Turing Categories and Computability

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Introduction

In this paper, we construct a turing category $\Bbbk$ and study the resulting implications on computability.

1 Preliminaries

1.1 Categories

Definition 1.1. A ***category*** $\mathscr{A}$ consists of:

1. A collection $\mathbf{ob}(\mathscr{A})$ of objects;
2. For each pair of objects $A, B \in \mathbf{ob}(\mathscr{A})$, a set $\mathscr{A}(A, B)$ of ***arrows*** or ***morphisms*** or ***maps*** from A to B;
3. For each $A, B, C \in \mathbf{ob}(\mathscr{A})$, a function

$$\begin{aligned} \circ_{A,B,C} : \mathscr{A}(B, C) \times \mathscr{A}(A, B) &\to \mathscr{A}(A, C) \\ (f, g) &\mapsto f \circ g \end{aligned}$$

   called ***composition***; where $(f \circ g)(x) = f(g(x))$ for all $x \in A$.
4. For each $A \in \mathbf{ob}(\mathscr{A})$, an arrow $\mathsf{id}_A \in \mathscr{A}(A, A)$ called the ***identity*** on A;

such that the following axioms hold:

1. **associativity**: for all $f \in \mathscr{A}(A,B)$, $g \in \mathscr{A}(B,C)$, and $h \in \mathscr{A}(C,D)$, $(h \circ g) \circ f = h \circ (g \circ f)$.

2. **identity laws**: for all $f \in \mathscr{A}(A,B)$, $f \circ \mathsf{id}_A = f = \mathsf{id}_B \circ f$.

Remark 1.2. As simplifications, we write:

(a) $A \in \mathscr{A}$ to mean $A \in \mathbf{ob}(\mathscr{A})$;

(b) $f : A \to B$ or $A \xrightarrow{f} B$ to mean $f \in \mathscr{A}(A,B)$;

(c) fg for $f \circ g$;

$\Diamond$

Examples 1.3. 1. There is a category Set, where

(a) $\mathbf{ob}(\mathsf{Set})$ is the collection of all sets;

(b) $\mathsf{Set}(A,B)$ is the set of all functions from A to B;

(c) composition is ordinary function composition;

(d) the identity on A is the identity function on A.

2. There is a category Grp, where

(a) $\mathbf{ob}(\mathsf{Grp})$ is the collection of all groups;

(b) $\mathsf{Grp}(G,H)$ is the set of all group homomorphisms from G to H;

(c) composition is ordinary function composition;

(d) the identity on G is the identity homomorphism on G.

3. There is a category Top of topological space and continuous maps.

4. For each field k, there is a category Vect_k of vector spaces over k and linear maps between them.

$\Diamond$

Definition 1.4. A map $f : A \to B$ in a category $\mathscr{A}$ is an ***isomorphism*** if there exists a map $g : B \to A$ such that $fg = \mathsf{id}_A$ and $gf = \mathsf{id}_B$. Ee call g the ***inverse*** of f and write $f^{-1} = g$, and say that A and B are ***isomorphic*** if there exists an isomorphism between them.

Examples 1.5. 1. In Set, isomorphisms are bijections.

2. In Grp and Ring, isomorphisms are group and ring isomorphisms respectively.

3. In Vect_k, isomorphisms are linear isomorphisms.

$\Diamond$

1.2 Restriction Categories

Definition 1.6. A ***restriction category*** is a category $\mathscr{A}$ with a ***restriction*** operation that assigns to each arrow $f : A \to B$ an arrow $\bar{f} : A \to A$ such that:

1. $\bar{f} \circ f = f$;
2. $\bar{f} \circ \bar{g} = \bar{g} \circ \bar{f}$ whenever $\mathbf{dom}\,(f) = \mathbf{dom}\,(g)$;
3. $\overline{f \circ \bar{g}} = \bar{g} \circ \bar{f}$ whenever $\mathbf{dom}\,(f) = \mathbf{dom}\,(g)$.
4. $\bar{g} \circ f = \bar{g} \circ f \circ \bar{g}$ whenever $\mathbf{dom}\,(f) = \mathbf{range}\,(g)$.

Remark 1.7. It follows from the definition that $\bar{f}$ is ***idempotent***. That is, $\bar{f} \circ \bar{f} = \bar{f}$. Furthermore, the operation $f \mapsto \bar{f}$ is also monotonic, with $\bar{\bar{f}} = \bar{f}$. ◊

Examples 1.8. Here are a few examples of restriction categories. [1]

1. All categories admit the trivial restriction operation that maps $f : A \to B$ to $\bar{f} = \mathsf{id}_A$.
2. The category Par of partial functions between sets admits a restriction operation that maps $f : A \rightharpoonup B$ to $\bar{f} = \mathsf{id}_{\mathbf{dom}(f)}$.

◊

2 Turing Categories

References

[1] Tom Leinster, *Basic category theory*, 2016.