Math 69: Logic | **Winter '23**

Homework assigned January 19, 2023

Prof. Marcia Groszek | ***Student: Amittai Siavava***

Problem 1.

Show that tautological equivalence is an equivalence relation on the set of wffs of sentential logic; that is, if we define

$$\alpha \equiv \beta \iff \alpha \vDash\Dashv \beta,$$

then $\equiv$ is an equivalence relation on the set of wffs.

(i) Show that $\equiv$ is reflexive.

Let α be a under some truth assignment v.

If $\overline{v}(\alpha) = T$, then $\alpha \vDash \alpha$. We also have that $\alpha \Dashv \alpha$ by the same argument.

If $\overline{v}(\alpha) = F$, then $\alpha \vDash\vDash \alpha$ since we may conclude anything from a false statement. The same argument applies to $\alpha \Dashv \alpha$ since the deduction itself is the assignment to α, and α has been assigned to F. *If we were deducing a different variable, say γ, then if $\overline{v}(\alpha) = F$, and $\overline{v}(\gamma) = T$, we would have $\overline{v}(\alpha \vDash \gamma) = T$ but $\overline{v}(\gamma \vDash \alpha) = F$, so $\overline{v}(\alpha \vDash\Dashv \gamma) = F$ and $\alpha \not\equiv \gamma$.*

(ii) Show that $\equiv$ is symmetric.

Let $\alpha \equiv \beta$ under some truth assignment v. By definition, $\alpha \vDash\Dashv \beta$. This implies that:

- if $\overline{v}(\alpha) = T$, then $\overline{v}(\beta) = T$;
- if $\overline{v}(\alpha) = F$, then $\overline{v}(\beta) \neq T$, as that would imply $\overline{v}(\alpha \Dashv \beta) = F$. Therefore, $\overline{v}(\beta) = F$.

Therefore, $\overline{v}(\alpha) = \overline{v}(\beta)$, so $\overline{v}(\beta \vDash\Dashv \alpha) = T$, and $\beta \equiv \alpha$.

(iii) Show that $\equiv$ is transitive.

Let $\alpha \equiv \beta$ and $\beta \equiv \gamma$ under some truth assignment v. By definition, $\alpha \vDash\Dashv \beta$ and $\beta \vDash\Dashv \gamma$. As shown in part (ii), If $\overline{v}(x \vDash\Dashv y) = T$, then $\overline{v}(x) = \overline{v}(y)$. Therefore, $\overline{v}(\alpha) = \overline{v}(\beta)$ and $\overline{v}(\beta) = \overline{v}(\gamma)$, implying that $\overline{v}(\alpha) = \overline{v}(\gamma)$. Therefore, $\alpha \vDash\Dashv \gamma$, so $\alpha \equiv \gamma$.

Problem 3.

Let X be the set of all wffs of sentential logic and $\equiv$ be tautological equivalence. Define a binary (2-place) function on equivalence classes, which we could call conjunction, by

$$[\alpha] \bigwedge [\beta] = [\alpha \wedge \beta]$$

Prove that this function is well-defined.

As you do this, at some point you are going to have to prove that two wffs are tautologically equivalent. For this exercise, please do this by showing explicitly that any truth assignment that satisfies one of the formulas also satisfies the other, and conversely.
You may think it's obvious that these wffs are tautologically equivalent. I agree, and after this proof, you can get away with saying so, or giving a more informal explanation, in similar circumstances.

Let v be a truth assignment.
Suppose that $v(\alpha) = x \in \{T, F\}$, then $v(\alpha_i) = T$ for all α_i in the equivalence relation $[\alpha]$.
Similarly, suppose that $v(\beta) = y \in \{T, F\}$, then $v(\beta_j) = T$ for all β_j in the equivalence relation $[\beta]$.
Then;

$$\overline{v}(\alpha_i \wedge \beta_j) = \overline{v}(\alpha \wedge \beta) = x \wedge y = \begin{cases} T & \text{if } x = y = T \\ F & \text{otherwise} \end{cases}$$

Therefore, any truth assignment assigns the same value to $(\alpha_i \wedge \beta_j)$ for all $\alpha_i \in [\alpha]$ and for all $\beta_j \in [\beta]$, so the function is well defined.

Problem 5.

Let X be the set of all wffs of sentential logic and $\equiv$ be tautological equivalence. Define a binary (2-place) relation on equivalence classes by

$$[\alpha] \vDash [\beta] \iff \alpha \vDash \beta.$$

Determine whether this relation is well-defined and prove your answer is correct.

Let v be a truth assignment.

Suppose that $v(\alpha) = x \in \{T, F\}$, then $v(\alpha_i) = T$ for all α_i in the equivalence relation $[\alpha]$.

Similarly, suppose that $v(\beta) = y \in \{T, F\}$, then $v(\beta_j) = T$ for all β_j in the equivalence relation $[\beta]$.

Then;

$$\overline{v}(\alpha_i \vDash \beta_j) = \overline{v}(\alpha \vDash \beta) = (x \vDash y) = \begin{cases} F & \text{if } x = T \text{ and } y = F \\ T & \text{otherwise} \end{cases}$$

Since the value of $[\alpha] \vDash [\beta]$ is always the same as that of $\alpha \vDash \beta$, the relation is well-defined.

Proof of correctness: In our universe, suppose all people who live on Mars do not use electricity.

Let:

- Mx mean "Person x lives on Mars", and
- Nx mean "Person x does not use electricity".

We can say that $Mx \vDash Nx$.

Or, if person x lives on Mars, then x does not use electricity.

In extending this to equivalence classes, let $[x]$ be the equivalence class of all people who live on Mars, and let $[y]$ be the equivalence class of all people who do not use electricity. Then any member of $[x]$ is a person who lives on Mars, and they may not be using electricity, so we deduce $[y]$. Therefore, $[x] \vDash [y]$.