CS 83: Computer Vision **Winter 2024**

Quiz 7 — 02/29/2024

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Credit Statement

I discussed solution ideas with:

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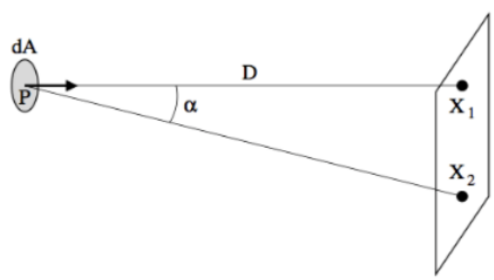
2. Angelic McPherson

However, all typed work is my own, with reference to class notes.

Problem 1.

A small Lambertian source $\mathrm{d}A$ is centered at P and emits radiance L. The orientation of this patch is the same as that of a plane containing two points, X_1 and X_2. The point X_1 is the point on this plane that is closest to P, and the distance from P to X_1 is D as shown.



(i) Calculate the solid angle subtended by $\mathrm{d}A$ at points X_1 and X_2.

The solid angle subtended by an area $\mathrm{d}A$ at a point $\mathbf{x}$ that is a distance r from $\mathrm{d}A$ is given by

$$\mathrm{d}\omega = \frac{\mathrm{d}A\cos\theta}{r^2},$$

where θ is the angle between the normal to $\mathrm{d}A$ and the line from $\mathbf{x}$ to $\mathrm{d}A$. In this case;

1. Since $\mathrm{d}A$ has the same orientation as the plane containing X_1 and X_2, and we are given that X_1 is the point on this plane that is closest to P, then the normal at $\mathrm{d}A$ passes through P. Therefore, $\theta = 0$ and $\cos\theta = 1$. Therefore, $\mathrm{d}\omega_{X_1} = \dfrac{\mathrm{d}A}{D^2}$.
2. As established in part **1.**, the normal at $\mathrm{d}A$ passes through P. Therefore, the angle between the normal to $\mathrm{d}A$ and the line from X_2 to $\mathrm{d}A$ is α as shown in the image. Let $D_2 = \|X_2 - P\|$; then:

$$\begin{aligned}\cos\alpha &= \frac{D}{D_2}\\ \implies D_2 &= \frac{D}{\cos\alpha}.\end{aligned}$$

   Therefore;

$$\begin{aligned}\mathrm{d}\omega_{X_2} &= \frac{\mathrm{d}A\cos\alpha}{D_2^2}\\ &= \frac{\mathrm{d}A\cos\alpha}{D^2/\cos^2\alpha}\\ &= \frac{\mathrm{d}A\cos^3\alpha}{D^2}.\end{aligned}$$

(ii) Calculate the irradiance E incident on the plane at points X_1 and X_2, and calculate the ratio $E(X_1)/E(X_2)$.

The irradiance E incident at a point $\mathbf{x}$ is given by

$$E(\mathbf{x}) = \int_{H^2} L(\mathbf{x}, \omega) \cos\theta \, \mathrm{d}\omega.$$

To find the irradiance at single points X_1 and X_2, we do not have to integrate over the entire hemisphere but simply calculate

$$E(\mathbf{x}) = L \cos\theta \, \mathrm{d}\omega$$

at the two points:

1. The irradiance $E(X_1)$ incident on X_1 is given by

$$\begin{aligned} E(X_1) &= L \cdot \cos\theta_{X_1} \cdot \omega_{X_1} \\ &= L \cdot \cos 0 \cdot \frac{\mathrm{d}A}{D^2} \\ &= \frac{L \cdot \mathrm{d}A}{D^2}. \end{aligned}$$

2. The irradiance $E(X_2)$ incident on X_2 is given by

$$\begin{aligned} E(X_2) &= L \cdot \cos\theta_{X_2} \cdot \omega_{X_2} \\ &= L \cdot \cos\alpha \cdot \frac{\mathrm{d}A \cos^3\alpha}{D^2} \\ &= \frac{L \cdot \mathrm{d}A \cos^4\alpha}{D^2} \end{aligned}$$

3. The ratio $E(X_1)/E(X_2)$ is given by

$$\begin{aligned} \frac{E(X_1)}{E(X_2)} &= \frac{\frac{L \cdot \mathrm{d}A}{D^2}}{\frac{L \cdot \mathrm{d}A \cos^4\alpha}{D^2}} \\ &= \frac{L \cdot \mathrm{d}A}{D^2} \cdot \frac{D^2}{L \cdot \mathrm{d}A \cos^4\alpha} \\ &= \frac{1}{\cos^4\alpha} \\ &= \sec^4\alpha. \end{aligned}$$

Problem 2.

As we discussed in class, the Lambertian and the specular BRDF are the two most commonly used reflectance models in physics-based vision.

(i) For Lambertian surfaces, the BRDF is a constant function of the input and the output directions. For such a material, we often describe the reflectance in terms of its *albedo*, which is given the symbol ρ. For a Lambertian surface, the BRDF and the albedo are related by $f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) = \rho/\pi$. Using conservation of energy, prove that $0 \leq \rho \leq 1$.

First, note that the albedo is a measure of the fraction of incident light that is reflected by the surface. Therefore, it cannot be negatie, so $0 \leq \rho$. We shall show that $\rho \leq 1$ by using the energy conservation principle in reflectance which stipulates that the energy reflected by a surface cannot exceed the energy incident on the surface. In other words, the integral of the BRDF over the hemisphere must be less than or equal to 1:

$$\int_{H^2} f_r(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_r) \cos\theta \, \mathrm{d}\vec{\omega}_i \leq 1.$$

Substituting in the known values gives:

$$\begin{aligned}
\int_{H^2} \frac{\rho}{\pi} \cos\theta \, \mathrm{d}\vec{\omega}_i &\leq 1 \\
\frac{\rho}{\pi} \int_{H^2} \frac{\mathrm{d}A \cos\theta}{r^2} &\leq 1 && \left(\text{since } \mathrm{d}\vec{\omega}_i = \frac{\mathrm{d}A}{r^2}\right) \\
\frac{\rho}{\pi} \int_{H^2} \frac{r^2 \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi}{r^2} &\leq 1 && (\text{since } \mathrm{d}A = r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi) \\
\frac{\rho}{\pi} \int_{H^2} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi &\leq 1 \\
\frac{\rho}{\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi &\leq 1 \\
\frac{\rho}{\pi} \int_0^{2\pi} \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \mathrm{d}\varphi &\leq 1 && \left(\text{since } \int \cos\theta \sin\theta \, \mathrm{d}\theta = \frac{\sin^2\theta}{2}\right) \\
\frac{\rho}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi &\leq 1 \\
\frac{\rho}{2\pi} \cdot [\varphi]_0^{2\pi} &\leq 1 && \left(\text{since } \int 1 \, \mathrm{d}\varphi = \varphi\right) \\
\frac{\rho}{2\pi} \cdot 2\pi &\leq 1 \\
\rho &\leq 1.
\end{aligned}$$

Therefore, $0 \leq \rho \leq 1$.

(ii) A specular surface perfectly reflects *radiance* in the *mirror direction*. Concretely, consider a (non-absorbing) specular surface patch with normal $\hat{\mathbf{n}}$. For any incident direction $\hat{\mathbf{v}}_i$, the mirror direction equals $\hat{\mathbf{v}}_s = 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}} - \hat{\mathbf{v}}_i$, and $L(\hat{\mathbf{v}}_s) = L(\hat{\mathbf{v}}_i)$. Given this property, derive an expression for the specular BRDF.

Since the surface is non-absorbing, the Fresnel term $F_r(\vec{\omega}_i) = 1$. The BRDF of an ideal specular reflection is a Dirac delta function as follows:

$$\begin{aligned}
f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) &= F_r(\vec{\omega}_i) \frac{\delta(\vec{\omega}_i - R(\vec{\omega}_r, \hat{\mathbf{n}}))}{\cos\theta_i} \\
&= F_r(\vec{\omega}_i) \frac{\delta(\vec{\omega}_i - R(\vec{\omega}_r, \hat{\mathbf{n}}))}{\cos\theta_i} \\
&= \frac{\delta(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_s) \cdot |\hat{\mathbf{v}}_i| \, |\mathbf{n}|}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}} \\
&= \frac{\delta(\hat{\mathbf{v}}_i - 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}} + \hat{\mathbf{v}}_i) \cdot |\hat{\mathbf{v}}_i| \, |\mathbf{n}|}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}} \\
&= \frac{\delta(2\hat{\mathbf{v}}_i - 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}}) \cdot |\hat{\mathbf{v}}_i| \, |\mathbf{n}|}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}}
\end{aligned}$$

If the direction vectors are normalized, then $|\hat{\mathbf{v}}_i| = |\hat{\mathbf{v}}_s| = 1$. Therefore,

$$= \frac{\delta(2\hat{\mathbf{v}}_i - 2(\hat{\mathbf{n}}^\top \hat{\mathbf{v}}_i)\hat{\mathbf{n}})}{\hat{\mathbf{v}}_i \cdot \hat{\mathbf{n}}}$$