CS 83: Computer Vision	Winter '23 Year

Quiz 3 — 02/02/2023

Prof. Pediredla — *Student: Amittai Siavava*

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(i) **Computer Vision: Algorithms and Applications** by **Richard Szeliski**

Problem 1.

Given a set N of points $\mathbf{p}_i = (x_i, y_i), i \in \{1, \ldots, N\}$, in the image plane, we wish to find the best line passing through those points.

(a) One way to solve this problem is to find (a, b) that most closely satisfy the equations $y_i = ax_i + b$, in a least-squares sense. Write these equations in the form of a *heterogeneous* least-squares problem $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (a, b)^T$, and give an expression for the least-squares estimate of $\mathbf{x}$. Give a geometric interpretation of the error being minimized, and use a simple graph to visualize the error. Does this make sense when fitting a line to points in an image?

> The error being minimized is the sum of the squared distances of the points from the line. This makes sense when fitting a line to points in an image, since the error is the sum of the squared distances of the points from the line.

(b) Another way to solve this problem is to find $\ell = (a, b, c)$, defined up to scale, that most closely satisfies the equations $ax_i + by_i + c = 0$, in a least-squares sense. Write these equations in the form of a *homogeneous* least-squares problem $A\ell = 0$, where $\ell = (a, b, c)^T$ and $\ell \neq 0$. This problem has a trivial solution (zero vector) which is not of much use. Describe some ways of avoiding this trivial solution and corresponding algorithms for solving the resulting optimization problem. Is this approach more or less useful than the previous one? Why?

Hint: Think about how we can express the distance of a point from a line.

Problem 2.

The equation for a conic in the plane using inhomogeneous coordinates (x, y) is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{2.1}$$

(a) Suppose you have a given a set of inhomogeneous points $\mathbf{x}_i = (x_i, y_i), i \in \{1, \ldots, N\}$. Derive an expression for the least squares estimate of a conic $\mathbf{c} = (a, b, c, d, e, f)$ passing through those points.

Note: your expression may take the form of a null vector or eigenvector of a matrix. If so, you must provide expressions for the matrix elements.

To determine the matching conic, we need to solve the equation

$$\mathbf{A}\mathbf{x} = 0,$$

where we know the matrix $\mathbf{A}$ (the points given), and we know the $\vec{0}$ vector, but we need to determine the vector $\mathbf{x}$ (the conic coefficients).

We compute $\mathbf{A}$ to be a matrix where the ith row, R_i, is computed from the ith point $\mathbf{x}_i = (x_i, y_i)$ as follows:

$$R_i = \begin{bmatrix} x_i^2 & x_i y_i & y_i^2 & x_i & y_i & 1 \end{bmatrix}$$

We solve the equation $\mathbf{A}\mathbf{x} = 0$ by converting $\mathbf{A}$ into its singular value decomposition

$$A = U\Sigma V^T$$

so that

$$U\Sigma V^T \mathbf{x} = 0.$$

From the singular value decomposition, we compute V (the transpose of V^T) and find the column corresponding to the smallest singular value. This column contains the coefficients of the conic fitting the points.

(b) In general, what is the minimum value of N that allows a unique solution for $\mathbf{c}$?

The minimum number of required points is 5, since there are 5 degrees of freedom.

(c) Homogenize equation 2.1 by making the substitutions $x \leftarrow x_1/x_3$ and $y \leftarrow y_1/y_3$, and show that in terms of homogeneous coordinates ($\mathbf{x} = (x_1, x_2, x_3)$) the conic can be expressed in matrix form

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0,$$

where $\mathbf{C}$ is a symmetric matrix.

$$\begin{aligned} ax^2 + bxy + cy^2 + dx + ey + f &= 0 \\ a\left(\frac{x_1}{x_3}\right)^2 + b\frac{x_1 y_1}{x_3 y_3} + c\left(\frac{y_1}{y_3}\right)^2 + d\frac{x_1}{x_3} + e\frac{y_1}{y_3} + f &= 0 \\ \frac{x_1^2}{x_3^2}a + \frac{x_1 y_1}{x_3 y_3}b + \frac{y_1^2}{y_3^2}c + \frac{x_1}{x_3}d + \frac{y_1}{y_3}e + f &= 0 \\ \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} &= 0 \end{aligned}$$

(d) Suppose we apply a projective transformation to our points $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$. The transformed points x_i' will lie on a transformed conic represented by a new symmetric matrix $\mathbf{C}'$. Write an equation that specifies the relationship between $\mathbf{C}'$ and $\mathbf{C}$ in terms of the homography $\mathbf{H}$.

$$\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$$

Problem 3.

The affine transform in heterogeneous coordinates is given by

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.1}$$

Affine transformations are combinations of:

(i) Arbitrary linear transformations with 4 degrees of freedom (a, b, d, e).

(ii) Translations with 2 degrees of freedom (c, f).

Does affine transformation apply translation first followed by arbitrary linear transformation, or the other way around? Prove your answer mathematically.

Hint: check if

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$$

By computing the matrix products, we see that:

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

$$\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & ac+bf \\ d & e & dc+ef \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

Our desired affine transformation matrix (3.1) matches the first matrix product (3.2).

Therefore, we achieve the affine transformation by doing the equivalent matrix multiplications in order, i.e. doing translation first then doing linear transformation.