Math 69: Logic	**Winter '23**

Homework assigned February 08, 2023

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Problem 2.

Prove the equivalence of parts (a) and (b) of the completeness theorem.

Suggestion: $\Gamma \vDash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable. And Δ is satisfiable iff $\Delta \nvdash \perp$, where $\perp$ is some unsatisfiable, refutable formula like $\neg\forall x x = x$. (Similarly, the soundness theorem is equivalent to the statement that every satisfiable set of formulas is consistent.)

Completeness Theorem:

(a) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

(b) Any consistent set of formulas is satisfiable.

We shall prove this in two parts: (i) that (a) implies (b), and (ii) that (b) implies (a).

Suppose Γ is a set of formulas such that (a) holds, then we claim that Γ is consistent if and only if it is satisfiable.

Assume Γ is not satisfiable, then there exists some formula α such that $\Gamma \vDash \alpha$ and $\Gamma \vDash \neg\alpha$. By part (a) of the completeness theorem, this implies that $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$. But deductions are finite, so there must exist some finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \alpha$, and there must exist some finite subset $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_1 \vdash \neg\alpha$. Therefore:

- Either Γ_0 deduces a sentence that says α, or Γ_0 or deduces two sentences γ_1 and γ_2 where γ_1 says $\gamma_2 \rightarrow \alpha$.
- Either Γ_1 deduces a sentence that says $\neg\alpha$, or Γ_0 deduces two sentences γ_3 and γ_4 where γ_3 says $\gamma_4 \rightarrow \neg\alpha$.

Since $\Gamma \supset \Gamma_0 \cup \Gamma_1$, Γ deduces each of $\gamma_1, \gamma_2, \gamma_3$, and γ_4, so Γ deduces both α and $\neg\alpha$, so it must not be consistent.

Therefore, given that if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$, for Γ to be consistent it must be satisfiable.

Next, suppose Γ is a consistent set of wffs and (b) holds — that is, Γ is consistent and satisfiable. We show that whenever $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

Let φ be a wff and $\Gamma \vDash \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable, meaning there exists a deduction of φ from $\Gamma \cup \{\neg\varphi\}$, and there also exists a deduction of $\neg\varphi$ from $\Gamma \cup \{\neg\varphi\}$. Deductions are finite, so there must exist some finite subset $\Gamma_0 \subseteq \Gamma \cup \{\neg\varphi\}$ so that $\Gamma_0 \vdash \varphi$ and there must exist some finite subseteq $\Gamma_1 \subseteq \Gamma \cup \{\neg\varphi\}$ so that $\Gamma_1 \vdash \neg\varphi$.

But since the original set Γ is consistent (therefore has no contradicting deductions): It must be the addition of $\neg\varphi$ to Γ that causes the contradiction. But $\{\neg\varphi\}$ deduces $\neg\varphi$, so Γ must have had no deduction of $\neg\varphi$. Additionally, $\neg\varphi \notin \Gamma_0$, since $\Gamma_0 \vdash \varphi$ and $\Gamma_0 \nvdash \neg\varphi$. Since $\Gamma_0 \subseteq \Gamma \cup \{\neg\varphi\}$ and $\neg\varphi \notin \Gamma_0$, $\Gamma_0 \subseteq \Gamma$. Therefore, $\Gamma \vdash \varphi$ (since its subset deduces φ).

Therefore, given that any consistent set of formulas is satisfiable, if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.

Problem 8.

Assume the language (with equality) has just the parameters $\forall$ and P, where P is a two-place predicate symbol. Let $\mathfrak{A}$ be the structure with $|\mathfrak{A}| = \mathbb{Z}$, the set of the integers (positive, negative, zero) and with $\langle a, b\rangle \in P^{\mathfrak{A}}$ iff $|a-b| = 1$.

Thus, $\mathfrak{A}$ looks like an infinite graph:

$$\cdots \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \cdots$$

Show that there is an elementarily equivalent structure $\mathfrak{B}$ that is not connected (being *connected* means that for every two members of $|\mathfrak{B}|$, there is a path between them. A path — of length n — is a sequence $\langle p_0, p_1, \ldots, p_n\rangle$ of elements of $|\mathfrak{B}|$ such that $a = p_0$ and $b = p_n$ and $p_i, \vec{p}_{i+1} \in P^{\mathfrak{B}}$ for each i).

Write down sentences saying c and d are far apart, applying compactness.

Let π be a new constant symbol, then we define $\mathfrak{B}$ to be the structure with $|\mathfrak{B}| = \mathbb{Z} \cup \{\pi\}$. Next, we specify that π does not belong to $P^{\mathfrak{B}}$ by adding a sentence to the language saying:

$$\forall p\, \neg P^{\mathfrak{B}}\ \pi p.$$

Then $\mathfrak{B}$ is not connected, since the new constant π does not have a path of length 1 to any other element of $|\mathfrak{B}|$, and the existence of a path of higher length depends on the existence of a path of a lower length. We can specify this formally with an extra set of sentences saying that there does not exist any path of any length from a point p to the constant π:

$$\left\{\forall d \forall p_1 \forall p_2 \ldots \forall p_n \neg((P^{\mathfrak{B}} d p_1) \wedge (P^{\mathfrak{B}}\ p_1 p_2) \wedge (P^{\mathfrak{B}}\ p_2 p_3) \wedge \ldots \wedge (P^{\mathfrak{B}}\ p_{n-1} p_n) \wedge (P^{\mathfrak{B}}\ p_n \pi)) \mid d \in |\mathfrak{B}|, n \in \mathbb{Z}\right\}$$

However, $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$ since whenever $\mathfrak{A} \vDash P^{\mathfrak{A}} p_i p_j$ then $\mathfrak{B} \vDash P^{\mathfrak{B}} p_i p_j$ and vice versa.