Math 63: Real Analysis **Winter 2024**

PSET 1 — 01/07/2024

Prof. Erchenko ***Student: Amittai Siavava***

Problems

Problem 1.

Let A, B, C be subsets of a set S. Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in (B \cap C)$.

1. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$.
2. If $x \in (B \cap C)$, then x is in **both** B and C.
   Therefore, $x \in (A \cup B)$ and $x \in (A \cup C)$.
   Therefore, $x \in (A \cup B) \cap (A \cup C)$.

(ii) Let $x \in (A \cup B) \cap (A \cup C)$.

Then $x \in (A \cup B)$ and $x \in (A \cup C)$. Since $x \in (A \cup B)$, either $x \in A$ or $x \in B$. Similarly, since $x \in (A \cup C)$, either $x \in A$ or $x \in C$. Thus, either $x \in A$ or x is in **both** B and C.

1. If $x \in A$, then $x \in A \cup (B \cap C)$.
2. If x is in **both** B and C, then $x \in (B \cap C)$, so $x \in A \cup (B \cap C)$.

Problem 2.

If A, B, C are sets, show that

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in (A \cup B)$ and $x \notin (A \cap B)$.

  1. If $x \in A$, then $x \in (A \setminus B)$.

  2. If $x \in B$, then $x \in (B \setminus A)$.

 Therefore, $x \in (A \setminus B) \cup (B \setminus A)$.

(ii) Let $x \in (A \setminus B) \cup (B \setminus A)$. Then either $x \in (A \setminus B)$ or $x \in (B \setminus A)$.

  1. If $x \in (A \setminus B)$, then $x \in A$ and $x \notin B$. Therefore, $x \in (A \cup B)$ and $x \notin (A \cap B)$.

  2. If $x \in (B \setminus A)$, then $x \in B$ and $x \notin A$. Therefore, $x \in (A \cup B)$ and $x \notin (A \cap B)$.

 Therefore, $x \in (A \cup B) \setminus (A \cap B)$.

Problem 3.

Let $f : X \to Y$ be a function. Let C and D be subsets of Y. Prove that

$$f^{-1}(X \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

To show that the two sets are equal, we will show that each set is a subset of the other. That is, any arbitrary element in the first set must be in the second set, and vice versa.

(i) Let $x \in f^{-1}(X \cup D)$. Then $f(x) \in X \cup D$.

  1. If $f(x) \in X$, then $x \in f^{-1}(X)$.

  2. If $f(x) \in D$, then $x \in f^{-1}(D)$.

  Therefore, $x \in f^{-1}(C) \cup f^{-1}(D)$.

(ii) Let $x \in f^{-1}(C) \cup f^{-1}(D)$. Then either $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$.

  1. If $x \in f^{-1}(C)$, then $f(x) \in C$. Therefore, $f(x) \in X \cup D$.

  2. If $x \in f^{-1}(D)$, then $f(x) \in D$. Therefore, $f(x) \in X \cup D$.

  Therefore, $x \in f^{-1}(X \cup D)$.

Problem 4.

Let $A = \{1, 2, \ldots, n\}$.

(a) Show that the cardinality of $\mathcal{P}(A)$ is 2^n.

(b) How many functions are there from A to A? Explain.

(c) How many onto (surjective) functions are there from A to A? Explain.

Problem 5.

Give an example of a countably infinite collection of infinite sets $A_1, A_2, A_3, \ldots$ with $A_j \cap A_k$ being inifinite for all j and k such that $\bigcap_{j=1}^{\infty} A_j$ is nonempty and finite.

Let $A = \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$ be the set of all nonnegative integers. For each $i \in \mathbb{N}$, define $A_i = \{a \in A \mid i \text{ divides } a\}$. That is, the A_i is the set of all nonnegative integers that are divisible by i. Then:

$$A_1 = \{0, 1, 2, 3, \ldots\} = A$$
$$A_2 = \{0, 2, 4, 6, \ldots\}$$
$$A_3 = \{0, 3, 6, 9, \ldots\}$$
$$A_4 = \{0, 4, 8, 12, \ldots\}$$
$$A_5 = \{0, 5, 10, 15, \ldots\}$$
$$\vdots$$

(i) First, we'll show that $A_j \cap A_k$ is infinite for all j and k. For any arbitrary $j, k \in \mathbb{N}$;

$$(A_j \cap A_k) = \{0, jk, 2 \cdot jk, 3 \cdot jk, \ldots\} = \{n \cdot jk \mid n \in \mathbb{Z}_{\geq 0}\}.$$

Since $\mathbb{Z}_{\geq 0}$ is infinite, $(A_j \cap A_k)$ is also infinite.

(ii) We'll then show that $\bigcap_{j=1}^{\infty} A_j$ is nonempty yet finite.

1. $\bigcap_{j=1}^{\infty} A_j$ is nonempty. Since $0 \in A_j$ for all $j \in \mathbb{N}$, then $0 \in \bigcap_{j=1}^{\infty} A_j$.

2. $\bigcap_{j=1}^{\infty} A_j$ is finite. Precisely, $\bigcap_{j=1}^{\infty} A_j = \{0\}$. Suppose, for contradiction, that $\bigcap_{j=1}^{\infty} A_j$ contains a non-zero element x. This implies that x is in every A_j, thus x is divisible by every positive integer j. Take $y = 2x$, then clearly $\frac{x}{y} = \frac{x}{2x} = \frac{1}{2}$, which is not an integer, so x is not divisible by y, contradicting the assumption that x is divisible by every positive integer.

Problem 6.

Prove that a function is invertible if and only if it is a bijection.

Problem 7.

Let $\mathbb{F}$ be a field. Prove that for any $a \in \mathbb{F}$ there exists a unique $b \in \mathbb{F}$ such that $a + b = 0$.

Definition 7.1. A *field* is a set $\mathbb{F}$ with two binary operations $+$ and $\times$ such that:

- $(\mathbb{F}, +)$ is an abelian group with identity 0.
- $(\mathbb{F} \setminus \{0\}, \times)$ is an abelian group with identity 1.
- $\times$ distributes over $+$.

Useful results in a field $\mathbb{F}$:

- $0 \times a = a \times 0 = 0$ for all $a \in \mathbb{F}$.

(i) First, we show that for any element $a \in \mathbb{F}$, there exists some element $b \in \mathbb{F}$ such that $a + b = 0$. Let $a \in \mathbb{F}$ be arbitrary. Then:

$$\begin{aligned} a \times 0 &= 0 \\ a \times (1 + (-1)) &= 0 \\ a \times 1 + a \times (-1) &= 0 \\ a + (-a) &= 0 \end{aligned}$$

(ii) Next, we show that this element b is unique by demonstrating that if any two elements $b_1, b_2 \in \mathbb{F}$ satisfy $a+b_1 = 0$ and $a + b_2 = 0$, then $b_1 = b_2$. Let $b_1, b_2 \in \mathbb{F}$ be selected as above. Then:

$$\begin{aligned} b_1 &= b_1 + 0 \\ &= b_1 + (a + b_2) \\ &= (b_1 + a) + b_2 \\ &= 0 + b_2 \\ &= b_2 \end{aligned}$$