Math 29: Computability Theory	**Spring 2024**

PSET 5 — 05/12/2024

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Problem 5.

Given a countable sequence of sets $\{A_i\}_{i\in\omega}$, define the *infinite join* $A = \bigoplus_{i\in\omega} A_i$ by

$$A = \{\langle i, n\rangle \mid n \in A_i\}.$$

Prove that there are sequences $\{A_i\}_{i\in\omega}$ and $\{B_i\}_{i\in\omega}$ such that $A_i \equiv_T B_i$ for all i but $A \not\equiv_T B$. In other words, this operation is defined on sets, but not on degrees (unlike the finite joins).

Hint: make A computable but B not computable.

For each $i \in \omega$, let

$$A_i = \{i\}$$

and

$$B_i = \{e \quad | \quad e = i \text{ and } \varphi_i(i)\downarrow\}$$

First, we show that each $A_i \equiv_T B_i$.

1. **$A_i \leq_T B_i$.** Note that each A_i is computable, since it is a singleton set containing i. Thus, for any A_i and B_i, we can define $\Phi_e^{B_i}$ to be an oracle machine that, given n, ignores B_i and computes χ_A as follows:
$$\chi_{A_i}(n) = \begin{cases} 1 & \text{if } n = i, \\ 0 & \text{otherwise.} \end{cases}$$
2. **$B_i \leq_T A_i$.** Define $\Phi_e^{A_i}$ to be an oracle machine that, given n, first checks if $n \in A_i$ then simulates $\varphi_n(n)$ and checks if it converges.
$$\chi_{B_i}(n) = \begin{cases} 1 & \text{if } n \in A_i \wedge \varphi_n(n)\downarrow \\ 0 & \text{otherwise.} \end{cases}$$

Next, we show that $A \not\equiv_T B$.

Let

$$A = \{\langle i, n\rangle \mid n \in A_i\} = \{\langle i, i\rangle \mid i \in \omega\}$$

and

$$B = \{\langle i, n\rangle \mid n \in B_i\} = \{\langle i, i\rangle \mid \varphi_i(i)\downarrow\} = \{\langle i, i\rangle \mid i \in K\}.$$

A is computable since it is a set of all pairs of the form $\langle i, i \rangle$. Given a pair $\langle i, n \rangle$, we can computably check if $n = i$. However, B is not computable since it is the set of all pairs $\langle i, i \rangle$ such that $\varphi_i(i)$ converges, which is equivalent to the halting problem. Thus, $A \not\equiv_T B$.