Math 69: Logic	**Winter '23**

Homework assigned February 10, 2023

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For this assignment, $\mathcal{L}$ is the language of first-order logic with equality, countably many constant symbols, $c_0, c_1, \ldots, c_n, \ldots$, and no other predicate, constant, or function symbols. We will find all the complete theories of $\mathcal{L}$. This is a single problem in five parts. You may use completeness, soundness, and compactness.

If $\mathfrak{A}$ is a structure for $\mathcal{L}$, define an equivalence relation on the set $C = \{c_0, c_1, \ldots, c_n, \ldots\}$ of constant symbols of $\mathcal{L}$ by

$$c_m \equiv_{\mathfrak{A}} c_n \iff c_m^{\mathfrak{A}} = c_n^{\mathfrak{A}},$$

That is, two constant symbols are equivalent if and only if they name the same element of $\mathfrak{A}$.

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are structures such that $\equiv_{\mathfrak{A}}$ is the same as $\equiv_{\mathfrak{B}}$. Then two constant symbols name the same element in $\mathfrak{A}$ if and only if they name the same element in $\mathfrak{B}$.

Let $\equiv$ be any equivalence relation on C. Then there is a structure $\mathfrak{A}$ for $\mathcal{L}$ such that $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv$. Namely, let the universe of the structure be the set of equivalence classes of constant symbols, and let each constant symbol name its own equivalence class:

$$|\mathfrak{A}| = C/\equiv, \quad \text{and} \quad c_n^{\mathfrak{A}} = [c_n].$$

The most complicated part of this is the notation. The relation $\equiv$ on C specifies whether constants c_n and c_m are to refer to the same element of a structure or to different elements. Therefore, as long as $\equiv$ actually is an equivalence relation, you can create a structure obeying those rules.

Problem 1.

Show that if $\mathfrak{A}$ is any finite structure for $\mathcal{L}$, there is a countable structure $\mathfrak{B}$ such that $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$, and in $\mathfrak{B}$, infinitely many elements are not named by constant symbols. In other words, we have that

$$\left\{b \in |\mathfrak{B}| \mid \forall n(b \neq c_n^{\mathfrak{B}})\right\}$$

is infinite. (*Hint:* Use compactness.)

Compactness Theorem: If $\Gamma \vDash \varphi$, then for some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vDash \varphi$. (In other words, a set Γ has a model iff every finite subset has a model)

Elementary equivalence: Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* if $\vDash_{\mathfrak{A}} \alpha \iff \vDash_{\mathfrak{B}} \alpha$,

Let $\mathfrak{A}$ be a *finite* structure for $\mathcal{L}$ such that $|\mathfrak{A}| \neq \infty$.

Define $\mathfrak{B}$ as such

(i) Include translations of the constant symbols from C in $|\mathfrak{B}|$. That is; for each $c_n \in C$, $c_n^{\mathfrak{B}} \in |\mathfrak{B}|$.
Whenever $c_m^{\mathfrak{A}} = c_n^{\mathfrak{A}}$, let $c_m^{\mathfrak{B}} = c_n^{\mathfrak{B}}$ so that $c_m \equiv_{\mathfrak{B}} c_n$ if and only if $c_m =_{\mathfrak{A}} c_n$.

(ii) Define a new set of countably many elements $B = \{b_0, b_1, b_2, \dots, b_n, \dots\}$ such that:
- $\forall i \forall j \neg(b_i = c_j)$.
- $\forall i \forall j (i = j \vee \neg(b_i = b_j))$.

(iii) For each $b_i \in B$, let $b_i^{\mathfrak{B}} \in |\mathfrak{B}|$.

Then $\mathfrak{B}$ is an infinite structure for $\mathcal{L}$ that is elementarily equivalent to $\mathfrak{A}$.

Problem 2.

Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are countable structures for $\mathcal{L}$ in which infinitely many elements sre not named by constant symbols, and $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$, then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

Since $\equiv A$ is the same relation as $\equiv B$, $\mathfrak{A}$ and $\mathfrak{B}$ agree on which of the constants in C name the same elements.

Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ be the set of countable elements of $\mathfrak{A}$ that are not named by constant symbols, and let $B = \{b_1, b_2, \ldots, b_n, \ldots\}$ be the set of countable elements of $\mathfrak{B}$ that are not named by constant symbols.

Define a bijection from $|\mathfrak{A}|$ to $|\mathfrak{B}|$ as follows:

$$\psi : |\mathfrak{A}| \to |\mathfrak{B}|$$

$$c_n^{\mathfrak{A}} \mapsto c_n^{\mathfrak{B}}$$

$$a_i \mapsto b_i$$

Then ψ is an isomorphism, since:

$$c_n^{\mathfrak{A}} \equiv_{\mathfrak{A}} c_m^{\mathfrak{A}} \qquad \Longleftrightarrow \qquad c_n^{\mathfrak{B}} \equiv_{\mathfrak{B}} c_m^{\mathfrak{B}},$$

but $c_n^{\mathfrak{B}} = \psi(c_n^{\mathfrak{A}})$ and $c_m^{\mathfrak{B}} = \psi(c_m^{\mathfrak{A}})$, so

$$c_n^{\mathfrak{A}} \equiv_{\mathfrak{A}} c_m^{\mathfrak{A}} \qquad \Longleftrightarrow \qquad \psi(c_n^{\mathfrak{A}}) \equiv_{\mathfrak{B}} \psi(c_m^{\mathfrak{A}}).$$

Since every element in A and every element in B is not named by any constant symbol, and each element is distinct from all other elements in the set, each $a \in A$ is in its own equivalence class and so is each $b \in B$. ψ matches element a_i in $\mathfrak{A}$ to element b_i in $\mathfrak{B}$, so it guarantees that any element in $[a_i]$ maps to an element in $[b_i]$ (since $[a_i]$ only contains the single element a_i, and $\psi(a_i) = b_i$). Therefore, $a_i \equiv_{\mathfrak{A}} a_j$ if and only if $b_i \equiv_{\mathfrak{B}} b_j$, since in both cases it must be that $i = j$.

Therefore ψ is an isomorphism, and $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

Problem 3.

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two structures for $\mathcal{L}$, each of which is countable (or possibly finite). Which of the following conditions imply which others? In each case, either explain or give a counter-example.

(i) $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

(ii) $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$.

(iii) $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$.

- (i) implies (ii) and (iii).

  If $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ under some isomorphism ψ, then $\psi(c_i^{\mathfrak{A}}) \equiv_{\mathfrak{B}} \psi(c_j^{\mathfrak{A}})\mathfrak{A}$ whenever $c_i^{\mathfrak{A}} \equiv_{\mathfrak{A}} c_j^{\mathfrak{A}}$ for every constant symbol, and, similarly, $\psi(a_i) \equiv_{\mathfrak{B}} \psi(a_j)$ whenever $a_i \equiv_{\mathfrak{A}} a_j$ symbol a_i in $|A|$ that is not in C. Therefore, $\equiv_{\mathfrak{A}}$ and $\equiv_{\mathfrak{B}}$ are the same relation since they agree on all elements in the universes of the structures. Furthermore, whenever $\mathfrak{A}$ tautologically implies that $\alpha \equiv_{\mathfrak{A}} \beta$ for some $\alpha, \beta \in |\mathfrak{A}|$, $\mathfrak{B}$ also tautologically implies that $\psi(\alpha) \equiv_{\mathfrak{B}} \psi(\beta)$, so $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent.

- (ii) implies (iii), but does not necessarily imply (i).

  If $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$, then $c_i \equiv_{\mathfrak{A}} c_j$ if and only if $c_i \equiv_{\mathfrak{B}} c_j$, so $\equiv_{\mathfrak{A}}$ and $\equiv_{\mathfrak{B}}$ are the same relation.

  However, $\mathfrak{A}$ and $\mathfrak{B}$ may not be isomorphic.

  If both $\mathfrak{A}$ and $\mathfrak{B}$ have a countably infinite set of non-constant symbols, or they both have the same finite number of non-constant symbols, then they are isomorphic (since there exists a bijection between $|\mathfrak{A}|$ and $|\mathfrak{B}|$ that preserves the relation).

  If both $\mathfrak{A}$ and $\mathfrak{B}$ have a finite number of non-constant symbols but the number of non-constant symbols in $\mathfrak{A}$ does not equal the number of non-constant symbols in $\mathfrak{B}$, or if one of $\mathfrak{A}$ has an infinite number of non-constant symbols while the other has a finite number then there is no possible bijection between the sets of non-constant symbols so $\mathfrak{A}$ cannot be isomorphic to $\mathfrak{B}$.

- (iii) implies (ii) but does not necessarily imply (i)

  If $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$, then $c_i \equiv_{\mathfrak{A}} c_j$ if and only if $c_i \equiv_{\mathfrak{B}} c_j$, so $\mathfrak{A}$ and $\mathfrak{B}$ always agree on any two constant symbols being in the same equivalence class or being in different equivalence classes. Therefore, $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent.

  However, as in the previous part, it is impossible for $\mathfrak{A}$ and $\mathfrak{B}$ to be isomorphic unless they either have the same number of non-constant symbols or they both have a countably infinite set of non-constant symbols.

Problem 4.

Suppose $\equiv$ is an equivalence relation on C. Define

$$\Sigma_{\equiv} = \{c_n = c_m \mid c_m \equiv c_n\} \cup \{c_n \neq c_m \mid c_n \not\equiv c_m\}.$$

Show that if $\equiv$ has infinitely many equivalence classes, then $Cn\Sigma_{\equiv}$ is a complete theory.

Suggestion: Explain why $\mathfrak{A}$ is a model of $Cn\Sigma_{\equiv}$ if and only if $\equiv_{\mathfrak{B}}$ is the same relation as $\equiv$. Then use (2) and (3) to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are any two models for $Cn\Sigma_{\equiv}$, then there are structures $\mathfrak{A}^*$ and $\mathfrak{B}^*$ elementarily equivalent to $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}^*$ and $\mathfrak{B}^*$ are isomorphic.

Suppose $\mathfrak{A}$ is a model for $\Sigma_{\equiv}$. If $\mathfrak{A}$ satisfies $\Sigma_{\equiv}$ under some variable assignment s, then for any two constant symbols c_m and c_n,

(i) If $\models_{\mathfrak{A}} c_m \equiv c_n[s]$, then $\Sigma_{\equiv}$ contains a sentence that says $c_m = c_n$. So, by definition of $\equiv_{\mathfrak{A}}$, we also have that $c_m \equiv_{\mathfrak{A}} c_n$.

(ii) If $\models_{\mathfrak{A}} c_m \not\equiv c_n[s]$, then $\Sigma_{\equiv}$ contains a sentence that says $c_m \neq c_n$, so $c_m \not\equiv_{\mathfrak{A}} c_n$.

Therefore, if $[c_i]_{\equiv}$ is the equivalence class of c_i under $\equiv$, and $[c_i]_{\equiv_{\mathfrak{A}}}$ is the equivalence class of c_i under $\equiv_{\mathfrak{A}}$, then $[c_i]_{\equiv} = [c_i]_{\equiv_{\mathfrak{A}}}$ since the two equivalence relations agree on any two constants being in the same equivalence class or not being in the same class. Therefore, $\equiv$ and $\equiv_{\mathfrak{A}}$ are the same relation.

Now, suppose $\mathfrak{A}$ and $\mathfrak{B}$ are models for $Cn\Sigma_{\equiv}$, then $\equiv_{\mathfrak{A}}$ is the same same relation as $\equiv$, and $\equiv_{\mathfrak{B}}$ is also the same relation as $\equiv$, meaning $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$. Extend $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathfrak{A}^*$ and $\mathfrak{B}^*$ respectively by adding new, distinct symbols as in the proof of problem (2). That is, for each $i \in \mathbb{N}$, we define a_i to be a new symbol that does not yet exist in $|\mathfrak{A}|$, and b_i to be a new symbol that does not yet exist in $|\mathfrak{B}|$. We then define $|\mathfrak{A}^*| = |\mathfrak{A}| \cup \{a_i \mid i \in \mathbb{N}\}$ and $|\mathfrak{B}^*| = |\mathfrak{B}| \cup \{b_i \mid i \in \mathbb{N}\}$. Then $\mathfrak{A}^*$ and $\mathfrak{B}^*$ are models for $Cn\Sigma_{\equiv}$, $\mathfrak{A}^*$ is elementarily equivalent to $\mathfrak{A}$, $\mathfrak{B}^*$ is elementarily equivalent to $\mathfrak{B}$, and $\mathfrak{A}^*$ is isomorphic to $\mathfrak{B}^*$ since they have the same cardinality and $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$.

Problem 5.

Suppose $\equiv$ is an equivalence relation on C with finitely many equivalence classes. Describe all the complete (consistent) theories of T with the property that $Cn\Sigma_{\equiv} \subset T$, by saying what sentences you need to add to $\Sigma_{\equiv}$ to produce the set of axioms for T.

Hint: Consider the possible ways to get finite or infinite countable models of $Cn\Sigma_{\equiv}$ that are not isomorphic. Then consider whether these non-isomorphic structures have different theories.

Since every complete theory of $\mathcal{L}$ contains some $Cn\Sigma_{\equiv}$ (because every structure for $\mathfrak{L}$ satisfies some $\Sigma_{\equiv}$), problems (4) and (5) together describe all the complete theories of $\mathcal{L}$.

First, T must satisfy all the consequences of $\Sigma_{\equiv}$, i.e. $Cn\Sigma_{\equiv} \subset T$, so the set of axioms of T must contain all the axioms in $\Sigma_{\equiv}$.

One way to generate axioms for T is by adding axioms to $\Sigma_{\equiv}$ that specify additional elements but do not alter the conditions that $\Sigma_{\equiv}$ places on the constant symbols.

For example:

(i) $\Sigma_1 = \Sigma_{\equiv} \cup \left\{\exists x_1 \forall c_i^{\mathfrak{A}} \neg(x_1 = c_i^{\mathfrak{A}})\right\}$ specifies that $|\mathfrak{A}|$ contains translations of all the constant symbols and $|\mathfrak{A}|$ also contains at least one non-constant symbol.

(ii) $\Sigma_2 = \Sigma_1 \cup \{\exists x_1 \exists x_2 (\neg(x_1 = x_2) \wedge \forall c_i (\neg(x_1 = c_i) \wedge \neg(x_2 = c_i)))\}$ specifies that $\mathfrak{A}$ contains translation of the constant symbols and it also contains at least two non-constant symbols.

(iii) Inductively, we define Σ_n as follows:

$$\Sigma_n = \Sigma_{n-1} \cup \{\exists x_1 \ldots \exists x_n (\neg(x_1 = x_2) \wedge \ldots \wedge \neg(x_{n-1} = x_n) \wedge \forall c_i (\neg(x_1 = c_i) \wedge \ldots \wedge \neg(x_n = c_i)))\}$$

Define x_n to be the new non-constant symbol added at step n, then Σ_n is countable. It is also finite for all finite n.

Σ_1 and Σ_2 are not isomorphic since they have different cardinalities. Likewise, all such finite models of $\mathfrak{A}$ that have different cardinalities are not isomorphic and they have different theories.