Math 69: Logic	**Winter '23**

Homework assigned February 06, 2023

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Problem 1.

Let $\mathcal{L}$ be a language for first-order logic, including the equality symbol (and any number of other symbols), and $\mathcal{L}^*$ be $\mathcal{L}$ without the equality symbol, but with an additional 2-place predicate symbol E. For any formula α of $\mathcal{L}$, define α^* to be the formula of $\mathcal{L}^*$ obtained fromα by replacing every occurrence of = with E. If Σ is a set of formulas of $\mathcal{L}$, define

$$\Sigma^* = \left\{a^* \mid a \in \Sigma\right\}.$$

(a) This is a short answer problem.

For which logical axiom groups is it true that α is a logical axiom in that group if and only if α^* is a logical axiom in that group?

> Groups 5 and 6.

(b) This is a short answer problem.

Which, if any, of the following are true?

(i) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a deduction from Σ in the language $\mathcal{L}$ then $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ is a deduction from Σ^* in the language $\mathcal{L}^*$

> True

(ii) If $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ is a deduction from Σ^* in the language $\mathcal{L}^*$ then $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a deduction from Σ in the language $\mathcal{L}$

> False

(c) Show that if Σ is a consistent set of formulas in $\mathcal{L}$ then Σ^* is a consistent set of formulas in $\mathcal{L}^*$.

You may use your answers in (a) and (b).

Hint: Recall that Σ is inconsistent if there is some α such that $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$.

> Suppose Σ is consistent and s is a variable assignment that satisfies Σ, then $\overline{s}(\sigma) = T$ for all $\sigma \in \Sigma$. Then there exists some deduction $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ with $\alpha_n = \alpha$ and each α_i either occurring in Σ or having some $j, k < i$ such that $\Sigma \vDash \alpha_j$ and α_k is $\alpha_j \to \alpha_i$. Therefore, the deduction $\langle \alpha_1^*, \alpha_2^*, \ldots, \alpha_n^* \rangle$ also exists in Σ^* with $\alpha_n^* = \alpha^*$, meaning $\Sigma^* \vDash \alpha^*$.
>
> Therefore, if Σ is consistent and some truth assignment satisfies all rules in Σ, then Σ^* is also consistent.

(d) Show by example that it is possible for Σ to be inconsistent but Σ^* to be consistent.

> Suppose both $\mathcal{L}$ and $\mathcal{L}^*$ contain two constant symbols c_1 and c_2 defined such that $c_1 \neq c_2$, and a variable symbol x. Suppose Σ contains the two sentences $a = c_2$ and $a = c_2$, then Σ is not consistent since it deduces two contradicting results. For instance, if α is the assertion that $a = c_1$ then $\Sigma \vDash \alpha$ and $\Sigma \vDash \neg\alpha$.

On the other hand, if we change equality in $\mathcal{L}$ with a predicate symbol E in $\mathcal{L}^*$, then Σ^* contains the two sentences Ea_ic_1 and Ea_ic_2. Thus, $\Sigma \vDash Ea_ic_1$ and $\Sigma \vDash Ea_ic_2$. However, there is no rule in Σ^* that says $\forall a(Eac_1 \rightarrow \neg Eac_2)$, so Σ^* is still consistent.

Problem 2.

Let $\mathcal{L}$ have equality, a one-place function symbol f, and a two-place predicate symbol P.

Define $\mathcal{L}^*$ as in problem 1.

Suppose $\mathfrak{A}$ is a structure for $\mathcal{L}^*$ with the property that if α is a logical axiom of $\mathcal{L}$ in group 5 or group 6 then $\mathfrak{A}^*$ satisfies α^*.

You showed in last-week's homework that in this case, $E^{\mathfrak{A}^*}$ defines an equivalence relation on $|\mathfrak{A}^*|$, and $f^{\mathfrak{A}^*}$ and $P^{\mathfrak{A}^*}$ induce a well-defined function and a well-defined relation on equivalence classes.

Let $\mathfrak{B}^*$ be a structure for $\mathcal{L}^*$ defined by setting

$$|\mathfrak{B}| = |\mathfrak{A}|/E^{\mathfrak{A}^*} = \left\{[a] \mid a \in |\mathfrak{A}^*|\right\}$$

where $[a]$ denotes the equivalence class of a under the equivalence relation $E^{\mathfrak{A}^*}$, and letting $f^{\mathfrak{B}^*}$, $P^{\mathfrak{B}^*}$, and $E^{\mathfrak{B}^*}$ be induced by $f^{\mathfrak{A}^*}$, $P^{\mathfrak{A}^*}$, and $E^{\mathfrak{A}^*}$, such that:

$$f^{\mathfrak{B}^*}([a]) = [f^{\mathfrak{A}^*}(a)]$$

$$\langle [a], [b] \rangle \in P^{\mathfrak{B}^*} \iff \langle a, b \rangle \in P^{\mathfrak{A}^*}$$

$$\langle [a], [b] \rangle \in E^{\mathfrak{B}^*} \iff \langle a, b \rangle \in E^{\mathfrak{A}^*}$$

The second and third condition can be rewritten (informally) as:

$$[a] P^{\mathfrak{B}^*} [b] \iff a P^{\mathfrak{A}^*} b$$

$$[a] E^{\mathfrak{B}^*} [b] \iff a E^{\mathfrak{A}^*} b$$

Define a function h from $|\mathfrak{A}^*|$ to $|\mathfrak{B}^*|$ by $h(a) = [a]$.

(a) Show that h is a surjective homomorphism from $\mathfrak{A}^*$ onto $\mathfrak{B}^*$.

Let b be an element in $|\mathfrak{B}|$, then, from the definition

$$|\mathfrak{B}| = \left\{[a] \mid a \in |\mathfrak{A}^*|\right\},$$

$b = [a]$ for some $a \in |\mathfrak{A}^*|$. Since $h : |\mathfrak{A}^*| \to |\mathfrak{B}^*|$ is defined as $h(a) = [a]$, for every $b \in |\mathfrak{B}^*|$, there exists some $a \in |\mathfrak{A}^*|$ such that $h(a) = b$, so h is surjective.

(b) Show that if α is any formula of $\mathcal{L}$ and s is a variable assignment for $|\mathfrak{A}^*|$, then

$$\mathfrak{A}^* \vDash \alpha^*[s] \iff \vDash \alpha^*[h \circ s]$$

$\Longrightarrow$

Suppose α is a formula in $\mathcal{L}$ and s is a variable assignment for $|\mathfrak{A}^*|$ such that $\mathfrak{A}^* \vDash \alpha^*[s]$. Note that $h \circ s$ is the function that sends each element x to the equivalence class $[\overline{s}(x)]$.

If $\mathfrak{A}^* \vDash \alpha^*[s]$, then $\overline{s}(\alpha^*) = T$, so $a^* = [T]$, i.e. a^* under $h \circ s$ is in the equivalence class of formulas that are always true. This implies that α^* is a tautology in $\mathfrak{B}^*$ under $h \circ s$, or $\vDash \alpha^*[h \circ s]$.

$\Longleftarrow$

Suppose $\models \alpha^*[h \circ s]$. Then α is a tautology under the assignment defined by $h \circ s$. Therefore, $\alpha^* = [T]$ under $h \circ s$, meaning $\overline{s}(\alpha^*) = T$, s0 $\mathfrak{A}^* \models \alpha^*[s]$.

(c) Show that

$$E^{\mathfrak{B}^*} = \left\{([a],[b]) \in \left|\mathfrak{B}^*\right| \times \left|\mathfrak{B}^*\right| \mid [a] = [b]\right\}$$

$E^{\mathfrak{A}^*}$ is an equivalence relation, and $\mathfrak{B}$ is defined as

$$\mathfrak{B} = \mathfrak{A}/E^{\mathfrak{A}^*} = \left\{[a] \mid a \in \left|\mathfrak{A}^*\right|\right\}.$$

Let $E^{\mathfrak{B}^*}$ be the translation of $E^{\mathfrak{A}^*}$ into $\mathfrak{B}^*$. Then, for every $a, b \in |\mathfrak{A}^*|$, $E^{\mathfrak{A}}ab \iff E^{\mathfrak{B}^*}[a][b]$. But since $E^{\mathfrak{A}^*}$ is an equivalence relation (the same one used to define $|\mathfrak{B}|$), $E^{\mathfrak{A}^*}ab \iff [x] = [y]$.

The translation $E^{\mathfrak{B}^*}$ is satisfied when the original equivalence relation is satisfied, i.e. $\mathfrak{B} \models E^{\mathfrak{B}^*}$ if and only if $\mathfrak{A} \models E^{\mathfrak{A}^*}$, and $\mathfrak{A} \models E^{\mathfrak{A}^*}$ if and only if $(a, y) = T$ if and only if $[a] = [y]$. Therefore, $E^{fB^*} = \{([a],[b]) \in |\mathfrak{B}^*| \times |\mathfrak{B}^*| \mid [a] = [b]\}$.

Problem 3.

With the same definitions as in problem (2), let $\mathfrak{B}$ be the *reduct* of $\mathfrak{B}^*$ to a structure for $\mathcal{L}$. This means that:

$$|\mathfrak{B}| = |\mathfrak{B}^*|$$
$$f^{\mathfrak{B}} = f^{\mathfrak{B}^*}$$
$$P^{\mathfrak{B}} = P^{\mathfrak{B}^*}$$

In other words, $\mathfrak{B}$ translates every parameter $\mathcal{L}$ in exactly the same way as $\mathfrak{B}^*$. It merely has no translation for E, since E is not a symbol of $\mathcal{L}$. (Of course, $\mathfrak{B}$ must translate = as equality).

Because $\mathfrak{B}^*$ translated E as equality, we can conclude that for any wff α of $\mathcal{L}$ and variable assignment $\overline{s}$ for $\mathfrak{B}$, since α^* is α with E replaced by =,

$$\mathfrak{B} \vDash \alpha[\overline{s}] \iff \mathfrak{B}^* \vDash \alpha^*[\overline{s}].$$

This makes sense because $\mathfrak{B}$ and $\mathfrak{B}^*$ have the same universe, so $\overline{s}$ is also a variable assignment for $\mathfrak{B}$.

By problem (2),

$$\mathfrak{B}^* \vDash \alpha[h \circ s] \iff \mathfrak{A}^* \vDash \alpha^*[s],$$

and therefore

$$\mathfrak{B} \vDash \alpha[h \circ s] \iff \mathfrak{A}^* \vDash \alpha^*[s].$$

We conclude that if Σ is any set of formulas of $\mathcal{L}$ including all the logical axioms in groups 5 and 6, and the set Σ^* (obtained by replacing = with E in every element of Σ) is satisfiable by some structure $\mathfrak{A}^*$ and variable assignment s, then the original set Σ is also satisfiable by the structure $\mathfrak{B}$ and variable assignment $h \circ s$ as defined in problem (2).

Using problem (1) and the further context, show that if every consistent set of sentences in $\mathcal{L}^*$ is satisfiable then every consistent set of sentences in $\mathcal{L}$ is satisfiable.

We shall prove the contrapositive: suppose some consistent set of sentences in $\mathcal{L}$ is not satisfiable. Then there is some set Σ of sentences in $\mathcal{L}^*$ that must not be satisfiable.

Let γ be a variable in $\mathcal{L}$ that is not satisfied under an assignment s, then $\mathfrak{A}^* \vDash \neg a[s]$.

Therefore, $\mathfrak{B} \vDash \neg a[h \circ s]$. By the compactness theorem, there exists some finite subset of $\mathfrak{B}_0 \subseteq |\mathfrak{B}|$ such that $\mathfrak{B}_0 \vdash \gamma$. Then either $\mathfrak{B}_0$ contains $\neg\gamma$ or $\mathfrak{B}_0$ contains or deduces some formulae α and β such that α = is $\beta \to \gamma$. so $\mathfrak{B}$ is also not satisfiable.