CS 83: Computer Vision **Winter 2024**

Quiz 5 — 02/15/2024

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Credit Statement

I discussed solution ideas with:

1. Ivy (Aiwei) Zhang
2. Angelic McPherson

However, all typed work is my own, with reference to class notes.

Problem 1.

As we discused in class, two cameras are said to form a *rectified pair* if their camera coordinate systems differ only by a translation of their origins (the camera centers) along a direction that is parallel to either the x- or y-axis of their coordinate systems.

(i) Derive an expression for the essential matrix $\mathbf{E}$ of a rectified pair.

Let $C1$ and $C2$ be two rectified pair cameras, with $\mathbf{T} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$ as the translation vector from $C1$ to $C2$. The essential matrix $\mathbf{E} \coloneqq [\mathbf{T}]_\times \mathbf{R}$, with $\mathbf{R} = \ell_3$ (since there is no rotation between the two cameras), is then given by:

$$\begin{aligned}\mathbf{E} &= [\mathbf{T}]_\times \mathbf{R} \\ &= \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\end{aligned}$$

As stated in the problem, the translation will be along the x- or the y-axis, meaning that $t_z = 0$, and either $t_x = 0$ or $t_y = 0$. Therefore, the essential matrix $\mathbf{E}$ will be equivalent to:

$$\mathbf{E}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{E}_y = \begin{bmatrix} 0 & 0 & t_y \\ 0 & 0 & 0 \\ -t_y & 0 & 0 \end{bmatrix}$$

(ii) Prove that the epipolar lines of a rectified pair are parallel to the axis of translation.

The epipolar lines given by $\ell' = \mathbf{E}\mathbf{x}$ and $\ell = \mathbf{E}^\top \mathbf{x}'$.

1. For $\mathbf{E}_x$: The epipolar lines are given by

$$\ell' = \mathbf{E}_x \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -t_x \\ t_x y \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \frac{-t_x}{t_x y} \\ 1 \end{bmatrix}$$

$$\ell = \mathbf{E}_x^\top \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t_x \\ 0 & -t_x & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ t_x \\ -t_x y' \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \frac{-t_x}{t_x y'} \\ 1 \end{bmatrix}$$

Thus the lines are given by

$$\ell'_y = \frac{-t_x}{t_x y} \quad \text{and} \quad \ell_y = \frac{-t_x}{t_x y'}.$$

Since the original points do not change, the epipolar lines are of the form $y = k$, where k is a constant, and are therefore parallel to the x-axis.

2. For $\mathbf{E}_y$: The epipolar lines are given by

$$\ell' = \mathbf{E}_y \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t_y \\ 0 & 0 & 0 \\ -t_y & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} t_y \\ 0 \\ -t_y x \end{bmatrix} \equiv \begin{bmatrix} \frac{-t_y}{t_y x} \\ 0 \\ 1 \end{bmatrix}$$

$$\ell = \mathbf{E}_y^\top \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -t_y \\ 0 & 0 & 0 \\ t_y & 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -t_y \\ 0 \\ t_y x' \end{bmatrix} \equiv \begin{bmatrix} \frac{-t_y}{t_y x'} \\ 0 \\ 1 \end{bmatrix}$$

Thus the lines are given by

$$\ell'_x = \frac{-t_y}{t_y x} \quad \text{and} \quad \ell_x = \frac{-t_y}{t_y x'}.$$

Since the original points do not change, the epipolar lines are of the form $x = k$, where k is a constant, and are therefore parallel to the y-axis.

Problem 2.

Suppose two cameras fixate on a point P (see Figure 1) in space such that their optical axes intersect at that point. Show that if the image coordinates are normalized so that the coordinate system origin $(0, 0)$ coincides with the principal point, the $\mathbf{F}_{33}$ element of the fundamental matrix $\mathbf{F}$ is zero.

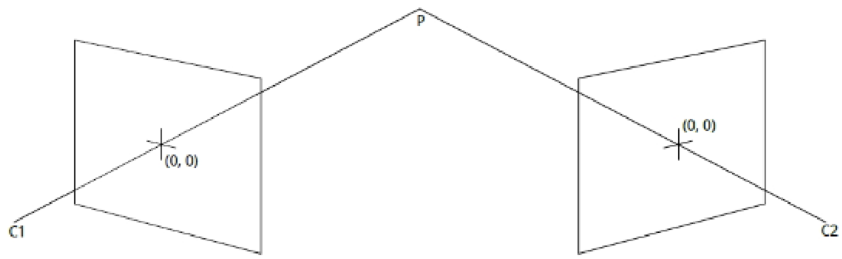

FIGURE 1. C_1 and C_2 are the optical centers. The principal axes intersect at point P.

The fundamental matrix $\mathbf{F}$ is given by

$$\mathbf{F} = (\mathbf{K}')^{-\top}[\mathbf{T}]_\times \mathbf{R}\mathbf{K}^{-1},$$

where $\mathbf{K}$ and $\mathbf{K}'$ are the intrinsic matrices of the two cameras, $\mathbf{T}$ is the translation vector from C_1 to C_2, and $\mathbf{R}$ is the rotation matrix from C_1 to C_2. Two points $\mathbf{x}$ and $\mathbf{x}'$ in the two images are related by $\mathbf{x}'^\top \mathbf{F}\mathbf{x} = \mathbf{0}$. When the image coordinates are normalized so that the coordinate system origin $(0, 0)$ coincides with the principal point, we have $\mathbf{x} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ and $\mathbf{x}' = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$, which implies that

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{F} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} \\ \mathbf{F}_{21} & \mathbf{F}_{22} & \mathbf{F}_{23} \\ \mathbf{F}_{31} & \mathbf{F}_{32} & \mathbf{F}_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{13} \\ \mathbf{F}_{23} \\ \mathbf{F}_{33} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{F}_{33} = \mathbf{0}.$$

Problem 3.

Consider three images I_1, I_2 and I_3 that have been captured by a system of three cameras, and suppose the fundamental matrices $\mathbf{F}_{13}$ and $\mathbf{F}_{23}$ are known. (Notation: the matrix $\mathbf{F}_{ij}$ satisfies the equation $\mathbf{x}_j^T \mathbf{F}_{ij} \mathbf{x}_i = 0$ for any correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_j$ between images I_i and I_j). In general, given a point $\mathbf{x}_1$ in I_1 and a corresponding point $\mathbf{x}_2$ in I_2, the corresponding point in $\mathbf{x}_3$ in I_3 is uniquely determined by the fundamental matrices $\mathbf{F}_{13}$ and $\mathbf{F}_{23}$.

(i) Write an expression for $\mathbf{x}_3$ in terms of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{F}_{13}$ and $\mathbf{F}_{23}$.

For a point $x_1 \in I_1$, we can write the epipolar line in I_3 as

$$\ell_{31} = \mathbf{F}_{13}\mathbf{x}_1.$$

For a point $x_2 \in I_2$, we can write the epipolar line in I_3 as

$$\ell_{32} = \mathbf{F}_{23}\mathbf{x}_2.$$

The point $\mathbf{x}_3$ in I_3 is the intersection of these two lines, meaning that $\mathbf{x}_3$ satisfies both the equations

$$\mathbf{x}_3^T \mathbf{F}_{13}\mathbf{x}_1 = 0 \quad \text{and} \quad \mathbf{x}_3^T \mathbf{F}_{23}\mathbf{x}_2 = 0.$$

To find $\mathbf{x}_3$, use the cross product of the two epipolar lines:

$$\mathbf{x}_3 = \ell_{31} \times \ell_{32} = \mathbf{F}_{13}\mathbf{x}_1 \times \mathbf{F}_{23}\mathbf{x}_2.$$

Since $\mathbf{x}_3$ is a point in the image that can be written as $\mathbf{x}_3 = \begin{bmatrix} x_3 \\ y_3 \\ 1 \end{bmatrix}$, we can write the expression for $\mathbf{x}_3$ as

$$\mathbf{x}_3 = \begin{bmatrix} x_3 \\ y_3 \\ 1 \end{bmatrix} = \mathbf{F}_{13} \begin{bmatrix} x_{13} \\ y_{13} \\ 1 \end{bmatrix} \times \mathbf{F}_{23} \begin{bmatrix} x_{23} \\ y_{23} \\ 1 \end{bmatrix}.$$

Thus, we have two unknowns, x_3 and y_3, and two corresponding equations to solve for them, so the point $\mathbf{x}_3$ is uniquely determined by the fundamental matrices $\mathbf{F}_{13}$ and $\mathbf{F}_{23}$.

(ii) Describe a degenerate configuration of three cameras for which the point $\mathbf{x}_3$ cannot be uniquely determined by this expression.

A degenerate configuration occurs when the fundamental matrices do not provide enough constraints to solve for $\mathbf{x}_3$. This happens when:

1. The cameras are collinear, meaning all three cameras are placed along the same line. This is an issue because the epipolar lines ℓ_{31} and ℓ_{32} will be parallel, and will never intersect (aka intersect at infinity), or intersect at multiple points.
2. The cameras are coplanar, meaning all three cameras are placed in the same exact plane. This is an issue because the epipolar lines ℓ_{31} and ℓ_{32} will be parallel and either never intersect (aka intersect at infinity) or intersect at multiple points.

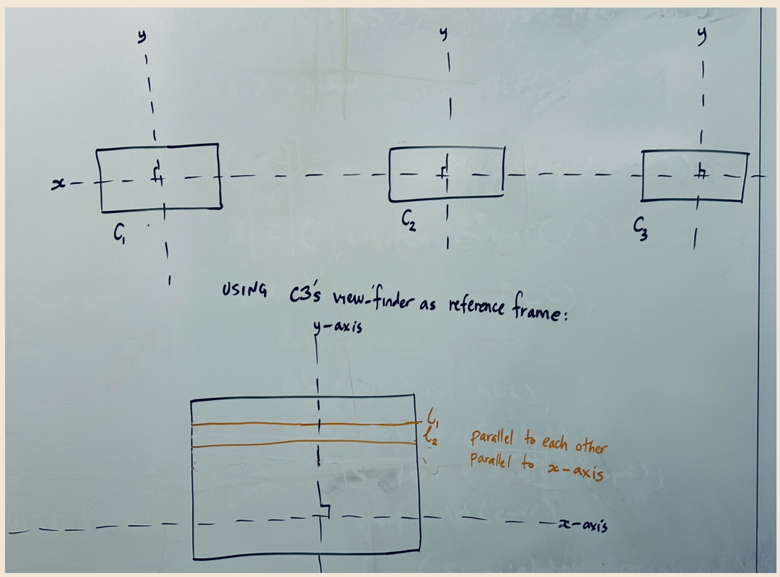

FIGURE 2. Epipolar geometry for degenerate configuration.

Hint: Consider the epipolar geometry of the situation. Draw a picture!