Math 69: Logic	**Winter '23**

Exam 2 — 01/30/2023

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You may consult your textbook, notes, class handouts, and returned homework as you work on this exam, but you should not discuss the exam with anybody other than the professor, or look in other textbooks or on the internet (except on the course web page).

It is still okay to discuss class worksheets and homework problems with each other, even if they are related to exam problems, as long as you do not discuss any possible relevance to the exam.

Please ask the professor if you have any questions about the exam. You can use anything from the portions of the text we have covered, including the results of homework problems that were assigned for graded homework. (If you want to use the result of a homework problem that wasn't assigned, you must first solve the problem, and include the solution in your answer.) You can use material from class handouts and worksheets, including the results of problems. You can also use earlier parts of an exam problem in the solutions to later parts of that same problem, even if you were not able to solve the earlier parts.

Your exam paper should follow the following format rules: Identify each problem by number, and also repeat or restate the problem before giving a solution.

The exam will be graded on the clarity and completeness of your explanations, and the correct use of mathematical notation and terminology, as well as on the content of your answers.

All the problems in this exam concern first order logic.

Throughout this exam, you may use $\exists$ and all the connectives of sentential logic, omit or add parentheses for clarity and readability, write binary predicates in infix notation (write $t_1 = t_2$ instead of $= t_1 t_2$), and use abbreviations such as $\neq$ (write $t_1 \neq t_2$ instead of $(\neg\, t_1 = t_2)$).

Problem 1.

Show carefully and formally, directly from the formal definition of satisfaction [1], that the sentence

$$\forall x\, Pxfc \to \exists x Pxfx$$

(where c is a constant symbol, f is a one-place function symbol, and P is a two-place predicate symbol) is logically valid.

[1] I repeat, directly from the formal, recursive definition of satisfaction, given on the bottom of page 83 and the top of page 84 in the textbook. In particular, this problem is about logical validity, and not about deductions.

Problem 2.

In each case, either show without using the Completeness Theorem that $\varphi \vdash \psi$, or else show that $\varphi \nvDash \psi$. Here P and Q are one-place predicate symbols.

To show $\varphi \vdash \psi$, you may use any of the metatheorems of Section 2.4. You do not need to give an actual deduction.

For this problem, if you want to give an example of a structure and variable assignment satisfying some particular formula γ, it is enough to specify the structure and variable assignment. You do not have to formally prove that γ is satisfied.

(i) φ is $\forall x\,(Px \vee Qx)$, and ψ is $\forall x\, Px \vee \forall x\, Qx$.

(ii) φ is $\forall x\, Px \vee \forall x\, Qx$, and ψ is $\forall x\,(Px \vee Qx)$.

(iii) φ is $\forall x\,(Px \vee Qy)$, and ψ is $\forall x\, Px \vee \forall x\, Qy$.

Problem 3.

You are free to use any of the results in the textbook, including the Soundness, Completeness, and Compactness Theorems.

Let $\mathcal{L}$ be a reasonable [2] language for first-order logic, and let T_1 and T_2 be two theories of $\mathcal{L}$.

(a) Show that $T_1 \cap T_2$ is also a theory.

(Recall that a set of sentences T is a theory iff, for every sentence σ, we have $(T \vdash \sigma \implies \sigma \in T)$.)

(b) Suppose that T_1 and T_2 are both axiomatizable, complete, consistent theories of $\mathcal{L}$. Which of the following must be true?

If true, explain why; if false, give a counterexample.

(i) $T_1 \cap T_2$ is consistent.

(ii) $T_1 \cap T_2$ is complete.

(iii) $T_1 \cap T_2$ is decidable.

[2] This means reasonable in Enderton's sense; see page 142.

Problem 4.

Write formulas that define the following sets in $\mathfrak{A}$.

Be sure your formulas have the correct free variables. Also be sure to notice that this language has only two nonlogical symbols other than $\forall$, namely 0 and $<$.

This is a short answer problem; you do not need to prove that your formulas define the sets they are supposed to define.

(a) $\{-2\}$.

(b) $\{0, -2\}$.

(c) $\{(n, m) \mid m = n + 1\}$.

(d) $\{n \mid n \geq -2\}$.

Problem 5.

Show there is a countable structure $\mathfrak{B}$ for $\mathcal{L}$ that is elementarily equivalent to $\mathfrak{A}$ and has the following property: There is an element $a \in |\mathfrak{B}|$ such that $\{b \in |\mathfrak{B}| \mid 0^{\mathfrak{B}} <^{\mathfrak{B}} b <^{\mathfrak{B}} a\}$ is infinite.

For this problem, you do not have to be too formal about justifying claims like "every model of σ has property Φ" or "$\mathfrak{C}$ with variable assignment s satisfies α."

For example, if σ is $\exists x \, \exists y \, x \neq y$, it is obvious that every model of σ has size at least 2 and that $\mathfrak{A}$ is a model of σ; you need not prove this.