Math 69: Logic	Winter '23

Exam 1 — 01/30/2023

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You may consult your textbook, notes, class handouts, and returned homework as you work on this exam, but you should not discuss the exam with anybody other than the professor, or look in other textbooks or on the internet (except on the course web page).

It is still okay to discuss class worksheets and homework problems with each other, even if they are related to exam problems, as long as you do not discuss any possible relevance to the exam.

Please ask the professor if you have any questions about the exam. You can use anything from the portions of the text we have covered, including the results of homework problems that were assigned for graded homework. (If you want to use the result of a homework problem that wasn't assigned, you must first solve the problem, and include the solution in your answer.) You can use material from class handouts and worksheets, including the results of problems. You can also use earlier parts of an exam problem in the solutions to later parts of that same problem, even if you were not able to solve the earlier parts.

Your exam paper should follow the following format rules: Identify each problem by number, and also repeat or restate the problem before giving a solution.

The exam will be graded on the clarity and completeness of your explanations, and the correct use of mathematical notation and terminology, as well as on the content of your answers.

Problem 1.

This is a problem in sentential logic.

Let v be a truth assignment on the set of sentence symbols. For any wff α, let;

(i) $f(\alpha)$ denote the number of occurrences of $\leftrightarrow$ in α.

(ii) $g(\alpha)$ denote the number of occurrences in α of sentence symbols A_i for which $v(A_i) = T$.

For example, if $v(A_1) = T$ and $v(A_2) = F$, and α is $((A_1 \wedge A_2) \rightarrow \neg A_1)$, then $g(\alpha) = 2$.

(a) Give careful definitions of $f(\alpha)$ and $g(\alpha)$ by recursion on α.

(b) Prove (carefully and formally) by induction on α that if $\leftrightarrow$ is the only connective symbol occurring in α, then $\overline{v}(\alpha) = T$ if and only if $f(\alpha)$ and $g(\alpha)$ have the opposite parity (i.e., one is even and the other is odd).

Problem 2.

This is a problem in sentential logic.

Suppose that α, β, and γ are wffs such that $\alpha \vDash \beta$ and $\alpha \vDash \neg\gamma$. Show that $\alpha \vDash \neg\gamma$.

Please do this formally, by showing that every truth assignment that satisfies α *also satisfies* $(\neg\beta)$. *You will be graded on whether you have a correct proof of this kind.*

Problem 3.

This is a problem in sentential logic.

Show that the Compactness Theorem can be proven from the Soundness Theorem and the Completeness Theorem.

Problem 4.

Define a relation $\equiv$ on $(\mathbb{N}^+)^2 = \{(x,y) : x, y \in \mathbb{N}^+\}$ by

$$(x_1, y_1) \equiv (x_2, y_2) \iff x_1 y_2 = x_2 y_1.$$

(a) Show that $\equiv$ is an equivalence relation on $(\mathbb{N}^+)^2$.

(b) Describe the equivalence class of $(3,3)$ (do this without mentioning the equivalence relation $\equiv$).

(c) Suppose that we try to define a function on equivalence classes by

$$f[(x,y)] = [(2x^2, 2y^2)].$$

Either show that this function is well-defined or show that it is not.

Problem 5.

We define the difference of two sets X and Y, written $X - Y$, to be the set of all members of X that are not members of Y. Suppose X is an effectively enumerable set of expressions and Y is a decidable set of expressions.

(a) Show that $X - Y$ is effectively enumerable.

(b) Suppose that X is not decidable, and $X \subseteq Y$, show that $Y - X$ is not effectively enumerable.

Problem 6.

This is a problem in first order logic.

It is also a short answer problem. That is, giving a correct answer is sufficient; no explanation is needed.

Feel free to use the following abbreviations and conventions: You may use any of our five sentential logic connectives, write =, <, +, − using infix notation (e.g. $x = y$ instead of $= xy$), use the quantifier $\exists$, call your variables x and y (or even δ and ε), and omit parenthesis, include extra parentheses, and use other kinds or sizes of parentheses (such as [] and) to enhance readability.

However, "$(\forall\ \varepsilon > 0)$," "≤," and "≠", for example, are not abbreviations we have defined.

You may, however, define your own abbreviations. For example, you may say something like: "for any terms t_1 and t_2, let $t_1 > t_2$ be an abbreviation for the formula $t_2 < t_1$.

If you do this, be careful to be technically correct. For example, do noart say "Let ≤ be an abbreviation for < ∨ =".