Math 69: Logic	Winter '23

Homework assigned February 13, 2023

Prof. Marcia Groszek — ***Student: Amittai Siavava***

Problem 2.

Let T_1 and T_2 be theories (in the same language $\mathcal{L}$) such that:

(i) $T_1 \subseteq T_2$.

(ii) T_1 is complete.

(iii) T_2 is satisfiable.

Show that $T_1 = T_2$.

T_1 and T_2 are both theories, meaning if T_1 deduces any wff φ then φ is a member of T_1 and, similarly, if T_2 deduces any wff φ then φ is a member of T_2. Furthermore, T_1 is complete, meaning that for any wff γ definable in the language $\mathcal{L}$, either T_1 deduces γ or T_1 deduces $\neg\gamma$. By (1), for every such γ definable in $\mathcal{L}$, T_1 either contains γ or T_1 contains $\neg\gamma$.

$T_1 \subseteq T_2$, so for every γ definable in $\mathcal{L}$, T_2 must also contain either γ or $\neg\gamma$ (whichever is in T_1). Since T_2 is satisfiable, it cannot deduce (hence, cannot contain, since it is a theory) both γ and $\neg\gamma$ for any wff γ. It can also not contain any sentences not true in $\mathcal{L}$ since it is a theory for $\mathcal{L}$. Therefore, T_2 must only contain the same wffs as those contained in the subset T_1, so $T_2 = T_1$.

Problem 3.

Establish the following facts;

(a) $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \text{Mod}\,\Sigma_1 \subseteq \text{Mod}\,\Sigma_2$.
$\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \text{Th}\,\mathcal{K}_1 \subseteq \text{Th}\,\mathcal{K}_2$.

Given $\mathfrak{A}$ is a model of Σ_2, then each sentence in Σ_2 is true in $\mathfrak{A}$. However, since $\Sigma_1 \subseteq \Sigma_2$, then each sentence in Σ_1 is also true in $\mathfrak{A}$, so $\mathfrak{A}$ must also be a model for Σ_1. On the other hand, every model of Σ_1 satisfies a subset of Σ_2, so there exists some extension that makes it a model of Σ_2.

Therefore, if Σ_1 is a subset of (or equal to) Σ_2 then every model of Σ_1 is some subset of (or equal to) some model of Σ_2.

The same argument applies to $\mathcal{K}_1 \subseteq \mathcal{K}_2$: Suppose T_2 is a theory for $\mathcal{K}_2$, then T_2 contains every sentence that is true in K_2. Since $\mathcal{K}_1 \subseteq \mathcal{K}_2$, then T_2 also contains every sentence that is true in K_1.

On the other hand, every theory for $\mathcal{K}_1$ contains some subset of the sentences that are true in $\mathcal{K}_2$, so it can be extended to a theory for $\mathcal{K}_2$. Therefore, every theory for $\mathcal{K}_1$ is a subset of (or equal to) some theory for $\mathcal{K}_2$.

(b) $\Sigma \subseteq \text{Th Mod}\,\Sigma$ and $\mathcal{K} \subseteq \text{Mod Th}\,\mathcal{K}$.

Let $\mathfrak{A}$ be a model of Σ. Then every sentence in Σ is true in $\mathfrak{A}$. Since Th $\mathfrak{A}$ contains every sentence that is true in $\mathfrak{A}$, then Th $\mathfrak{A}$ must contain every sentence in Σ, so $\Sigma \subseteq \text{Th}\,\mathfrak{A}$. But $\mathfrak{A} = \text{Mod}\,\Sigma$, so $\Sigma \subseteq \text{Th Mod}\,\Sigma$.

Let Γ be a theory for $\mathcal{K}$. Then Γ contains every wff that is true in $\mathcal{K}$. Let $\mathfrak{A}$ be a model for Γ, then every sentence in Γ s true in $\mathfrak{A}$, so every sentence true in $\mathcal{K}$ is true in $\mathfrak{A}$. Therefore, $\mathcal{K} \subseteq \text{Mod Th}\,\mathcal{K}$.

(c) $\text{Mod}\,\Sigma = \text{Mod Th Mod}\,\Sigma$ and $\text{Th}\,\mathcal{K} = \text{Th Mod Th}\,\mathcal{K}$.
Hint: Part (c) follows from (a) and (b).

By (b), $\Sigma \subseteq \text{Th Mod}\,\Sigma$, so, by (a), $\text{Mod}\,\Sigma \subseteq \text{Mod Th Mod}\,\Sigma$.
Mod Th Mod Σ is a model for Th Mod Σ, so every sentence true in Mod Th Mod Σ is tautologically implied by (or... deducible from) Th Mod Σ. But Th Mod Σ is a theory, so any such deduction σ from Th Mod Σ must be contained in Th Mod Σ, so it must also be true in $\text{mod}\ \Sigma$. Therefore, any sentence true in Mod Th Mod Σ must also be true in Mod Σ, so $\text{Mod Th Mod}\,\Sigma \subseteq \text{Mod}\,\Sigma$.

Therefore, $\text{Mod}\,\Sigma \subseteq \text{Mod Th Mod}\,\Sigma$ and $\text{Mod Th Mod}\,\Sigma \subseteq \text{Mod}\,\Sigma$, so $\text{Mod}\,\Sigma = \text{Mod Th Mod}\,\Sigma$.

By (b), $\mathcal{K} \subseteq \text{Mod Th}\,\mathcal{K}$, so, by (a), $\text{Th}\,\mathcal{K} \subseteq \text{Th Mod Th}\,\mathcal{K}$.
Th Mod Th $\mathcal{K}$ contains all the wffs true in Mod Th $\mathcal{K}$. But Mod Th $\mathcal{K}$ is a model for Th $\mathcal{K}$, meaning sentences true in Mod Th $\mathcal{K}$ are tautologically implied by (or... deducible from) Th $\mathcal{K}$. But $\mathcal{K}$ is a theory (for $\mathcal{K}$), meaning any such deduction σ must be in Th $\mathcal{K}$.

Therefore, $\text{Th Mod Th}\,\mathcal{K} \subseteq \text{Th}\,\mathcal{K}$ and $\text{Th}\,\mathcal{K} \subseteq \text{Th Mod Th}\,\mathcal{K}$, meaning $\text{Th}\,\mathcal{K} = \text{Th Mod Th}\,\mathcal{K}$.

Problem 7.

Consider a language with a two-place predicate symbol $<$, and let $\mathfrak{N} = (\mathbb{N}; <)$ be the structure consisting of the natural numbers with their usual ordering. Show that there is some $\mathfrak{A}$ elementarily equivalent to $\mathfrak{N}$ such that $<^{\mathfrak{A}}$ has a descending chain. That is, there must be $a_0, a_1, \ldots$ such that $\langle a_{i+1}, a_i \rangle \in <^{\mathfrak{A}}$ for all i.

Hint: Apply the compactness theorem.

Let $\mathfrak{A} = (\mathbb{N}; <, c)$ be a structure consisting of the natural numbers under their usual ordering, with an additional constant symbol $c \notin \mathbb{N}$. Let

$$\Gamma = \{x < c \mid x \in \mathbb{N}\} = \left\{0^{\mathfrak{A}} < c, 1^{\mathfrak{A}} < c, 2^{\mathfrak{A}} < c, \ldots\right\}.$$

For any finite subset of some size k, Mapping $x^{\mathfrak{A}} \mapsto x^{\mathfrak{N}}$ for all $x \in \mathbb{N}$, $x < k$ and mapping $c^{\mathfrak{A}} \mapsto k^{\mathfrak{N}}$ gives a model of Γ that contains a descending chain of length k.

Since every such finite subset of $\Gamma \cup \text{Th}\,\mathfrak{N}$ has a model, then by the compactness theorem $\Gamma \cup \text{Th}\,\mathfrak{N}$ also has a model. But since the size of $\mathbb{N}$ is infinite, there exists a descending chain in $\mathfrak{A}$, starting with c, that has infinite length.