Math 69: Logic **Winter '23**

Exam 1 — 01/30/2023

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You may consult your textbook, notes, class handouts, and returned homework as you work on this exam, but you should not discuss the exam with anybody other than the professor, or look in other textbooks or on the internet (except on the course web page).

It is still okay to discuss class worksheets and homework problems with each other, even if they are related to exam problems, as long as you do not discuss any possible relevance to the exam.

Please ask the professor if you have any questions about the exam. You can use anything from the portions of the text we have covered, including the results of homework problems that were assigned for graded homework. (If you want to use the result of a homework problem that wasn't assigned, you must first solve the problem, and include the solution in your answer.) You can use material from class handouts and worksheets, including the results of problems. You can also use earlier parts of an exam problem in the solutions to later parts of that same problem, even if you were not able to solve the earlier parts.

Your exam paper should follow the following format rules: Identify each problem by number, and also repeat or restate the problem before giving a solution.

The exam will be graded on the clarity and completeness of your explanations, and the correct use of mathematical notation and terminology, as well as on the content of your answers.

Problem 1.

This is a problem in sentential logic.

Let v be a truth assignment on the set of sentence symbols. For any wff α, let;

(i) $f(\alpha)$ denote the number of occurrences of $\leftrightarrow$ in α.

(ii) $g(\alpha)$ denote the number of occurrences in α of sentence symbols A_i for which $v(A_i) = T$.

For example, if $v(A_1) = T$ and $v(A_2) = F$, and α is $((A_1 \wedge A_2) \rightarrow \neg A_1)$, then $g(\alpha) = 2$.

(a) Give careful definitions of $f(\alpha)$ and $g(\alpha)$ by recursion on α.

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a sentence symbol} \\ f(\alpha_1) & \text{if } \alpha \text{ is } (\neg\alpha_1) \text{ for some wff } \alpha_1 \\ f(\alpha_1) + f(\alpha_2) + 1 & \text{if } \alpha \text{ is } (\alpha_1 \leftrightarrow \alpha_2) \text{ for some wffs } \alpha_1 \text{ and } \alpha_2 \\ f(\alpha_1) + f(\alpha_2) & \text{if } \alpha \text{ is } (\alpha_1 * \alpha_2)\text{, where } * \text{ is a binary logical connective.} \end{cases}$$

$$g(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a sentence symbol and } v(\alpha) = F \\ 1 & \text{if } \alpha \text{ is a sentence symbol and } v(\alpha) = T \\ g(\alpha_1) & \text{if } \alpha \text{ is } (\neg\alpha_1) \text{ for some wff } \alpha_1 \\ g(\alpha_1) + g(\alpha_2) & \text{if } \alpha \text{ is } (\alpha_1 * \alpha_2)\text{, where } * \text{ is a binary logical connective.} \end{cases}$$

(b) Prove (carefully and formally) by induction on α that if $\leftrightarrow$ is the only connective symbol occurring in α, then $\overline{v}(\alpha) = T$ if and only if $f(\alpha)$ and $g(\alpha)$ have the opposite parity (i.e., one is even and the other is odd).

(See next page.)

Claim 1.1. *If α is a wff with no other connective symbols than $\leftrightarrow$, then $\overline{v}(\alpha) = T$ if and only if $f(\alpha)$ and $g(\alpha)$ have the opposite parity.*

Proof. Let α be a wff with no other connective symbols than $\leftrightarrow$. We shall prove by induction on the structure of α.

Base Case: If α is a sentence symbol, then $f(\alpha) = 0$. There are two cases:

(i) If $v(\alpha) = T$, then $g(\alpha) = 1$, so $f(\alpha)$ and $g(\alpha)$ have opposing parity and the claim holds.

(ii) If $v(\alpha) = F$, then $g(\alpha) = 0$, so $f(\alpha)$ and $g(\alpha)$ have the same parity and the claim holds.

Inductive Step: If α is $(\alpha_1 \leftrightarrow \alpha_2)$ for some wffs α_1 and α_2 and the claim is true for α_1 and α_2, we show that the claim is also true for α. There are 16 possibilities:

$f(\alpha_1)$	$g(\alpha_1)$	$\overline{v}(\alpha_1)$	$f(\alpha_2)$	$g(\alpha_2)$	$\overline{v}(\alpha_2)$	$f(\alpha_1 \leftrightarrow \alpha_2)$	$g(\alpha_1 \leftrightarrow \alpha_2)$	$\overline{v}(\alpha_1 \leftrightarrow \alpha_2)$
even	even	F	even	even	F	odd	even	T
even	even	F	even	odd	T	odd	odd	F
even	even	F	odd	even	T	even	even	F
even	even	F	odd	odd	F	even	odd	T
even	odd	T	even	even	F	odd	odd	F
even	odd	T	even	odd	T	odd	even	T
even	odd	T	odd	even	T	even	odd	T
even	odd	T	odd	odd	F	even	even	F
odd	even	T	even	even	F	even	even	F
odd	even	T	even	odd	T	even	odd	T
odd	even	T	odd	even	T	odd	even	T
odd	even	T	odd	odd	F	odd	odd	F
odd	odd	F	even	even	F	even	odd	T
odd	odd	F	even	odd	T	even	even	F
odd	odd	F	odd	even	T	odd	odd	F
odd	odd	F	odd	odd	F	odd	even	T

As we can infer from the table, $f(\alpha_1 \leftrightarrow \alpha_2)$ and $g(\alpha_1 \leftrightarrow \alpha_2)$ always have opposing parity whenever $\overline{v}(\alpha_1 \leftrightarrow \alpha_2) = T$. $\square$

Problem 2.

This is a problem in sentential logic.

Suppose that α, β, and γ are wffs such that $\alpha \vDash \gamma$ and $\beta \vDash \neg\gamma$. Show that $\alpha \vDash \neg\beta$.

Please do this formally, by showing that every truth assignment that satisfies α *also satisfies* $(\neg\beta)$. *You will be graded on whether you have a correct proof of this kind.*

Let α be a wff and v be a truth assignment satisfying α. Then $\overline{v}(\alpha) = T$ and, since $\alpha \vDash \gamma$, $\overline{v}(\gamma) = T$.

Consider the truth assignment of v on β. There are two possible assignments; either T or F. If $\overline{v}(\beta) = T$, then $\beta \vDash \neg\gamma$ would imply that $\overline{v}(\neg\gamma) = T$ (so $\overline{v}(\gamma) = T$), contradicting the deduction $\overline{v}(\gamma) = T$ from the assignment $\overline{v}(\alpha) = T$. Therefore, whenever a truth assignment v satisfies α, it must also assign the value $\overline{v}(\beta) = F$ (i.e. it must satisfy $(\neg\beta)$) to avoid a contradicting deduction of γ.

Problem 3.

This is a problem in sentential logic.

Show that the Compactness Theorem can be proven from the Soundness Theorem and the Completeness Theorem.

Compactness Theorem: If Σ is an infinite set of wffs, then Σ is satisfiable if Σ is finitely satisfiable.

Completeness Theorem: If $\Gamma \vDash \alpha$ then $\Gamma \vdash \alpha$.

Soundness Theorem: If $\Gamma \vdash \alpha$ then $\Gamma \vDash \alpha$.

Let Σ be an infinite set of wffs.
We aim to show that if Σ is finitely satisfiable then it is satisfiable. An easier route is to prove the contrapositive: if Σ is not satisfiable, then it is not finitely satisfiable.

Suppose Σ is not satisfiable, then $\Sigma \vDash (\sigma \wedge \neg\sigma)$ for some wff σ. By the completeness theorem, $\Sigma \vdash (\sigma \wedge \neg\sigma)$. This implies that there exists a finite $\Sigma_0 \subset \Sigma$, such that $\Sigma_0 \vdash (\sigma \wedge \neg\sigma)$. By the soundness theorem, we have that $\Sigma_0 \vDash (\sigma \wedge \neg\sigma)$. Therefore, Σ_0 is not satisfiable, hence Σ cannot be finitely satisfiable since Σ_0 is a subset of Σ.

Problem 4.

Define a relation $\equiv$ on $(\mathbb{N}^+)^2 = \{(x, y) : x, y \in \mathbb{N}^+\}$ by

$$(x_1, y_1) \equiv (x_2, y_2) \iff x_1 y_2 = x_2 y_1.$$

(a) Show that $\equiv$ is an equivalence relation on $(\mathbb{N}^+)^2$.

To prove that $\equiv$ is an equivalence relation on $(\mathbb{N}^+)^2$, we need to show that it is reflexive, symmetric, and transitive.

(i) **Reflexivity:** By definition, $(x_1, y_1) \equiv (x_2, y_2)$ if and only if $x_1 y_2 = x_2 y_1$. For any arbitrary element $\alpha = (x, y)$, we always have that $xy = xy$ so $\alpha \equiv \alpha$.

(ii) **Symmetry:** Let $\alpha = (x_1, y_1)$ and $\beta = (x_2, y_2)$ such that $\alpha \equiv \beta$, then $x_1 y_2 = x_2 y_1$. Since moving the left-hand side of the equation to the right-hand side and the right-hand side to the left-hand side does not change the equality, we also have that $x_2 y_1 = x_1 y_2$ so $\beta \equiv \alpha$.

(iii) **Transitivity:** Let $a = (x_1, y_1)$, $b = (x_2, y_2)$, and $c = (x_3, y_3)$ such that $a \equiv b$ and $b \equiv c$. Then $x_1 y_2 = x_2 y_1$ and $x_2 y_3 = x_3 y_2$. Therefore;

$$a \equiv b \iff x_1 y_2 = x_2 y_1, \quad \text{so} \quad \frac{x_1 y_2}{x_2 y_1} = 1$$

$$b \equiv c \iff x_2 y_3 = x_3 y_2, \quad \text{so} \quad \frac{x_2 y_3}{x_3 y_2} = 1$$

$$\therefore \frac{x_1 y_2}{x_2 y_1} \cdot \frac{x_2 y_3}{x_3 y_2} = 1$$

$$\therefore \frac{x_1 y_2 x_2 y_3}{x_2 y_1 x_3 y_2} = 1$$

$$\therefore \frac{x_1 y_3}{x_3 y_1} = 1$$

$$\therefore x_1 y_3 = x_3 y_1$$

$$\therefore a \equiv c$$

(b) Describe the equivalence class of $(3, 3)$

Do this without mentioning the equivalence relation '≡'.

The equivalence class of $(3, 3)$ is the set of all pairs (x, y) such that $3x = 3y$. This is the set

$$\{(x, y) : x, y \in \mathbb{N}^+ \text{ and } x = y\}$$

(c) Suppose that we try to define a function on equivalence classes by

$$f[(x, y)] = [(2x^2, 2y^2)].$$

Either show that this function is well-defined or show that it is not.

Claim 4.2. *This function is well-defined.*

Proof. To show that f is well-defined on the equivalence classes, we are going to show that any two members of an equivalence class are mapped to elements of the same equivalence class (although it may be different from the original equivalence class) — i.e.., that $f(a) \equiv f(b)$ whenever $a \equiv b$. Since any two equivalence classes are always either equal or disjoint, this implies that $f[a] = [f(a)] = [f(b)] = f[b]$.

Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$ be two elements chosen from the same equivalence class without loss of generality, then $x_1 y_2 = x_2 y_1$.

Let $[a'] = f[a] = [(2x_1^2, 2y_1^2)]$ and $[b'] = f[b] = [(2x_2^2, 2y_2^2)]$, such that $a' = (2x_1^2, 2y_1^2)$ and $b' = (2x_2^2, 2y_2^2)$.

To show that $[a'] = [b']$, we need to show that $a' \equiv b'$ as follows:

$$\begin{aligned} 2x_1^2 \cdot 2y_2^2 &= 4x_1^2 y_2^2 \\ &= (2x_1 y_2)^2 \\ &= (2x_2 y_1)^2 && \text{since } x_1 y_2 = x_2 y_1 \\ &= 4x_2^2 y_1^2 \\ &= 2x_2^2 \cdot 2y_1^2 \end{aligned}$$

Therefore, whenever a and b are in the same equivalence class, $f[a] = f[b]$, so f is well-defined. ☐

Problem 5.

We define the difference of two sets X and Y, written $X - Y$, to be the set of all members of X that are not members of Y. Suppose X is an effectively enumerable set of expressions and Y is a decidable set of expressions.

(a) Show that $X - Y$ is effectively enumerable.

Since X is effectively enumerable, we can write a program to list all the elements of X, although the program may never halt. On the other hand, Y is decidable, so given any wff α, we can determine whether α is a member of Y.

To determine the members of $X - Y$, we can proceed as follows:

(i) Start the algorithm to list members of X.

(ii) Each moment a member is listed, check whether the listed wff is a member of Y (since Y is decidable).

- If it is a member of Y, discard the wff.
- If it is not a member of Y, list it as a member of $X - Y$.

(iii) Repeat from step (i).

Since all members of $X - Y$ are members of X, every such wff will be listed as a member of X, therefore $X - Y$ is effectively enumerable.

(b) Suppose that X is not decidable, and $X \subseteq Y$, show that $Y - X$ is not effectively enumerable.

Since all decidable sets are effectively enumerable, we can effectively list all the members of Y, although an program to do so may never halt.

We could have proceeded as above, only swapping the roles of X and Y. However, to determine that a member of Y is a member of $Y - X$, we must determine that they are not a member of X, and X is not decidable so we cannot do so. Therefore, even though we can list members of Y, we cannot determine the membership of all such members in $Y - X$, so $Y - X$ is not effectively enumerable.

Note: the fact that both X and Y are effectively enumerable does not help us in this case. To effectively list all the members of $Y - X$, we would have to somehow list all the members of Y, list all the members of X, then compare the two. However, the programs to list all the members of Y and X are not guaranteed to halt, so we cannot compare the two.

Problem 6.

This is a problem in first order logic.

It is also a short answer problem. That is, giving a correct answer is sufficient; no explanation is needed.

Feel free to use the following abbreviations and conventions: You may use any of our five sentential logic connectives, write $=, <, +, -$ using infix notation (e.g. $x = y$ instead of $= xy$), use the quantifier $\exists$, call your variables x and y (or even δ and ε), and omit parenthesis, include extra parentheses, and use other kinds or sizes of parentheses (such as [ ] and) to enhance readability. However, "$(\forall\varepsilon > 0)$," "$\leq$," and "$\neq$", for example, are not abbreviations we have defined. You may, however, define your own abbreviations. For example, you may say something like: "for any terms t_1 and t_2, let $t_1 > t_2$ be an abbreviation for the formula $t_2 < t_1$. If you do this, be careful to be technically correct. For example, do not say "Let $\leq$ be an abbreviation for $< \vee =$".

Let $\mathfrak{L}$ be the language for first order logic that has equality, constant symbols 0 and r, one-place function symbols f and a, two-place function symbols $+$, $-$, and $\cdot$, and a two-place predicate symbol $<$. Translate the parameters of this language as follows. Note: "Translate r as a real number" means that r is the name of some specific real number, we just aren't saying which one; similarly for "translate f as a function."

Symbol	Translation
$\forall$	for all real numbers
0	zero
r	a real number
f	a function from $\mathbb{R}$ to $\mathbb{R}$
a	the absolute value function
$+$	addition
$-$	subtraction
$\cdot$	multiplication
$<$	less than

TABLE 1. Translation of symbols

(a) Write down sentences that mean:

(i) The function f is strictly increasing.

$$\forall x \forall y(\ (x < y) \to (f(x) < f(y))\ )$$

(ii) Every square is the absolute value of some number.

For any x, we define x^2 to be an abbreviation for $x \cdot x$.

For any x, y, we define

$$\forall x\ (\exists y(\neg(((x^2 - ay) < 0) \lor ((ay - x^2) < 0))))$$

(b) Translate the following sentences into English:

(i) $\forall x \forall y (= fx\, fy \to = xy)$

If f maps any two real numbers x and y to the same real number, then x and y are equal.

(ii) $(\neg \forall y(\neg = fy\, r))$

The function f maps some real number y to the number r.

(c) Write down the sentence of $\mathfrak{L}$ meaning "f is not continuous at r." (Remember from calculus that f is continuous at r iff $\lim_{x \to r} f(x) = f(r)$, and $\lim_{x \to b} F(x) = L$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever x is within a distance δ of b (and $x \neq b$) then $f(x)$ is within a distance ε of L.)

$$\exists \varepsilon \forall \delta\ (((0 < \varepsilon) \land (0 < \delta)) \land ((\varepsilon < (a\ (f\ (r - \delta) - f(r)))) \lor (\varepsilon < (a\ (f\ (r + \delta) - f(r))))))$$