Math 69: Logic **Winter '23**

Exam 2 — 02/20/2023

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You may consult your textbook, notes, class handouts, and returned homework as you work on this exam, but you should not discuss the exam with anybody other than the professor, or look in other textbooks or on the internet (except on the course web page).

It is still okay to discuss class worksheets and homework problems with each other, even if they are related to exam problems, as long as you do not discuss any possible relevance to the exam.

Please ask the professor if you have any questions about the exam. You can use anything from the portions of the text we have covered, including the results of homework problems that were assigned for graded homework. (If you want to use the result of a homework problem that wasn't assigned, you must first solve the problem, and include the solution in your answer.) You can use material from class handouts and worksheets, including the results of problems. You can also use earlier parts of an exam problem in the solutions to later parts of that same problem, even if you were not able to solve the earlier parts.

Your exam paper should follow the following format rules: Identify each problem by number, and also repeat or restate the problem before giving a solution.

The exam will be graded on the clarity and completeness of your explanations, and the correct use of mathematical notation and terminology, as well as on the content of your answers.

All the problems in this exam concern first order logic.

Throughout this exam, you may use $\exists$ and all the connectives of sentential logic, omit or add parentheses for clarity and readability, write binary predicates in infix notation (write $t_1 = t_2$ instead of $= t_1 t_2$), and use abbreviations such as $\neq$ (write $t_1 \neq t_2$ instead of $(\neg\, t_1 = t_2)$).

Problem 1.

Show carefully and formally, directly from the formal definition of satisfaction [1], that the sentence

$$\forall x\, Pxfc \to \exists x Pxfx$$

(where c is a constant symbol, f is a one-place function symbol, and P is a two-place predicate symbol) is logically valid.

Formal definition of satisfaction:

$\models_{\mathfrak{A}} \varphi\ [s]$ if and only if the translation of φ determined by $\mathfrak{A}$ is true.

- Terms.
  (i) For each variable x, $\overline{s}(x) = s(x)$.
  (ii) For each constant c, $\overline{s}(c) = c^{\mathfrak{A}}$.
  (iii) If $t_1, \ldots, t_n$ are terms and f is an n-place function symbol, then $\overline{s}(ft_1 \cdots t_n) = f^{\mathfrak{A}}(\overline{s}(t_1), \ldots, \overline{s}(t_n))$.
- Atomic formulas.
  (i) $\models_{\mathfrak{A}} = t_1 t_2$ iff $\overline{s}(t_1) = \overline{s}(t_2)$.
  (ii) For an n-place predicate parameter P, $\models_{\mathfrak{A}} Pt_1 \cdots t_n$ iff $\langle \overline{s}(t_1), \ldots, \overline{s}(t_n) \rangle \in P^{\mathfrak{A}}$.
- Other wffs
  (i) Atomic formulas as defined above.
  (ii) $\models_{\mathfrak{A}} \neg\varphi\ [s]$ iff $\not\models_{\mathfrak{A}} \varphi\ [s]$.
  (iii) $\models_{\mathfrak{A}} (\varphi \to \psi)\ [s]$ iff either $\not\models_{\mathfrak{A}} \varphi[s]$ or $\models_{\mathfrak{A}} \psi[s]$, or both.
  (iv) $\models_{\mathfrak{A}} \forall x\ \varphi\ [s]$ iff for every $d \in |\mathfrak{A}|$, we have $\models_{\mathfrak{A}} \varphi\ [s(x \mid d)]$.

Let $\mathfrak{A}$ be a structure and s a variable assignment.

Let $\gamma_1 = \forall x\, Pxfc$ and $\gamma_2 = \exists x Pxfx$, such that the given sentence is equivalent to $(\gamma_1 \to \gamma_2)$. Then the sentence is valid iff either γ_1 is not valid, or γ_2 is valid (or both).

When γ_1 is not valid, it follows from the definition of the "$\to$" connective that $(\gamma_1 \to \gamma_2)$ is valid.

Suppose γ_1 is valid, i.e. $\forall x\, Pxfc = T$, we show that γ_2 must also be valid.
By definition of satisfaction, $\forall x\, Pxfc$ is valid iff for every $d \in |\mathfrak{A}|$, we have $\models_{\mathfrak{A}} Pxfc\ [s(x \mid d)]$.
This means that $\overline{s}(Pxfc) = P^{\mathfrak{A}} s(x) f^{\mathfrak{A}} c^{\mathfrak{A}} = T$ whenever $s(x) = d$ for some $d \in |\mathfrak{A}|$. By definition of satisfaction, $\overline{s}(c) = c^{\mathfrak{A}}$, the translation of c into an element in $|\mathfrak{A}|$. When we map x to $c^{\mathfrak{A}}$, we get that $(Pxfc)^x_{c^{\mathfrak{A}}}$ is valid (since $(Pxfc)^x_d$ is valid for all $d \in |\mathfrak{A}|$).
This evaluates to $P^{\mathfrak{A}} c^{\mathfrak{A}} f^{\mathfrak{A}} c^{\mathfrak{A}} = \overline{s}(Pcfc)$, so $Pcfc$ must also be valid, and $\exists x Pxfx$ is valid, particularly when $x = c$.

[1] I repeat, directly from the formal, recursive definition of satisfaction, given on the bottom of page 83 and the top of page 84 in the textbook. In particular, this problem is about logical validity, and not about deductions.

Problem 2.

In each case, either show without using the Completeness Theorem that $\varphi \vdash \psi$, or else show that $\varphi \nvdash \psi$. Here P and Q are one-place predicate symbols.

To show $\varphi \vdash \psi$, you may use any of the metatheorems of Section 2.4. You do not need to give an actual deduction.

For this problem, if you want to give an example of a structure and variable assignment satisfying some particular formula γ, it is enough to specify the structure and variable assignment. You do not have to formally prove that γ is satisfied.

(i) φ is $\forall x\,(Px \vee Qx)$, and ψ is $\forall x\,Px \vee \forall x\,Qx$.

> In this case, $\varphi \nvDash \psi$.
>
> Let $\mathfrak{A}$ be a structure with $|\mathfrak{A}| = \{p, q\}$ such that $P^{\mathfrak{A}} = \{p\}$ and $Q^{\mathfrak{A}} = \{q\}$. Let s be a variable assignment.
>
> Then the sentence $\forall x\,(Px \vee Qx)$ is satisfied by s since $\vDash_{\mathfrak{A}} (Px \vee Qx)[s(x \mid p)]$ and $\vDash_{\mathfrak{A}} (Px \vee Qx)[s(x \mid q)]$.
>
> However, $\forall x\,Px$ is not satisfied by s since $\nvDash_{\mathfrak{A}} Px[s(x \mid q)]$ because $q \notin P^{\mathfrak{A}}$. Likewise, $\forall x\,Qx$ is not satisfied by s since $\nvDash_{\mathfrak{A}} Qx[s(x \mid p)]$ because $p \notin Q^{\mathfrak{A}}$.
>
> Therefore, $\forall x\,Px \vee \forall x\,Qx$ is not satisfied, so $\varphi \nvDash \psi$.

(ii) φ is $\forall x\,Px \vee \forall x\,Qx$, and ψ is $\forall x\,(Px \vee Qx)$.

> In this case, $\varphi \vdash \psi$.
>
> Let $\mathfrak{A}$ be a structure and s a variable assignment such that $\vDash_{\mathfrak{A}} (\forall x\,Px \vee \forall x\,Qx)[s]$.
>
> Since $\overline{s}(\forall x\,Px \vee \forall x\,Qx) = \overline{s}(\forall x\,Px) \vee \overline{s}(\forall x\,Qx)$, either $\vDash_{\mathfrak{A}} \forall x\,Px[s]$ or $\vDash_{\mathfrak{A}} \forall x\,Qx[s]$.
>
> (i) When $\vDash_{\mathfrak{A}} \forall x\,Px[s]$ is satisfied, then $\vDash_{\mathfrak{A}} Px[s(x \mid d)]$ for every $d \in |\mathfrak{A}|$.
> Therefore, $\vDash_{\mathfrak{A}} Px \vee Qx[s(x \mid d)]$ for every $d \in |\mathfrak{A}|$, meaning $\forall x\,Px \vee Qx$ is also satisfied by s.
>
> (ii) When $\vDash_{\mathfrak{A}} \forall x\,Qx[s]$ is satisfied, then $\vDash_{\mathfrak{A}} Qx[s(x \mid d)]$ for every $d \in |\mathfrak{A}|$.
> Therefore, $\vDash_{\mathfrak{A}} Px \vee Qx[s(x \mid d)]$ for every $d \in |\mathfrak{A}|$, meaning $\forall x\,Px \vee Qx$ is also satisfied by s.
>
> in the first case, $\vDash_{\mathfrak{A}} \forall x\,Px[s]$, so $\mathfrak{A}$ satisfies $\forall x\,Px$.

(iii) φ is $\forall x\,(Px \vee Qy)$, and ψ is $\forall x\,Px \vee \forall x\,Qy$.

> In this case, $\varphi \vdash \psi$.
>
> Since y occurs free, $\forall x\,(Px \vee Qy)$ is satisfied by s iff $\vDash_{\mathfrak{A}} \forall x Px[s]$ or $\vDash_{\mathfrak{A}} Qy[s]$ (by the generalization theorem, if $\mathfrak{A} \vDash \forall x Qy$ then $\mathfrak{A} \vDash Qy$).
>
> In the first case, $\vDash_{\mathfrak{A}} \forall x Px$ deduces that that $\forall x Px$ is satisfied by s, which tautologically implies that $\forall x\,(Px \vee Qy)$ is satisfied by s.
>
> In the second case, $\vDash_{\mathfrak{A}} Qy$ deduces that that $\forall x\,(Px \vee Qy)$ is satisfied by s.

Problem 3.

You are free to use any of the results in the textbook, including the Soundness, Completeness, and Compactness Theorems.

Let $\mathcal{L}$ be a reasonable [2] language for first-order logic, and let T_1 and T_2 be two theories of $\mathcal{L}$.

(a) Show that $T_1 \cap T_2$ is also a theory.

(Recall that a set of sentences T is a theory iff, for every sentence σ, we have $(T \vdash \sigma \implies \sigma \in T)$.)

Let σ be a sentence such that $T_1 \cap T_2 \vdash \sigma$, for $T_1 \cap T_2$ to be a theory, then $\sigma \in T_1 \cap T_2$.

If $T_1 \cap T_2 \vdash \sigma$, then $T_1 \vdash \sigma$ (since $T_1 \cap T_2 \subseteq T_1$) and $T_2 \vdash \sigma$ (since $T_1 \cap T_2 \subseteq T_2$). However, if $T_1 \vdash \sigma$, then $\sigma \in T_1$ (since T_1 is a theory), and if $T_2 \vdash \sigma$, then $\sigma \in T_2$ (since T_2 is a theory). It follows that $\sigma \in T_1 \cap T_2$.

Therefore, $T_1 \cap T_2 \vdash \sigma$, iff $\sigma \in T_1 \cap T_2$, so $T_1 \cap T_2$ is a theory.

(b) Suppose that T_1 and T_2 are both axiomatizable, complete, consistent theories of $\mathcal{L}$. Which of the following must be true?

If true, explain why; if false, give a counterexample.

(i) $T_1 \cap T_2$ is consistent.

True.

Since T_1 is a theory for $\mathcal{L}$, every sentence in T_1 is true in $\mathcal{L}$. Therefore, every sentence in $T_1 \cap T_2$ must be true in $\mathcal{L}$ since $(T_1 \cap T_2) \subseteq T_1$. T_2 is also a theory for $\mathcal{L}$, so every sentence in T_2 is true in $\mathcal{L}$ and the same argument can be made using T_2. Therefore, every deduction from $T_1 \cap T_2$ is also deduced by T_1 and T_2. Since T_1 and T_2 are consistent, $T_1 \cap T_2$ must also consistent.

If $T_1 \cap T_2$ were inconsistent, then it would deduce — hence contain — some sentence σ and its negation, $\neg\sigma$, which would mean T_1 and T_2 would both contain σ and $\neg\sigma$, thus contradicting the consistency of T_1 and T_2.

(ii) $T_1 \cap T_2$ is complete.

True.

Since T_1 is complete, for every sentence σ, either $\sigma \in T_1$ or $(\neg\sigma) \in T_1$. Likewise, since T_2 is complete, for every sentence σ, either $\sigma \in T_2$ or $(\neg\sigma) \in T_2$.

Since both T_1 and T_2 are theories for $\mathcal{L}$, we cannot have $\sigma \in T_1$ and $(\neg\sigma) \in T_2$ (since all sentences in each of the theories have to be true in $\mathcal{L}$, and both theories are consistent). Therefore, if σ is a valid sentence in $\mathcal{L}$, it must be in T_1 *and* it must be in T_2, so it must be in $T_1 \cap T_2$. On the other hand, if σ is not a valid sentence in $\mathcal{L}$, then $\neg\sigma$ must be in T_1 *and* $\neg\sigma$ must be in T_2, so $\neg\sigma$ must be in $T_1 \cap T_2$.

Therefore, $T_1 \cap T_2$ is also complete.

[2] This means reasonable in Enderton's sense; see page 142.

(iii) $T_1 \cap T_2$ is decidable.

False.

That the language $\mathcal{L}$ is reasonable means we can effectively enumerate the set of valid sentences in $\mathcal{L}$. However, this set is not necessarily decidable so, given any formula γ, we may not readily determine if it is valid or not valid in $\mathcal{L}$. Therefore, we may not be able to determine if $\gamma \in T_1 \cap T_2$ or $\neg\gamma \in T_1 \cap T_2$, so $T_1 \cap T_2$ may not be decidable.

Problem 4.

Let $\mathcal{L}$ be the language with the equality symbol, a constant symbol 0, and a two-place predicate symbol <. Let the structure $\mathfrak{A} = \langle Z; 0, < \rangle$ be the integers with distinguished element 0 and the usual ordering; that is $|\mathfrak{A}| = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, $0^{\mathfrak{A}} = 0$, and $<^{\mathfrak{A}} = \{(m, n) \mid m < n\}$.

Write formulas that define the following sets in $\mathfrak{A}$.

Be sure your formulas have the correct free variables. Also be sure to notice that this language has only two nonlogical symbols other than $\forall$, namely 0 and <.

This is a short answer problem; you do not need to prove that your formulas define the sets they are supposed to define.

(a) $\{-2\}$.

$$\exists y\,((v_1 < y) \wedge (y < 0) \wedge \forall z\,((z < 0) \rightarrow ((\neg(z < y) \wedge \neg(y < z)) \vee (\neg(z < v_1) \wedge \neg(v_1 < z)) \vee (z < v_1))))$$

(b) $\{0, -2\}$.

$$(\neg(v_1 < 0) \wedge \neg(0 < v_1))$$

$$\vee$$

$$\exists y\,((v_1 < y) \wedge (y < 0) \wedge \forall z\,((z < 0) \rightarrow ((\neg(z < y) \wedge \neg(y < z)) \vee (\neg(z < v_1) \wedge \neg(v_1 < z)) \vee (z < v_1))))$$

(c) $\{(n, m) \mid m = n + 1\}$.

$$(v_1 < v_2) \wedge \forall x((v_1 < x) \rightarrow (\neg(v_2 < x) \wedge \neg(x < v_2)) \vee (v_2 < x))$$

(d) $\{n \mid n \geq -2\}$.

$$\exists \alpha\,(\exists y\,((\alpha < y) \wedge (y < 0) \wedge \forall z\,((z < 0) \rightarrow ((\neg(z < y) \wedge \neg(y < z)) \vee (\neg(z < \alpha) \wedge \neg(\alpha < z)) \vee (z < \alpha))))$$

$$\wedge$$

$$((\neg(v_1 < \alpha) \wedge \neg(\alpha < v_1)) \vee (\alpha < v_1)))$$

Problem 5.

Let $\mathcal{L}$ be the language with the equality symbol, a constant symbol 0, and a two-place predicate symbol <. Let the structure $\mathfrak{A} = \langle Z; 0, < \rangle$ be the integers with distinguished element 0 and the usual ordering; that is $|\mathfrak{A}| = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, $0^{\mathfrak{A}} = 0$, and $<^{\mathfrak{A}} = \{(m, n) \mid m < n\}$.

Show there is a countable structure $\mathfrak{B}$ for $\mathcal{L}$ that is elementarily equivalent to $\mathfrak{A}$ and has the following property: There is an element $a \in |\mathfrak{B}|$ such that $\{b \in |\mathfrak{B}| \mid 0^{\mathfrak{B}} <^{\mathfrak{B}} b <^{\mathfrak{B}} a\}$ is infinite.

For this problem, you do not have to be too formal about justifying claims like "every model of σ has property Φ" or "$\mathfrak{C}$ with variable assignment s satisfies α."

For example, if σ is $\exists x\, \exists y\, x \neq y$, it is obvious that every model of σ has size at least 2 and that $\mathfrak{A}$ is a model of σ; you need not prove this.

$\mathfrak{A}$ is a model of a discrete linear order without endpoints.

Define $\mathfrak{B}$ such that $|\mathfrak{B}| = \mathbb{Z} \times \{0, 1\}$, $\mathfrak{B}$ has two distinguished elements $(0, 0)$ and $(0, 1)$, $0^{\mathfrak{B}} = (0, 0)$, and following the usual ordering of $\mathbb{Z}$ on the first coordinate of elements in $|\mathfrak{B}|$, i.e. $|\mathfrak{B}| =$

$$\{\ldots, (-3,0), (-2,0), (-1,0), (0,0), (0,1), (1,0), (2,0), (3,0), \ldots\}$$

$$\cup$$

$$\{\ldots, (-3,1), (-2,1), (-1,1), (0,1), (1,1), (2,1), (3,1), \ldots\}$$

Define $<^{\mathfrak{B}} = \{((a_1, b_1), (a_2, b_2)) \mid (b_1 < b_2) \vee (\neg(b_2 < b_1) \wedge (a_1 < a_2))\}$.

Then $\mathfrak{B}$ is countable and there exists an element $a \in |\mathfrak{B}|$ such that the set $S = \left\{b \in |\mathfrak{B}| \mid 0^{\mathfrak{B}} <^{\mathfrak{B}} b <^{\mathfrak{B}} a\right\}$ is infinite (take $a = (0, 1)$, for example, then $S = \{(n, 0) \mid n \in \mathbb{Z}, n > 0\}$, which is infinite).