Math 69: Logic | **Winter '23**

Homework assigned January 6, 2023

Prof. Marcia Groszek | ***Student: Amittai Siavava***

Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***A Mathematical Introduction to Logic*** by **Herbert Enderton**.

Problem 2.

(a) Is $(((P \to Q) \to P) \to P)$ a tautology?

Yes. Let's start by constructing a simple truth table for the connective $(\to)$.

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

Suppose $\overline{v}(((P \to Q) \to P) \to P) = F$.

From the truth table, we can infer that $v((P \to Q) \to P) = T$ and $v(P) = F$.

But if $\overline{v}((P \to Q) \to P) = T$ and $v(P) = F$, then $\overline{v}(P \to Q) = F$.

However, if $v(P) = F$ implies that $\overline{v}(P \to Q) = T$ irrespective of the value of Q, which contradicts the deduction that $\overline{v}(P \to Q) = F$.

Therefore, $\overline{v}(((P \to Q) \to P) \to P) = T$ for all possible values of P and Q.

(b) Define σ_k recursively as follows:

$$\sigma_0 = (P \to Q)$$

$$\sigma_{k+1} = (\phi_k \to P).$$

For which values of k is σ_k a tautology? *Note: Part A corresponds to* $k = 2$.

We can prove that σ_k whenever (and only when) k is a non-zero positive integer by induction on k.

Base Cases:

(i) $k = 0$: $\sigma_0 = (P \to Q)$ is not a tautology, since $\overline{v}(P \to Q) = F$ whenever $v(P) = T$ and $v(Q) = F$.

(ii) $k = 1$: $\sigma_0 \to P$ is also not a tautology; if $v(P) = F$, then $\overline{v}(\sigma_0) = T$ and $\overline{v}(\sigma_0 \to P) = F$.

(iii) However, σ_2 is a tautology (see part (a) for proof).

Inductive Step:

Suppose σ_k is a tautology, then $\overline{v}(\sigma_k) = T$ for all values of P and Q.

First, consider $\sigma_{k+1} = (\sigma_k \to P)$. Since σ_k is a tautology, $\sigma_{k+1} = (T \to v(P))$. Therefore, $\sigma_{k+1} = T$ whenever $v(P) = T$, and $\sigma_{k+1} = F$ whenever $v(P) = F$ (or, $\sigma_{k+1}) = v(P)$. When $v(P) = F$, $\sigma_{k+1} = F$, therefore σ_{k+1} is not a tautology.

Next, consider $\sigma_{k+2} = (\sigma_{k+1} \to P)$. As demonstrated above, whenever σ_k is a tautology, we have that $\sigma_{k+1} = v(P)$. This means $\sigma_{k+2} = (v(P) \to P)$, which evaluates to T for all possible values of P. Therefore, σ_{k+2} is a tautology.

By induction, we can conclude that whenever σ_k is a tautology, then σ_{k+1} is not a tautology, but σ_{k+2} is a tautology. Since the first tautology in the sequence is σ_2, the set of tautologies will be the set $\{\sigma_n \mid n \in \{2, 4, 6, 8, \ldots\}\}$ — that is, σ_n is a tautology whenever n is an even positive integer.

Problem 4.

Recall that $\Sigma; \alpha = \Sigma \cup \{\alpha\}$, *the set* Σ *together with the one possibly new member* α.

Show that the following hold:

(a) $\Sigma; \alpha \vDash \beta \iff \Sigma \vDash (\alpha \to \beta)$.

(i) $\Sigma; \alpha \vDash \beta \implies \Sigma \vDash (\alpha \to \beta)$

Suppose $\Sigma; \alpha \vDash \beta$. Let v be a truth assignment satisfying Σ.

If $\overline{v}(\alpha) = T$, then v satisfies $\Sigma; \alpha$ (since v already satisfies Σ), and $\Sigma; \alpha \vDash \beta$, implying that $\overline{v}(\beta) = T$. Therefore, $\overline{v}(\alpha \to \beta) = (T \to T) = T$.

If $\overline{v}(\alpha) = F$, then $\overline{v}(\alpha \to \beta) = (F \to \overline{v}(\beta)) = T$.

(ii) $\Sigma; \alpha \vDash \beta \Longleftarrow \Sigma \vDash (\alpha \to \beta)$

Suppose $\Sigma \vDash (\alpha \to \beta)$ but $\Sigma; \alpha \nvDash \beta$.

Let v be a truth assignment satisfying Σ. Suppose $\overline{v}(\alpha) = T$.

- First, we can note that $\overline{v}(\alpha) = T$ implies that v satisfies $\Sigma; \alpha$, which further implies that $\overline{v}(\beta) = F$, since $\Sigma; \alpha \nvDash \beta$.
- Next, since $\Sigma \vDash (\alpha \to \beta)$, $\overline{v}(\alpha) = T$ implies $\overline{v}(\beta) = T$. This is a contradiction.

Therefore, it must be the case that $\Sigma \vDash (\alpha \to \beta) \implies \Sigma; \alpha \vDash \beta$

(b) $\alpha \vDash\Dashv \beta \iff \vDash (\alpha \leftrightarrow \beta)$.

Let v be any truth assignment to α and β satisfying the wff $\alpha \vDash\Dashv \beta$. Then:

- $\overline{v}(\beta) = T$ whenever $\overline{v}(\alpha) = T$ (since $\alpha \vDash \beta$).
- $\overline{v}(\alpha) = T$ whenever $\overline{v}(\beta) = T$ (since $\beta \vDash \alpha$).
- Consequently, $\neg(\overline{v}(\alpha)) \leftrightarrow \neg(\overline{v}(\beta))$, implying that $\overline{v}(\alpha \leftrightarrow \beta) = T$.

Therefore, any truth assignment to α and β satisfying the wff $\alpha \vDash\Dashv \beta$ also satisfies $\vDash (\alpha \leftrightarrow \beta)$.

Problem 5.

Prove or refute each of the following assertions:

(a) If either $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$, then $\Sigma \vDash (\alpha \vee \beta)$.

Yes.

(i) Note that if $\Sigma \vDash \alpha$ then $\Sigma; \alpha$ is finitely satisfiable. Similarly, if $\Sigma \vDash \beta$ then $\Sigma; \beta$ is finitely satisfiable.

(ii) However, if $\Sigma \nvDash (\alpha \vee \beta)$ then $\Sigma; (\alpha \vee \beta)$ is *not* finitely satisfiable.

(iii) Suppose that either $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$, but $\Sigma \nvDash (\alpha \vee \beta)$. By finite satisfiability, either there exists some finite subset $\Sigma_\alpha \subseteq \Sigma$ such that $\Sigma_\alpha; \alpha$ is satisfiable, or there exists some finite subset $\Sigma_\beta \subseteq \Sigma$ such that $\Sigma_\beta; \beta$ is satisfiable.

(iv) Let $\Sigma_\gamma = \Sigma_\alpha \cup \Sigma_\beta$ (substituting the empty set for any nonexisting set), then either $\Sigma_\gamma; \alpha$ is satisfiable or $\Sigma_\gamma; \beta$ is satisfiable, implying that $\Sigma_\gamma; (\alpha \vee \beta)$ is satisfiable.

(v) But if $\Sigma_\gamma; (\alpha \vee \beta)$ is satisfiable, and $\Sigma_\gamma \subseteq \Sigma$, then $\Sigma; (\alpha \vee \beta)$ is finitely satisfiable.

(vi) Comparing step (b) and step (e), we see a clear contradiction, implying that whenever either $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$, it must be the case that $\Sigma \vDash (\alpha \vee \beta)$.

(b) If $\Sigma \vDash (\alpha \vee \beta)$, then either $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$.

Yes.

(i) Note that if $\Sigma \vDash (\alpha \vee \beta)$ then $\Sigma; (\alpha \vee \beta)$ is finitely satisfiable.

(ii) However, if $\Sigma \nvDash \alpha$ and $\Sigma \nvDash \beta$, then $\Sigma; \alpha$ and $\Sigma; \beta$ are *not* finitely satisfiable.

(iii) Suppose that $\Sigma \vDash (\alpha \vee \beta)$, but $\Sigma \nvDash \alpha$ and $\Sigma \nvDash \beta$. By finite satisfiability, either there exists some finite subset $\Sigma_\alpha \subseteq \Sigma$ such that $\Sigma_\alpha; \alpha$ is satisfiable, or there exists some finite subset $\Sigma_\beta \subseteq \Sigma$ such that $\Sigma_\beta; \beta$ is satisfiable.

(iv) Let $\Sigma_\gamma = \Sigma_\alpha \cup \Sigma_\beta$ (substituting the empty set for any nonexisting set), then either $\Sigma_\gamma; \alpha$ is satisfiable or $\Sigma_\gamma; \beta$ is satisfiable, implying that $\Sigma_\gamma; (\alpha \vee \beta)$ is satisfiable.

(v) But if $\Sigma_\gamma; (\alpha \vee \beta)$ is satisfiable, and $\Sigma_\gamma \subseteq \Sigma$, then $\Sigma; (\alpha \vee \beta)$ is finitely satisfiable.

(vi) Comparing step (b) and step (e), we see a clear contradiction, implying that whenever $\Sigma \vDash (\alpha \vee \beta)$, it must be the case that either $\Sigma \vDash \alpha$ or $\Sigma \vDash \beta$.