CS 83: Computer Vision **Winter 2024**

Quiz 1 — 01/11/2024

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Problem 1.

The continuous convolution of two functions $f(x)$ and $g(x)$ is given as

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\ \mathrm{d}y.$$

(i) Prove that the convolution of two functions is commutative, i.e., changing the order of operands produces the same result.

$$(f * g) = (g * f)$$

Hint: Perform integration by substitution.

By definition,

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\ \mathrm{d}y$$

Let $u = x - y$, then $\mathrm{d}u = -\,\mathrm{d}y$ and $y = x - u$. Furthermore, when $y = -\infty$, $u = x - (-\infty) \approx +\infty$, and when $y = +\infty$, $u = x - (+\infty) \approx -\infty$.

$$\begin{aligned}
(f * g)(x) &= \int_{+\infty}^{-\infty} f(x-u)\, g(u)\, (-\,\mathrm{d}u) \\
&= -\int_{+\infty}^{-\infty} f(x-u)\, g(u)\ \mathrm{d}u \\
&= \int_{-\infty}^{+\infty} f(x-u)\, g(u)\ \mathrm{d}u \\
&= \int_{-\infty}^{+\infty} g(u)\, f(x-u)\ \mathrm{d}u \\
&= (g * f)(x)
\end{aligned}$$

(ii) Prove that the convolution operand is also associative, i.e., rearranging the parentheses on two or more occurrences of the convolution operator produces the same result:

$$(f * g) * h = f * (g * h)$$

Hint: Be careful with variables. Understand which variable should be integrated, and why.

By definition,

$$(\varphi * \zeta)(x) = \int_{-\infty}^{+\infty} \varphi(y)\,\zeta(x-y)\,\mathrm{d}y.$$

Plugging in $(f * g)$ for φ and h for ζ, we get:

$$((f * g) * h)(x) = \int_{-\infty}^{+\infty} (f * g)(y)\,h(x-y)\,\mathrm{d}y$$

Let us expand $(f * g)(y)$:

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(z)\,g(y-z)\,\mathrm{d}z \right) h(x-y)\,\mathrm{d}y$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z)\,g(y-z)\,h(x-y)\,\mathrm{d}z\,\mathrm{d}y$$

Rearranging the integrals:

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z)\,g(y-z)\,h(x-y)\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_{-\infty}^{+\infty} f(z) \left(\int_{-\infty}^{+\infty} g(y-z)\,h(x-y)\,\mathrm{d}y \right) \mathrm{d}z$$

To simplify the inner integral, substitute $u = y - z$, then $\mathrm{d}u = \mathrm{d}y$ and $y = z + u$.

$$= \int_{-\infty}^{+\infty} f(z) \left(\int_{-\infty}^{+\infty} g(u)\,h(x-(z+u))\,\mathrm{d}u \right) \mathrm{d}z$$

$$= \int_{-\infty}^{+\infty} f(z) \left(\int_{-\infty}^{+\infty} g(u)\,h((x-z)-u)\,\mathrm{d}u \right) \mathrm{d}z$$

$$= \int_{-\infty}^{+\infty} f(z)\,(g * h)(x-z)\,\mathrm{d}z$$

$$= (f * (g * h))(x)$$

Problem 2.

In class, we talked about finite-difference approximation to the derivative of the univariate function $f(x)$. Using Taylor polynomial approximations of $f(x+h)$ and $f(x-h)$, we can easily show that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2),$$

so that the derivative can be approximated by convolving a discrete version of $f(x)$ — a vector of values $(\ldots,\ f(x_o - \Delta), f(x_o), f(x_o + \Delta), \ldots)$ with kernel $(1/2, 0, -1/2)$. This is termed a central difference because its interval is symmetric about a sample point.

(i) Derive a higher order central-difference approximation to $f'(x)$ such that the truncation error tends to zero as h^4 instead of h^2. *Hint* : consider Taylor polynomial approximations of $f(x \pm 2h)$ in addition to $f(x \pm h)$. (7 points)

Taylor Polynomial Approximation of $f(x+\varepsilon)$:

$$f(x+\varepsilon) = f(x) + \varepsilon f'(x) + \frac{\varepsilon^2}{2!} f''(x) + \frac{\varepsilon^3}{3!} f'''(x) + \frac{\varepsilon^4}{4!} f^{(4)}(x) + \frac{\varepsilon^5}{5!} f^{(5)}(x) + \ldots$$

Once we fix x, we can treat $f(x)$, $f'(x)$, $f''(x)$, etc. as constants. To simplify the equations, let's replace $f''(x)$ and higher order derivatives and their corresponding coefficients with $c_i = \dfrac{f^{(i)}(x)}{i!}$:

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + c_5\varepsilon^5 + \ldots.$$

Plugging in $h, 2h, -h$, and $-2h$, respectively:

$$f(x+h) = f(x) + f'(x)h + c_2h^2 + c_3h^3 + c_4h^4 + c_5h^5 + \ldots$$

$$f(x+2h) = f(x) + 2f'(x)h + 4c_2h^2 + 8c_3h^3 + 16c_4h^4 + 32c_5h^5 + \ldots$$

$$f(x-h) = f(x) - f'(x)h + c_2h^2 - c_3h^3 + c_4h^4 - c_5h^5 + \ldots$$

$$f(x-2h) = f(x) - 2f'(x)h + 4c_2h^2 - 8c_3h^3 + 16c_4h^4 - 32c_5h^5 + \ldots$$

Then the estimations for $f'(x)$ using h and $2h$ are:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \left[c_2h^2 + c_4h^4 + c_6h^6 + \ldots\right]. \tag{2.1}$$

$$f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} - \left[4c_2h^2 + 16c_4h^4 + 64c_6h^6 + \ldots\right] \tag{2.2}$$

We can eliminate the h^2 term in $f'(x)$ by subtracting 2.2 from 4 times 2.1:

$$
\begin{aligned}
3f'(x) &= 4\left[\frac{f(x+h)-f(x-h)}{2h}\right] - \left[\frac{f(x+2h)-f(x-2h)}{4h}\right] - \left[(4h^2-4h^2)+O(h^4)\right] \\
3f'(x) &= \frac{8\left[f(x+h)-f(x-h)\right] - \left[f(x+2h)-f(x-2h)\right]}{4h} + O(h^4) \\
f'(x) &= \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h} + O(h^4) \\
f'(x) &= \frac{-1}{12}\left[f(x+2h)\right] + \frac{2}{3}\left[f(x+h)\right] + 0\left[f(x)\right] + \frac{-2}{3}\left[f(x-h)\right] + \frac{1}{12}\left[f(x-2h)\right] + O(h^4)
\end{aligned}
$$

We get a Taylor approximation of $f'(x)$ with a truncation error of $\mathcal{O}(h^4)$.

(ii) What is the corresponding convolution (not correlation!) kernel? (3 points)

The approximation has a correlation kernel of

$$\left[-\tfrac{1}{12},\quad \tfrac{8}{12},\quad 0,\quad -\tfrac{8}{12},\quad \tfrac{1}{12}\right].$$

The convolution kernel is the same as the correlation kernel, but flipped:

$$\left[\tfrac{1}{12},\quad -\tfrac{8}{12},\quad 0,\quad \tfrac{8}{12},\quad -\tfrac{1}{12}\right].$$