Math 29: Computability Theory

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PSET 7 — 05/24/2024

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Problem 1.

Are there sets A and B such that $A' \leq_T B'$ but $A \not\leq_T B$? Justify your answer.

Yes.

Pick $B = \emptyset$ — which is computable — and let A be any non-computable set that is **low**, meaning $\emptyset' \equiv_T A'$ As we showed in class using the Low Basis Theorem and Sacks' Splitting Theorem, such sets exist (and in fact, there are many of them). Then:

- **1.** Since B is computable but A is non-computable, $A \not\leq_T B$.
- **2.** However, $A' = \emptyset'$ and $B' \equiv_T \emptyset'$, meaning $A' \equiv_T B'$. Therefore, $A' \leq_T B'$.

Problem 2.

(a) Prove that if there is $g \leq_T X$ such that $\varphi_{g(x)} \neq \varphi_x$, then there is $h \leq_T X$ such that $h(e) \neq \varphi_e(e)$ for all e.

This means that g is fixed-point free. Define h as follows:

$$h(e) = g(\varphi_e(e)).$$

- 1. First, note that $\varphi_{g(x)} \neq \varphi_x$ means that $g(x) \neq x$ for all x, as that would trivially imply $\varphi_{g(x)} = \varphi_x$ for some x. Therefore, $h(e) = g(\varphi_e(e)) \neq \varphi_e(e)$ for all e.
- 2. Next, we show that $h \leq_T X$. Since $g \leq_T X$, we can use X as an oracle to compute g(e) for any e. Specifically, there exists an oracle machine k that uses X as an oracle to compute g, such that $\Phi_k^X(e) = g(e)$ for all e. We can compute h(e) by simulating $\Phi_k^X(\varphi_e(e))$ and returning the result, thus h can also be computed by an oracle machine that takes X as an oracle and, on input e, simulates Φ_k^X on $\varphi_e(e)$.
- (b) Given $h \leq_T X$ such that $h(e) \neq \varphi_e(e)$ for all e, show that there is $f \leq_T X$ such that $W_{f(e)} \neq W_e$ for all e.

Hint: make $|W_{f(e)}| = 1$.

For each e, define e' such that:

$$\varphi_{e'}(x) = \begin{cases} h(e) & \text{if } x = e \\ \uparrow & \text{otherwise.} \end{cases}$$

This ensures that for any e, $\varphi_{e'}(e) = h(e) \neq \varphi_{e}(e)$, and $W_{e'} = \{e\}$ for all e and corresponding e'.

Finally, define f as follows:

$$f(x) = (x+1)'.$$

- **1.** First, since f(e) = (e+1)', $W_{f(e)} = \{e+1\} \neq \{e\} = W_e$.
- 2. Next, we show that $f \leq_T X$. Since X is an oracle for h, we can use X to compute h(e) for any e. Specifically, there exists an oracle machine k that uses X as an oracle to compute h, such that $\Phi_k^X(e) = h(e)$ for all e. We can compute f as follows:

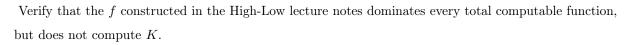
 On input e, construct an oracle machine that, on input x, simulates $\Phi_k^X(x)$ if x = e + 1,

On input e, construct an oracle machine that, on input x, simulates $\Phi_k^{\Lambda}(x)$ if x = e + 1, and otherwise diverges.

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Return the code of this machine as f(e).

Problem 3.



Problem 4.

Prove that not ML-random set has an infinite c.e. subset. (Hint: use a lemma from the class notes).