Math 69: Logic	**Winter '23**

Homework assigned January 30, 2023

Prof. Marcia Groszek — ***Student: Amittai Siavava***

Problem 5.

Show that the formula

$$x = y \to Pzfx \to Pzfy$$

(where f is a one-place function symbol and P is a two-place predicate symbol) is valid.

Suppose the formula is not valid. Then there exists some structure $\mathfrak{A}$ and some variable assignment s such that $\nvDash_{\mathfrak{A}} x = y \to Pzfx \to Pzfy$, meaning:

$$\vDash_{\mathfrak{A}} (x = y)[s] \tag{5.1}$$

$$\vDash_{\mathfrak{A}} Pzfx[s] \tag{5.2}$$

$$\nvDash_{\mathfrak{A}} Pzfy[s] \tag{5.3}$$

Since $\vDash_{\mathfrak{A}} (x = y)[s]$, we have that $\overline{s}(x) = \overline{s}(y)$.

However, since $\overline{s}(x) = \overline{s}(y)$, we have that $\overline{s}(fx) = \overline{s}(fy)$:

$$\overline{s}(fx) = f^{\mathfrak{A}}(\overline{s}(x)) = f^{\mathfrak{A}}(\overline{s}(y)) = \overline{s}(fy).$$

Therefore, $Pzfx[s]$ and $Pzfy[s]$ are logically equivalent, so 5.2 and 5.3 are a contradiction, meaning that the formula is valid and any structure which does not satisfy the formula is inconsistent.

Problem 26.

(a) Consider a fixed structure $\mathfrak{A}$ and define its *elementary type* to be the class of structures elementarily equivalent to $\mathfrak{A}$. Show that this class is EC_Δ.

Hint: Show that it is Mod Thm $\mathfrak{A}$

By definition, two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if whenever $\vDash_{\mathfrak{A}} \varphi$, we also have that $\vDash_{\mathfrak{B}} \varphi$. That is, whenever φ is tautologically implied by $\mathfrak{A}$ then it is also tautologically implied by $\mathfrak{B}$.

Let $[\mathfrak{A}]$ be the class of all structures elementarily equivalent to $\mathfrak{A}$, and let Thm $\mathfrak{A}$ be the set of all consequences of $\mathfrak{A}$. Then every structure $\mathfrak{B}$ in $[\mathfrak{A}]$ is a model of Thm $\mathfrak{A}$, since it must satisfy all the consequences of $\mathfrak{A}$. Therefore, $[\mathfrak{A}]$ = Mod Thm $\mathfrak{A}$, so it is EC_Δ.

(b) Call a class $\mathcal{K}$ of structures *elementarily closed* or ECL if whenever a structure belongs to $\mathcal{K}$ then all elementarily equivalent structures also belong. Show that any such class is a union of EC_Δ classes. (A class that is a union of EC_Δ classes is said to be an $\mathrm{EC}_{\Delta\Sigma}$ class; this notation is derived from topology.)

Let K be an ECL class, and $\mathfrak{A}$ be a structure in K. Since $\mathfrak{A} \in K$, K satisfies all the consequences of $\mathfrak{A}$, i.e. Mod Thm $\mathfrak{A} \subseteq K$.

Let $\mathfrak{B}$ be an elementarily equivalent structure to $\mathfrak{A}$. This means that $\mathfrak{B}$ is a model of Thm $\mathfrak{A}$. However, K is elementarily closed, so $\mathfrak{B}$ must also be in K, meaning Mod Thm $\mathfrak{B} \subseteq K$.

By extension, every other structure $\mathfrak{D}_i$ that is elementarily equivalent to some structure $\mathfrak{D}$ that belongs in K must also belong in K. Therefore;

$$\mathfrak{K} = \text{Mod Thm } \mathfrak{A} \cup \text{Mod Thm } \mathfrak{B} \cup \text{Mod Thm } \mathfrak{D}_1 \cup \text{Mod Thm } \mathfrak{D}_2 \cup \dots$$

for every other structure $\mathfrak{D}_i$ in K, so it is $\mathrm{EC}_{\Delta\Sigma}$.

(c) Conversely show that any class that is the union of EC_Δ classes is ECL.

Let K be a class that is $\mathrm{EC}_{\Delta\Sigma}$. Then

$$K = \bigcup_{\mathfrak{A} \text{ belongs in } K} \text{Mod Thm } \mathfrak{A}.$$

Consequently, whenever a structure $\mathfrak{A}$ belongs in K, Mod Thm $\mathfrak{A} \subseteq K$.

Take any structure $\mathfrak{B}$ that is elementarily equivalent to $\mathfrak{A}$, then $\mathfrak{B}$ tautologically implies the same set of consequences as $\mathfrak{A}$, i.e. Mod Thm $\mathfrak{B}$ = Mod Thm $\mathfrak{A}$. Therefore, Mod Thm $\mathfrak{B} \subseteq K$ so $\mathfrak{B}$ also belongs in K. Therefore, K is ECL.

Problem 27.

Let $\mathcal{L}$ be the language for first-order logic with two-place predicate symbols E and P and one-place function symbol f. (We are not assuming that $\mathcal{L}$ has the equality symbol. On the other hand, we are not ruling out the possibility that $\mathcal{L}$ has the equality symbol and/or any number of parameter symbols in addition to $\forall$, E, P, and f. Other symbols are not relevant to this question.)

Suppose $\mathfrak{A}$ is a structure for $\mathcal{L}$ that is a model of the sentence

$$\forall x Exx$$

and of every sentence of the form

$$\forall x \forall y \forall z_1 \forall z_2 \ldots \forall z_n (Exy \to (\alpha \to \alpha')).$$

where α is an atomic formula with variables included among $\{x, y, z_1, z_2, \ldots, z_n\}$, and α' is obtained from α by replacing some (possibly none, possibly some but not all, possibly all) occurrences of x by y. Examples of sentences of this form are

$$\forall x \forall y (Exy \to (Exx \to Eyx)) \text{ and } \forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy)).$$

An example of a sentence *not* of this form is

$$\forall x \forall y \forall z (Exy \to (Ezfy \to Ezfx)).$$

Show that $E^{\mathfrak{A}}$ is an equivalence relation on the universe $|\mathfrak{A}|$, that $P^{\mathfrak{A}}$ induces a well-defined relation on equivalence classes, and that $f^{\mathfrak{A}}$ induces a well-defined function on equivalence classes.

Let s be a variable assignment satisfying the conditions as stated. Then:

(a) $E^{\mathfrak{A}}$ is an equivalence relation on the universe $|\mathfrak{A}|$.

(i) E is reflexive: This follows from the sentence $\forall x Exx$.

(ii) E is symmetric: Suppose $\models_{\mathfrak{A}} Exy[s]$. Take the following sentence, which is valid in $\mathfrak{A}$:

$$\forall x \forall y (Exy \to (Exx \to Eyx)).$$

Since we have that $\forall x Exx$, for any such x, y pair we have that:

- $\bar{s}(Exy) = T$
- $\bar{s}(Exx) = T$

we can deduce Eyx as follows:

$$Exy \quad \text{(from choice of } x \text{ and } y\text{)} \tag{27.1}$$

$$\forall x \forall y (Exy \to (Exx \to Eyx)) \tag{27.2}$$

$$(Exx \to Eyx) \quad \text{(modus ponens on 27.1 and 27.2)} \tag{27.3}$$

$$Exx \quad \text{(from the first sentence in the model)} \tag{27.4}$$

$$Eyx \quad \text{(modus ponens on 27.3 and 27.4)} \tag{27.5}$$

(iii) Transitive: the following sentence is valid in $\mathfrak{A}$:

$$\forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy)) \tag{27.6}$$

Let $\vDash_{\mathfrak{A}} Exy[s]$ and take $z \in |\mathfrak{A}|$ such that $\vDash_{\mathfrak{A}} Ezx$. Define a one-place function $f : |\mathfrak{A}| \to |\mathfrak{A}|$ by

$$f(a) = a$$

Then, sentence 27.6 is equivalent to

$$\forall x \forall y \forall z (Exy \to (Ezx \to Ezy)).$$

Since E is symmetric, this means whenever $\vDash_{\mathfrak{A}} Eyx$ and $\vDash_{\mathfrak{A}} Exz$, then $\vDash_{\mathfrak{A}} Eyz$, so E is transitive.

(b) $P^{\mathfrak{A}}$ is a well-defined relation on equivalence classes.

Let $x, y \in |\mathfrak{A}|$ such that $\vDash_{\mathfrak{A}} Exy[s]$. We show that $P^{\mathfrak{A}}xa = P^{\mathfrak{A}}ya$ for all $a \in |\mathfrak{A}|$.

Suppose not, then there exists some $z \in |\mathfrak{A}|$ such that $\vDash_{\mathfrak{A}} Pxa[s]$ and $\nvDash_{\mathfrak{A}} Pya[s]$.

Define a function $f : |\mathfrak{A}| \to |\mathfrak{A}|$ by

$$f(n) = Ean,$$

then clearly $f(x) = T$ and $f(y) = F$, so $f(x) \neq f(y)$.

However, we have the sentence

$$\forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy)),$$

which does not hold when

(c) $f^{\mathfrak{A}}$ is a well-defined function on equivalence classes.

Let $x, y \in |\mathfrak{A}|$ be such that $\vDash_{\mathfrak{A}} Exy$. First, $E^{\mathfrak{A}}$ is symmetric, so $E^{\mathfrak{A}}xy \leftrightarrow E^{\mathfrak{A}}yx$.

We also have the sentence $\forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy))$ as a valid axiom. But since $E^{\mathfrak{A}}xy \leftrightarrow E^{\mathfrak{A}}yx$, whenever the above formula is valid the same for Eyx is also valid, i.e. $\forall x \forall y \forall z (Eyx \to (Ezfy \to Ezfx))$.

This means $\overline{s}(Ezfx) = \overline{s}(Ezfy)$ whenever $\overline{s}(Exy) = T$. Since $E^{\mathfrak{A}}$ is transitive, this means $E(fx)(fy) = T$, therefore $f(x)$ and $f(y)$ are in the same equivalence class whenever x and y are in the same equivalence class so $f^{\mathfrak{A}}$ is well-defined.