Math 29: Computability Theory **Spring 2024**

PSET 7 — 05/24/2024

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Problem 1.

Are there sets A and B such that $A' \leq_T B'$ but $A \not\leq_T B$? Justify your answer.

Yes.

Pick $B = \varnothing$ — which is computable — and let A be any non-computable set that is **low**, meaning $\varnothing' \equiv_T A'$ *As we showed in class using the Low Basis Theorem and Sacks' Splitting Theorem, such sets exist (and in fact, there are many of them).* Then:

1. Since B is computable but A is non-computable, $A \not\leq_T B$.
2. However, $A' = \varnothing'$ and $B' \equiv_T \varnothing'$, meaning $A' \equiv_T B'$. Therefore, $A' \leq_T B'$.

Problem 2.

(a) Prove that if there is $g \leq_T X$ such that $\varphi_{g(x)} \neq \varphi_x$, then there is $h \leq_T X$ such that $h(e) \neq \varphi_e(e)$ for all e.

This means that g is fixed-point free. Define h as follows:

$$h(e) = g(\varphi_e(e)).$$

1. First, note that $\varphi_{g(x)} \neq \varphi_x$ means that $g(x) \neq x$ for all x, as that would trivially imply $\varphi_{g(x)} = \varphi_x$ for some x. Therefore, $h(e) = g(\varphi_e(e)) \neq \varphi_e(e)$ for all e.
2. Next, we show that $h \leq_T X$. Since $g \leq_T X$, we can use X as an oracle to compute $g(e)$ for any e. Specifically, there exists an oracle machine k that uses X as an oracle to compute g, such that $\Phi_k^X(e) = g(e)$ for all e. We can compute $h(e)$ by simulating $\mathit{\Phi}_k^X(\varphi_e(e))$ and returning the result, thus h can also be computed by an oracle machine that takes X as an oracle and, on input e, simulates $\mathit{\Phi}_k^X$ on $\varphi_e(e)$.

(b) Given $h \leq_T X$ such that $h(e) \neq \varphi_e(e)$ for all e, show that there is $f \leq_T X$ such that $W_{f(e)} \neq W_e$ for all e.
Hint: make $\left|W_{f(e)}\right| = 1$.

For each e, define e' such that:

$$\varphi_{e'}(x) = \begin{cases} h(e) & \text{if } x = e \\ \uparrow & \text{otherwise.} \end{cases}$$

This ensures that for any e, $\varphi_{e'}(e) = h(e) \neq \varphi_e(e)$, and $W_{e'} = \{e\}$ for all e and corresponding e'.

Finally, define f as follows:

$$f(x) = (x+1)'.$$

1. First, since $f(e) = (e+1)'$, $W_{f(e)} = \{e+1\} \neq \{e\} = W_e$.
2. Next, we show that $f \leq_T X$. Since X is an oracle for h, we can use X to compute $h(e)$ for any e. Specifically, there exists an oracle machine k that uses X as an oracle to compute h, such that $\Phi_k^X(e) = h(e)$ for all e. We can compute f as follows:
On input e, construct an oracle machine that, on input x, simulates $\Phi_k^X(x)$ if $x = e+1$, and otherwise diverges.
Return the code of this machine as $f(e)$.

Problem 3.

Verify that the f constructed in the High-Low lecture notes dominates every total computable function, but does not compute K.

Problem 4.

Prove that not ML-random set has an infinite c.e. subset. (*Hint: use a lemma from the class notes*).