - f(a))\,.$$

Problem 2.

Use Problem 1 (Cauchy Mean Value Theorem) to prove L'Hôpital's Rule:

Let $U = (a, b) \subset \mathbb{R}$, and let f and g be differentiable real-valued functions on U, with g and g' nowhere zero on U. Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a+} g(x) = 0$. Then,

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(x)}{g'(x)}$$

if the limit exists.

Given that f and g are differentiable on $U = (a, b)$ and g' is nowhere zero on U, and that $\lim_{x\to a} f(x) = \lim_{x\to a+} g(x) = 0$, we aim to prove $\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(x)}{g'(x)}$ if the limit exists.

Consider a sequence $\{x_n\}$ in U converging to a from the right. For each n, by the Cauchy Mean Value Theorem, there exists $c_n \in (a, x_n)$ such that

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x_n) - f(a)}{g(x_n) - g(a)}.$$

As $n \to \infty$ and $x_n \to a+$, we have:

(i) As $n \to \infty$, $x_n \to a+$; so $c_n \to a+$.

(ii) Thus, $f(c_n) \to f(a) = 0$ and $g(c_n) \to g(a) = 0$.

$f(x_n) \to 0$ and $g(x_n) \to 0$, so the right-hand side becomes $\frac{f(a)}{g(a)}$. Thus, we have

$$\lim \frac{f'(c_n)}{g'(c_n)} = \frac{f(a)}{g(a)}.$$

Thus, we can rewrite $\frac{f(a)}{g(a)} = \lim_{c_n\to a+} \frac{f(c_n)}{g(c_n)}$. This means that

$$\lim_{n\to\infty} \frac{f'(c_n)}{g'(c_n)} = \frac{f(a)}{g(a)}$$

implies that

$$\lim_{x\to a+} \frac{f'(x)}{g'(x)} = \lim_{x\to a+} \frac{f(x)}{g(x)}$$

if the limit exists.

Problem 3.

Use Taylor's Theorem to prove the "binomial theorem" for $n \in \mathbb{N}$:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \cdots + x^n.$$

We will use Taylor's Theorem to prove the binomial theorem for $n \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^n$.

1. First, consider the value of f at $x = a$:

$$f(a) = a^n$$

2. Next, consider the derivatives of f:

$$\begin{aligned}
f'(x) &= n(x)^{n-1} && = \frac{n}{(n-1)!}(x)^{n-1} \\
f''(x) &= n(n-1)(x)^{n-2} && = \frac{n}{(n-2)!}(x)^{n-2} \\
f'''(x) &= n(n-1)(n-2)(x)^{n-3} && = \frac{n}{(n-3)!}(x)^{n-3} \\
&\vdots \\
f^{(k)}(x) &= n(n-1)\cdots(n-k+1)(x)^{n-k} && = \frac{n!}{(n-k)!}(x)^{n-k} \\
&\vdots \\
f^{(n)}(x) &= n! && = n!
\end{aligned}$$

3. Now, consider the Taylor expansion of $f(x)$ about a point $a \in \mathbb{R}$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

4. Plugging in $f = (x)^n$ into the Taylor expansion gives, with center a and point $a + x$:

$$f(a+x) = (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}(a+a)^{n-2}x^2 + \cdots + x^n,$$

which is the binomial theorem for $n \in \mathbb{N}$.

Problem 4.

Let f, g, f_n be real-valued functions on $[a, b] \subset \mathbb{R}$ for all $n \in \mathbb{N}$. Assume $f_n \in C^1([a, b])$ for all $n \in \mathbb{N}$. Suppose that $f_n \to f$ pointwise and $f_n' \rightrightarrows g$ uniformly as $n \to \infty$. Show that $f \in C^1([a, b])$, $f' = g$, and $f_n' \rightrightarrows f'$ uniformly as $n \to \infty$.

We are given that $f_n \to f$ pointwise and $f_n' \rightrightarrows g$ uniformly as $n \to \infty$. We need to show that $f \in C^1([a, b])$, $f' = g$, and $f_n' \rightrightarrows f'$ uniformly as $n \to \infty$.

1. First, we show that $f \in C^1([a, b])$ and $f' = g$.

Since $f_n \in C^1([a, b])$ for all $n \in \mathbb{N}$, each f_n is differentiable with a continuous derivative. The uniform convergence of $f_n' \rightrightarrows g$ and the pointwise convergence of $f_n \to f$ implies that f is differentiable and $f' = g$. This follows from the fact that the uniform limit of continuous functions is continuous (hence g is continuous), and the limit of the derivatives is the derivative of the limit function.

2. Next, we show that $f_n' \to f'$ uniformly as $n \to \infty$.

We are given that $f_n' \to g$ uniformly as $n \to \infty$ and, from Part **1.**, we have established that $f' = g$.

Therefore, $f_n' \to f'$ uniformly.

Do I need to show more for this? I wasn't sure.

Problem 5.

Compute $\int_0^1 x \, \mathrm{d}x$ directly from the definition of the Riemann integral.

Hint: Consider the partition $0 = x_0 < x_1 < \ldots < x_n = 1$ where $x_i = \frac{i}{n}$ for $i = 0, 1, \ldots, n$.

Consider the upper Riemann integral of the function $f(x) = x$ over the interval $[0, 1]$;

$$I(x) = \sum_{i=1}^{n} S(x_i)\Delta x_i, \qquad \text{where} \qquad S(x_i) = \mathbf{sup}\,\{f(x) \mid x \in [x_{i-1}, x_i]\} \qquad \text{and} \qquad \Delta x_i = x_i - x_{i-1}.$$

Given $f(x) = x$, we have $S(x_i) = \mathbf{sup}\,\{x \mid x \in [x_{i-1}, x_i]\} = x_i$ for all $x \in [x_{i-1}, x_i]$. Thus, $I(x) = \sum_{i=1}^{n} x_i \Delta x_i$. Using the partition $0 = x_0 < x_1 < \ldots < x_n = 1$ where $x_i = \frac{i}{n}$ for $i = 0, 1, \ldots, n$, we have $\Delta x_i = \frac{1}{n}$ for all i, so

$$\begin{aligned} I(x) = \sum_{i=1}^{n} x_i \cdot \frac{1}{n} &= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2} \sum_{i=1}^{n} i \\ &= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \\ &= \frac{n+1}{2n}. \end{aligned}$$

As we make the partition finer and finer, the limit of n goes to infinity. We then have:

$$\int_0^1 x \, \mathrm{d}x = \lim_{n \to \infty} I(x) = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2}.$$