Math 29: Computability Theory **Spring 2024**

PSET 6 — 05/17/2024

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Problem 1.

Define a set X such that X computes $\varnothing^{(n)}$ for all n uniformly, i.e. there is an e such that

$$\Phi_e^X(n,k) = \chi_{\varnothing^{(n)}}(k)$$

for all n, k. Justify your answer.

We define X to be the set of all possible turing jumps of $\varnothing$:

$$X = \left\{ \langle n, k \rangle \mid n \in \omega, k \in \varnothing^{(n)} \right\}.$$

We claim that X computes $\varnothing^{(n)}$ for all n. Define

$$\Phi_e^X(n,k) = \begin{cases} 1 & \text{if } \langle n,k \rangle \in X \\ 0 & \text{otherwise.} \end{cases}$$

For all n, k;

1. If $k \in \varnothing^{(n)}$, then $\langle n,k \rangle \in X$, so $\Phi_e^X(n,k) = 1 = \chi_{\varnothing^{(n)}}(k)$.

2. If $k \notin \varnothing^{(n)}$, then $\langle n,k \rangle \notin X$, so $\Phi_e^X(n,k) = 0 = \chi_{\varnothing^{(n)}}(k)$.

3. Therefore, $\chi_{\varnothing^{(n)}}(k)$ is X-computable.

Problem 2.

Prove that, for all n and $f : \omega \to \omega$, there is a computable function $g : \omega^{n+1} \to \omega$ such that

$$f(x) = \lim_{s_0 \to \inf} \lim_{s_1 \to \inf} \cdots \lim_{s_{n-1} \to \inf} g(x, s_0, s_1, \ldots, s_{n-1})$$

if and only if $f \leq_T \varnothing^{(n)}$.

We will use the ***limit lemma***, which states that a function $f : \omega \to \omega$ is limit computable if and only if $f \leq_T \varnothing'$.

Suppose $f(x) = \lim_{s_0 \to \inf} \lim_{s_1 \to \inf} \cdots \lim_{s_{n-1} \to \inf} g(x, s_0, s_1, \ldots, s_{n-1})$.

First, we see that f is Σ^0_{n+1}:

$$\begin{aligned}
&\exists\, s_0 \forall\ (s_{0'} > s_0) \\
&\quad \exists\, s_1 \forall\ (s_{1'} > s_1) \\
&\qquad \ddots \\
&\qquad\quad \exists\, s_{n-1} \forall\ (s_{n-1'} > s_{n-1}) \\
&\qquad\qquad g(x, s_{0'}, s_{1'}, \ldots, s_{(n-1)'}) = f(x).
\end{aligned}$$

Similarly, f is Π^0_{n+1}:

$$\begin{aligned}
&\forall\, s_0 \exists\ (s_{0'} > s_0) \\
&\quad \forall\, s_1 \exists\ (s_{1'} > s_1) \\
&\qquad \ddots \\
&\qquad\quad \forall\, s_{n-1} \exists\ (s_{n-1'} > s_{n-1}) \\
&\qquad\qquad g(x, s_{0'}, s_{1'}, \ldots, s_{(n-1)'}) = f(x).
\end{aligned}$$

Thus, $f(x)$ is Δ^0_{n+1}.

By the limit lemma (Lemma 4), f is computable from $\varnothing'$ *if and only if* f is Δ^0_2. Since the limit lemma relativizes, f is computable from $\varnothing^{(n)}$ *if and only if* f is Δ^0_{n+1}.

Furthermore, since every computable function is c.e., and every c.e. function is limit computable, f is n-limit computable

Finally, by relativizing Lemma 2. of the limit lemma, f is n-limit computable *if and only if* $f \leq_T \varnothing^{(n)}$.

Problem 3.

Give an example of a set X such that $X \perp_T \varnothing^{(n)}$ for all $n > 1$.

Hint: we are only required to perform (priority) constructions computably.

We use a priority construction to define X.

Define the requirements R_e and Q_e as follows:

$$R_e : \chi_X \neq \Phi_e^{\varnothing^{(e)}}$$

Let $X_0 = \varnothing$. At each step $s+1$, pick $x \notin X_s$. Simulate $\Phi_x^{\varnothing^{(s)}}(x)$. If $\Phi_x^{\varnothing^{(s)}}(x) \downarrow = 0$, then set $X_{s+1} = X_s \cup \{x\}$. Otherwise, repeat this step until such an x is found.

For each $n > 1$, let x_n be the n-th element that wasadded to X, then $\Phi_x^{\varnothing^{(n)}}(x_n) \downarrow = 0$, so $\chi_X(x_n) = 1 \neq \Phi_x^{\varnothing^{(n)}}(x_n)$. Thus, $X \perp_T \varnothing^{(n)}$ for all $n > 1$.

Problem 4.

We say that $X =^* Y$ if X and Y agree on all but finitely many numbers. Show that there are sequences of sets $\{A_n\}_{n\in\omega}$ and $\{B_n\}_{n\in\omega}$ such that $A_n =^* B_n$ for all n, but $\bigoplus_{n\in\omega} A_n \neq^* \bigoplus_{n\in\omega} B_n$.

Let $A_n = \{2n\}$ and $B_n = \{2n+1\}$. Then $A_n =^* B_n$ for all n, since A_n and B_n are disjoint, but they are each singleton sets, meaning they agree on all elements except for two: $2n$ and $2n+1$.

However, $\bigoplus_{n\in\omega} A_n = \{0, 2, 4, 6, \ldots\}$ and $\bigoplus_{n\in\omega} B_n = \{1, 3, 5, 7, \ldots\}$, so $\oplus_{n\in\omega} A_n \neq^* \oplus_{n\in\omega} B_n$ since they disagree on infinitely many numbers.

Problem 5.

Show that HW3 Q5 relativizes. That is, show that A is X-computable if and only if A and A^c are both X-ce.

($\Longrightarrow$)

Suppose A is X-computable. Then there is an e such that $\Phi_e^X = \chi_A$. This means that for each n,

$$\Phi_e^X(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \qquad \text{(hence } n \in A^c\text{)} \end{cases}$$

A can be computably enumerated by a turing machine that goes through all $n = 1, 2, 3, \ldots$ and outputs n if $\Phi_e^X(n) = 1$.

TM 1: Enumerate A

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for n = 0, 1, 2, ... do
    if Φ_e^X(n) = 1 then
        output n
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Similarly, A^c can be computably enumerated by a turing machine that goes through all $n = 1, 2, 3, \ldots$ and outputs n if $\Phi_e^X(n) = 0$.

TM 2: Enumerate A^c

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for n = 0, 1, 2, ... do
    if Φ_e^X(n) = 0 then
        output n
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Therefore, A and A^c are both X-ce.

($\Longleftarrow$)

Suppose A and A^c are both X-ce. Then A is the domain of some X-computable function f, and A^c is the domain of some X-computable function g. We can define a function h that computes A as follows:

$$h(n) = \begin{cases} 1 & \text{if } f(n) \text{ is defined} \\ 0 & \text{if } g(n) \text{ is defined} \end{cases}$$

Specifically, let $f = \Phi_i^X$ and $g = \Phi_j^X$.
Then we can define Φ_h^X as follows:

TM **3:** Compute A

On input n:
for $k = 1, 2, 3, \ldots$ **do**
  if $\Phi_{i,k}^X(n)\downarrow$ **then**
    output 1
  if $\Phi_{j,k}^X(n)\downarrow$ **then**
    output 0

Since *both* A and A^c are X-ce and $A \cup A^c = \omega$, for any $n \in \omega$, eventually either one of $\Phi_i^X(n)$ or $\Phi_j^X(n)$, simulated for some finite k steps, will halt. Thus, the TM eventually halts and outputs either 1 or 0 for any $n \in \omega$, effectively computing χ_A.

Therefore, A is X-computable.