Math 69: Logic
Abelian Groups

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Abstract

This paper describes a set of axioms and a language for abelian groups and explores what the theory of abelian groups can tell us about completeness.

Contents

1. Introduction

This is the introduction.

2. Preliminary Questions

For this problem, we will start with the following definitions. An element g of a group G has order n if n is the smallest positive natural number such that

$$ng = \underbrace{g + \ldots + g}_{n \text{ times}} = 0.$$

For example, g has order 2 if $g \neq 0$ and $g + g = 0$.
An element is said to be *torsion free* if it does not have order n for any $n \in \mathbb{N}$ with $n > 0$. A group is said to be *torsion free* if each of its elements, other than the identity, is torsion free.
Lastly, we say that a group G is divisible if for each $g \in G$ and $n \in \mathbb{N}$ with $n > 0$, there exists $h \in G$ such that

$$nh = \underbrace{h + \ldots + h}_{n \text{ times}} = g.$$

2.1. **Definitions.**

For convenience, we define the following shorthands used in the rest of the writing:

(i) $\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\}$.
(ii) $(x \neq y)$ is shorthand for $\neg(x = y)$.

2.2. **Axiomatizing Abelian Groups.**

Define a language $\mathcal{L}$ and a set of axioms Σ such that any model that satisfies Σ is an abelian group. Next, define a set of axioms T such that any model which satisfies $\Sigma \cup T$ is a divisible torsion free abelian group.

Using additive notation for groups, we define $\mathcal{L}$ to specify group operation $(+)$ and the group identity, 0.

We specify $\mathcal{L}$ as follows:

$$\mathcal{L} = \langle 0, +, -, = \rangle \tag{2.1}$$

For any structure to be abelian, it needs to satisfy the group axioms (closure, associativity, identity, and inversion) and commutativity. We Σ to contain the following axioms:

For any structure to be abelian, it needs to satisfy the following group axioms:

(i) Closure. This does not need to be specified explicitly, since n-ary functions in first-order logic are always translated as n-ary functions from $|\mathfrak{A}|^n$ to a subset $S \subseteq |\mathfrak{A}|$.

(ii) Associativity:

$$\forall g\,\forall h\,\forall i\,((g + (h + i)) = ((g + h) + i)). \tag{2.2}$$

(iii) Existence of an identity element:

$$\exists \varepsilon\,\forall g(g + \varepsilon = g). \tag{2.3}$$

Using additive notation, we denote the identity element as 0.

(iv) Existence of inverses:

$$\forall g\,\exists h\,(g + h = 0). \tag{2.4}$$

Using additive notation, we denote the inverse of g as $-g$.

(v) For any structure to be abelian, it must be commutative. For this, we need an extra axiom:

$$\forall g\,\forall h\,(g + h = h + g). \tag{2.5}$$

2.3. **Scalar Multiplication.**

Let $n \in \mathbb{Q}$ be any rational number. For any group element h, we define the multiplication of h by n as follows.

1. When n is a natural number:

(i) If $n > 0$, then nh is the unique element

$$g = \underbrace{h + \ldots + h}_{n \text{ times}}.$$

(ii) if $n = 0$, then $nh = 0g = 0$.

(iii) If $n < 0$, then nh is the unique element

$$g = \underbrace{(-h) + \ldots + (-h)}_{|q| \text{ times}}.$$

2. When n is not a natural number, write $n = \frac{a}{b}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}_{>0}$ (Since n is rational, we can always do this). Then;

$$nh = \frac{a}{b}h = \frac{1}{b}(ah).$$

First, note the multiplication ah yields a unique element in the group by the definition of multiplication in **1.** above. Call this element h'. We then define the multiplication of h' by $\frac{1}{n}$ to be the element g such that $ng = h'$.

Claim 2.6. *If* $nh_1 = nh_2 = g$, *then* $h_1 = h_2$.

Proof. Take n, h_1, h_2, and g as in the claim. Then;

$$nh_1 = nh_2 = g$$

$$\underbrace{h_1 + \ldots + h_1}_{n \text{ times}} = \underbrace{h_2 + \ldots + h_2}_{n \text{ times}} = g$$

$$\therefore \left(\underbrace{h_1 + \ldots + h_1}_{n \text{ times}}\right) - \left(\underbrace{h_2 + \ldots + h_2}_{n \text{ times}}\right) = g - g = 0$$

$$\therefore \underbrace{(h_1 - h_2) + \ldots + (h_1 - h_2)}_{n \text{ times}} = 0$$

Since G is torsion-free, we know that $ng \neq 0$ for any scalar $n \neq 0$ given $g \neq 0$. Therefore, it must be that $h_1 - h_2 = 0$. From the definition and axiomatizing of the group (section 2.2), we know that we can deduce that $-h_2$ is equivalent to the unique element $-h_1$ such that $h_1 + (-h_1) = 0$, but if the inverse of h_1 and the inverse of h_2 are equivalent then h_1 and h_2 must be equivalent.

$\square$

Claim 2.7. *Scalar multiplication is well-defined.*

Proof. Let $f : G \times \mathbb{Q} \to G$ be the function

$$f(h, n) = nh.$$

That is, f takes a group element $g \in G$ and a rational number $n \in \mathbb{Q}$ and yields the element equivalent to the scalar multiplication of g by n.

f is clearly well-defined for $n \in \mathbb{N} \subset \mathbb{Q}$, since the element $g = \underbrace{h + \ldots + h}_{n \text{ times}}$ is unique for any $n \in \mathbb{N}$.

Suppose n is not a natural number. Since n is rational, write $n = \frac{a}{b}$ and apply multiplication as defined in

(i) Suppose $h \in G$ and $m, n \in \mathbb{N}$ such that $f(h, m) = g$ and $f(h, n) = g$. Then

$$\underbrace{h + \ldots + h}_{m \text{ times}} = \underbrace{h + \ldots + h}_{n \text{ times}} = g.$$

Let $k = \mathbf{max}(m, n)$ and $l = \mathbf{min}(m, n)$, then $k - l \geq 0$, and:

$$\begin{aligned} kh - lh &= g - g = 0 \\ \therefore (k - l)\, h &= 0 \\ \therefore k - l &= 0 && \text{(since } G \text{ is torsion-free)} \\ \therefore n &= m \end{aligned}$$

(ii) Suppose $h_1, h_2 \in G$ and $n \in \mathbb{N}_{>0}$ such that $f(h_1, n) = g$ and $f(h_2, n) = g$. Then

$$\begin{aligned} \underbrace{h_1 + \ldots + h_1}_{n \text{ times}} &= \underbrace{h_2 + \ldots + h_2}_{n \text{ times}} = g \\ \therefore \left(\underbrace{h_1 + \ldots + h_1}_{n \text{ times}}\right) - \left(\underbrace{h_2 + \ldots + h_2}_{n \text{ times}}\right) &= g - g = 0 \\ \therefore \underbrace{(h_1 - h_2) + \ldots + (h_1 - h_2)}_{n \text{ times}} &= 0 \end{aligned}$$

Since G is torsion-free, $ng \neq 0$ for any $g \in G$ given $g \neq 0$. Therefore, if $n(h_1 - h_2) = 0$, it follows that $h_1 - h_2 = 0$, so $h_1 = h_2$.

Therefore, scalar multiplication is well-defined. □

2.4. **Divisibility and Torsion-Free ness.**

We define an extra set of conditions, T, which must be satisfied (in addition to those in Σ) for a structure to be a divisible torsion-free abelian group:

$$T_1 = \{\forall g\,(g \neq 0 \rightarrow ng \neq 0) \mid n = 1, 2, 3, \ldots\} \tag{2.8}$$

$$T_2 = \{\forall g\,\exists h,\,(g = nh) \mid n = 1, 2, 3, \ldots\} \tag{2.9}$$

$$T = T_1 \cup T_2 \tag{2.10}$$

2.5. **Existence of $\mathbb{Q}$-vector space structures.**

Show that any divisible torsion free abelian group has a $\mathbb{Q}$-vector space structure.
Hint: Show that if G is such a group, $n \in \mathbb{N}$ with $n > 0$ and $g \in G$ then there is a unique $h \in G$ such that $nh = g$. Note that to show there is a $\mathbb{Q}$-vector space structure, you must define scalar multiplication (and prove it is well-defined).

2.5.1. *Scalar Multiplication (Definition).*

2.6. **Any divisible torsion-free abelian group has a $\mathbb{Q}$-vector space structure.**

As seen in the proof to 2.7, if a group G is divisible and torsion-free, then scalar multiplication is well-defined. Since G is divisible, then for every $g \in G$ and $n \in \mathbb{N}_{>0}$ there is a unique $h \in G$ such that $nh = g$. As seen in the proof to 2.7, for fixed n, any such h must be unique (likewise, for fixed h, any such n must be unique). Therefore, G has a $\mathbb{Q}$-vector space structure.

2.7. **Axiomatizing Abelian Groups Wherein Each Element Other Than the Identity Has Order 2.**

Define a set of axioms S such that any model that satisfies $\Sigma \cup S$ is an abelian group in which each element other than the identity has order two. Can we give a model for $\Sigma \cup S$ a vector space structure?
Hint: Be creative in your choice of the scalar field.

We define S to contain the following axioms:

$$\forall g\,((g \neq 0) \rightarrow ((g + g) = 0)) \qquad \text{(every element has order 2)} \tag{2.11}$$

For examples of models of $\Sigma \cup S$; first, let's consider finite abelian groups in which every non-identity element has order 2. The smallest such group is the cyclic group of order 2:

$$C_2 = (\{0, 1\}, +)$$

To generate larger groups where every non-identity element has order 2, we can take the direct product of C_2 with itself. For instance, the Klein-Four group V_4 is isomorphic to $C_2 \times C_2$. For an infinite such group, take an infinite sequence of direct products of C_2:

$$C_2^\infty = C_2 \times C_2 \times C_2 \times \ldots$$

Take any element $g \in C_2^\infty$ such that g is not the identity. Since g has order 2, we have that $3g = 2g + g = 0 + g = g$, and, more generally, for all $n \in \mathbb{N}_{>0}$, $ng = kg$ where $n \equiv k \mod 2$. Therefore, G cannot have a $\mathbb{Q}$-vector space structure under scalar multiplication with elements of $\mathbb{N}_{>0}$. One work-around is to limit the scalar field to a two-element field, such as $\mathbb{F}_2$.

$$\mathbb{F}_2 = (\{0, 1\}, +, \times).$$

Under multiplication with elements of $\mathbb{F}_2$, we have that $\forall g(0g = 0)$ and $\forall g(1g = g)$, thus we have a $\mathbb{Q}$-vector space structure.