Math 29: Computability Theory **Spring 2024**

Mid-Term Exam 05/03/2024

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Problem 1.

Give a register machine which, when given a natural number n in R_0 at ***start***, halts and outputs $x + 1$ in R_1 if there is some natural x satisfying $x^2 + n^2 = 2nx$, and returns 0 otherwise.

Problem 2.

Prove that no index set is immune.

Definition 2.1. A set S is *immune* if A is infinite but contains no infinite c.e. subset.

To show that no index set is immune, we will show that every index set must contain some infinite c.e. subset.

Let I be an index set. Here are some facts about I:

1. Since I is an index set, I is either empty or infinite.

2. For all $e, f \in \mathbb{N}$, if $\varphi_e = \varphi_f$, then $e \in I \iff f \in I$.

Problem 3.

Give an example of a partial computable function $i(x, y)$ such that:

- $i(x, y)\downarrow$ implies $i(x, y) = 0$ or $i(x, y) = 1$.
- If A is computable, then there exists an e such that $i(e, n) = \chi_A(n)$ for all n.
- $I_x := \{y \mid i(x, y)\downarrow > 0\}$ is computable for all x.

Why does $i(x, y)$ not contradict Homework 3, Question 2?

Definition of $i(x, y)$

Let $i(x, y)$ be the following partial computable function:

$$i(x, y) = \begin{cases} 1 & \text{if } \varphi_x(y)\downarrow = 1 \\ 0 & \text{if } \varphi_x(y)\downarrow = 0 \end{cases}$$

Problem 4.

Let M be the set

$$\{x \mid \forall\, y < x\, (\varphi_x \neq \varphi_y)\}\,.$$

That is, M is the set of minimal indices of computable functions: the smallest indices which define a given (partial) computable function.

(a) Is M immune? Simple?

(b) Is M productive? Creative?

Problem 5.

Prove that $W_e \leq_1 H$ for all e, then prove that $H \leq_1 K$.

Let e be given. Define the function $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} n & \text{if } n \in W_e \\ 0 & \text{otherwise.} \end{cases}$$

Then f is computable, and $W_e = f^{-1}(\mathbb{N})$. Therefore, $W_e \leq_1 H$.

Now, let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$f(n) = \begin{cases} n & \text{if } n \in K \\ 0 & \text{otherwise.} \end{cases}$$

Then f is computable, and $K = f^{-1}(\mathbb{N})$. Therefore, $H \leq_1 K$.

Problem 6.

Suppose C is the set of valid codes for machines based on our coding scheme defined in class. Give a total, computable bijection from ω to C, i.e. a computable function which associates every natural number to a unique machine.

Since C is c.e., we can enumerate it as $C = \{e_1, e_2, e_3, \ldots\}$. Define the function $f : \mathbb{N} \to C$ by $f(n) = e_n$. Then f is total, computable, and bijective.

Problem 7.

Show that there is an e such that

$$W_e = \left\{e + 1, e^2 + 4, e^3 + 9, \ldots\right\}.$$

Let f be the function defined by

$$\begin{aligned} f &: \mathbb{N}^2 \to \mathbb{N} \\ (a, b) &\mapsto a^b + b^2. \end{aligned}$$

Since f is computable, $f = \varphi_k$ for some k.

Consider the following Turing Machine:

TM 1: Compute $\psi(n) : \mathbb{N} \to \mathbb{N}$

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Let e be the code for this machine.
for i = 1, 2, 3, ... do
    y_i ← φ_k(e, i)
    if y_i = n then
        output 1
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The machine passes its own code to φ_k, with successive integers $1, 2, 3, \ldots$ as the second argument. It converges and outputs 1 *if and only if* $\varphi_k(e, i)$ returns some value equal to n, meaning $n \in \left\{e + 1, e^2 + 4, e^3 + 9, \ldots\right\}$. Otherwise, it diverges as it will keep searching forever.

$$\varphi_e(n) = \psi(n) = \begin{cases} 1 & \text{if } n = \varphi_k(e, i) \text{ for some } i \in \mathbb{N}_{>0} \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $W_e = \left\{e + 1, e^2 + 4, e^3 + 9, \ldots\right\}$.