Math 29: Computability Theory **Spring 2024**

PSET 6 — 05/17/2024

Prof. Miller ***Student: Amittai Siavava***

Problem 1.

Define a set X such that X computes $\varnothing^{(n)}$ for all n uniformly, i.e. there is an e such that

$$\Phi_e^X(n,k) = \chi_{\varnothing^{(n)}}(k)$$

for all n, k. Justify your answer.

We define X to be the set of all possible turing jumps of $\varnothing$:

$$X = \left\{ \langle n, k \rangle \mid n \in \omega, k \in \varnothing^{(n)} \right\}.$$

We claim that X computes $\varnothing^{(n)}$ for all n. Define

$$\Phi_e^X(n,k) = \begin{cases} 1 & \text{if } \langle n, k \rangle \in X \\ 0 & \text{otherwise.} \end{cases}$$

For all n, k;

1. If $k \in \varnothing^{(n)}$, then $\langle n, k \rangle \in X$, so $\Phi_e^X(n,k) = 1 = \chi_{\varnothing^{(n)}}(k)$.

2. If $k \notin \varnothing^{(n)}$, then $\langle n, k \rangle \notin X$, so $\Phi_e^X(n,k) = 0 = \chi_{\varnothing^{(n)}}(k)$.

3. Therefore, $\chi_{\varnothing^{(n)}}(k)$ is X-computable.

Problem 2.

Prove that, for all n and $f : \omega \to \omega$, there is a computable function $g : \omega^{n+1} \to \omega$ such that

$$f(x) = \lim_{s_0 \to \inf} \lim_{s_1 \to \inf} \cdots \lim_{s_{n-1} \to \inf} g(x, s_0, s_1, \ldots, s_{n-1})$$

if and only if $f \leq_T \varnothing^{(n)}$.

We will use induction to show that g is limit-computable.

1. For $i = 1$, we show that $f = \lim_{s_0 \to \infty} g(x, s_0)$ if and only if $f \leq_T \varnothing^{(1)}$.

(i) $\implies$:

Suppose $f = \lim_{s_0 \to \infty} g(x, s_0)$. Then there exists an e such that $\Phi_e(x, s_0) = g(x, s_0)$ and $\Phi_e(x, n) = f(x)$ for all $n \geq N$ for some $N \in \omega$. Therefore, $f \leq_T \varnothing^{(1)}$.

(ii) Suppose $f = \lim_{s_0 \to \infty} g(x, s_0)$. Then f is limit computable with 1 limit. By definition, f is a computable function, so f is c.e. and therefore limit computable with a single limit.

(iii) Suppose $f \leq_T \varnothing^{(0)}$. Then f is limit computable with 0 limits. By definition, f is a computable function, so f is c.e. and therefore limit computable with a single limit.

g is computable with one limit. By definition, g is a computable function, so g is c.e. and therefore limit computable with a single limit.

2. For $1 < i \leq n$, we assume that g is limit computable with $i - 1$ limits. We show that g is limit computable with i limits. By the induction hypothesis, g is limit computable with $i - 1$ limits, so g is c.e. and therefore limit computable with i limits.

Problem 3.

Give an example of a set X such that $X \perp_T \varnothing^{(n)}$ for all $n > 1$.

Hint: we are only required to perform (priority) constructions computably.

---

1. Start with $X = \varnothing$.

2. Define the requirement $R_{e,n}$:

$$R_{e,n} : \Phi_e^{\varnothing^{(n)}}(x_n) \neq 1$$

3. Define the requirement $Q_{e,n}$ as follows:

$$Q_{e,n} : \Phi_e^{\varnothing^{(n)}}(n) \neq x_n$$

4. For each $n \in 1, 2, 3, \ldots,$

(i) Pick $x' = \mathbf{max}\{x_1, x_2, \ldots, x_{n-1}\}$. If $n = 1$, set $x' = 0$.

(ii) Pick x_n to be the smallest $i \in \omega$ such that $x_n > x'$ and both $R_{e,n}$ and $Q_{e,n}$ are satisfied by the selection of x_n, for all $e \in \omega$.

(iii) Add x_n to X.

We claim that $X \perp_T \varnothing^{(n)}$ for all $n > 1$.

1. $X \not\leq_T \varnothing^{(n)}$:

For any $n > 1$, let x_n be the n-th item added to X, such that x_n satisfies $R_{e,n}$ for all $e \in \omega$, meaning $\Phi_e^{\varnothing^{(n)}}(x_n) \neq 1$. This ensures that $\Phi_e^{\varnothing^{(n)}}(x_n) \neq \chi_X(x_n)$.

2. $\varnothing^{(n)} \not\leq_T X$:

For any $n > 1$, let x_n be the n-th item added to X, such that x_n satisfies $Q_{e,n}$ for all $e \in \omega$, meaning $\Phi_e^{\varnothing^{(n)}}(n) \neq x_n$. This ensures that $\Phi_e^{\varnothing^{(n)}}$ cannot compute x_n.

Problem 4.

We say that $X =^* Y$ if X and Y agree on all but finitely many numbers. Show that there are sequences of sets $\{A_n\}_{n\in\omega}$ and $\{B_n\}_{n\in\omega}$ such that $A_n =^* B_n$ for all n, but $\bigoplus_{n\in\omega} A_n \not\equiv_T \bigoplus_{n\in\omega} B_n$.

$$A_n = \{n\}$$

$$B_n = \{e \mid (e = n) \land \varphi_n(n)\downarrow\}$$

Then;

1. For any given n, A_n contains only the number n, and B_n either contains n or is empty. Therefore, A_n and B_n disagree on either 1 or 0 numbers.

2. However, $\bigoplus_{n\in\omega} A_n = \{n \mid n \in \omega\} = \omega$, and $\bigoplus_{n\in\omega} B_n = \{n \mid (n \in \omega) \land \varphi_n(n)\downarrow\} = K$.
Since ω is computable but K is not computable, A and B have different Turing degrees, so $A \not\equiv_T B$.

Problem 5.

Show that HW3 Q5 relativizes. That is, show that A is X-computable if and only if A and A^c are both X-ce.

($\Longrightarrow$)

Suppose A is X-computable. Then there is an e such that $\Phi_e^X = \chi_A$. This means that for each n,

$$\Phi_e^X(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \qquad \text{(hence } n \in A^c\text{)} \end{cases}$$

A can be computably enumerated by a turing machine that goes through all $n = 1, 2, 3, \ldots$ and outputs n if $\Phi_e^X(n) = 1$.

TM 1: Enumerate A

for $n = 0, 1, 2, \ldots$ **do**
  if $\Phi_e^X(n) = 1$ **then**
    output n

Similarly, A^c can be computably enumerated by a turing machine that goes through all $n = 1, 2, 3, \ldots$ and outputs n if $\Phi_e^X(n) = 0$.

TM 2: Enumerate A^c

for $n = 0, 1, 2, \ldots$ **do**
  if $\Phi_e^X(n) = 0$ **then**
    output n

Therefore, A and A^c are both X-ce.

($\Longleftarrow$)

Suppose A and A^c are both X-ce. Then A is the domain of some X-computable function f, and A^c is the domain of some X-computable function g. We can define a function h that computes A as follows:

$$h(n) = \begin{cases} 1 & \text{if } f(n) \text{ is defined} \\ 0 & \text{if } g(n) \text{ is defined} \end{cases}$$

Specifically, let $f = \Phi_i^X$ and $g = \Phi_j^X$.
Then we can define Φ_h^X as follows:

TM 3: Compute A

On input n:
for $k = 1, 2, 3, \ldots$ **do**
  if $\Phi_{i,k}^X(n) \downarrow$ **then**
    output 1
  if $\Phi_{j,k}^X(n) \downarrow$ **then**
    output 0

Since *both* A and A^c are X-ce and $A \cup A^c = \omega$, for any $n \in \omega$, eventually either one of $\Phi_i^X(n)$ or $\Phi_j^X(n)$, simulated for some finite k steps, will halt. Thus, the TM eventually halts and outputs either 1 or 0 for any $n \in \omega$, effectively computing χ_A.

Therefore, A is X-computable.