Math 69: Logic	**Winter '23**

Homework assigned January 30, 2023

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Problem 5.

Show that the formula

$$x = y \to Pzfx \to Pzfy$$

(where f is a one-place function symbol and P is a two-place predicate symbol) is valid.

Suppose the formula is not valid. Then there exists some structure $\mathfrak{A}$ and some variable assignment s such that $\not\models_{\mathfrak{A}} x = y \to Pzfx \to Pzfy$, meaning:

$$\models_{\mathfrak{A}} (x = y)[s] \tag{5.1}$$

$$\models_{\mathfrak{A}} Pzfx[s] \tag{5.2}$$

$$\not\models_{\mathfrak{A}} Pzfy[s] \tag{5.3}$$

Since $\models_{\mathfrak{A}} (x = y)[s]$, we have that $\overline{s}(x) = \overline{s}(y)$.

However, since $\overline{s}(x) = \overline{s}(y)$, we have that $\overline{s}(fx) = \overline{s}(fy)$:

$$\overline{s}(fx) = f^{\mathfrak{A}}(\overline{s}(x)) = f^{\mathfrak{A}}(\overline{s}(y)) = \overline{s}(fy).$$

Therefore, $Pzfx[s]$ and $Pzfy[s]$ are logically equivalent, so 5.2 and 5.3 are a contradiction, meaning that the formula is valid and any structure which does not satisfy the formula is inconsistent.

Problem 26.

(a) Consider a fixed structure $\mathfrak{A}$ and define its *elementary type* to be the class of structures elementarily equivalent to $\mathfrak{A}$. Show that this class is EC_Δ.

Hint: Show that it is Mod Thm $\mathfrak{A}$

By definition, two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if whenever $\models_\mathfrak{A} \varphi$, we also have that $\models_\mathfrak{B} \varphi$. That is, whenever φ is tautologically implied by $\mathfrak{A}$ then it is also tautologically implied by $\mathfrak{B}$.

Let $[\mathfrak{A}]$ be the class of all structures elementarily equivalent to $\mathfrak{A}$, and let Thm $\mathfrak{A}$ be the set of all consequences of $\mathfrak{A}$. Then every structure $\mathfrak{B}$ in $[\mathfrak{A}]$ is a model of Thm $\mathfrak{A}$, since it must satisfy all the consequences of $\mathfrak{A}$. Therefore, $[\mathfrak{A}]$ = Mod Thm $\mathfrak{A}$, so it is EC_Δ.

(b) Call a class $\mathcal{K}$ of structures *elementarily closed* or ECL if whenever a structure belongs to $\mathcal{K}$ then all elementarily equivalent structures also belong. Show that any such class is a union of EC_Δ classes. (A class that is a union of EC_Δ classes is said to be an $\mathrm{EC}_{\Delta\Sigma}$ class; this notation is derived from topology.)

Let K be an ECL class, and $\mathfrak{A}$ be a structure in K. Since $\mathfrak{A} \in K$, K satisfies all the consequences of $\mathfrak{A}$, i.e. Mod Thm $\mathfrak{A} \subseteq K$.

Let $\mathfrak{B}$ be an elementarily equivalent structure to $\mathfrak{A}$. This means that $\mathfrak{B}$ is a model of Thm $\mathfrak{A}$. However, K is elementarily closed, so $\mathfrak{B}$ must also be in K, meaning Mod Thm $\mathfrak{B} \subseteq K$.

By extension, every other structure $\mathfrak{D}_i$ that is elementarily equivalent to some structure $\mathfrak{D}$ that belongs in K must also belong in K. Therefore;

$$\mathfrak{K} = \text{Mod Thm } \mathfrak{A} \cup \text{Mod Thm } \mathfrak{B} \cup \text{Mod Thm } \mathfrak{D}_1 \cup \text{Mod Thm } \mathfrak{D}_2 \cup \dots$$

for every other structure $\mathfrak{D}_i$ in K, so it is $\mathrm{EC}_{\Delta\Sigma}$.

(c) Conversely show that any class that is the union of EC_Δ classes is ECL.

Let K be a class that is $\mathrm{EC}_{\Delta\Sigma}$. Then

$$K = \bigcup_{\mathfrak{A} \text{ belongs in } K} \text{Mod Thm } \mathfrak{A}.$$

Consequently, whenever a structure $\mathfrak{A}$ belongs in K, Mod Thm $\mathfrak{A} \subseteq K$.

Take any structure $\mathfrak{B}$ that is elementarily equivalent to $\mathfrak{A}$, then $\mathfrak{B}$ tautologically implies the same set of consequences as $\mathfrak{A}$, i.e. Mod Thm $\mathfrak{B}$ = Mod Thm $\mathfrak{A}$. Therefore, Mod Thm $\mathfrak{B} \subseteq K$ so $\mathfrak{B}$ also belongs in K. Therefore, K is ECL.

Problem 27.

Let $\mathcal{L}$ be the language for first-order logic with two-place predicate symbols E and P and one-place function symbol f. (We are not assuming that $\mathcal{L}$ has the equality symbol. On the other hand, we are not ruling out the possibility that $\mathcal{L}$ has the equality symbol and/or any number of parameter symbols in addition to $\forall$, E, P, and f. Other symbols are not relevant to this question.)

Suppose $\mathfrak{A}$ is a structure for $\mathcal{L}$ that is a model of the sentence

$$\forall x Exx$$

and of every sentence of the form

$$\forall x \forall y \forall z_1 \forall z_2 \ldots \forall z_n (Exy \to (\alpha \to \alpha')).$$

where α is an atomic formula with variables included among $\{x, y, z_1, z_2, \ldots, z_n\}$, and α' is obtained from α by replacing some (possibly none, possibly some but not all, possibly all) occurrences of x by y. Examples of sentences of this form are

$$\forall x \forall y (Exy \to (Exx \to Eyx)) \text{ and } \forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy)).$$

An example of a sentence *not* of this form is

$$\forall x \forall y \forall z (Exy \to (Ezfy \to Ezfx)).$$

Show that $E^{\mathfrak{A}}$ is an equivalence relation on the universe $|\mathfrak{A}|$, that $P^{\mathfrak{A}}$ induces a well-defined relation on equivalence classes, and that $f^{\mathfrak{A}}$ induces a well-defined function on equivalence classes.

Let s be a variable assignment satisfying the conditions as stated. Then:

(a) $E^{\mathfrak{A}}$ is an equivalence relation on the universe $|\mathfrak{A}|$.

  (i) E is reflexive: This follows from the sentence $\forall x Exx$.

  (ii) E is symmetric: Suppose $\models_{\mathfrak{A}} Exy[s]$. Take the following sentence, which is valid in $\mathfrak{A}$:

$$\forall x \forall y (Exy \to (Exx \to Eyx)).$$

  We can deduce Eyx as follows:

$$Exy \quad \text{(from choice of } x \text{ and } y\text{)} \tag{27.1}$$

$$\forall x \forall y (Exy \to (Exx \to Eyx)) \quad \text{(As derived above)} \tag{27.2}$$

$$Exy \to (Exx \to Eyx) \quad \text{(Generalization theorem on 27.2)} \tag{27.3}$$

$$(Exx \to Eyx) \quad \text{(modus ponens on 27.1 and 27.3)} \tag{27.4}$$

$$\forall x Exx \quad \text{(first sentence in the model)} \tag{27.5}$$

$$Exx \quad \text{(Generalization theorem on 27.5)} \tag{27.6}$$

$$Eyx \quad \text{(modus ponens on 27.4 and 27.6)} \tag{27.7}$$

(iii) Transitive: the following sentence is valid in $\mathfrak{A}$:

$$\forall x \forall y \forall z (Exy \to (Ezx \to Ezy)) \tag{27.8}$$

Since E is symmetric; the formula is logically equivalent to

$$\forall x \forall y \forall z (Eyx \to (Exz \to Eyz)),$$

so $E^{\mathfrak{A}}$ is transitive.

(b) $P^{\mathfrak{A}}$ is a well-defined relation on equivalence classes.

Let $x, y \in |\mathfrak{A}|$ such that $\vDash_{\mathfrak{A}} Exy[s]$. We show that $E^{\mathfrak{A}}(Pxa)(Pya)$ for all $a \in |\mathfrak{A}|$.

Suppose not, then there exists some $z \in |\mathfrak{A}|$ such that $\neg E^{\mathfrak{A}}(Pxa)(Pya)$, meaning Pxa and Pya are not in the same equivalence class. However, we have a rule in our structure that

$$\forall x \forall y \forall z_1 \forall z_2 \dots \forall z_n (Exy \to (\alpha \to \alpha'))$$

where α' is obtained by replacing none, some, or all occurrences of x in α with y. This means

$$\forall x \forall y \forall z_1 \forall z_2 \dots \forall z_n (Exy \to (Pxa \to Pya))$$

is a valid axiom in $\mathfrak{A}$. Since E is symmetric, Exy implies Eyx, so we can also deduce that

$$\forall x \forall y \forall z_1 \forall z_2 \dots \forall z_n (Eyx \to (Pya \to Pxa)).$$

The two conditions only hold when Pya and Pxa are in the same equivalence class.

(c) $f^{\mathfrak{A}}$ is a well-defined function on equivalence classes.

Let $x, y \in |\mathfrak{A}|$ be such that $\vDash_{\mathfrak{A}} Exy$.

The sentence

$$\forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy))$$

as a valid axiom.

Since E is symmetric, Eyx is true whenever Exy is true, so the sentence

$$\forall x \forall y \forall z (Eyx \to (Ezfy \to Ezfx))$$

is also valid in $\mathfrak{A}$.

This means that whenever Exy is satisfied, $Ezfx$ is satisfied if and only if $Ezfy$ is satisfied, so fx and fy are in the same equivalence class and $f^{\mathfrak{A}}$ is well-defined on equivalence classes.