Math 69: Logic	**Winter '23**

Homework assigned January 30, 2023

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Problem 5.

Show that the formula

$$x = y \rightarrow Pzfx \rightarrow Pzfy$$

(where f is a one-place function symbol and P is a two-place predicate symbol) is valid.

Suppose the formula is not valid. Then there exists some structure $\mathfrak{A}$ and some variable assignment s such that $\not\models_{\mathfrak{A}} x = y \rightarrow Pzfx \rightarrow Pzfy$, meaning:

$$\models_{\mathfrak{A}} (x = y)[s] \tag{5.1}$$

$$\models_{\mathfrak{A}} Pzfx[s] \tag{5.2}$$

$$\not\models_{\mathfrak{A}} Pzfy[s] \tag{5.3}$$

Since $\models_{\mathfrak{A}} (x = y)[s]$, we have that $\overline{s}(x) = \overline{s}(y)$.

However, since $\overline{s}(x) = \overline{s}(y)$, we have that $\overline{s}(fx) = \overline{s}(fy)$:

$$\overline{s}(fx) = f^{\mathfrak{A}}(\overline{s}(x)) = f^{\mathfrak{A}}(\overline{s}(y)) = \overline{s}(fy).$$

Therefore, $Pzfx[s]$ and $Pzfy[s]$ are logically equivalent, so 5.2 and 5.3 are a contradiction, meaning that the formula is valid and any structure which does not satisfy the formula is inconsistent.

Problem 26.

(a) Consider a fixed structure $\mathfrak{A}$ and define its *elementary type* to be the class of structures elementarily equivalent to $\mathfrak{A}$. Show that this class is EC_Δ.

Hint: Show that is is Mod Thm $\mathfrak{A}$

By definition, two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if $\vDash_{\mathfrak{A}} \varphi \iff \vDash_{\mathfrak{B}} \varphi$ for all formulas φ. Therefore, the elementary type of $\mathfrak{A}$ is the class of all structures $\mathfrak{B}$ such that

(b) Call a class $\mathcal{K}$ of structures *elementarily closed* or ECL if whenever a structure belongs to $\mathcal{K}$ then all elementarily equivalent structures also belong. Show that any such class is a union of EC_Δ classes. (A class that is a union of EC_Δ classes is said to be an $\mathrm{EC}_{\Delta\Sigma}$ class; this notation is derived from topology.)

(c) Conversely show that any class that is the union of EC_Δ classes is ECL.

Problem 27.

Let $\mathcal{L}$ be the language for first-order logic with two-place predicate symbols E and P and one-place function symbol f. (We are not assuming that $\mathcal{L}$ has the equality symbol. On the other hand, we are not ruling out the possibility that $\mathcal{L}$ has the equality symbol and/or any number of parameter symbols in addition to $\forall$, E, P, and f. Other symbols are not relevant to this question.)

Suppose A is a structure for L that is a model of the sentence

$$\forall x Exx$$

and of every sentence of the form

$$\forall x \forall y \forall z_1 \forall z_2 \ldots \forall z_n (Exy \to (\alpha \to \alpha')).$$

where α is an atomic formula with variables included among $\{x, y, z_1, z_2, \ldots, z_n\}$, and α' is obtained from α by replacing some (possibly none, possibly some but not all, possibly all) occurrences of x by y. Examples of sentences of this form are

$$\forall x \forall y (Exy \to (Exx \to Eyx)) \text{ and } \forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy)).$$

An example of a sentence *not* of this form is

$$\forall x \forall y \forall z (Exy \to (Ezfy \to Ezfx)).$$

Show that $E^{\mathfrak{A}}$ is an equivalence relation on the universe $|\mathfrak{A}|$, that $P^{\mathfrak{A}}$ induces a well-defined relation on equivalence classes, and that $f^{\mathfrak{A}}$ induces a well-defined function on equivalence classes.

Let $\mathfrak{A}$ be a structure for $\mathcal{L}$ that is a model of the sentence

$$\forall x Exx$$

and of every sentence of the form

$$\forall x \forall y \forall z_1 \forall z_2 \ldots \forall z_n (Exy \to (\alpha \to \alpha')).$$

where $alpha$ is an atomic formula with variables included among $\{x, y, z_1, z_2, \ldots, z_n\}$, and α' is obtained from α by replacing some (possibly none, possibly some but not all, possibly all) occurrences of x by y. Let $s : V \to |\mathfrak{A}|$ be an assignment function such that $\vDash_{\mathfrak{A}} \forall x \forall y \forall z_1 \forall z_2 \ldots \forall z_n (Exy \to (\alpha \to \alpha'))$.

Let $a, b \in |\mathfrak{A}|$. We will show that $a \equiv b$ if and only if $E^{\mathfrak{A}}(a, b)$. Suppose $a \equiv b$. Then there exists $c \in |\mathfrak{A}|$ such that $a = c$ and $b = c$. Since $a \equiv b$, we have that $E^{\mathfrak{A}}(a, b)$. Since $a = c$ and $b = c$, we have that $E^{\mathfrak{A}}(a, c)$ and $E^{\mathfrak{A}}(b, c)$. Since $E^{\mathfrak{A}}(a, c)$ and $E^{\mathfrak{A}}(b, c)$, we have that $E^{\mathfrak{A}}(c, c)$. Since $E^{\mathfrak{A}}(c, c)$, we have that $E^{\mathfrak{A}}(a, b)$.

Suppose $E^{\mathfrak{A}}(a, b)$. Then there exists $c \in |\mathfrak{A}|$ such that $a = c$ and $b = c$. Since $E^{\mathfrak{A}}(a, b)$, we have that $E^{\mathfrak{A}}(a, c)$ and $E^{\mathfrak{A}}(b, c)$. Since $E^{\mathfrak{A}}(a, c)$ and $E^{\mathfrak{A}}(b, c)$, we have that $E^{\mathfrak{A}}(c, c)$. Since $E^{\mathfrak{A}}(c, c)$, we have that $a = c$ and $b = c$. Since $a = c$ and $b = c$, we have that $a \equiv b$.