Math 69: Logic **Winter '23**

Homework assigned January 20, 2023

Prof. Marcia Groszek ***Student: Amittai Siavava***

Problem 7.

Write down 4 sentences for a language $\mathfrak{L}$ such that any structure $\mathfrak{U} = \langle X, \leq \rangle$ is a linear ordering if and only if it satisfies those four sentences.

$$\forall\, x\; Pxx \qquad \text{(reflexive)}$$

$$\forall\, x\, \forall\, y\; ((Pxy \wedge Pyx) \to (x = y)) \qquad \text{(antisymmetric)}$$

$$\forall\, x\, \forall\, y\, \forall\, z\, ((Pxy \wedge Pyz) \to Pxz) \qquad \text{(transitive)}$$

$$\forall\, x\, \forall\, y\; (Pxy \vee Pyx) \qquad \text{(total)}$$

Problem 9.

Suppose that X is a set and $\leq$ is a preordering of X. Define a new binary relation on X by

$$x \equiv y \iff (x \leq y \wedge y \leq x).$$

Show that $\equiv$ is an equivalence relation on X, that $\leq$ induces a well-defined relation on equivalence classes, and that this induced relation is a partial ordering of $X/\equiv$.

Claim 9.1. $\equiv$ is an equivalence relation on X.

Proof. We need to show that $\equiv$ is reflexive, transitive, and symmetric.

Reflexivity: $x \leq x$ for all $x \in X$, so $x \equiv x$, so the equivalence relation is reflexive.

Transitivity: For $x, y, z \in X$, suppose $x \equiv y$ and $y \equiv z$, then:

$$x \equiv y \iff (x \leq y \wedge y \leq x) \tag{9.2}$$

$$y \equiv z \iff (y \leq z \wedge z \leq y) \tag{9.3}$$

$$9.2 \wedge 9.3 \iff (x \leq z \wedge z \leq x) \iff (x \equiv z)$$

Symmetry: For $x, y \in X$, suppose $x \equiv y$, then:

$$\begin{aligned} x \equiv y &\iff (x \leq y \wedge y \leq x) \\ &\iff (y \leq x \wedge x \leq y) \\ &\iff (y \equiv x) \end{aligned}$$

□

Claim 9.4. $\leq$ induces a well-defined relation on equivalence classes.

Proof. Let $x, y \in X$ such that $x \equiv y$. Take any $z \in X$ without loss of generality. If $z \equiv x$, then $z \equiv y$ since $x \equiv y$. On the other hand, suppose $z \not\equiv y$. Then it may not be the case that $z \equiv x$, as that would imply that $x \equiv y$ (since $z \equiv x$ and $x \equiv y$). This implies that, for any $z \in X$, either (1) $z \equiv x$, and $z \equiv x_i$ for all $x_i \equiv x$, or (2) $z \not\equiv x$, and $z \not\equiv x_i$ for all $x_i \equiv x$. Therefore, $\leq$ induces a well-defined relation on equivalence classes. □

Claim 9.5. The induced relation is a partial ordering of $X/\equiv$.

Proof. We need to show that $\equiv$ is reflexive, transitive, and antisymmetric when applied to equivalence classes of x.

Reflexivity: Suppose $[x]$ and $[y]$ are equivalence classes on X. Suppose $[x] \equiv [y]$. Then $x_i \equiv y_j$ for all $x_i \in [x]$ and $y_j \in [y]$. Since $\equiv$ is symmetric when applied to members of X, $y_j \equiv x_i$ for all $x_i \in [x]$ and $y_j \in [y]$, so $[x] \equiv [y]$ implies $[y] \equiv [x]$.

Transitivity: Suppose $[x] \equiv [y]$ and $[y] \equiv [z]$. Then $x_i \equiv y_j$ for all $x_i \in [x]$ and $y_j \in [y]$, and $y_j \equiv z_k$ for all $y_j \in [y]$ and $z_k \in [z]$, since $\equiv$ is transitive when applied to members of X. Therefore, $x_i \equiv z_k$ for all $x_i \in [x]$ and $z_k \in [z]$, so $[x] \equiv [z]$.

Antisymmetry: Suppose $[x] \equiv [y]$ and $[y] \equiv [x]$. As we saw in the proof to claim 9.4, this implies that every $x_i \in [x]$ is equivalent to every $y_j \in [y]$, so everything in $[x]$ is in the equivalence class of everything in $[y]$, which is only possible if $[x] = [y]$. □

Problem 11.

Define the notion of isomorphism between two equivalence relations

$$\mathfrak{A} = \langle X, \equiv_X \rangle \text{ and } \mathfrak{B} = \langle Y, \equiv_Y \rangle .$$

Let $f : X \to Y$ be a function. We say that f is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ if f is a bijection and if $f(\alpha) \equiv_Y f(\beta)$ whenever $\alpha \equiv_X \beta$. For instance, take $x \in X$, suppose $f(x) = y \in Y$. If f is an isomorphism, then f maps every element in the equivalence class $[x]$ to some element in the equivalence class $[y]$.