

**Quiz 1 — 01/11/2024**

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**Credit Statement**

I discussed solution ideas with:

1. Ivy (Aiwei) Zhang
2. Angelic McPherson
3. Lin Shi (Q2 – convolution vs. correlation)

However, all the typed work is my own, with reference to class notes especially on convolutions and correlations.

**Problem 1.**

The continuous convolution of two functions  $f(x)$  and  $g(x)$  is given as

$$[f * g](x) = \int_{-\infty}^{+\infty} f(y) g(x - y) \, dy.$$

- (i) Prove that the convolution of two functions is commutative, i.e., changing the order of operands produces the same result.

$$[f * g] = [g * f]$$

*Hint:* Perform integration by substitution.

By definition,

$$[f * g](x) = \int_{-\infty}^{+\infty} f(y) g(x - y) \, dy$$

Let  $u = x - y$ , then  $du = -dy$  and  $y = x - u$ . Furthermore, when  $y = -\infty$ ,  $u = x - (-\infty) \approx +\infty$ , and when  $y = +\infty$ ,  $u = x - (+\infty) \approx -\infty$ .

$$\begin{aligned} [f * g](x) &= \int_{+\infty}^{-\infty} f(x - u) g(u) (-du) \\ &= - \int_{+\infty}^{-\infty} f(x - u) g(u) \, du \\ &= \int_{-\infty}^{+\infty} f(x - u) g(u) \, du \\ &= \int_{-\infty}^{+\infty} g(u) f(x - u) \, du \\ &= [g * f](x) \end{aligned}$$

- (ii) Prove that the convolution operand is also associative, i.e., rearranging the parentheses on two or more occurrences of the convolution operator produces the same result:

$$[f * g] * h = f * [g * h]$$

*Hint:* Be careful with variables. Understand which variable should be integrated, and why.

By definition,

$$[\varphi * \zeta](x) = \int_{-\infty}^{+\infty} \varphi(y) \zeta(x - y) \, dy.$$

Plugging in  $[f * g]$  for  $\varphi$  and  $h$  for  $\zeta$ , we get:

$$([f * g] * h)(x) = \int_{-\infty}^{+\infty} [f * g](y) h(x - y) \, dy$$

Let us expand  $[f * g](y)$ :

$$\begin{aligned} &= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(z) g(y - z) \, dz \right) h(x - y) \, dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z) g(y - z) h(x - y) \, dz \, dy \end{aligned}$$

Rearranging the integrals:

$$\begin{aligned} &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z) g(y - z) h(x - y) \, dy \, dz \\ &= \int_{-\infty}^{+\infty} f(z) \left( \int_{-\infty}^{+\infty} g(y - z) h(x - y) \, dy \right) \, dz \end{aligned}$$

To simplify the inner integral, substitute  $u = y - z$ , then  $du = dy$  and  $y = z + u$ .

$$\begin{aligned} &= \int_{-\infty}^{+\infty} f(z) \left( \int_{-\infty}^{+\infty} g(u) h(x - (z + u)) \, du \right) \, dz \\ &= \int_{-\infty}^{+\infty} f(z) \underbrace{\left( \int_{-\infty}^{+\infty} g(u) h((x - z) - u) \, du \right)}_{[g * h](x - z)} \, dz \\ &= \int_{-\infty}^{+\infty} f(z) [g * h](x - z) \, dz \\ &= [f * [g * h]](x) \end{aligned}$$

### Problem 2.

In class, we talked about finite-difference approximation to the derivative of the univariate function  $f(x)$ . Using Taylor polynomial approximations of  $f(x+h)$  and  $f(x-h)$ , we can easily show that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2),$$

so that the derivative can be approximated by convolving a discrete version of  $f(x)$  — a vector of values  $(\dots, f(x_o - \Delta), f(x_o), f(x_o + \Delta), \dots)$  with kernel  $(1/2, 0, -1/2)$ . This is termed a central difference because its interval is symmetric about a sample point.

- (i) Derive a higher order central-difference approximation to  $f'(x)$  such that the truncation error tends to zero as  $h^4$  instead of  $h^2$ . *Hint*: consider Taylor polynomial approximations of  $f(x \pm 2h)$  in addition to  $f(x \pm h)$ . (7 points)

#### Taylor Polynomial Approximation of $f(x + \varepsilon)$ :

$$f(x + \varepsilon) = f(x) + \varepsilon f'(x) + \frac{\varepsilon^2}{2!} f''(x) + \frac{\varepsilon^3}{3!} f'''(x) + \frac{\varepsilon^4}{4!} f^{(4)}(x) + \frac{\varepsilon^5}{5!} f^{(5)}(x) + \dots$$

Once we fix  $x$ , we can treat  $f(x)$ ,  $f'(x)$ ,  $f''(x)$ , etc. as constants. To simplify the equations, let's replace  $f''(x)$  and higher order derivatives and their corresponding coefficients with  $c_i = \frac{f^{(i)}(x)}{i!}$ :

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + c_5\varepsilon^5 + \dots$$

Plugging in  $h, 2h, -h$ , and  $-2h$ , respectively:

$$f(x + h) = f(x) + f'(x)h + c_2h^2 + c_3h^3 + c_4h^4 + c_5h^5 + \dots$$

$$f(x + 2h) = f(x) + 2f'(x)h + 4c_2h^2 + 8c_3h^3 + 16c_4h^4 + 32c_5h^5 + \dots$$

$$f(x - h) = f(x) - f'(x)h + c_2h^2 - c_3h^3 + c_4h^4 - c_5h^5 + \dots$$

$$f(x - 2h) = f(x) - 2f'(x)h + 4c_2h^2 - 8c_3h^3 + 16c_4h^4 - 32c_5h^5 + \dots$$

Then the estimations for  $f'(x)$  using  $h$  and  $2h$  are:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - [c_2h^2 + c_4h^4 + c_6h^6 + \dots]. \quad (2.1)$$

$$f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} - [4c_2h^2 + 16c_4h^4 + 64c_6h^6 + \dots] \quad (2.2)$$

We can eliminate the  $h^2$  term in  $f'(x)$  by subtracting 2.2 from 4 times 2.1:

$$3f'(x) = 4 \left[ \frac{f(x+h) - f(x-h)}{2h} \right] - \left[ \frac{f(x+2h) - f(x-2h)}{4h} \right] - [(4h^2 - 4h^2) + O(h^4)]$$

$$3f'(x) = \frac{8[f(x+h) - f(x-h)] - [f(x+2h) - f(x-2h)]}{4h} + O(h^4)$$

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + O(h^4)$$

$$f'(x) = \frac{-1}{12h}f(x+2h) + \frac{2}{3h}f(x+h) + 0f(x) + \frac{-2}{3h}f(x-h) + \frac{1}{12h}f(x-2h) + O(h^4)$$

We get a Taylor approximation of  $f'(x)$  with a truncation error of  $\mathcal{O}(h^4)$ .

(ii) What is the corresponding convolution (not correlation!) kernel? (3 points)

The *convolution* kernel picks up the coefficients starting from the positive boundary,  $x + 2h$ , and steps down to the negative boundary,  $x - 2h$ :

$$\left[ -\frac{1}{12}, \quad \frac{2}{3}, \quad 0, \quad -\frac{2}{3}, \quad \frac{1}{12} \right].$$