Math 69: Logic **Winter '23**

Homework assigned January 11, 2023

Prof. Marcia Groszek ***Student: Amittai Siavava***

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) **A Mathematical Introduction to Logic** by **Herbert Enderton**.

Problem 3.

Show that from the corollary to the compactness theorem we can prove the compactness theorem itself (far more easily than we can starting from scratch).

Compactness Theorem: A set of wffs is satisfiable iff every finite subset is satisfiable.

Corollary: If $\Sigma \vDash \tau$, then there exists a finite subset Σ_0 such that $\Sigma_0 \vDash \{\tau\}$

For each element $\alpha \in \Sigma$, fix a finite set $\Sigma_\alpha \subseteq \Sigma$ such that $\alpha \in \Sigma_\alpha$. Since $\alpha \in \Sigma_\alpha$, $\Sigma_\alpha \cup \{\neg\alpha\}$ is not satisfiable, so $\Sigma_\alpha \vDash \alpha$. Therefore, any truth assignment v that satisfies Σ_α must assign $v(\alpha) = T$.

If every such subset Σ_α is satisfiable for all elements $\alpha \in \Sigma$, then there exists a truth assignment v such that $v(\alpha) = T$ for all $\alpha \in \Sigma$, implying that Σ is satisfiable.

Problem 4.

In 1977, it was proved that every planar mapcan be colored with four colors. Of course, the definition of "map" requires that there be only finitely many countries. But extending the concept, suppose we have an infinite (but countable) planar map with countries $C_1, C_2, C_3, \ldots$. Prove that this infinite planar map can still be colored with four colors.

Suggestion:

(i) Partition the sentence symbols into four parts. One sentence symbol, for example, can be used to translate, "Country C7 is colored red."

(ii) Form a set Σ_1 of wffs that say, for example, C_7 is exactly one of the colors.

(iii) Form another set Σ_2 of wffs that say, for each pair of adjacent countries, that they are not the same color.

(iv) Apply compactness to $\Sigma_1 \cup \Sigma_2$.

First, we define constraints to ensure every country is colored. Suppose we use the four colors red, green, blue, and yellow, with the symbols $\alpha_{(n,1)}$, $\alpha_{(n,2)}$, $\alpha_{(n,3)}$, and $\alpha_{(n,4)}$ used to indicate that country C_n is colored red, green, blue, or yellow respectively.

We need to define constraints to ensure every country is colored:

$$\Sigma_1 = \left\{ \left(\alpha_{(n,1)} \vee \alpha_{(n,2)} \vee \alpha_{(n,3)} \vee \alpha_{(n,4)} \right) : n \in \mathbb{Z}_{>0} \right\}.$$

We then need to ensure each country is only colored a single color:

$$\Sigma_2 = \left\{ \left(\alpha_{(n,i)} \to \neg\alpha_{(n,j)} \right) : n \in \mathbb{Z}_{>0}, 1 \leq i < j \leq 4 \right\}$$

Next, we need to make sure every pair of adjacent countries C_a and C_b are not colored the same color.

$$\Sigma_3 = \left\{ \left(\alpha_{(a,i)} \to \neg\alpha_{(b,i)} \right) : C_a \text{ and } C_b \text{ are adjacent} \right\}.$$

Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. As proven before, finite planar maps can be colored with four colors, so Σ is finitely satisfiable. The compactness theorem tells us that Σ is satisfiable iff Σ is finitely satisfiable, so it is possible to color every country in the infinite planar map.

Problem 5.

Where Σ is a set of wffs, define a deduction from Σ to be a finite sequence $\alpha_0, \alpha_1, \ldots, \alpha_n$ of wffs such that for each $k \leq n$, either:

(i) α_k is a tautology,

(ii) $\alpha_k \in \Sigma$, or

(iii) for some i and j less than k, α_i is $(\alpha_j \to \alpha_k)$.

In case 3, one says that α_k is obtained by *modus ponens* from α_i and α_j.

Give a deduction from the set $\{\neg S \vee R, R \to P, S\}$, the last component of which is P.

(i) $(\neg S \vee R) \in \Sigma$, implying $(S \to R)$.

(ii) $R \to P \in \Sigma$.

(iii) $S \in \Sigma$.

(iv) Therefore, R (by modus ponens on (i) and (iii)) .

(v) Therefore, P (by modus ponens on (ii) and (iv)).