Math 69: Logic

Abelian Groups

Amittai Siavava

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Abstract

This paper describes a set of axioms and a language for abelian groups and explores what the theory of abelian groups can tell us about completeness.

I will be expanding the abstract once I have worked mroe on the conclusions and have a better understanding of the ultimate shape/direction of the paper.

Contents

1. Introduction

Mathematical logic is interesting in itself, but its ability to effectively model and describe other fields in mathematics is, perhaps, even more intriguing and what makes the study of logic so useful to the regular mathematician. For instance, using logic one may model aspects of analysis, algebra, calculus, probability, geometry, and even game theory. This offers disparate and often insightful perspectives into these other fields, letting one apply the laws of logic to their study of the structures of other fields in mathematics and the resulting consequences. More importantly, modeling other structures is useful in the study of logic itself as it lets one interrogate the rules of logic from other perspectives.

In this paper, we will model the structure of abelian groups using first-order logic. We will then narrow down our study to groups that are *abelian*, *divisible*, and *torsion-free*. We will then study groups in this category as vector-spaces over the rational numbers, $\mathbb{Q}$, and study how the structure of these vector spaces can tell us about logic.

2. Definitions

Note: This section is included mostly as a prelude for the non-experienced mathematician, and may be skipped by readers well-versed in ***Group Theory****,* ***Abstract Vector Spaces****, and* ***First-Order Logic****.*

2.1. **Groups.**

A group $G = (S, *)$ is a set S paired with a binary operation, $* : S \to S$, satisfying the following axioms:

(i) Closure: $x * y \in S$ for any two elements $x, y \in S$.
(ii) Associativity: $(x * y) * z = x * (y * z)$ for any three elements $x, y, z \in S$.
(iii) Identity: there exists an element $\varepsilon \in S$ such that for any $x \in S$, $x * e = e * x = x$.
(iv) Inversion: for any $x \in S$, there exists an element $x^{-1} \in S$ such that $x * x^{-1} = x^{-1} * x = \varepsilon$.

There are two common notations for groups:

(i) Additive notation: $G = (S, 0, +)$, where 0 is the identity element and + is the binary operation. For any element g, we denote the inverse of g as $-g$.
(ii) Multiplicative notation: $G = (S, 1, \cdot)$, where 1 is the identity element and $\cdot$ is the binary operation. For any element g, the inverse of g is denoted as g^{-1}.

Indeed, some groups with numerical elements use modified versions of addition and multiplication as their binary operations. For instance, the group of integers modulo a positive integer n is denoted as $\mathbb{Z}/n\mathbb{Z}$, where $a + b$ in the group is equivalent to $a + b \mod n$. Multiplicative groups can also be formed this way, but require more attention since not every element less than some integer n necessarily has an inverse. We denote such groups as $(\mathbb{Z}/n\mathbb{Z})^\times$ and usually include only the elements co-prime to n — for instance, $(\mathbb{Z}/12\mathbb{Z})^\times$ contains the elements $1, 5, 7$, and 11, with each element being its own inverse.

In this paper, we will use additive notation for groups.

2.2. **Fields.**

A field $F = (S; 0, 1, +, \times)$ is a set S with two binary operations, + and $\cdot$, such that the pairing $(S, +)$ is an *abelian* group and the pairing $(S \setminus \{0\}, \cdot)$ is a group (not necessarily abelian). Some of the most common examples of a field is the real numbers $R = (\mathbb{R}, 0, 1, +, \cdot)$, the rational numbers $Q = (\mathbb{Q}, 0, 1, +, \cdot)$, and the complex numbers $C = (\mathbb{C}, (0,0), (1,0), +, \cdot)$ under their standard addition and multiplication operations.

2.3. **Vector Spaces over Fields.**

A vector space $V = (S, +)$ over a field F is an *abelian group* such that multiplication of elements of V by elements of the field F is defined and satisfies the following axioms:

(i) $0x = 0$ for any $x \in S$ and the additive identity $0 \in F$.

(ii) $1x = x$ for any $x \in S$ and the multiplicative identity $1 \in F$.

(iii) $nx = \underbrace{x + \ldots + x}_{n \text{ times}}$ for any $x \in S$ and any $n \in F$.

(iv) $(-n)x = \underbrace{-x + \ldots + -x}_{n \text{ times}}$ for any $x \in S$ and any $n \in F$.

(v) $n^{-1}x = y$ where y is the element in the field such that $ny = x$.

2.4. **Divisible and Torsion Free Groups.**

The order of any element g in a group G is the smallest positive integer n such that $ng = 0$. For instance, if $g + g + g = 0$, then g has order 3.

1. A group G is *divisible* if for any $g \in G$ and any $n \in \mathbb{N}$ with $n > 0$, there exists an element $h \in G$ such that $nh = g$.
2. An element $g \in G$ is *torsion free* if $ng \neq 0$ for any positive integer $n > 0$, and a group is *torsion free* if every non-identity element is torsion free.

3. Axiomatizing Groups

To model groups using logic, we need to specify a sufficiently capable language. The language dictates which symbols are valid in sentences/formulas we use to describe groups. We will use the following language:

$$\mathcal{L} = \langle 0, +, -, = \rangle \tag{3.1}$$

3.1. **Abelian Groups.**

For any structure to be an abelian group, it needs to satisfy the group axioms (closure, associativity, identity, and inversion) and commutativity (see section 2.1 for reference). We define the set Σ to contain the following abelian group axioms:

$$\forall g \, \forall h \, (g + h = h + g) \qquad \text{(commutativity)} \tag{3.2}$$

$$\forall g \, \forall h \, \forall i \, ((g + (h + i)) = ((g + h) + i)) \qquad \text{(associativity)} \tag{3.3}$$

$$\exists \varepsilon \, \forall g (g + \varepsilon = g) \qquad \text{(group identity)} \tag{3.4}$$

$$\forall g \, \exists h \, (g + h = 0) \qquad \text{(group inversion)} \tag{3.5}$$

Since n-ary functions in a structure $\mathfrak{A}$ in first-order logic always assume the domain $|\mathfrak{A}|^n$, and the range $|\mathfrak{A}|$, we do not need to explicitly specify an axiom for closure.

3.2. **Divisible, Torsion Free Abelian Groups.**

We define an extra set of conditions which must be satisfied (in addition to those in Σ) for a structure to be a divisible torsion-free abelian group. Recall that a group G is *divisible* if for any $g \in G$ and any $n \in \mathbb{N}$ with $n > 0$, there exists an element $h \in G$ such that $nh = g$. We can represent this using an infinite set of axioms

$$T_1 = \{\forall g \, \exists h, \, (g = nh) \mid n = 1, 2, 3, \ldots\}.$$

Recall that a group G is *torsion free* if every non-identity element is torsion free. That is, if $g \neq 0$ then $ng \neq 0$ for any positive integer $n > 0$. We can represent this using an infinite set of axioms

$$T_2 = \{\forall g \, (g \neq 0 \to ng \neq 0) \mid n = 1, 2, 3, \ldots\}.$$

Let $T = T_1 \cup T_2$, then any model satisfying $\Sigma \cup T$ is a divisible, torsion free abelian group.

3.3. **Abelian Groups Wherein Each Element Other Than the Identity Has Order 2.**

To specify that each element in an abelian group has order 2, define the set

$$S = \{\forall g \, (g \neq 0 \to (g + g = 0))\},$$

then any model satisfying $\Sigma \cup S$ is an abelian group wherein each element other than the identity has order 2.

4. Vector Space Structures

A group G has a *vector space structure* over some field F if multiplication of elements of G by scalars in F is defined

4.1. **Divisible, Torsion Free Abelian Groups over $\mathbb{Q}$.**

Claim 4.1. *The scalar multiplication of elements of a divisible, torsion free abelian group G by elements of the field $\mathbb{Q}$ is well-defined.*

Proof. The proof for integer multiplication is trivial since the sum $\underbrace{x + \ldots + x}_{n \text{ times}}$ always names a unique element in the vector space (and in the equivalent abelian group). Consider multiplication by non-integer rational numbers. Take $n \in \mathbb{Q}$ and $h_1, h_2, g \in G$ such that $nh_1 = nh_2 = g$, then;

$$\begin{aligned} nh_1 = nh_2 &= g \\ \underbrace{h_1 + \ldots + h_1}_{n \text{ times}} = \underbrace{h_2 + \ldots + h_2}_{n \text{ times}} &= g \\ \therefore \left(\underbrace{h_1 + \ldots + h_1}_{n \text{ times}}\right) - \left(\underbrace{h_2 + \ldots + h_2}_{n \text{ times}}\right) &= g - g = 0 \\ \therefore \underbrace{(h_1 - h_2) + \ldots + (h_1 - h_2)}_{n \text{ times}} &= 0 \end{aligned}$$

But G is torsion-free, meaning $ng \neq 0$ for any scalar $n \neq 0$ given $g \neq 0$. so $h_1 - h_2$ must be equivalent to 0. Group inverses are unique, so this means that $-h_1$ and $-h_2$ are the same element since $h_1 - h_2 = 0$. It then follows that h_1 and h_2 are equivalent. □

Claim 4.2. *Any divisible torsion-free abelian group has a $\mathbb{Q}$-vector space structure*

Proof. Let G be a divisible, torsion-free abelian group, and let $g \in G$. By definition, since G is divisible, then for every $n \in \mathbb{N}$ such that $n > 0$, there exists some element $h \in G$ such that $nh = g$. Furthermore, as seen in the proof to 4.1, if $nh_1 = g$ and $nh_2 = g$ then $h_1 = h_2$ — hence, for any $g \in G, n \in \mathbb{N}, n > 0$, not only does the element h exist such that $nh = g$, but the element h is also unique. □

Therefore, any divisible, torsion free abelian group forms a vector space over multiplication by rational numbers (i.e. has a $\mathbb{Q}$-vector space structure).

Claim 4.3. $\Sigma \cup T$ *is categorical over the cardinality of the reals.*

Proof. Any model $\mathfrak{A}$ of $\Sigma \cup T$ is a divisible, torsion free abelian group. As shown in the proof to 4.2, any such abelian group has a $\mathbb{Q}$-vector space structure. Define an isomorphism φ by: $\varphi : \mathbb{R} \to |\mathfrak{A}|$ by

$$\begin{aligned} \varphi : \mathbb{R} &\to |\mathfrak{A}| \\ 0 &\mapsto 0^{\mathfrak{A}} \\ 1 &\mapsto 1^{\mathfrak{A}} \\ n &\mapsto n^{\mathfrak{A}} \text{ for } n \in \mathbb{N} \\ \frac{p}{q} &\mapsto \frac{1}{q} \cdot p^{\mathfrak{A}} \text{ for } p, q \in \mathbb{N} \end{aligned}$$

φ must be a bijection since every element in G has infinite order and every element in G is divisible by every rational number in $\mathbb{Q}$, hence the cardinality of $|\mathfrak{A}| = |\mathbb{R}|$.

Therefore, there is, up to isomorphism, a unique model such model $\mathfrak{A}$ of $\Sigma \cup T$ that has the size of $|\mathbb{R}|$. □

Claim 4.4. *$\Sigma \cup T$ is not countably categorical.*

I struggled a bit with this proof and I'm not sure if it is correct. I would appreciate any help.

Proof. Let $\mathfrak{A}$ be a model for $\Sigma \cup T$. Then $\mathfrak{A}$ must be a divisible, torsion free abelian group. Suppose $\mathfrak{A}$ has a countable cardinality, then there exists an immediate successor of the element 0 in $\mathfrak{A}$. Let x be such an element. Since $\mathfrak{A}$ is divisible, there exists some element $y_i \in |\mathfrak{A}|$ for each corresponding integer $i \in \mathbb{N}_{>1}$ such that $iy_i = x$. But since $\mathfrak{A}$ is torsion free, all these elements must be between 0 and x, contradicting the assumption that x is the successor of 0. □

4.2. **Abelian Groups Wherein Each Element Other Than the Identity Has Order 2.**

We define S to contain the following axioms:

$$\forall g\,((g \neq 0) \to (g + g = 0)) \qquad \text{(every element has order 2)} \tag{4.5}$$

The easiest way to construct an infinite model for $\Sigma \cup S$ is to take direct products of a group in which every non-identity element has order 2, such as C_2. For instance, the Klein-Four group V_4 is a four-element group in which every non-identity element has order 2, and it is isomorphic to the direct product of C_2 with itself.

$$\begin{aligned} C_2 &= (\{0, 1\}, 0, +) \\ V_4 &= (\{(0,0), (0,1), (1,0), (1,1)\}, (0,0), +) \end{aligned}$$

For reference, let C_2^n be the direct product of C_2 with itself that yields a group with n elements. Clearly, the only possible values for n are powers of 2, but we can also generate a group with infinite elements by taking infinite number of direct products of C_2 with itself:

$$C_2^\infty = C_2 \times C_2 \times C_2 \times \dots$$

Claim 4.6. *C_2^∞ is countably categorical as a vector-space structure.*

Proof. For any $n = 2^k$ for some $k \in \mathbb{N}$, every non-identity element in C_2^n has order 2, so multiplication is only well-defined when the scalar field is limited to a two-element field, such as the field

$$\mathbb{F}_2 = (\{0, 1\}; 0, 1, +, \cdot),$$

with scalar multiplication defined as $\forall g(0g = 0)$ and $\forall g(1g = g)$.

This means there is a unique model of $\Sigma \cup S$ of cardinality n up to isomorphism (since every such direct product of C_2 containing n is clearly isomorphic to C_2^n), and it every corresponding 2-element field has to be isomorphic to $\mathbb{F}_2$. Therefore, $\Sigma \cup S$ is countably categorical.

I think I have to explicitly show here that all the vector-space models are isomorphic for different values of n, but I am not entirely sure how to go about it. I would appreciate any guidance or suggestions. □

Take any element $g \in C_2^\infty$ such that g is not the identity. Since g has order 2, we have that $3g = 2g + g = 0 + g = g$, and, more generally, for all $n \in \mathbb{N}_{>0}$, $ng = kg$ where $n \equiv k \mod 2$. Therefore, G cannot have a $\mathbb{Q}$-vector space structure under scalar multiplication with elements of $\mathbb{N}_{>0}$. One work-around is to limit the scalar field to a two-element field, such as $\mathbb{F}_2$.

$$\mathbb{F}_2 = (\{0, 1\}, +, \times).$$

Under multiplication with elements of $\mathbb{F}_2$, we have that $\forall g(0g = 0)$ and $\forall g(1g = g)$, thus we have a $\mathbb{Q}$-vector space structure.

5. Conclusions

I did not have enough time to work on the conclusions so if okay I would like to submit this section in the second draft (Friday).