

Quiz 7 — 02/29/2024

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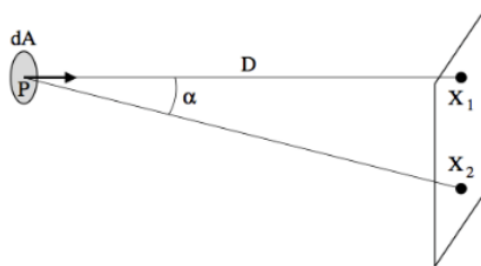
I discussed solution ideas with:

1. Ivy (Aiwei) Zhang
2. Angelic McPherson

However, all typed work is my own, with reference to class notes.

Problem 1.

A small Lambertian source dA is centered at P and emits radiance L . The orientation of this patch is the same as that of a plane containing two points, X_1 and X_2 . The point X_1 is the point on this plane that is closest to P , and the distance from P to X_1 is D as shown.



(i) Calculate the solid angle subtended by dA at points X_1 and X_2 .

The solid angle subtended by an area dA at a point \mathbf{x} that is a distance r from dA is given by

$$d\omega = \frac{dA \cos \theta}{r^2},$$

where θ is the angle between the normal to dA and the line from \mathbf{x} to dA . In this case;

1. Since dA has the same orientation as the plane containing X_1 and X_2 , and we are given that X_1 is the point on this plane that is closest to P , then the normal at dA passes through P . Therefore, $\theta = 0$ and $\cos \theta = 1$. Therefore, $d\omega_{X_1} = \frac{dA}{D^2}$.
2. As established in part 1., the normal at dA passes through P . Therefore, the angle between the normal to dA and the line from X_2 to dA is α as shown in the image. Let $D_2 = \|X_2 - P\|$; then:

$$\begin{aligned} \cos \alpha &= \frac{D}{D_2} \\ \implies D_2 &= \frac{D}{\cos \alpha}. \end{aligned}$$

Therefore;

$$\begin{aligned} d\omega_{X_2} &= \frac{dA \cos \alpha}{D_2^2} \\ &= \frac{dA \cos \alpha}{D^2 / \cos^2 \alpha} \\ &= \frac{dA \cos^3 \alpha}{D^2}. \end{aligned}$$

- (ii) Calculate the irradiance E incident on the plane at points X_1 and X_2 , and calculate the ratio $E(X_1)/E(X_2)$.

The irradiance E incident at a point \mathbf{x} is given by

$$E(\mathbf{x}) = \int_{H^2} L(\mathbf{x}, \omega) \cos \theta \, d\omega.$$

To find the irradiance at single points X_1 and X_2 , we do not have to integrate over the entire hemisphere but simply calculate

$$E(\mathbf{x}) = L \cos \theta \, d\omega$$

at the two points:

1. The irradiance $E(X_1)$ incident on X_1 is given by

$$\begin{aligned} E(X_1) &= L \cdot \cos \theta_{X_1} \cdot \omega_{X_1} \\ &= L \cdot \cos 0 \cdot \frac{dA}{D^2} \\ &= \frac{L \cdot dA}{D^2}. \end{aligned}$$

2. The irradiance $E(X_2)$ incident on X_2 is given by

$$\begin{aligned} E(X_2) &= L \cdot \cos \theta_{X_2} \cdot \omega_{X_2} \\ &= L \cdot \cos \alpha \cdot \frac{dA \cos^3 \alpha}{D^2} \\ &= \frac{L \cdot dA \cos^4 \alpha}{D^2} \end{aligned}$$

3. The ratio $E(X_1)/E(X_2)$ is given by

$$\begin{aligned} \frac{E(X_1)}{E(X_2)} &= \frac{\frac{L \cdot dA}{D^2}}{\frac{L \cdot dA \cos^4 \alpha}{D^2}} \\ &= \frac{L \cdot dA}{D^2} \cdot \frac{D^2}{L \cdot dA \cos^4 \alpha} \\ &= \frac{1}{\cos^4 \alpha} \\ &= \sec^4 \alpha. \end{aligned}$$

Problem 2.

As we discussed in class, the Lambertian and the specular BRDF are the two most commonly used reflectance models in physics-based vision.

- (i) For Lambertian surfaces, the BRDF is a constant function of the input and the output directions. For such a material, we often describe the reflectance in terms of its *albedo*, which is given the symbol ρ . For a Lambertian surface, the BRDF and the albedo are related by $f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) = \rho/\pi$. Using conservation of energy, prove that $0 \leq \rho \leq 1$.

First, note that the albedo is a measure of the fraction of incident light that is reflected by the surface. Therefore, it cannot be negative, so $0 \leq \rho$. We shall show that $\rho \leq 1$ by using the energy conservation principle in reflectance which stipulates that the energy reflected by a surface cannot exceed the energy incident on the surface. In other words, the integral of the BRDF over the hemisphere must be less than or equal to 1:

$$\int_{H^2} f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) \cos \theta \, d\vec{\omega}_i \leq 1.$$

Substituting in the known values gives:

$$\begin{aligned} \int_{H^2} \frac{\rho}{\pi} \cos \theta \, d\vec{\omega}_i &\leq 1 \\ \frac{\rho}{\pi} \int_{H^2} \frac{dA \cos \theta}{r^2} &\leq 1 \quad \left(\text{since } d\vec{\omega}_i = \frac{dA}{r^2} \right) \\ \frac{\rho}{\pi} \int_{H^2} \frac{r^2 \cos \theta \sin \theta \, d\theta \, d\varphi}{r^2} &\leq 1 \quad \left(\text{since } dA = r^2 \sin \theta \, d\theta \, d\varphi \right) \\ \frac{\rho}{\pi} \int_{H^2} \cos \theta \sin \theta \, d\theta \, d\varphi &\leq 1 \\ \frac{\rho}{\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\varphi &\leq 1 \\ \frac{\rho}{\pi} \int_0^{2\pi} \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/2} d\varphi &\leq 1 \quad \left(\text{since } \int \cos \theta \sin \theta \, d\theta = \frac{\sin^2 \theta}{2} \right) \\ \frac{\rho}{2\pi} \int_0^{2\pi} d\varphi &\leq 1 \\ \frac{\rho}{2\pi} \cdot [\varphi]_0^{2\pi} &\leq 1 \quad \left(\text{since } \int 1 \, d\varphi = \varphi \right) \\ \frac{\rho}{2\pi} \cdot 2\pi &\leq 1 \\ \rho &\leq 1. \end{aligned}$$

- (ii) A specular surface perfectly reflects *radiance* in the *mirror direction*. Concretely, consider a (non-absorbing) specular surface patch with normal $\hat{\mathbf{n}}$. For any incident direction $\hat{\mathbf{v}}_i$, the mirror direction equals $\hat{\mathbf{v}}_s = 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}} - \hat{\mathbf{v}}_i$, and $L(\hat{\mathbf{v}}_s) = L(\hat{\mathbf{v}}_i)$. Given this property, derive an expression for the specular BRDF.

Since the surface is non-absorbing, the Fresnel term $F_r(\bar{\omega}_i) = 1$. The BRDF of an ideal specular reflection is a Dirac delta function as follows:

$$\begin{aligned}
 f_r(\mathbf{x}, \bar{\omega}_i, \bar{\omega}_r) &= F_r(\bar{\omega}_i) \frac{\delta(\bar{\omega}_i - R(\bar{\omega}_r, \hat{\mathbf{n}}))}{\cos \theta_i} \\
 &= F_r(\bar{\omega}_i) \frac{\delta(\bar{\omega}_i - R(\bar{\omega}_r, \hat{\mathbf{n}}))}{\cos \theta_i} \\
 &= \frac{\delta(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_s) \cdot |\hat{\mathbf{v}}_i| |\mathbf{n}|}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}} \\
 &= \frac{\delta(\hat{\mathbf{v}}_i - 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}} + \hat{\mathbf{v}}_i) \cdot |\hat{\mathbf{v}}_i| |\mathbf{n}|}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}} \\
 &= \frac{\delta(2\hat{\mathbf{v}}_i - 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}}) \cdot |\hat{\mathbf{v}}_i| |\mathbf{n}|}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}}
 \end{aligned}$$

If the direction vectors are normalized, then $|\hat{\mathbf{v}}_i| = |\hat{\mathbf{v}}_s| = 1$. Therefore,

$$= \frac{\delta(2\hat{\mathbf{v}}_i - 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}})}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}}$$