Math 69: Logic **Winter '23**

Homework assigned January 25, 2023

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Problem 1.

(a) Show that $\Gamma; \alpha \vDash \varphi$ iff $\Gamma \vDash (\alpha \to \varphi)$.

Let $\mathfrak{A}$ be a model for Γ and let $s : V \to |\mathfrak{A}|$ be a function such that $\mathfrak{A}$ satisfies Γ with s. Then $\vDash_{\mathfrak{A}} \gamma[s]$ for all $\gamma \in \Gamma$. Then:

($\Rightarrow$) We shall prove the right implication via the contrapositive, i.e. if $\Gamma \nvDash (\alpha \to \varphi)$ then $\Gamma; \alpha \nvDash \varphi$.

Suppose $\Gamma \nvDash (\alpha \to \varphi)$. Since $\mathfrak{A}$ satisfies Γ with s and $\Gamma \nvDash (\alpha \to \varphi)$, it must be the case that $\overline{s}(\alpha) = T$ and $\overline{s}(\varphi) = F$ so that $\overline{s}(\alpha \to \varphi) = F$. Therefore, $\mathfrak{A}$ satisfies $\Gamma; \alpha$ with s, but since $\mathfrak{A}$ does not satisfy φ with s, it must be the case that $\Gamma; \alpha \nvDash \varphi$.

($\Leftarrow$) We shall prove this directly, i.e. by showing that if $\Gamma \vDash (\alpha \to \varphi)$ then $\Gamma; \alpha \vDash \varphi$.

Suppose $\Gamma \vDash (\alpha \to \varphi)$, we show that it is always the case that $\Gamma; \alpha \vDash \varphi$.

(i) If $\mathfrak{A}$ does not satisfy α with s, then $\nvDash_{\mathfrak{A}} \alpha[s]$. Therefore, $\overline{s}(\alpha) = F$, so $\overline{s}(\alpha \to \varphi) = T$ irrespective of the value of $\overline{s}(\varphi)$, so $\vDash_{\mathfrak{A}} (\alpha \to \varphi)[s]$. Since $\overline{s}(\alpha) = F$ and $a \to \varphi$, $\Gamma; \alpha \vDash \varphi$.

(ii) If $\mathfrak{A}$ satisfies α with s, then $\vDash_{\mathfrak{A}} \alpha[s]$, so $\overline{s}(\alpha) = T$. Therefore, $\overline{s}(\alpha \to \varphi) = \overline{s}(\varphi)$. Since $\Gamma \vDash (\alpha \to \varphi)$, and $\mathfrak{A}$ satisfies Γ with s, then $\mathfrak{A}$ satisfies φ with s. so $\vDash_{\mathfrak{A}} (\alpha \to \varphi)[s]$. Since $\overline{s}(\alpha) = T$ and $\overline{s}(\varphi) = T$, $\Gamma; \alpha \vDash \varphi$.

Problem 3.

Show that

$$\{\forall x(\alpha \to \beta), \forall x\alpha\} \vDash \forall x\beta$$

Let $\mathfrak{A}$ be a structure and s be a function from V to $|\mathfrak{A}|$ such that $\vDash_{\mathfrak{A}} (\forall x(\alpha \to \beta(x)))[s]$ and $\vDash_{\mathfrak{A}} (\forall x\alpha)[s]$.

For all $a \in \mathfrak{A}$, we can satisfy the conditions by sending a to x. That is:

$$\begin{aligned} \vDash_{\mathfrak{A}} (\forall x(\alpha \to \beta(x)))[s] &\quad \text{gives us} \quad \vDash_{\mathfrak{A}} (\alpha \to \beta)[s(a \mid x)] \\ \vDash_{\mathfrak{A}} (\forall x\alpha)[s] &\quad \text{gives us} \quad \vDash_{\mathfrak{A}} \alpha[s(a \mid x)] \\ \therefore \vDash_{\mathfrak{A}} \beta[s(a \mid x)] &\quad \text{by modus ponens} \end{aligned}$$

This condition holds for all $a \in |\mathfrak{A}|$, so $\vDash_{\mathfrak{A}} \forall x\beta$.