Math 69: Logic **Winter '23**

Homework assigned February 10, 2023

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For this assignment, $\mathcal{L}$ is the language of first-order logic with equality, countably many constant symbols, $c_0, c_1, \ldots, c_n, \ldots$, and no other predicate, constant, or function symbols. We will find all the complete theories of $\mathcal{L}$. This is a single problem in five parts. You may use completeness, soundness, and compactness.

If $\mathfrak{A}$ is a structure for $\mathcal{L}$, define an equivalence relation on the set $C = \{c_0, c_1, \ldots, c_n, \ldots\}$ of constant symbols of $\mathcal{L}$ by

$$c_m \equiv_{\mathfrak{A}} c_n \iff c_m^{\mathfrak{A}} = c_n^{\mathfrak{A}},$$

That is, two constant symbols are equivalent if and only if they name the same element of $\mathfrak{A}$.

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are structures such that $\equiv_{\mathfrak{A}}$ is the same as $\equiv_{\mathfrak{B}}$. Then two constant symbols name the same element in $\mathfrak{A}$ if and only if they name the same element in $\mathfrak{B}$.

Let $\equiv$ be any equivalence relation on C. Then there is a structure $\mathfrak{A}$ for $\mathcal{L}$ such that $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv$. Namely, let the universe of the structure be the set of equivalence classes of constant symbols, and let each constant symbol name its own equivalence class:

$$|\mathfrak{A}| = C/\equiv, \quad \text{and} \quad c_n^{\mathfrak{A}} = [c_n].$$

The most complicated part of this is the notation. The relation $\equiv$ on C specifies whether constants c_n and c_m are to refer to the same element of a structure or to different elements. Therefore, as long as $\equiv$ actually is an equivalence relation, you can create a structure obeying those rules.

Problem 1.

Show that if $\mathfrak{A}$ is any finite structure for $\mathcal{L}$, there is a countable structure $\mathfrak{B}$ such that $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$, and in $\mathfrak{B}$, infinitely many elements are not named by constant symbols. In other words, we have that

$$\left\{ b \in |\mathfrak{B}| \mid \forall n (b \neq c_n^{\mathfrak{B}}) \right\}$$

is infinite. (*Hint:* Use compactness.)

Compactness Theorem: If $\Gamma \vDash \varphi$, then for some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vDash \varphi$.

(In other words, a set Γ has a model iff every finite subset has a model)

Problem 2.

Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are countable structures for $\mathcal{L}$ in which infinitely many elements sre not named by constant symbols, and $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$, then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

Problem 3.

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two structures for $\mathcal{L}$, each of which is countable (or possibly finite). Which of the following conditions conditions imply which others? In each case, either explain or give a counter-example.

(i) $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

(ii) $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$.

(iii) $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$.

Problem 4.

Suppose $\equiv$ is an equivalence relation on C. Define

$$\Sigma_{\equiv} = \{c_n = c_m \mid c_m \equiv c_n\} \cup \{c_n \neq c_m \mid c_n \not\equiv c_m\}.$$

Show that if $\equiv$ has infinitely many equivalence classes, then $Cn\Sigma_{\equiv}$ is a complete theory.

Suggestion: Explain why $\mathfrak{A}$ is a model of $Cn\Sigma_{\equiv}$ if and only if $\equiv_{\mathfrak{B}}$ is the same relation as $\equiv$. Then use (2) and (3) to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are any two models for $Cn\Sigma_{\equiv}$, then there are structures $\mathfrak{A}^*$ and $\mathfrak{B}^*$ elementarily equivalent to $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}^*$ and $\mathfrak{B}^*$ are isomorphic.

Problem 5.

Suppose $\equiv$ is an equivalence relation on C with finitely many equivalence classes. Describe all the complete (consistent) theories of T with the property that $Cn\Sigma_{\equiv} \subset T$, by saying what sentences you need to add to $\Sigma_{\equiv}$ to produce the set of axioms for T.

Hint: Consider the possible ways to get finite or infinite countable models of $Cn\Sigma_{\equiv}$ that are not isomorphic. Then consider whether these non-isomorphic structures have different theories.

Since every complete theory of $\mathcal{L}$ contains some $Cn\Sigma_{\equiv}$ (because every structure for fL satisfies some $\Sigma_{\equiv}$), problems (4) and (5) together describe all the complete theories of $\mathcal{L}$.