CS 83: Computer Vision **Winter 2024**

Quiz 1 — 01/11/2024

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Credit Statement

I discussed solution ideas with:

1. Ivy (Aiwei) Zhang
2. Angelic McPherson
3. Lin Shi (Q2 – convolution vs. correlation)

However, all the typed work is my own, with reference to class notes especially on convolutions and correlations.

Problem 1.

The continuous convolution of two functions $f(x)$ and $g(x)$ is given as

$$[f * g](x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\ \mathrm{d}y.$$

(i) Prove that the convolution of two functions is commutative, i.e., changing the order of operands produces the same result.

$$[f * g] = [g * f]$$

Hint: Perform integration by substitution.

By definition,

$$[f * g](x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\ \mathrm{d}y$$

Let $u = x - y$, then $\mathrm{d}u = -\,\mathrm{d}y$ and $y = x - u$. Furthermore, when $y = -\infty$, $u = x - (-\infty) \approx +\infty$, and when $y = +\infty$, $u = x - (+\infty) \approx -\infty$.

$$\begin{aligned}
[f * g](x) &= \int_{+\infty}^{-\infty} f(x-u)\, g(u)\, (-\,\mathrm{d}u) \\
&= -\int_{+\infty}^{-\infty} f(x-u)\, g(u)\ \mathrm{d}u \\
&= \int_{-\infty}^{+\infty} f(x-u)\, g(u)\ \mathrm{d}u \\
&= \int_{-\infty}^{+\infty} g(u)\, f(x-u)\ \mathrm{d}u \\
&= [g * f](x)
\end{aligned}$$

(ii) Prove that the convolution operand is also associative, i.e., rearranging the parentheses on two or more occurrences of the convolution operator produces the same result:

$$[f * g] * h = f * [g * h]$$

Hint: Be careful with variables. Understand which variable should be integrated, and why.

By definition,

$$[\varphi * \zeta](x) = \int_{-\infty}^{+\infty} \varphi(y)\, \zeta(x-y)\ \mathrm{d}y.$$

Plugging in $[f * g]$ for φ and h for ζ, we get:

$$([f * g] * h)(x) = \int_{-\infty}^{+\infty} [f * g](y)\, h(x-y)\ \mathrm{d}y$$

Let us expand $[f * g](y)$:

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(z)\, g(y-z)\ \mathrm{d}z \right) h(x-y)\ \mathrm{d}y$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z)\, g(y-z)\, h(x-y)\ \mathrm{d}z\ \mathrm{d}y$$

Rearranging the integrals:

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z)\, g(y-z)\, h(x-y)\ \mathrm{d}y\ \mathrm{d}z$$

$$= \int_{-\infty}^{+\infty} f(z) \left(\int_{-\infty}^{+\infty} g(y-z)\, h(x-y)\ \mathrm{d}y \right) \mathrm{d}z$$

To simplify the inner integral, substitute $u = y - z$, then $\mathrm{d}u = \mathrm{d}y$ and $y = z + u$.

$$= \int_{-\infty}^{+\infty} f(z) \left(\int_{-\infty}^{+\infty} g(u)\, h(x - (z+u))\ \mathrm{d}u \right) \mathrm{d}z$$

$$= \int_{-\infty}^{+\infty} f(z) \underbrace{\left(\int_{-\infty}^{+\infty} g(u)\, h((x-z) - u)\ \mathrm{d}u \right)}_{[g*h](x-z)} \mathrm{d}z$$

$$= \int_{-\infty}^{+\infty} f(z)\, [g * h](x-z)\ \mathrm{d}z$$

$$= [f * [g * h]]\,(x)$$

Problem 2.

In class, we talked about finite-difference approximation to the derivative of the univariate function $f(x)$. Using Taylor polynomial approximations of $f(x+h)$ and $f(x-h)$, we can easily show that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2),$$

so that the derivative can be approximated by convolving a discrete version of $f(x)$ — a vector of values $(\ldots,\ f(x_o - \Delta), f(x_o), f(x_o+\Delta), \ldots)$ with kernel $(1/2, 0, -1/2)$. This is termed a central difference because its interval is symmetric about a sample point.

(i) Derive a higher order central-difference approximation to $f'(x)$ such that the truncation error tends to zero as h^4 instead of h^2. *Hint* : consider Taylor polynomial approximations of $f(x \pm 2h)$ in addition to $f(x \pm h)$. (7 points)

Taylor Polynomial Approximation of $f(x+\varepsilon)$:

$$f(x+\varepsilon) = f(x) + \varepsilon f'(x) + \frac{\varepsilon^2}{2!} f''(x) + \frac{\varepsilon^3}{3!} f'''(x) + \frac{\varepsilon^4}{4!} f^{(4)}(x) + \frac{\varepsilon^5}{5!} f^{(5)}(x) + \ldots$$

Once we fix x, we can treat $f(x)$, $f'(x)$, $f''(x)$, etc. as constants. To simplify the equations, let's replace $f''(x)$ and higher order derivatives and their corresponding coefficients with $c_i = \dfrac{f^{(i)}(x)}{i!}$:

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + c_5\varepsilon^5 + \ldots.$$

Plugging in $h, 2h, -h$, and $-2h$, respectively:

$$f(x+h) = f(x) + f'(x)h + c_2h^2 + c_3h^3 + c_4h^4 + c_5h^5 + \ldots$$

$$f(x+2h) = f(x) + 2f'(x)h + 4c_2h^2 + 8c_3h^3 + 16c_4h^4 + 32c_5h^5 + \ldots$$

$$f(x-h) = f(x) - f'(x)h + c_2h^2 - c_3h^3 + c_4h^4 - c_5h^5 + \ldots$$

$$f(x-2h) = f(x) - 2f'(x)h + 4c_2h^2 - 8c_3h^3 + 16c_4h^4 - 32c_5h^5 + \ldots$$

Then the estimations for $f'(x)$ using h and $2h$ are:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \left[c_2h^2 + c_4h^4 + c_6h^6 + \ldots\right]. \tag{2.1}$$

$$f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} - \left[4c_2h^2 + 16c_4h^4 + 64c_6h^6 + \ldots\right] \tag{2.2}$$

We can eliminate the h^2 term in $f'(x)$ by subtracting 2.2 from 4 times 2.1:

$$3f'(x) = 4\left[\frac{f(x+h) - f(x-h)}{2h}\right] - \left[\frac{f(x+2h) - f(x-2h)}{4h}\right] - \left[(4h^2 - 4h^2) + O(h^4)\right]$$

$$3f'(x) = \frac{8\left[f(x+h) - f(x-h)\right] - \left[f(x+2h) - f(x-2h)\right]}{4h} + O(h^4)$$

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + O(h^4)$$

$$f'(x) = \frac{-1}{12h}f(x+2h) + \frac{2}{3h}f(x+h) + 0f(x) + \frac{-2}{3h}f(x-h) + \frac{1}{12h}f(x-2h) + O(h^4)$$

We get a Taylor approximation of $f'(x)$ with a truncation error of $\mathcal{O}(h^4)$.

(ii) What is the corresponding convolution (not correlation!) kernel? (3 points)

The *convolution* kernel picks up the coefficients starting from the positive boundary, $x + 2h$, and steps down to the negative boundary, $x - 2h$:

$$\left[-\tfrac{1}{12}, \quad \tfrac{2}{3}, \quad 0, \quad -\tfrac{2}{3}, \quad \tfrac{1}{12}\right].$$