Math 69: Logic **Winter '23**

Homework assigned January 27, 2023

Prof. Marcia Groszek ***Student: Amittai Siavava***

Problem 8.

Assume that Σ is a set of sentences such that for any sentence τ, either $\Sigma \models \tau$ or $\Sigma \models \neg\tau$. Assume that $\mathfrak{A}$ is a model of Σ.

Show that for any sentence τ, we have $\models_{\mathfrak{A}} \tau$ *iff* $\Sigma \models \tau$.

Since $\mathfrak{A}$ is a model for Σ, $\mathfrak{A}$ must agree wth Σ on all sentences in Σ.

($\Rightarrow$) For any sentence τ, suppose $\models_{\mathfrak{A}} \tau$. Since $\mathfrak{A}$ is a model for Σ, there must be a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \tau$. This implies that $\Sigma_0 \models \tau$, so $\Sigma \models \tau$.

($\Leftarrow$) Now suppose $\not\models_{\mathfrak{A}} \tau$. Since $\mathfrak{A}$ is a model for Σ, this implies that no finite subset $\Sigma_0 \subseteq \Sigma$ can deduce τ, so $\Sigma_0 \nvdash \tau$ implying $\Sigma_0 \not\models \tau$ for all $\Sigma_0 \in \mathcal{P}(\Sigma)$. However, if no finite subset of Σ can deduce τ, then Σ cannot deduce τ. Therefore, $\Sigma \not\models \tau$.

Problem 11.

For each of the following relations, give a formula which defines it in $(\mathbb{N}, +)$.

The language is assumed to have equality and the parameters $\forall, +, \cdot$.

(a) $\{0\}$.

$$f_1(x) = \forall y(x + y = y)$$

(b) $\{1\}$.

$$f_2(x) = \forall y(x \cdot y = y)$$

(c) $\{\langle m, n\rangle : \ n$ is the successor of m in $\mathbb{N}\}$.

$$f_3(m, n) = \exists x(\forall y(x \cdot y = y) \wedge (m + x = n))$$

(d) $\{\langle m, n\rangle : \ m < n$ in $\mathbb{N}\}$.

$$f_4(m, n) = \exists x(\neg\forall y(x + y = y) \wedge (m + x = n))$$