CS 83: Computer Vision	**Winter 2024**

Quiz 6 — 02/22/2024

Prof. Pediredla | ***Student: Amittai Siavava***

Credit Statement

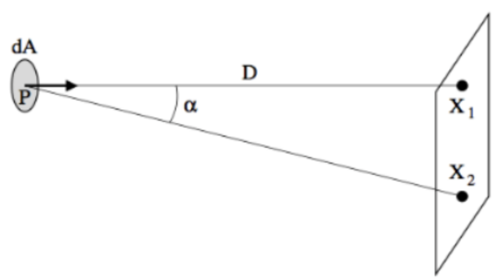
I discussed solution ideas with:

1. Ivy (Aiwei) Zhang
2. Angelic McPherson

However, all typed work is my own, with reference to class notes.

Problem 1.

A small Lambertian source $\mathrm{d}A$ is centered at P and emits radiance L. The orientation of this patch is the same as that of a plane containing two points, X_1 and X_2. The point X_1 is the point on this plane that is closest to P, and the distance from P to X_1 is D as shown.



(i) Calculate the solid angle subtended by $\mathrm{d}A$ at points X_1 and X_2.

(ii) Calculate the irradiance E incident on the plane at points X_1 and X_2, and calculate the ratio $E(X_1)/E(X_2)$.

Problem 2.

As we discussed in class, the Lambertian and the specular BRDF are the two most commonly used reflectance models in physics-based vision.

(i) For Lambertian surfaces, the BRDF is a constant function of the input and the output directions. For such a material, we often describe the reflectance in terms of its *albedo*, which is given the symbol ρ. For a Lambertian surface, the BRDF and the albedo are related by $f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) = \rho/\pi$. Using conservation of energy, prove that $0 \leq \rho \leq 1$.

First, note that the albedo is a measure of the fraction of incident light that is reflected by the surface. Therefore, it cannot be negatie, so $0 \leq \rho$. We shall show that $\rho \leq 1$ by using the energy conservation principle in reflectance which stipulates that the energy reflected by a surface cannot exceed the energy incident on the surface. In other words, the integral of the BRDF over the hemisphere must be less than or equal to 1:

$$\int_\Omega f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) \cos\theta \, \mathrm{d}\omega \leq 1.$$

Substituting in the known values gives:

$$\begin{aligned}
\int_\Omega \frac{\rho}{\pi} \cos\theta_i \, \mathrm{d}\omega &\leq 1 \\
\frac{\rho}{\pi} \int_\Omega \cos\theta \frac{\mathrm{d}A}{r^2} &\leq 1 && \left(\text{since } \omega = \frac{A}{r^2} \implies \frac{\mathrm{d}A}{r^2}\right) \\
\frac{\rho}{\pi} \int_\Omega \cos\theta \frac{r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi}{r^2} &\leq 1 && \left(\text{since } \mathrm{d}A = r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi\right) \\
\frac{\rho}{\pi} \int_\Omega \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi &\leq 1 \\
\frac{\rho}{\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi &\leq 1 \\
\frac{\rho}{\pi} \int_0^{2\pi} \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \mathrm{d}\varphi &\leq 1 && \left(\text{since } \int \cos\theta \sin\theta \, \mathrm{d}\theta = \frac{\sin^2\theta}{2}\right) \\
\frac{\rho}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi &\leq 1 \\
\frac{\rho}{2\pi} \cdot [\varphi]_0^{2\pi} &\leq 1 && \left(\text{since } \int 1 \, \mathrm{d}\varphi = \varphi\right) \\
\frac{\rho}{2\pi} \cdot 2\pi &\leq 1 \\
\rho &\leq 1.
\end{aligned}$$

(ii) A specular surface perfectly reflects *radiance* in the *mirror direction*. Concretely, consider a (non-absorbing) specular surface patch with normal $\hat{\mathbf{n}}$. For any incident direction $\hat{\mathbf{v}}_i$, the mirror direction equals $\hat{\mathbf{v}}_s = 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}} - \hat{\mathbf{v}}_i$, and $L(\hat{\mathbf{v}}_s) = L(\hat{\mathbf{v}}_i)$. Given this property, derive an expression for the specular BRDF.

The BRDF is defined as the ratio of the radiance reflected in the direction $\hat{\mathbf{v}}_r$ to the irradiance incident from the direction $\hat{\mathbf{v}}_i$. For a specular surface, the BRDF is a Dirac delta function centered at the mirror direction:

$$f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) = k\delta(\hat{\mathbf{v}}_r - \hat{\mathbf{v}}_s),$$

where k is a constant. The constant k is chosen so that the integral of the BRDF over the hemisphere is 1. Therefore, we have:

$$\int_\Omega f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) \cos\theta \, \mathrm{d}\omega = 1.$$

Substituting in the known values gives:

$$\begin{aligned}
\int_\Omega k\delta(\hat{\mathbf{v}}_r - \hat{\mathbf{v}}_s) \cos\theta_i \, \mathrm{d}\omega &= 1 \\
k \int_\Omega \delta(\hat{\mathbf{v}}_r - \hat{\mathbf{v}}_s) \cos\theta_i \, \mathrm{d}\omega &= 1 \\
k \cos\theta_i &= 1 \\
k &= \frac{1}{\cos\theta_i}.
\end{aligned}$$

Therefore, the BRDF for a specular surface is:

$$f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) = \frac{1}{\cos\theta_i}\delta(\hat{\mathbf{v}}_r - \hat{\mathbf{v}}_s).$$