Math 29: Computability Theory **Spring 2024**

PSET 6 — 05/17/2024

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Problem 3.

Give an example of a set X such that $X \perp_T \varnothing^{(n)}$ for all $n > 1$.

Hint: we are only required to perform (priority) constructions computably.

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1. Start with $X = \varnothing$.

2. Define the requirement $R_{e,n}$:

$$R_{e,n} : \Phi_e^{\varnothing^{(n)}}(x_n) \neq 1$$

3. Define the requirement $Q_{e,n}$ as follows:

$$Q_{e,n} : \Phi_e^{\varnothing^{(n)}}(n) \neq x_n$$

4. For each $n \in 1, 2, 3, \ldots,$

(i) Pick $x' = \mathbf{max}\,\{x_1, x_2, \ldots, x_{n-1}\}$. If $n = 1$, set $x' = 0$.

(ii) Pick x_n to be the smallest $i \in \omega$ such that $x_n > x'$ and both $R_{e,n}$ and $Q_{e,n}$ are satisfied by the selection of x_n, for all $e \in \omega$.

(iii) Add x_n to X.

We claim that $X \perp_T \varnothing^{(n)}$ for all $n > 1$.

1. $X \not\leq_T \varnothing^{(n)}$:

For any $n > 1$, let x_n be the n-th item added to X, such that x_n satisfies $R_{e,n}$ for all $e \in \omega$, meaning $\Phi_e^{\varnothing^{(n)}}(x_n) \neq 1$. This ensures that $\Phi_e^{\varnothing^{(n)}}(x_n) \neq \chi_X(x_n)$.

2. $\varnothing^{(n)} \not\leq_T X$:

For any $n > 1$, let x_n be the n-th item added to X, such that x_n satisfies $Q_{e,n}$ for all $e \in \omega$, meaning $\Phi_e^{\varnothing^{(n)}}(n) \neq x_n$. This ensures that $\Phi_e^{\varnothing^{(n)}}$ cannot compute x_n.

Problem 4.

We say that $X =^* Y$ if X and Y agree on all but finitely many numbers. Show that there are sequences of sets $\{A_n\}_{n\in\omega}$ and $\{B_n\}_{n\in\omega}$ such that $A_n =^* B_n$ for all n, but $\bigoplus_{n\in\omega} A_n \not\equiv_T \bigoplus_{n\in\omega} B_n$.

$$A_n = \{n\}$$

$$B_n = \{e \mid (e = n) \wedge \varphi_n(n)\downarrow\}$$

Then;

1. For any given n, A_n contains only the number n, and B_n either contains n or is empty. Therefore, A_n and B_n disagree on either 1 or 0 numbers.

2. However, $\bigoplus_{n\in\omega} A_n = \{n \mid n \in \omega\} = \omega$, and $\bigoplus_{n\in\omega} B_n = \{n \mid (n \in \omega) \wedge \varphi_n(n)\downarrow\} = K$.
Since ω is computable but K is not computable, A and B have different Turing degrees, so $A \not\equiv_T B$.
