Math 63: Real Analysis **Winter 2024**

PSET 8 — 02/28/2024

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***Introduction to Analysis*** by Maxwell Rosenlicht

Problem 1.

Prove that $\int_0^1 f(x)\,\mathrm{d}x = 0$ if $f\left(\frac{1}{n}\right) = 1$ for all $n \in \mathbb{N}$ and $f(x) = 0$ for all other x.

Problem 2.

Prove that if f is a continuous real-valued function on the interval $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $f(x) > 0$ for some $x \in [a, b]$, then $\int_a^b f(x)\, \mathrm{d}x > 0$.

Problem 3.

Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (i.e. not rational).} \\ \frac{1}{q} & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime.} \end{cases}$$

Show that $\int_0^1 f(x)\,\mathrm{d}x$ exists and is not equal to 0.

Hint: Use the Lebesgue criterion for integrability. In particular, you need to determine at what points f is continuous.

Problem 4.

Prove that if the real-valued function f on the interval $[a, b]$ is integrable on $[a, b]$, then so is f, and

$$\left| \int_a^b f(x) \, \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x.$$

Problem 5.

Prove integration by parts. That is, suppose F and G are continuously differentiable functions on $[a, b]$. Then, prove that

$$\int_a^b F(x)G'(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,\mathrm{d}x.$$

Problem 6.

Let $g, f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on any interval $[a, b] \subset \mathbb{R}$. Is it true that $g \circ f$ is also Riemann integrable on any interval $[a, b] \subset \mathbb{R}$?

Hint: Consider g such that $g(x) = 0$ if $x = 0$ and $g(x) = 1$ if $x \neq 0$, and f as in Problem 3.