

PSET 7 — 02/21/2024

Prof. Erchenko

Student: Amittai Siavava

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

- (a) *Introduction to Analysis* by Maxwell Rosenlicht

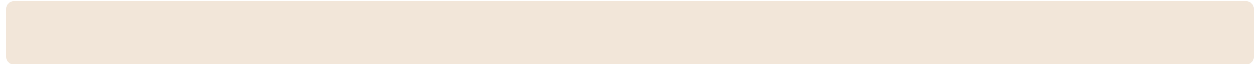
Problem 1.

Let $a, b \in \mathbb{R}$, $a < b$, and let f, g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b) . Prove that there exists a number $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

Hint: Consider the function

$$F(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a))$$



Problem 2.

Use Problem 1 (Cauchy Mean Value Theorem) to prove L'Hôpital's Rule:

Let $U = (a, b) \subset \mathbb{R}$, and let f and g be differentiable real-valued functions on U , with g and g' nowhere zero on U . Suppose that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a+} g(x) = 0$. Then,

$$\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a+} \frac{f'(x)}{g'(x)}$$

if the limit exists.



Problem 3.

Use Taylor's Theorem to prove the “binomial theorem” for $n \in \mathbb{N}$:

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \cdots + x^n.$$

Problem 4.

Let f, g, f_n be real-valued functions on $[a, b] \subset \mathbb{R}$ for all $n \in \mathbb{N}$. Assume $f_n \in C^1([a, b])$ for all $n \in \mathbb{N}$. Suppose that $f_n \rightarrow f$ pointwise and $f'_n \rightarrow g$ uniformly as $n \rightarrow \infty$. Show that $f \in C^1([a, b])$, $f' = g$, and $f'_n \rightarrow f'$ uniformly as $n \rightarrow \infty$.

Problem 5.

Compute $\int_0^1 x \, dx$ directly from the definition of the Riemann integral.

Hint: Consider the partition $0 = x_0 < x_1 < \dots < x_n = 1$ where $x_i = \frac{i}{n}$ for $i = 0, 1, \dots, n$.