CS 83: Computer Vision	**Winter '23 2024**

Quiz 1 — 01/09/2024

Prof. Pediredla	***Student: Amittai Siavava***

Problem 1.

The continuous convolution of two functions $f(x)$ and $g(x)$ is given as

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\ \mathrm{d}y.$$

(i) Prove that the convolution of two functions is commutative, i.e., changing the order of operands produces the same result.

$$(f * g) = (g * f)$$

Hint: Perform integration by substitution.

By definition,

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{+\infty} f(y)\, g(x-y)\ \mathrm{d}y \\ &= \end{aligned}$$

(ii) Prove that the convolution operand is also associative, i.e., rearranging the parentheses on two or more occurrences of the convolution operator produces the same result:

$$(f * g) * h = f * (g * h)$$

Hint: Be careful with variables. Understand which variable should be integrated, and why.

Problem 2.

In class, we talked about finite-difference approximation to the derivative of the univariate function $f(x)$. Using Taylor polynomial approximations of $f(x+h)$ and $f(x-h)$, we can easily show that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h2),$$

so that the derivative can be approximated by convolving a discrete version of $f(x)$ — a vector of values $(\dots,\ f(x_o - \Delta), f(x_o), f(x_o + \Delta), \dots)$ with kernel (1/2, 0, -1/2). This is termed a central difference because its interval is symmetric about a sample point.

(i) Derive a higher order central-difference approximation to $f'(x)$ such that the truncation error tends to zero as h^4 instead of h^2. *Hint : consider Taylor polynomial approximations of* $f(x \pm 2h)$ *in addition to* $f(x \pm h)$. (7 points)

Let c_i denote the coefficient for each term containing h^i in the full Taylor polynomial expansion, then we may write the current estimation of $f'(x)$ as

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \left[c_2 h^2 + c_4 h^4 + c_6 h^6 + \dots\right]. \tag{2.1}$$

Our goal is to eliminate the h^2 term. Consider the approximations using $f(x \pm 2h)$ by plugging $2h$ into the formula:

$$f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} - \left[4c_2 h^2 + 16 c_4 h^4 + 64 c_6 h^6 + \dots\right] \tag{2.2}$$

We notice that the h^2 term in 2.2 is 4 times larger than the h^2 term in 2.1. We can eliminate the h^2 term in $f'(x)$ by subtracting 2.2 from 4 times 2.1:

$$\begin{aligned}
3f'(x) &= 4\frac{f(x+h) - f(x-h)}{2h} - \frac{f(x+2h) - f(x-2h)}{4h} - \left[0h^2 + O(h^4)\right] \\
3f'(x) &= \frac{8\left(f(x+h) - f(x-h)\right) - \left(f(x+2h) - f(x-2h)\right)}{4h} + O(h^4) \\
3f'(x) &= \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{4h} + O(h^4) \\
f'(x) &= \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + O(h^4)
\end{aligned} \tag{2.3}$$

We get equation 2.3 as a Taylor approximation of $f'(x)$ with a truncation error of $\mathcal{O}(h^4)$.

(ii) What is the corresponding convolution (not correlation!) kernel? (3 points)

The approximation has a correlation kernel of

$$\left(-\frac{1}{12}, \quad \frac{8}{12}, \quad -\frac{8}{12}, \quad \frac{1}{12}\right).$$

The convolution kernel is the same as the correlation kernel, but flipped.

$$\left(\frac{1}{12}, \quad -\frac{8}{12}, \quad \frac{8}{12}, \quad -\frac{1}{12}\right).$$