

## PSET 8 — 02/28/2024

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## Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

1. *Introduction to Analysis* by Maxwell Rosenlicht

## Problem 1.

Prove that  $\int_0^1 f(x) \, dx = 0$  if  $f(\frac{1}{n}) = 1$  for all  $n \in \mathbb{N}$  and  $f(x) = 0$  for all other  $x$ .

**Claim 1.1.**  $f$  is Riemann integrable on  $[0, 1]$ .

*Proof.* Since  $f = 0$  at all points  $x \in [0, 1] \setminus \{1/n : n \in \mathbb{N}\}$ ,  $f$  is continuous at all such  $x$ . Therefore, we can consider the points of the form  $1/n, n \in \mathbb{N}$  as discontinuities of  $f$ . However, since  $\mathbb{N}$  has measure zero (since it is countable),  $\{1/n : n \in \mathbb{N}\}$  also has measure zero. By the Lebesgue criterion for Riemann integrability,  $f$  is Riemann integrable, so  $\int_0^1 f(x) \, dx$  exists.  $\square$

**Claim 1.2.**  $\int_0^1 f(x) \, dx = 0$ .

*Proof.* Given a partition  $P_n$  of  $[0, 1]$  into  $n$  subintervals,

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i \leq \int_0^1 f(x) \, dx \leq \sum_{i=1}^n M_i \Delta x_i = U(f, P),$$

where  $m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$  and  $M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$ . Furthermore;

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^n 0 \cdot \Delta x_i = 0$$

and

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i.$$

Since the set of points where  $f$  is nonzero has measure zero, as we make the partitions finer and finer,  $M_i \Delta x_i$  will either be zero or approach 0 for all  $i$ , so  $U(f, P)$  will also approach 0. On the other hand,

$L(f, P)$  will always be 0. Thus, we can make  $U(f, P) - L(f, P)$  arbitrarily small, so  $\int_0^1 f(x) \, dx = 0$ .  $\square$

**Problem 2.**

Prove that if  $f$  is a continuous real-valued function on the interval  $[a, b]$  such that  $f(x) \geq 0$  for all  $x \in [a, b]$  and  $f(x) > 0$  for some  $x \in [a, b]$ , then  $\int_a^b f(x) \, dx > 0$ .

**Claim 2.1.**  $\int_a^b f(x) \, dx > 0$ .

*Proof.* First, note that  $f$  is Riemann integrable on  $[a, b]$  since it is continuous on  $[a, b]$  and  $[a, b]$  is a closed interval. We are given that  $f(x) > 0$  for some  $x \in [a, b]$ . Since  $f$  is continuous over  $[a, b]$ , for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x_1, x_2 \in [a, b]$ , if  $|x_1 - x_2| < \delta$  then  $|f(x_1) - f(x_2)| < \varepsilon$ . Let  $y = \inf \{f(x) \mid x \in [a, b]\}$ , with  $\xi$  as the corresponding value for  $x$ . Since  $f > 0$  at some point over  $[a, b]$ ,  $y > 0$ . Pick  $\varepsilon = y/2$ , and pick  $\delta$  as above. Then, for all  $x \in [\xi - \delta, \xi + \delta] \cap [a, b]$ ,  $f(x) > y - \varepsilon = y/2 > 0$ . Therefore,

$$\text{Therefore, } \int_{\max(a, \xi - \varepsilon)}^{\min(b, \xi + \varepsilon)} f(x) \, dx > 0.$$

Since  $f(x) \geq 0$  for all  $x \in [a, b]$ , and  $[a, b]$  is a superset of  $[\max(a, \xi - \varepsilon), \min(b, \xi + \varepsilon)]$ , it follows that

$$\int_a^b f(x) \, dx \geq \int_{\max(a, \xi - \varepsilon)}^{\min(b, \xi + \varepsilon)} f(x) \, dx > 0.$$

$$\text{Therefore, } \int_a^b f(x) \, dx > 0.$$

□

### Problem 3.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  with

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (i.e. not rational).} \\ \frac{1}{q} & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0. \end{cases}$$

Show that  $\int_0^1 f(x) \, dx$  exists and is equal to 0.

Hint: Use the Lebesgue criterion for integrability. In particular, you need to determine at what points  $f$  is continuous.

**Claim 3.1.**  $\int_0^1 f(x) \, dx$  exists.

*Proof.* We are given that  $f$  is only nonzero for rational numbers of the form  $p/q$ , where  $p, q \in \mathbb{N}$  and  $p, q$  are coprime. Since the rational numbers are countable,  $\mathbb{Q}$  has Lebesgue measure zero, meaning that the set of points where  $f$  is nonzero, which is a subset of  $\mathbb{Q}$ , also has measure zero. By the Lebesgue criterion,  $f$  is Riemann integrable on  $[0, 1]$  since the set of points where  $f$  is discontinuous has measure zero, so  $\int_0^1 f(x) \, dx$  exists.  $\square$

**Claim 3.2.**  $\int_0^1 f(x) \, dx = 0$ .

*Proof.* Given a partition  $P_n$  of  $[0, 1]$  into  $n$  subintervals,

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i \leq \int_0^1 f(x) \, dx \leq \sum_{i=1}^n M_i \Delta x_i = U(f, P),$$

where  $m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$  and  $M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$ . Furthermore;

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^n 0 \cdot \Delta x_i = 0$$

and

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i.$$

Since the set of points where  $f$  is nonzero has measure zero, as we make the partitions finer and finer,  $M_i \Delta x_i$  will approach 0 for all  $i$ , so  $U(f, P)$  will also approach 0 (since only a smaller subset will have nonzero  $M_i$ ). On the other hand,  $L(f, P)$  will always be 0. For any  $\varepsilon > 0$ , take  $P_\varepsilon$  to be a partition such that  $U(f, P_\varepsilon) < \varepsilon$ , then  $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$ . Therefore,  $\int_0^1 f(x) \, dx = 0$ .  $\square$

**Problem 4.**

Prove that if the real-valued function  $f$  on the interval  $[a, b]$  is integrable on  $[a, b]$ , then so is  $|f|$ , and

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

**Claim 4.1.** Suppose  $f$  is integrable on  $[a, b]$ , then so is  $|f|$ .

*Proof.* Since  $f$  is integrable on  $[a, b]$ , for any  $\varepsilon > 0$ , there exists a partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Since  $|a| - |b| \leq |a - b|^a$  for all  $a, b \in \mathbb{R}$ , we have  $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \varepsilon$ , so  $|f|$  is also integrable on  $[a, b]$ .  $\square$

<sup>a</sup>This can be shown using Cauchy-Schwarz inequality:

$$\begin{aligned} |a| &= |(a - b) + b| \leq |a - b| + |b| && \text{(triangle inequality)} \\ |a| - |b| &\leq |a - b| && \text{(deduct } |b| \text{ to both sides)} \end{aligned}$$

**Claim 4.2.** Given that both  $f$  and  $|f|$  are integrable on  $[a, b]$ , then  $\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx$ .

*Proof.* Since  $f$  is integrable on  $[a, b]$ , for any  $\varepsilon > 0$ , there exists a partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  such that

$$U(f, P) - L(f, P) < \varepsilon,$$

and

$$L(f, P) \leq R(f, P) \leq U(f, P),$$

with  $R(f, P) = \sum_{i=1}^n f(c_i) \Delta x_i$  for some  $c_i \in [a_{i-1}, a_i]$ . As  $\|P\| \rightarrow 0$ ,  $\varepsilon \rightarrow 0$ , so the Riemann sum  $R(f, P)$  converges to the integral;

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) \, dx \quad \text{and} \quad \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n |f(c_i)| \Delta x_i = \int_a^b |f(x)| \, dx.$$

Thus, our original claim is equivalent to:

$$\begin{aligned} \left| \int_a^b f(x) \, dx \right| &= \left| \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i \right| && \text{(for some } c_i \in [x_{i-1}, x_i]) \\ &\leq \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n |f(c_i)| \Delta x_i && \text{(triangle inequality)} \end{aligned}$$

$$= \int_a^b |f(x)| \, dx,$$

□

**Problem 5.**

Prove integration by parts. That is, suppose  $F$  and  $G$  are continuously differentiable functions on  $[a, b]$ . Then, prove that

$$\int_a^b F(x)G'(x) \, dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x) \, dx.$$

*Proof.* Since  $F$  and  $G$  are continuously differentiable on  $[a, b]$ , they are also continuous on  $[a, b]$ . Thus, by the fundamental theorem of calculus, we have Let  $H(x) = F(x)G(x)$ , then

$$\begin{aligned} H'(x) &= \frac{d}{dx} [F(x)G(x)] \\ &= F'(x)G(x) + F(x)G'(x) \quad (\text{by the product rule}). \end{aligned}$$

What happens if we integrate both sides of this equation?

$$\begin{aligned} \int_a^b H'(x) \, dx &= \int_a^b [F'(x)G(x) + F(x)G'(x)] \, dx \\ H(b) - H(a) &= \int_a^b F'(x)G(x) \, dx + \int_a^b F(x)G'(x) \, dx \end{aligned}$$

Rearranging this gives us

$$H(b) - H(a) - \int_a^b F'(x)G(x) \, dx = \int_a^b F(x)G'(x) \, dx$$

Since  $H(x) = F(x)G(x)$ , we have

$$\int_a^b F(x)G'(x) \, dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x) \, dx$$

□

**Problem 6.**

Let  $g, f : \mathbb{R} \rightarrow \mathbb{R}$  be Riemann integrable on any interval  $[a, b] \subset \mathbb{R}$ . Is it true that  $g \circ f$  is also Riemann integrable on any interval  $[a, b] \subset \mathbb{R}$ ?

Hint: Consider  $g$  such that  $g(x) = 0$  if  $x = 0$  and  $g(x) = 1$  if  $x \neq 0$ , and  $f$  as in Problem 3:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (i.e. not rational).} \\ \frac{1}{q} & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0. \end{cases}$$

**Claim 6.1.** The composition of two Riemann integrable functions is not necessarily Riemann integrable.

*Proof.* Consider the functions  $g, f : \mathbb{R} \rightarrow \mathbb{R}$  as defined above, and their composition  $g \circ f$ . Both  $g$  and  $f$  are Riemann integrable on any interval  $[a, b] \subset \mathbb{R}$ , as priorly shown. However, Let's look at  $g \circ f$ :

$$(g \circ f)(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (i.e. not rational).} \\ 1 & \text{if } x := \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0. \end{cases}$$

Note that  $g \circ f$  is not continuous at any point  $x \in \mathbb{Q}$ , since there exists a rational number between any two distinct irrationals <sup>a</sup> and there exists an irrational number between any two distinct rationals <sup>b</sup>. Consequently, if we take any  $0 < \varepsilon < 1$ , then there is no value for  $\delta > 0$  that satisfies the  $\varepsilon$ - $\delta$  criterion for continuity at any point  $x \in \mathbb{R}$  since. Take  $x_2$  to be any number in the interval  $(x - \delta, x + \delta)$ , then:

1. If  $x \in \mathbb{Q}$  and  $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ , then  $(g \circ f)(x) = 1$  and  $(g \circ f)(x_2) = 0$ , so

$$|(g \circ f)(x) - (g \circ f)(x_2)| = 1 > \varepsilon.$$

2. If  $x \in \mathbb{Q}$  and  $x_2 \in \mathbb{Q}$ , then there exists some irrational number  $x_3$  between  $x$  and  $x_2$ , then  $(g \circ f)(x) = 1$  and  $(g \circ f)(x_3) = 0$ . Therefore,

$$|(g \circ f)(x) - (g \circ f)(x_3)| = 1 > \varepsilon.$$

3. If  $x \in \mathbb{R} \setminus \mathbb{Q}$  and  $x_2 \in \mathbb{Q}$ , then  $(g \circ f)(x) = 0$  and  $(g \circ f)(x_2) = 1$ , so

$$|(g \circ f)(x) - (g \circ f)(x_2)| = 1 > \varepsilon.$$

4. If  $x \in \mathbb{Q}$  and  $x_2 \in \mathbb{Q}$ , then there exists some irrational number  $x_3$  between  $x$  and  $x_2$ , then  $(g \circ f)(x) = 1$  and  $(g \circ f)(x_3) = 0$ . Therefore,

$$|(g \circ f)(x) - (g \circ f)(x_3)| = 1 > \varepsilon.$$

Thus,  $g \circ f$  is not continuous at any point  $x \in \mathbb{R}$ , therefore not Riemann integrable on any interval  $[a, b] \subset \mathbb{R}$ . □



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**<sup>a</sup>Proof that there exists a rational number between any two distinct irrationals.**

Let  $a, b \in \mathbb{R} \setminus \mathbb{Q}$  with  $a < b$ . Let  $c = b - a$ . By the properties of  $\mathbb{R}$ , there exists  $n \in \mathbb{N}$  such that  $n > 1/c$ , which implies that  $cn > 1$ . Since we took  $c = b - a$ , this implies that  $nb - na > 1$ . Therefore, there exists some integer  $N$  such that  $na < N < nb$ . Dividing by  $n$ , we get  $a < N/n < b$ . Thus,  $N/n$  is a rational number between  $a$  and  $b$ .

**<sup>b</sup>Proof that there exists an irrational number between any two distinct rationals.**

Let  $a, b \in \mathbb{Q}$  with  $a < b$ . Then  $b - a > 0$ ,  $b - a > \frac{b - a}{\sqrt{2}}$ , and  $\frac{b - a}{\sqrt{2}} \notin \mathbb{Q}$ . Therefore  $a + \frac{b - a}{\sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$ , and it is contained in the interval  $(a, b)$ .