

PSET 1 — 04/05/2024

Prof. Miller

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Problem 1.

The rational numbers, \mathbb{Q} , are $\left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q > 0\right\}$. The Gaussian rationals are complex numbers of the form $\{r + si \mid r, s \in \mathbb{Q}\}$. Provide a bijection between the Gaussian rationals and \mathbb{N} .

Prove that it is a bijection.

We can provide a bijection between the Gaussian rationals and \mathbb{N} by using the Cantor Pairing Function. The Cantor Pairing Function is defined as $p : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ with

$$p(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y.$$

p is a bijection^a. We can construct a correspondence between the Gaussian rationals to \mathbb{N} as follows. First, write $r + si$ in the form $\frac{a}{b} + \frac{c}{d}i$ where $a, b, c, d \in \mathbb{Z}$ such that $b, d > 0$, a, b are coprime, and c, d are coprime. Since $r, s \in \mathbb{Q}$, this is possible. Define the map $\psi : \mathbb{Q}[i] \rightarrow \mathbb{N}$ as follows:

$$\begin{aligned} \psi : \mathbb{Q}[i] &\rightarrow \mathbb{N} \\ \frac{a}{b} + \frac{c}{d}i &\mapsto p(p(a, b), p(c, d)). \end{aligned}$$

To show **bijection**, we must show that ψ is both **injective** and **surjective**.

1. ψ is injective:

$$\text{Suppose } \psi\left(\frac{a_1}{b_1} + \frac{c_1}{d_1}i\right) = \psi\left(\frac{a_2}{b_2} + \frac{c_2}{d_2}i\right).$$

This means that $p(p(a_1, b_1), p(c_1, d_1)) = p(p(a_2, b_2), p(c_2, d_2))$. However, the Cantor Pairing Function is a bijection, meaning that $p(p(a_1, b_1), p(c_1, d_1)) = p(p(a_2, b_2), p(c_2, d_2))$ if and only if $p(a_1, b_1) = p(a_2, b_2)$ and $p(c_1, d_1) = p(c_2, d_2)$, which in turn implies that $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, and $d_1 = d_2$. Therefore,

$$\psi\left(\frac{a_1}{b_1} + \frac{c_1}{d_1}i\right) = \psi\left(\frac{a_2}{b_2} + \frac{c_2}{d_2}i\right) \implies \frac{a_1}{b_1} + \frac{c_1}{d_1}i = \frac{a_2}{b_2} + \frac{c_2}{d_2}i.$$

2. ψ is surjective:

Let $n \in \mathbb{N}$. Since the Cantor Pairing Function is surjective, $\exists x, y \in \mathbb{N}$ such that $n = p(x, y)$.

Likewise, $\exists a, b, c, d \in \mathbb{N}$ such that $x = p(a, b)$ and $y = p(c, d)$. Therefore, there exists some

$\gamma := \frac{a}{b} + \frac{c}{d}i$ in $\mathbb{Q}[i]$ such that

$$\psi(\gamma) = \psi\left(\frac{a}{b} + \frac{c}{d}i\right) = p(p(a, b), p(c, d)) = p(x, y) = n.$$

^aWe used this in class. Do I need to show that it is a bijection?

Problem 2.

Give a register machine which converges if there is a 2 in R_0 and diverges otherwise.

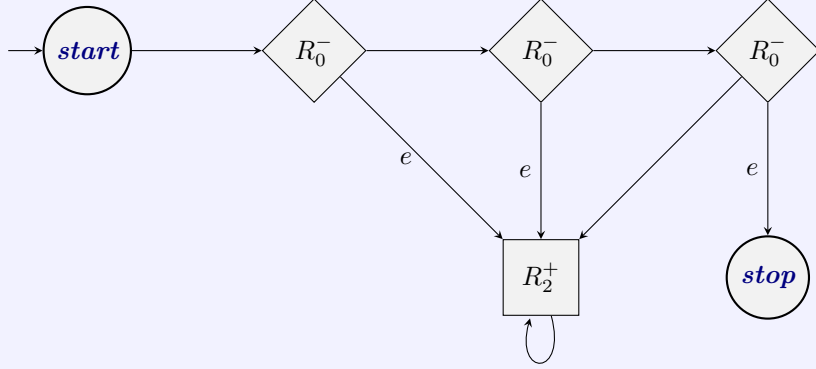


Figure 1: Converge if there is a 2 in R_0 and diverge otherwise.

Idea

The register machine in Figure 1 runs as follows:

1. Start at the **start** state. Let the initial value of R_0 be n .
2. Decrement R_0 by 1.
 - If impossible, go to the looping state.
 - If possible, the current value of R_0 is $n - 1$.
3. If step 2. was possible, decrement R_0 by 1 again.
 - If impossible, this means the value of R_0 before the decrement was $n - 1 = 0$, therefore $n = 1$, so $n \neq 2$. Go to the looping state.
 - If possible, the current value of R_0 is $n - 2$.
4. If step 3. was possible, decrement R_0 by 1 again.
 - If impossible, this means the value of R_0 before the decrement was $n - 2 = 1$, therefore $n = 2$. Go to the **stop** state.
 - If possible, the current value of R_0 is $n - 3 \geq 0$, meaning $n \geq 3 > 2$. Therefore, $n \neq 2$. Go to the looping state.
5. Finally, the looping node just increments R_2 *ad-infinitum*.

Problem 3.

Prove that the squaring function is computable by providing a register machine which takes in n in R_0 and outputs n^2 in R_1 . You may use the multiplication function $x(n, m)$ — which starts with n in R_0 and m in R_1 and outputs $n \cdot m$ in R_2 — as a black box function labeled M .

Register Machine

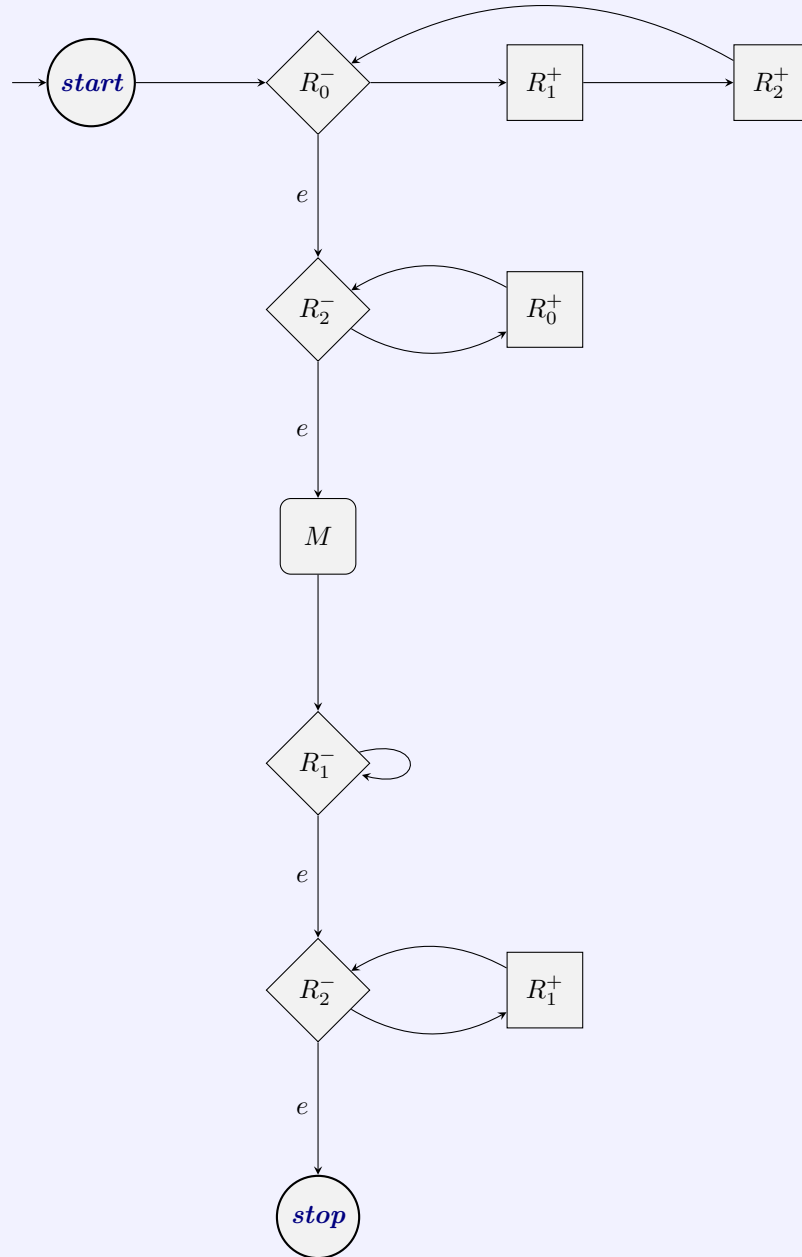


Figure 2: Calculate $f : n \mapsto n^2$ for $n \in \mathbb{N}$.

Idea

I found it useful to categorize the machine into five general parts (by height in Figure 2):

1. *start*
2. Copy n to R_1 and R_2 , which sets $R_0 \leftarrow 0$.
3. Copy n from R_2 back to R_0 .
4. Call M to calculate $R_2 \leftarrow R_0 \cdot R_1 = n \cdot n = n^2$.
5. Set $R_1 \leftarrow 0$.
6. Move n^2 from R_2 to R_1 .
7. *stop*

Algorithm

I wrote this to convince myself that the machine works as I intended.

Please disregard if irrelevant.

Algorithm 1: Calculate $f : n \mapsto n^2$ for $n \in \mathbb{N}$.

```
1 start
2 while (status :=  $R_0 - 1$ )  $\neq e$  do           ▷ Copy  $n$  to  $R_1$  and  $R_2$ , which sets  $R_0 = 0$ 
3   |  $R_1 \leftarrow R_1 + 1$ 
4   |  $R_2 \leftarrow R_2 + 1$ 
5 while (status :=  $R_2 - 1$ )  $\neq e$  do           ▷ Move  $n$  from  $R_2$  back to  $R_0$ 
6   |  $R_0 \leftarrow R_0 + 1$ 
7 call  $M$                                        ▷ Call  $M$  to populate  $R_2$  with  $n^2 := R_0 \cdot R_1$ 
8 while (status :=  $R_1 - 1$ )  $\neq e$  do           ▷ Set  $R_1 \leftarrow 0$ 
9   | loop
10 while (status :=  $R_2 - 1$ )  $\neq e$  do           ▷ Move  $n^2$  from  $R_2$  to  $R_1$ 
11   |  $R_1 \leftarrow R_1 + 1$ 
12 stop
```

Problem 4.

Prove that the set of ordered pairs of natural numbers (x, y) such that $x \leq y$ is computable by providing a register machine which takes n and m as inputs in R_0 and R_1 respectively and outputs 1 if $n \leq m$ and 0 in R_2 otherwise.

Register Machine

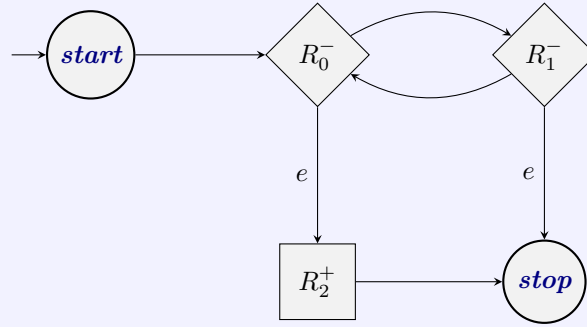


Figure 3: Calculate $f : (x, y) \mapsto \begin{cases} 1 & \text{if } x \leq y. \\ 0 & \text{otherwise.} \end{cases}$

Idea

The register machine in Figure 3 runs as follows:

1. **start**
2. Deduct 1 from R_0
 - If impossible, $R_0 = 0$. Since we have deducted the same times from R_1 as we have from R_0 and R_1 has not run out before now, $R_1 \geq R_0$, meaning $R_0 \leq R_1$. Increment R_2 and go to the **stop** state.
 - If possible, go to step 3.
3. Deduct 1 from R_1
 - If impossible, $R_1 = 0$. Since $R_0 \geq 0$ (as we have not branched out in the previous deduction from R_0 and gone to the **stop** state), $R_1 < R_0$, meaning $R_0 \not\leq R_1$. Go to the **stop** state (leaving the default value of 0 in R_2).
 - If possible, go to step 2.