Math 69: Logic

Abelian Groups

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Abstract

This paper explores what the theory of abelian groups can tell us about completeness.

Contents

1. Preliminary Questions

For this problem, we will start with the following definitions. An element g of a group G has order n if n is the smallest positive natural number such that

$$ng = \underbrace{g + \ldots + g}_{n \text{ times}} = 0.$$

For example, g has order 2 if $g \neq 0$ and $g + g = 0$.
An element is said to be *torsion free* if it does not have order n for any $n \in \mathbb{N}$ with $n > 0$. A group is said to be *torsion free* if each of its elements, other than the identity, is torsion free.
Lastly, we say that a group G is divisible if for each $g \in G$ and $n \in \mathbb{N}$ with $n > 0$, there exists $h \in G$ such that

$$nh = \underbrace{h + \ldots + h}_{n \text{ times}} = g.$$

1.1. **Axiomatizing Abelian Groups.**

Define a language $\mathcal{L}$ and a set of axioms Σ such that any model that satisfies Σ is an abelian group.
Next, define a set of axioms T such that any model which satisfies $\Sigma \cup T$ is a divisible torsion free abelian group.

Using additive notation for groups, we define $\mathcal{L}$ to specify group operation $(+)$ and the group identity, 0. We also define element equality in the group as a two-place predicate. Precisely, two elements g and h in the group are considered equal if and only if $g + x = h + x$ for every other element x in the group.
Thus, if $\mathfrak{A}$ is a model for $\mathcal{L}$, then

$$(=^{\mathfrak{A}}) = \{(g, h) \mid \forall x(g + x = h + x)\}.$$

For convenience, we also define the two-place predicate symbol "$\neq$" to be an abbreviation such that:

$$(x \neq y) = \neg(x = y).$$

We define $\mathcal{L}$ to contain the following symbols:

$$\mathcal{L} = \langle 0, +, = \rangle \tag{1.1}$$

We define Σ to contain the following axioms:

$$\forall g((g+0)=g) \qquad \text{(0 as the group identity)} \tag{1.2}$$

$$\forall g\,\forall h\,((g+h)=(h+g)) \qquad \text{(commutativity)} \tag{1.3}$$

$$\forall g\,\forall h\,\forall i\,((g+(h+i))=((g+h)+i)) \qquad \text{(associativity)} \tag{1.4}$$

$$\forall g\,\exists h\,((g+h)=0) \qquad \text{(existence of inverses)} \tag{1.5}$$

We define T to contain the following axioms, which my must be satisfied *in addition to* the axioms in Σ for any model to be a divisible torsion-free abelian group:

$$\forall g\,\forall n\,((g\neq 0)\wedge(0<n)\rightarrow(ng\neq 0)) \qquad \text{(torsion-free)} \tag{1.6}$$

$$\forall g\,\forall n\,\exists h,((n\neq 0)\rightarrow(g=nh)) \qquad \text{(divisibility)} \tag{1.7}$$

1.2. Existence of $\mathbb{Q}$-vector space structures.

Show that any divisible torsion free abelian group has a $\mathbb{Q}$-vector space structure.

Hint: Show that if G is such a group, $n\in\mathbb{N}$ with $n>0$ and $g\in G$ then there is a unique $h\in G$ such that $nh=g$. Note that to show there is a $\mathbb{Q}$-vector space structure, you must define scalar multiplication (and prove it is well-defined).

1.2.1. *Definition of Scalar Multiplication.*

For any group element $h\in G$ and a scalar $n\in\mathbb{N}$, we define the scalar multiplication of h by n to be the unique element $g\in G$ such that

$$g=\underbrace{h+\ldots+h}_{n\text{ times}}.$$

1.3. Axiomatizing Abelian Groups Wherein Each Element Other Than the Identity Has Order 2.

Define a set of axioms S such that any model that satisfies $\sigma\cup S$ is an abelian group in which each element other than the identity has order two. Can we give a model for $\Sigma\cup S$ a vector space structure?

Hint: Be creative in your choice of the scalar field.