CS 39: Theory of Computation	Winter '23

PSET 1 — 2023-01-15

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) **Introduction to the Theory of Computation** by **Michael Sipser**.

(b) **A Mathematical Introduction to Logic** by **Herbert Enderton**.

Problem 1.

For a language L over alphabet Σ, define

$$\textsc{Cycle}(L) = \{yx : x, y \in \Sigma^* \text{ and } xy \in L\},$$

$$\textsc{Half}(L) = \{x \in \Sigma^* : \exists y \in \Sigma^* (|x| = |y| \text{ and } xy \in L)\},$$

$$\textsc{HalfPalindrome}(L) = \left\{x \in \Sigma^* : xx^R \in L\right\}.$$

(a) Let $A = \{(01)^n : n \geq 0\}$ and $B = \{0^n 1^n : n > 0\}$. What is $\textsc{Cycle}$(A) and $\textsc{Cycle}$(B)?

$$\textsc{Cycle}(A) = \{(01)^m : m \geq 0\} \cup \{(10)^n : n \geq 0\}$$

$$\textsc{Cycle}(B) = \{0^m 1^m : m \geq 0\} \cup \{1^n 0^n : n \geq 0\}$$

(b) Let $C = \{0^p 1 6 q 0^r : p, q, r \geq 0 \text{ and } q = p + r\}$. What is $\textsc{Half}$(C) and $\textsc{HalfPalindrome}$(C)?

Problem 2.

A string u is a *proper prefix* of a string v if u is a prefix of v and $u \neq v$. For a language L over alphabet Σ, define

$$\textsc{Min}(L) = \{x \in \Sigma^* : x \in L \text{ and no proper prefix of } x \in L\},$$

$$\textsc{Max}(L) = \{x \in \Sigma^* : x \in L \text{ and } x \text{ is not a proper prefix of any string in } L\}.$$

(a) Let $A = \{0^m 1^n : m, n \geq 0\}$. What is $\textsc{Min}(A)$ and $\textsc{Max}(A)$?

(b) Let $B = 0^m 1^n, m, n > 0$. Read carefully; $B \neq A$. What is $\textsc{Min}(B)$ and $\textsc{Max}(B)$?

(c) Specify an infinite language L such that $\textsc{Min}(L) = \textsc{Max}(L)$.

Problem 3.

For each of the languages below over the alphabet $\Sigma = \{0, 1\}$, specify a simple DFA by drawing a state diagram. Simplicity is important here: don't design unnecessarily complicated DFAs.

(a) The language that accepts only the empty string.

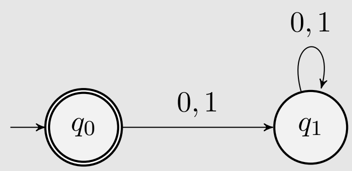

FIGURE 3.1. Only accept the empty string.

(b) The set of all strings that in Σ^* except 100 and 101.

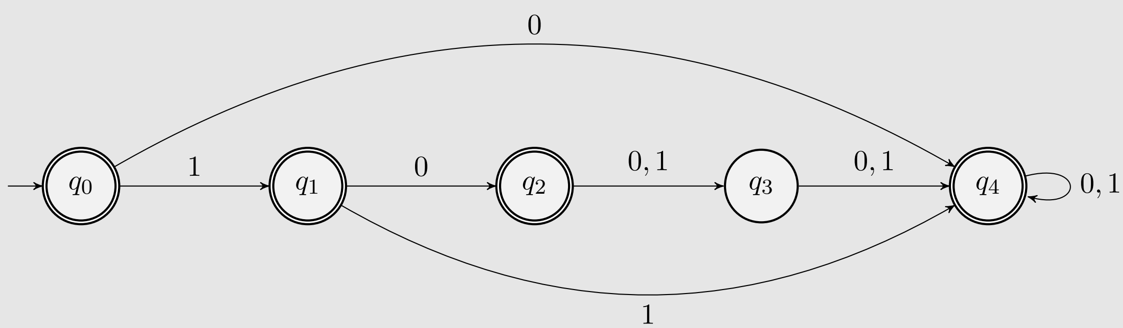

FIGURE 3.2. Accept all strings in $\{0, 1\}^*$ except 100 and 101.

(c) $\{x \in \Sigma^* : x \text{ does not contain the substring } 110\}$.

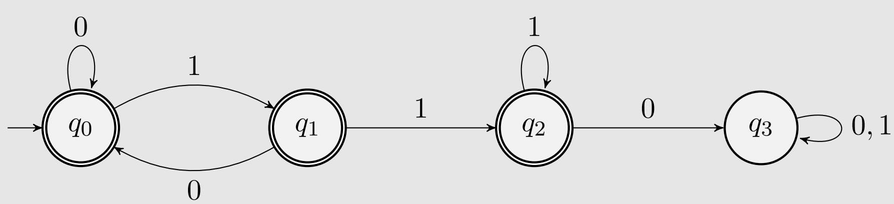

FIGURE 3.3. Accept all strings in $\{0, 1\}^*$ except those containing 110.

Problem 4.

Draw a state diagram of a DFA that recognizes the language consisting of all strings in $\{0,1\}^*$ such that each string is of length at least three and every block of three consecutive symbols has at least one 0.
(Thus, for example, 0011001 *is in the language, but* 0011100 *is not.)*

Explain, in one or two sentences, the idea behind your DFA construction

First, let's build a DFA to accept all strings of length at least 3.

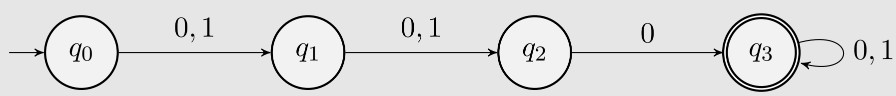

Figure 4.1. Accept all strings of length at least 3.

Next, let's build a DFA to accept all strings with at least one 0 in each block of 3 letters.

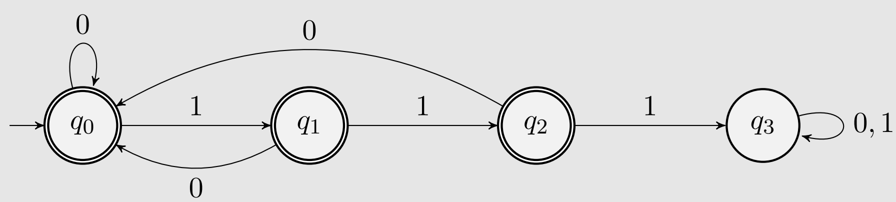

Figure 4.2. Accept all strings with at least one 0 in each block of 3.

Next, let's combine the two into a single SFA that accepts all strings that are of length at least 3 *and* have at least one 0 in each block of 3.

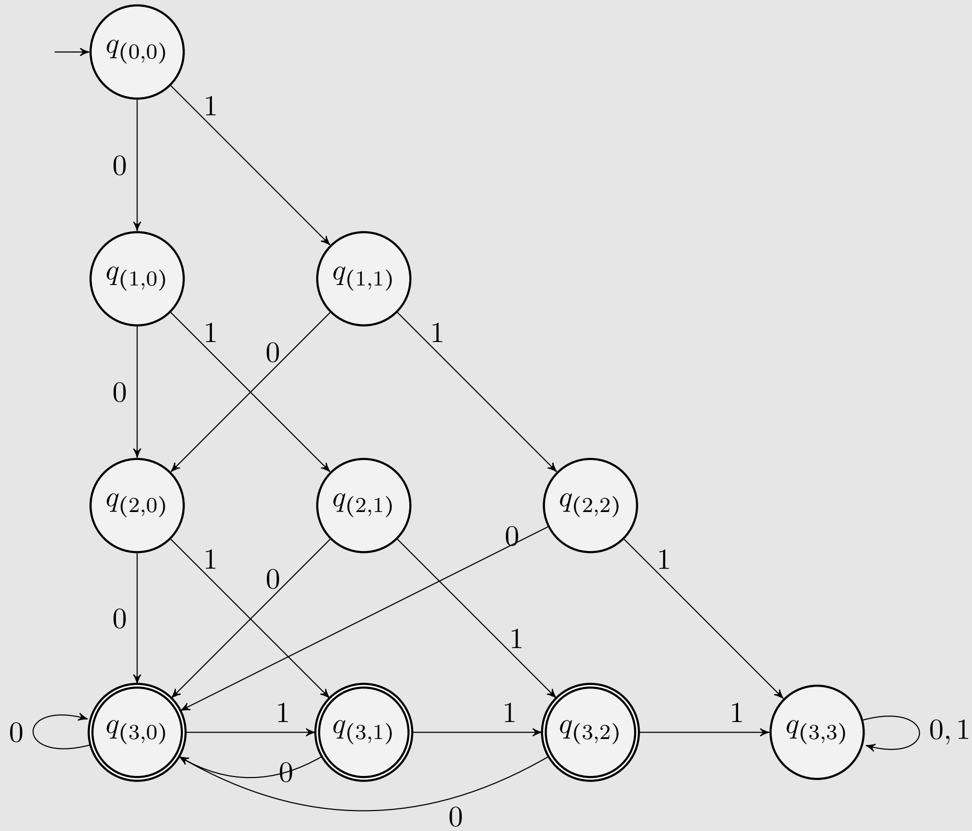

FIGURE 4.3. Accept all strings with length at least 3 and at least one 0 in each block of 3.

DFA Specification

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F) \text{ where:}$$

$$Q = \left\{q_{(i,j)} : 0 \le i \le 3, 0 \le j \le i\right\}$$

$$\Sigma = \{0, 1\}$$

$$\delta(q_{(i,j)}, x) = \begin{cases} q_{(i+1,\ 0)} & \text{if } i < 3 \text{ and } x = 0 \\ q_{(i+1,\ j+1)} & \text{if } i < 3 \text{ and } x = 1 \\ q_{(i,\ 0)} & \text{if } i = 3 \text{ and } j = 0 \\ q_{(i,j+1)} & \text{otherwise} \end{cases}$$

$$q_0 = q_{(0,0)}$$

$$F = \left\{q_{(i,\ j)} : i = 3,\ 0 \le j < 3\right\}$$

Problem 5.

For a string $x \in 0,1^*$, let $\beta(x)$ denote the integer obtained by interpreting x as a binary number. Thus, for example, $\beta(11001) = 25$ and $\beta(0011) = 3$. We also define $\beta(\epsilon) = 0$ for convenience.

Design a DFA for the language $L = \left\{x \in \{0,1\}^* : \beta(x) \text{ is divisible by } 5\right\}$. *Draw the state diagram and also specify the DFA formally.*

Hint: For integers m, n, p, we have $(mn + p) \mod 5 = ((m \mod 5) \cdot n + p) \mod 5$

State Diagram

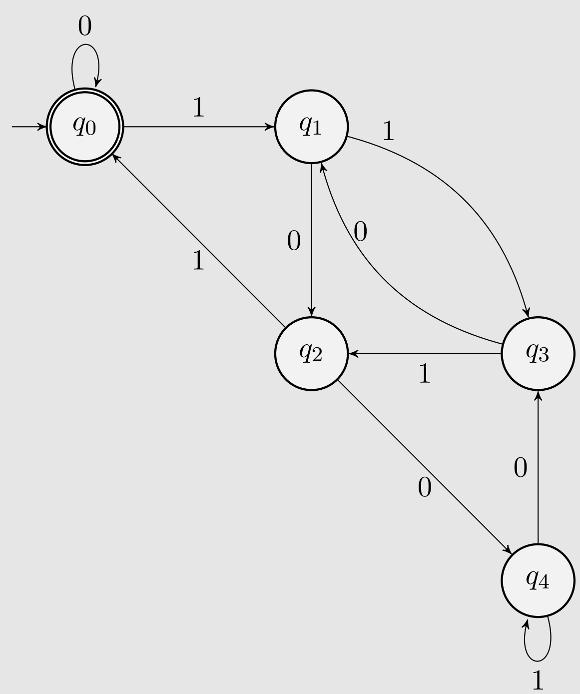

FIGURE 5.1. Accept all strings x where $\beta(x) \equiv 0 \pmod 5$.

DFA Specification

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F) \text{ where:}$$

$$Q = \{q_i : 0 \leq i < 5\}$$

$$\Sigma = \{0, 1\}$$

$$\delta(q_i, x) = q_j \text{ where } j = (2i + x) \mod 5$$

$$q_0 = q_0$$

$$F = \{q_0\}$$

Problem 6.

Formally specify a DFA for the set of all strings in $\Sigma = \{0, 1\}^*$ such that each string is of length at least 2000 and every block of 2000 consecutive symbols has at least ten 0s. Explain the design in one or two sentences.