Math 29: Computability Theory **Spring 2024**

PSET 1 — 04/05/2024

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Problem 1.

The rational numbers, $\mathbb{Q}$, are $\left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q > 0\right\}$. The Gaussian rationals are complex nunbers of the form $\{r + si \mid r, s \in \mathbb{Q}\}$. Provide a bijection between the Gaussian rationals and ω.

Prove that it is a bijection.

This... is a bijection.

Problem 2.

Give a register machine which converges if there is a 2 in R_0 and diverges otherwise.

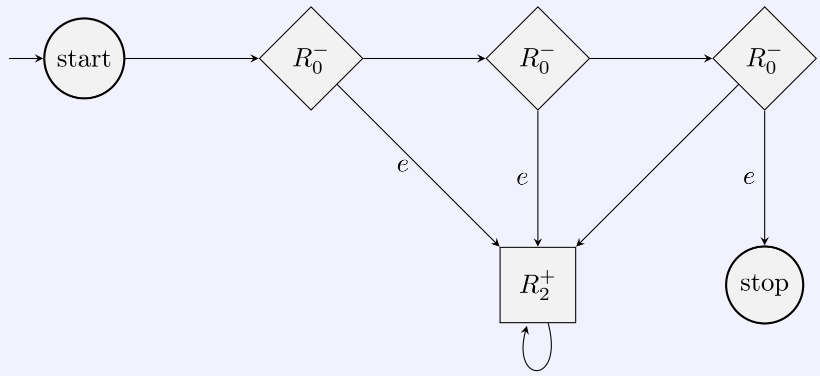

Figure 1: Converge if there is a 2 in R_0 and diverge otherwise.

Algorithm 1: Converge if there is a 2 in R_0 and diverge otherwise.

while $R_0 \neq 0$ **do** ▷ `Decrement` n
    $R_0 \leftarrow R_0 - 1$
if $R_0 = 0$ **then** ▷ `If` $n = 0$`, then` $m \geq n$
    $R_2 \leftarrow R_2 + 1$

Problem 3.

Prove that the squaring function is computable by providing a register machine which takes in n in R_0 and outputs n^2 in R_1. You may use the multiplication function $x(n, m)$ — which starts with n in R_0 and m in R_1 and outputs $n \cdot m$ in R_2 — as a black box function labeled M.

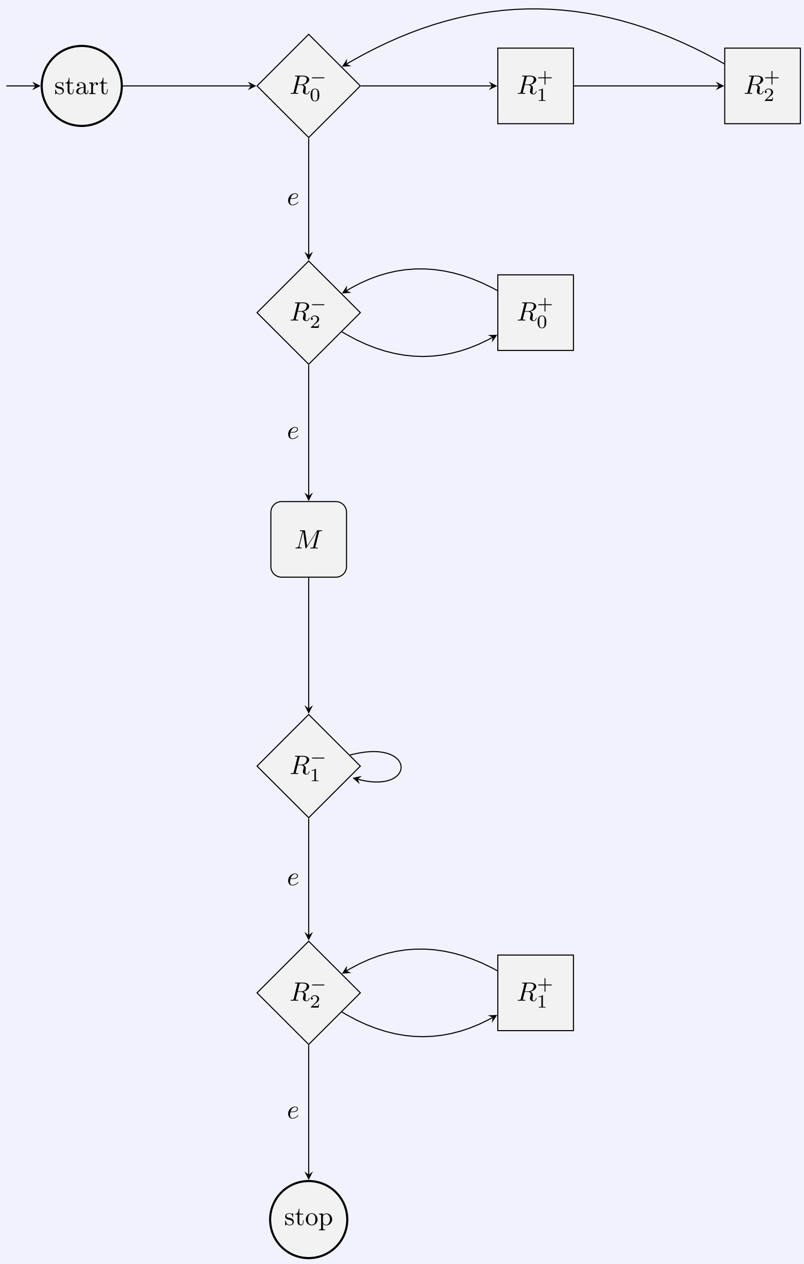

Figure 2: Calculate $f : n \mapsto n^2$ for $n \in \omega$.

Algorithm 2: Calculate $f : n \mapsto n^2$ for $n \in \omega$.

while $(\textit{status} \coloneqq R_0 - 1) \neq e$ **do** ▷ `Copy` n `to` R_1 `and` R_2`, which sets` $R_0 = 0$

  $R_1 \leftarrow R_1 + 1$

  $R_2 \leftarrow R_2 + 1$

while $(\textit{status} \coloneqq R_2 - 1) \neq e$ **do** ▷ `Move` n `from` R_2 `back to` R_0

  $R_0 \leftarrow R_0 + 1$

call M ▷ `Call` M `to populate` R_2 `with` $n^2 \coloneqq R_0 \cdot R_1$

while $(\textit{status} \coloneqq R_1 - 1) \neq e$ **do** ▷ `Set` $R_1 \leftarrow 0$

  continue

while $(\textit{status} \coloneqq R_2 - 1) \neq e$ **do** ▷ `Move` n^2 `from` R_2 `to` R_1

  $R_1 \leftarrow R_1 + 1$

Problem 4.

Prove that the set of ordered pairs of natural numbers (x, y) such that $x \leq y$ is computable by providing a register machine which takes n and m as imputs in R_0 and R_1 respectively and outputs 1 if $n \leq m$ and 0 in R_2 otherwise.

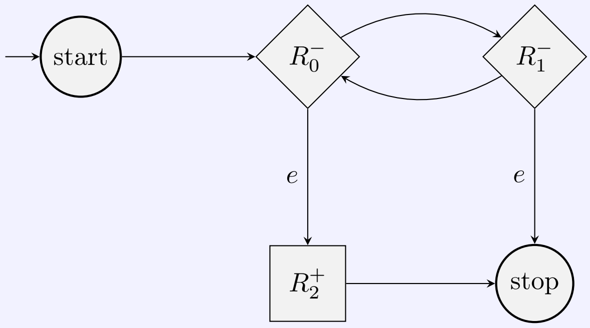

Figure 3: Calculate $f : (x, y) \mapsto \begin{cases} 1 & \text{if } x \leq y. \\ 0 & \text{otherwise.} \end{cases}$

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while $R_0 \neq 0$ **do** ▷ `Decrement` n

  $R_0 \leftarrow R_0 - 1$

  $R_1 \leftarrow R_1 - 1$

if $R_1 = 0$ **then** ▷ `If` $n = 0$`, then` $m \geq n$

  $R_2 \leftarrow R_2 + 1$
