CS 83: Computer Vision	**Winter 2024**

Quiz 6 — 02/22/2024

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Credit Statement

I discussed solution ideas with:

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2. Angelic McPherson

However, all typed work is my own, with reference to class notes.

Problem 1.

You will derive the Lucas Kanade (or forward-additive) image alignment algorithm by replicating the derivation learned in class. Consider first a *warp function* $\mathbf{W}(\mathbf{x};\mathbf{p})$ that maps coordinate vectors $\mathbf{x} \in \mathbb{R}^2$ to other coordinate vectors in $\mathbb{R}^2$, with the mapping depending on a set of parameters $\mathbf{p} \in \mathbb{R}^N$. Given an image $I(\mathbf{x})$ and a template $T(\mathbf{x})$, we want to find the parameters $\mathbf{p}$ such that the warp function $\mathbf{W}(\mathbf{x};\mathbf{p})$ best aligns the image with the template in terms of sum-of-squared differences (SSD) error. That is, we want to find the parameters $\mathbf{p}$ that minimize the loss function:

$$\min_{\mathbf{p}} \sum_{\mathbf{x}} \left[I(\mathbf{W}(\mathbf{x};\mathbf{p})) - T(\mathbf{x})\right]^2 \tag{1.1}$$

The Lucas-Kanade alignment algorithm minimizes Equation (1.1) using the Gauss-Newton algorithm. To this end, given some initial set of parameters $\mathbf{p}_0$, they are updated iteratively as:

$$\mathbf{p}^{t+1} = \mathbf{p}^t + \Delta\mathbf{p}^t \tag{1.2}$$

for $t = 0, 1, 2, \ldots, T$, where the number of iterations T can be selected based on any of the common convergence criteria. Then, the Gauss-Newton algorithm corresponds to selecting a specific form for the update vector $\Delta\mathbf{p}^t$, which you will derive below step-by-step.

(i) Use the first-order Taylor expansion to linearize the composite function $I(\mathbf{W}(\mathbf{x};\mathbf{p}))$ with respect to $\mathbf{p}$ around the value $\mathbf{p}^t$. Write out the expression for this Taylor expansion.

The general form of the first-order Taylor expansion of a function f around a point x is given by:

$$f(x+h) \approx f(x) + f'(x)\cdot h.$$

Plugging in the composite function $I(\mathbf{W}(\mathbf{x};\mathbf{p}))$ and centering the expansion at $\mathbf{p}^t$, the approximation for $\mathbf{p}$ in the neighborhood of $\mathbf{p}^t$ is given by:

$$I(\mathbf{W}(\mathbf{x};\mathbf{p})) \approx I(\mathbf{W}(\mathbf{x};\mathbf{p}^t)) + \frac{\mathrm{d}I}{\mathrm{d}\mathbf{p}}(\mathbf{W}(\mathbf{x};\mathbf{p}^t))\left[\mathbf{p}-\mathbf{p}^t\right]. \tag{1.3}$$

(ii) Combine the Taylor expansion expression with Equation (1.2) to obtain an approximation for $I(\mathbf{W}(\mathbf{x};\mathbf{p}^t+\Delta\mathbf{p}^t))$.

Substituting $\mathbf{p} := \mathbf{p}^t + \Delta\mathbf{p}^t$ into the first-order Taylor expansion (1.3) gives:

$$I(\mathbf{W}(\mathbf{x};\mathbf{p}^{t+1})) = I(\mathbf{W}(\mathbf{x};\mathbf{p}^t+\Delta\mathbf{p}^t)) \approx I(\mathbf{W}(\mathbf{x};\mathbf{p}^t)) + \frac{\mathrm{d}I}{\mathrm{d}\mathbf{p}}(\mathbf{W}(\mathbf{x};\mathbf{p}^t))\Delta\mathbf{p}^t, \tag{1.4}$$

where $\frac{\mathrm{d}I}{\mathrm{d}\mathbf{p}}$ is the image gradient with respect to the warp parameters, equivalent to $\nabla I\left[\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}\mathbf{p}}\right]$.

(iii) Show that, using this approximation, the approximation problem of Equation (1.1) can be written in the form:

$$\min_{\Delta\mathbf{p}^t}\left\|\mathbf{A}\Delta\mathbf{p}^t-\mathbf{b}\right\|^2 \tag{1.5}$$

for some matrix $\mathbf{A}$ and vector $\mathbf{b}$.

Substituting the approximation from Equation (1.4) into the loss function of Equation (1.1) gives:

$$\begin{aligned}
\min_{\Delta\mathbf{p}^t}\left\|I(\mathbf{W}(\mathbf{x};\mathbf{p}^t+\Delta\mathbf{p}^t))-T(\mathbf{x})\right\|^2 &\approx \min_{\Delta\mathbf{p}^t}\left\|I(\mathbf{W}(\mathbf{x};\mathbf{p}^t))+\frac{\mathrm{d}I}{\mathrm{d}\mathbf{p}}(\mathbf{W}(\mathbf{x};\mathbf{p}^t))\Delta\mathbf{p}^t-T(\mathbf{x})\right\|^2\\
&\approx \min_{\Delta\mathbf{p}^t}\left\|\frac{\mathrm{d}I}{\mathrm{d}\mathbf{p}}(\mathbf{W}(\mathbf{x};\mathbf{p}^t))\Delta\mathbf{p}^t-\left[T(\mathbf{x})-I(\mathbf{W}(\mathbf{x};\mathbf{p}^t))\right]\right\|^2\\
&\approx \min_{\Delta\mathbf{p}^t}\left\|\mathbf{A}\Delta\mathbf{p}^t-\mathbf{b}\right\|^2,
\end{aligned}$$

where

$$\mathbf{A} := \frac{\mathrm{d}I}{\mathrm{d}\mathbf{p}}(\mathbf{W}(\mathbf{x};\mathbf{p}^t)) = \begin{bmatrix}\frac{\partial I_x}{\partial p_1} & \frac{\partial I_x}{\partial p_2} & \dots & \frac{\partial I_x}{\partial p_N}\\ \frac{\partial I_y}{\partial p_1} & \frac{\partial I_y}{\partial p_2} & \dots & \frac{\partial I_y}{\partial p_N}\end{bmatrix}(\mathbf{W}(\mathbf{x};\mathbf{p}^t)) \qquad \text{and} \qquad \mathbf{b} := T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}^t)).$$

Note that $x \in \mathbb{R}^2$ and $\mathbf{p} \in \mathbb{R}^N$, so $\mathbf{A}$ is not necessarily square!

(iv) Show how to solve the optimization problem of Equation (1.5) for the parameter update $\Delta\mathbf{p}^t$, and write out an expression for this solution.

The solution to the optimization problem of Equation (1.5) is given by the least-squares solution to the linear system $\mathbf{A}\Delta\mathbf{p}^t = \mathbf{b}$. This solution is given by:

$$
\begin{aligned}
\mathbf{A}\Delta\mathbf{p}^t - \mathbf{b} &= 0 \\
\mathbf{A}\Delta\mathbf{p}^t &= \mathbf{b} \\
\underbrace{\mathbf{A}^T\mathbf{A}}_{2\times 2}\Delta\mathbf{p}^t &= \mathbf{A}^T\mathbf{b} \\
\Delta\mathbf{p}^t &= \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{b}.
\end{aligned}
$$

Thus, we can solve for the parameter update $\Delta\mathbf{p}^t$ using the expression

$$
\Delta\mathbf{p}^t = \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{b}. \tag{1.6}
$$

(v) Finally, explain how this expression for $\Delta\mathbf{p}^t$ can be evaluated, using image convolutions, warps, element-wise operations, and matrix-vector operations. You can either explain this in words or provide pseudocode, but make sure to explain each step clearly.

The expression for $\Delta\mathbf{p}^t$ in Equation (1.6) can be evaluated using the following steps:

1. Warp the image $I(\mathbf{x})$ using the current warp parameters $\mathbf{p}^t$ to obtain the warped image $I(\mathbf{W}(\mathbf{x};\mathbf{p}^t))$.
2. Compute the image gradients $\nabla I(\mathbf{x}') := \begin{bmatrix} I_x(\mathbf{x}') \\ I_y(\mathbf{x}') \end{bmatrix}$ by convolving the warped image with the appropriate filters, such as the Sobel kernel. Note that $\mathbf{x}' = \mathbf{W}(\mathbf{x};\mathbf{p}^t)$.
3. Compute the Jacobian $\frac{\partial \mathbf{W}}{\partial \mathbf{p}}$, by taking the partial derivatives of the warp function $\mathbf{W}(\mathbf{x};\mathbf{p}^t)$ with respect to the warp parameters $\mathbf{p}^t$.
4. Compute the *matrix* of image gradients with respect to the warp parameters, $\frac{\mathrm{d}I}{\mathrm{d}\mathbf{p}} = (\nabla I)\frac{\partial \mathbf{W}}{\partial \mathbf{p}}$.
5. Compute the matrix $\mathbf{A}$ by taking the partial derivatives of the image I with respect to the warp parameters $\mathbf{p}$. This can be done by convolving the image with the appropriate kernel, such as the Sobel kernel, to compute the image gradients I_x and I_y. Then, the Jacobian matrix is the cross-product of the image gradients with the partial-derivative operators for the warp parameters:
$$\mathbf{A} = \begin{bmatrix} I_x(\mathbf{W}(\mathbf{x};\mathbf{p}^t)) \\ I_y(\mathbf{W}(\mathbf{x};\mathbf{p}^t)) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial p_1} & \frac{\partial}{\partial p_2} & \cdots & \frac{\partial}{\partial p_N} \end{bmatrix} (\mathbf{W}(\mathbf{x};\mathbf{p})).$$
6. Compute the residual vector $\mathbf{b}$ by taking the difference between the template $T(\mathbf{x})$ and the warped image $I(\mathbf{W}(\mathbf{x};\mathbf{p}^t))$.
7. Compute the matrix $\mathbf{A}^T\mathbf{A}$ and the vector $\mathbf{A}^T\mathbf{b}$.
8. Solve the linear system $\mathbf{A}\Delta\mathbf{p}^t = \mathbf{b}$ by solving Equation (1.6);
$$\Delta\mathbf{p}^t = \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{b}.$$

Problem 2.

In this question, we will derive the Baker-Matthews (or inverse compositional) image alignment algorithm for a simpler case. Consider first the *warp function* $\mathbf{W}(\mathbf{x};\Delta\mathbf{p})$ that maps coordinate vectors $\mathbf{x} \in \mathbb{R}^2$ to other coordinate vectors in $\mathbb{R}^2$:

$$\mathbf{W}(\mathbf{x};\Delta\mathbf{p}) = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_1 x \\ p_2 y \end{bmatrix} \qquad \text{where} \qquad \mathbf{p} := \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \mathbf{x} := \begin{bmatrix} x \\ y \end{bmatrix}, \tag{2.1}$$

where the mapping depends on a set of parameters $\mathbf{p} \in \mathbb{R}^2$. Given an image $I(\mathbf{x})$ and a template $T(\mathbf{x})$, we want to find the parameters $\mathbf{p}$ such that the warp function $\mathbf{W}(\mathbf{x};\Delta\mathbf{p})$ (representing a scaling operation) best aligns the image with the template in terms of sum-of-squared differences (SSD) error.

The idea behind the Baker-Matthews alignment algorithm is to iteratively find incremental warp parameter $\Delta\mathbf{p}$ that minimize the following objective:

$$\min_{\Delta\mathbf{p}} \sum_{\mathbf{x}} \left[T(\mathbf{W}(\mathbf{x};\Delta\mathbf{p})) - I(\mathbf{W}(\mathbf{x};\Delta\mathbf{p})) \right]^2 . \tag{2.2}$$

We will use the Gauss-Newton algorithm to solve this non-linear and non-parametric least-squares problem to compute the incremental warp parameters $\Delta\mathbf{p}$.

(i) The Baker-Matthews alignment algorithm assumes that $\mathbf{W}(\mathbf{x};\mathbf{0})$ is the identity warp, i.e., $\mathbf{W}(\mathbf{x};\mathbf{0}) = \mathbf{x}$. However, the current warp function in Equation (2.1) does not satisfy this property. Propose a modified form of this scaling warp function that satisfies the identity assumption and write out its corresponding Jacobian $\frac{\partial \mathbf{W}}{\partial \mathbf{p}}$.
(Note that there are multiple valid solutions for this part of the quiz.)

The current warp function $\mathbf{W}(\mathbf{x};\Delta\mathbf{p})$ in Equation (2.1) has the problem that

$$\mathbf{W}(\mathbf{x};\mathbf{0}) = \begin{bmatrix} 0x \\ 0y \end{bmatrix} = \mathbf{0} \neq \mathbf{x}.$$

To make it satisfy the identity assumption, we can add a translation term to the warp function:

$$\begin{aligned} \mathbf{W}(\mathbf{x};\Delta\mathbf{p}) &= \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} p_1 + 1 & 0 \\ 0 & p_2 + 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

The Jacobian of this modified warp function with respect to the warp parameters $\mathbf{p}$ is given by:

$$\frac{\partial \mathbf{W}}{\partial \mathbf{p}} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \tag{2.3}$$

(ii) Use the first-order Taylor expansion to linearize the composite function $T(\mathbf{W}(\mathbf{x}; \Delta\mathbf{p}))$ with respect to $\Delta\mathbf{p}$ around the value $\mathbf{0}$. Write out the expression for this Taylor expansion. Plug this expression back into Equation (2.2).

The general form of the first-order Taylor expansion of a function f around a point x is given by:

$$f(x+h) \approx f(x) + f'(x) \cdot h.$$

Plugging in the composite function $T(\mathbf{W}(\mathbf{x}; \Delta\mathbf{p}))$ and centering the expansion at $\mathbf{0}$, the approximation for $\Delta\mathbf{p}$ in the neighborhood of $\mathbf{0}$ is given by:

$$\begin{aligned} T(\mathbf{W}(\mathbf{x}; \Delta\mathbf{p})) &\approx T(\mathbf{W}(\mathbf{x}; \mathbf{0})) + \frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{W}(\mathbf{x}; \mathbf{0}))\Delta p \\ &\approx T(\mathbf{x}) + \frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{x})\Delta\mathbf{p} \qquad \text{since } \mathbf{W}(\mathbf{x}; \mathbf{0}) = \mathbf{x} \end{aligned}$$

where $\frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}$ is the template gradient with respect to the warp parameters, equivalent to:

$$\frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{x}) = \nabla T(\mathbf{x}) \frac{\partial \mathbf{W}}{\mathrm{d}\mathbf{p}} = \nabla T(\mathbf{x}) \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}.$$

Thus, we can solve the optimization problem of Equation (2.2) by solving the following optimization problem:

$$\begin{aligned} \mathbb{E} &= \min_{\Delta\mathbf{p}} \sum_{\mathbf{x}} \left\| T(\mathbf{W}(\mathbf{x}; \Delta\mathbf{p})) - I(\mathbf{W}() \right\|^2 \\ &= \min_{\Delta\mathbf{p}} \sum_{\mathbf{x}} \left\| T(\mathbf{W}(\mathbf{x}; \mathbf{0})) + \frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{W}(\mathbf{x}; \mathbf{0}))\Delta\mathbf{p} - I(\mathbf{W}(\mathbf{x}; \Delta\mathbf{p})) \right\|^2 \\ &= \min_{\Delta\mathbf{p}} \sum_{\mathbf{x}} \left\| T(\mathbf{x}) + \frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{x})\Delta\mathbf{p} - I(\mathbf{W}(\mathbf{x}; \Delta\mathbf{p})) \right\|^2 \qquad \text{since } \mathbf{W}(\mathbf{x}; \mathbf{0}) = \mathbf{x}. \end{aligned}$$

(iii) Show that this approximation to the optimization problem can be rewritten in the form:

$$\min_{\Delta\mathbf{p}} \|\mathbf{A}\Delta\mathbf{p} - \mathbf{b}\|^2, \tag{2.4}$$

for some matrix $\mathbf{A}$ and vector $\mathbf{b}$. Show how to solve the optimization problem of Equation (2.4) for the parameter update $\Delta\mathbf{p}$, and write out an expression for this solution.

Substituting the approximation from Equation (2.5) into the loss function of Equation (2.2) gives:

$$\begin{aligned}\min_{\Delta\mathbf{p}} \left\| T(\mathbf{x}) + \frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{x})\Delta\mathbf{p} - I(\mathbf{W}(\mathbf{x};\Delta\mathbf{p}))\right\|^2 &\approx \min_{\Delta\mathbf{p}} \left\| \frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{x})\Delta\mathbf{p} - (I(\mathbf{W}(\mathbf{x};\Delta\mathbf{p})) - T(\mathbf{x}))\right\|^2 \\ &\approx \min_{\Delta\mathbf{p}} \|\mathbf{A}\Delta\mathbf{p} - \mathbf{b}\|^2,\end{aligned}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the image point and $\mathbf{A}$ is the Jacobian matrix $\frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{x})$ given by

$$\mathbf{A} := \frac{\mathrm{d}T}{\mathrm{d}\mathbf{p}}(\mathbf{x}) = \nabla T(\mathbf{x}) \begin{bmatrix} \frac{\partial \mathbf{W}}{\partial p_1} \\ \frac{\partial \mathbf{W}}{\partial p_2} \end{bmatrix} = \nabla T(\mathbf{x}) \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} T_x(\mathbf{x}) \cdot x \\ T_y(\mathbf{x}) \cdot y \end{bmatrix}$$

and $\mathbf{b}$ is the residual vector

$$\mathbf{b} := I(\mathbf{W}(\mathbf{x};\Delta\mathbf{p})) - T(\mathbf{x}).$$

The solution to the optimization problem of Equation (2.4) is given by:

$$\Delta\mathbf{p} = \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{b}, \tag{2.5}$$

which can be solved using singular value decomposition (SVD) or inverting the matrix $A^T A$ if it is invertible.

(iv) Finally, you will need to compute your new warp parameter $\mathbf{p}'$ by using the following update rule for the inverse compositional algorithm:

$$\mathbf{W}(\mathbf{x};\mathbf{p}') \leftarrow \mathbf{W}(\mathbf{x};\mathbf{p}) \circ \mathbf{W}(\mathbf{x};\Delta\mathbf{p})^{-1}. \tag{2.6}$$

Given your modified warp function from part 1, write down an expression for the new parameters $\mathbf{p}'$ in terms of $\mathbf{p}$ and $\Delta\mathbf{p}$.

The warp function $\mathbf{W}(\mathbf{x};\mathbf{p}')$ is given by:

$$\begin{aligned}
\mathbf{W}(\mathbf{x};\mathbf{p}') &= \mathbf{W}(\mathbf{x};\mathbf{p}) \circ \mathbf{W}(\mathbf{x};\Delta\mathbf{p})^{-1} \\
&= \mathbf{W}(\mathbf{W}(\mathbf{x};\Delta\mathbf{p});\mathbf{p}) \\
&= \mathbf{W}(\begin{bmatrix} \Delta p_1 + 1 & 0 \\ 0 & \Delta p_2 + 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix};\mathbf{p}) \\
&= \mathbf{W}(\begin{bmatrix} \Delta p_1 x + x \\ \Delta p_2 y + y \end{bmatrix};\mathbf{p}) \\
&= \begin{bmatrix} p_1 + 1 & 0 \\ 0 & p_2 + 1 \end{bmatrix}\begin{bmatrix} \Delta p_1 x + x \\ \Delta p_2 y + y \end{bmatrix} \\
&= \begin{bmatrix} (p_1 + 1)(\Delta p_1 x + x) \\ (p_2 + 1)(\Delta p_2 y + y) \end{bmatrix} \\
&= \begin{bmatrix} p_1 \Delta p_1 x + p_1 x + \Delta p_1 x + x \\ p_2 \Delta p_2 y + p_2 y + \Delta p_2 y + y \end{bmatrix} \\
&= \begin{bmatrix} (p_1 \cdot \Delta p_1 + p_1 + \Delta p_1)x + x \\ (p_2 \cdot \Delta p_2 + p_2 + \Delta p_2)y + y \end{bmatrix} \\
&= \begin{bmatrix} (p_1 \cdot \Delta p_1 + p_1 + \Delta p_1) + 1 \\ (p_2 \cdot \Delta p_2 + p_2 + \Delta p_2) + 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \mathbf{W}(\mathbf{x};\mathbf{p}\Delta\mathbf{p} + \mathbf{p} + \Delta\mathbf{p})
\end{aligned}$$

Thus, the new warp parameters $\mathbf{p}'$ are given by

$$\mathbf{p}' = \mathbf{p}\Delta\mathbf{p} + \mathbf{p} + \Delta\mathbf{p}. \tag{2.7}$$

where $(\cdot)$ denotes element-wise multiplication of the vectors (i.e. $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ab \\ cd \end{bmatrix}$).