Math 69: Logic

Abelian Groups

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Abstract

We study what the properties of vector spaces over specific fields can tell us regarding the corresponding theories of those vector spaces.

Contents

1. Introduction

Mathematical logic is interesting in itself, but its ability to effectively model and describe other fields in mathematics is, perhaps, even more intriguing and what makes the study of logic so useful to the regular mathematician. For instance, using logic one may model aspects of analysis, algebra, calculus, probability, geometry, and even game theory. This offers disparate and often insightful perspectives into these other fields, letting one apply the laws of logic to their study of the structures of other fields in mathematics and the resulting consequences. More importantly, modeling other structures is useful in the study of logic itself as it lets one interrogate the rules of logic from other perspectives.

In this paper, we will model the structure of abelian groups using first-order logic. We will then narrow down our study to groups that are *abelian*, *divisible*, and *torsion-free*. We will then study groups in this category as vector-spaces over the rational numbers, $\mathbb{Q}$, and study what the structure of these vector spaces can tell us about logic.

2. Preliminaries

*Note: This section is included mostly as a prelude for the non-experienced mathematician, and may be skipped by readers well-versed in **Group Theory**, **Abstract Vector Spaces**, and **First-Order Logic**.*

2.1. Groups.

Definition 2.1. A group $G = (S, *)$ is a set S paired with a binary operation, $* : S \to S$, satisfying the following axioms:

(i) Closure: $x * y \in S$ for any two elements $x, y \in S$.
(ii) Associativity: $(x * y) * z = x * (y * z)$ for any three elements $x, y, z \in S$.
(iii) Identity: there exists an element $\varepsilon \in S$ such that for any $x \in S$, $x * e = e * x = x$.
(iv) Inversion: for any $x \in S$, there exists an element $x^{-1} \in S$ such that $x * x^{-1} = x^{-1} * x = \varepsilon$.

There are two common notations for groups:

(i) Additive notation: $G = (S, 0, +)$, where 0 is the identity element and + is the binary operation. For any element g, we denote the inverse of g as $-g$.
(ii) Multiplicative notation: $G = (S, 1, \cdot)$, where 1 is the identity element and $\cdot$ is the binary operation. For any element g, the inverse of g is denoted as g^{-1}.

However, the group operation need not be normal addition or multiplication. For instance, the group of integers modulo a positive integer n, denoted $\mathbb{Z}/n\mathbb{Z}$, defines its binary operation on any two elements a and b as $(a + b) \mod n$ — so, for instance, $5 + 7 = 0$ in $\mathbb{Z}/12\mathbb{Z}$. More absurd operations can be defined in some groups, including rotations and reflections.

Henceforth, we will use additive notation for groups.

Definition 2.2. A group G is *abelian* if $x + y = y + x$ for any two elements $x, y \in G$.

Definition 2.3. The order of any element g in a group G is the smallest positive integer n such that $\underbrace{g + \ldots + g}_{n \text{ times}} = 0$.

For instance, if an element γ is such that $\gamma + \gamma + \gamma = 0$, then γ has order 3.

2.2. **Fields.**

A field $F = (S; 0, 1, +, \times)$ is a set S with two binary operations, + and ·, such that the pairing $(S, +)$ is an abelian group and the pairing $(S \setminus \{0\}, \cdot)$ is also an abelian group. The · operation is distributive over over +, meaning

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Some common examples of fields are the real numbers $\mathbf{R} = (\mathbb{R}, 0, 1, +, \cdot)$, the rational numbers $\mathbf{Q} = (\mathbb{Q}, 0, 1, +, \cdot)$, and the complex numbers $\mathbf{C} = (\mathbb{C}, (0, 0), (1, 0), +, \cdot)$. The · operation is often referred to as *multiplication* while the + operation is often referred to as *addition*, since they correspond to multiplication and addition in many fields with numerical elements.

2.3. **Vector Spaces over Fields.**

A vector space V over a field F is an *abelian group* whose elements can be multiplied by elements in the field. We refer to the elements of the vector space as *vectors* and the elements of the field as *scalars*. The multiplication of a vector by a scalar is called *scaling*. For any vector space V over a field F, the scaling operation has to satisfy the following axioms:

(i) $1x = x$ for any $x \in V$, where 1 is the multiplicative identity in F.
(ii) $a(bx) = (ab)x$ for any vector $x \in V$ and any two scalars $a, b \in F$.
(iii) $(a + b)x = ax + bx$ for any vector $x \in V$ and any two scalars $a, b \in F$.
(iv) $a(x + y) = ax + ay$ for any two vectors $x, y \in V$ and any scalar $a \in F$.

Corollary 2.4. *Scaling any vector by* 0 *yields the zero vector. Equivalently,* $0x = 0$ *for any* $x \in V$.

Proof. Let $x \in V$, then $0x + x = (0 + 1)x = x$. Therefore, $0x$ must be the identity in the vector space (the zero vector). □

Corollary 2.5. *Scaling any vector by* -1 *yields the additive inverse of that vector. Equivalently,* $(-1)x = -x$.

Proof. For any $x \in V$, $(1)x + x = (-1)x + 1x = (-1 + 1)x = 0$.
Therefore, $(-1)x$ must be the additive inverse of x. □

Corollary 2.6. *Scaling any vector by* $-n$ *yields the same result as scaling the inverse of that vector by* n*. Equivalently,* $(-n)x = n(-x)$ *for any* $x \in V$ *and any* $n \in F$.

Proof.

$$(-n)x = (n \cdot (-1))x = n \cdot (-1)(x) = n(-x).$$

□

Corollary 2.7. *Scaling any vector $x \in V$ by $-n$ yields the additive inverse to the result of scaling the same vector by n.*

Proof. For any $x \in V$ and any $n \in F$,

$$(-n)x = (-1 \cdot n)x = (-1)(nx).$$

By corollary 2.5, $(-1)nx$ is equivalent to the additive inverse of nx. □

Definition 2.8. A *linear combination* of vectors in a set $S \subseteq V$ where V is a vector space defined over a field F is a sum

$$\sum_{v \in S} k_v v$$

where each $k_v \in F$.

Definition 2.9. A *basis* of a vector space V over a field F is a minimal set of vectors in V that spans V. This means that;

(i) Every vector in V can be written as a linear combination of vectors in the basis.

(ii) No vector in in the basis can be written as a linear combination of any other vectors in the basis (except itself).

Definition 2.10. The *dimension* of a vector space V (written $\mathbf{dim}(V)$) over a field F is the cardinality of a basis of V. While a vector space may have multiple subsets that are bases of V, all such bases must have the same cardinality.

Claim 2.11. *Two finite-dimensional vector spaces of the same dimension, defined over the same field, are isomorphic.*

Proof. Let U and V be two finite-dimensional vector spaces of the same dimension defined over the same field, and let $(u_1, \ldots, u_k)$ be a basis of U and $(v_1, \ldots, v_k)$ be a basis of V. Then there exists an isomorphism from U to V that maps u_i to v_i for each i. □

2.4. **Divisible and Torsion Free Abelian Groups.**

Definition 2.12. We say a group G is *divisible* if for any $g \in G$ and any $n \in \mathbb{N}$ with $n > 0$, there exists an element $h \in G$ such that $\underbrace{h + \ldots + h}_{n \text{ times}} = g$.

Definition 2.13. An element $g \in G$ is *torsion free* if $\underbrace{h + \ldots + h}_{n \text{ times}} \neq 0$ for any positive integer $n > 0$. A group is *torsion free* if every non-identity element is torsion free.

Lemma 2.14. *Where F is a field with numeric elements and G is a group, define scalar multiplication of an element $x \in G$ and a scalar $n \in F$ as follows:*

$$nx = \begin{cases} 0 & \textit{if } n = 0. & (2.15) \\ \underbrace{x + \ldots + x}_{n \textit{ times}} & \textit{if } n \textit{ is an integer and } n > 0. & (2.16) \\ \underbrace{-x + \ldots + -x}_{n \textit{ times}} & \textit{if } n \textit{ is an integer and } n < 0. & (2.17) \\ y \quad \textit{where } x = qy & \textit{if } n = q^{-1} \textit{ for some integer } q \in F. & (2.18) \\ p\left(q^{-1} \cdot x\right) & \textit{if } n = pq^{-1} \textit{ with } q \neq 0. & (2.19) \end{cases}$$

For scalar multiplication (per our definition) to be well-defined:

1. For every element $g \in G$ and every positive integer $n > 0$, definable in F, the multiplication of n^{-1} by g must be well-defined. Therefore, so there must exist an element $h \in G$ such that $nh = g$. By our definition of scalar multiplication of an element in the group by integer scalars, $nh = \underbrace{h + \ldots + h}_{n \text{ times}}$, hence this condition is satisfied for every positive integer if the group is divisible.
2. If $pq^{-1} = mn^{-1}$, then the scalar multiplication of any group element x by pq^{-1} should equal the scalar multiplication of x by mn^{-1}. Since $pq^{-1} = mn^{-1}$, $pq^{-1} - mn^{-1} = 0$. Therefore, $(pq^{-1} - mn^{-1})x = 0x = 0$. Since multiplication is distributive,

$$(pq^{-1} - mn^{-1})x = pq^{-1}x - mn^{-1}x = 0.$$

Therefore, $pq^{-1}x$ and $-mn^{-1}x$ are additive inverses, which can only happen if $pq^{-1}x$ and $mn^{-1}x$ are equivalent.

3. Analysis of some Vector Space Structures

Recall that for any arbitrary structure $\mathfrak{A}$ to be a vector space over a field F, $\mathfrak{A}$ must be an *abelian* group, and the multiplication of elements of $\mathfrak{A}$ by elements of F must be well-defined and satisfy the vector space axioms. When both of these conditions are satisfied, we say that $\mathfrak{A}$ has a vector space structure over F. In this section, we shall see two examples of abelian groups that have a vector space structure over some field and look at why they do not have a vector space structure over some other fields.

3.1. **Divisible, Torsion Free Abelian Groups over $\mathbb{Q}$.**

Recall that an abelian group G is divisible if for every $n \in \mathbb{N}_{>0}$, every element $g \in G$ element can be written as nh for some other element $h \in G$. Furthermore, a group G is torsion free if every non-identity element is torsion free — that is, if g is not the identity element, then scaling it by any non-zero factor $n \in \mathbb{N}_{>0}$ yields a non-zero element.

Note that in the field $\mathbb{Q}$, the additive inverse of any element g is the element $-g$ and the multiplicative inverse of any element g is the element $1/g$ — for example, the additive inverse inverse of 7 is -7, and the multiplicative inverse of 7 is $1/7$.

Claim 3.1. *The scalar multiplication of elements of a divisible, torsion free abelian group G by elements of the field $\mathbb{Q}$ is well-defined.*

Proof. Using the definition of multiplication(2.15):

1. The proof for multiplication by positive integers is straightforward since the sum $\underbrace{x + \ldots + x}_{n \text{ times}}$ always names a unique element in the group.
2. The proof for multiplication by negative integers is also straightforward since the sum $\underbrace{-x + \ldots + -x}_{n \text{ times}}$ always names a unique element in the group.
3. The proof for multiplication by 0 is trivial; $0 \cdot x$ always evaluates to 0.
4. For multiplication by a rational number $n = p/q$, we have $nh = (p/q)h = p(\frac{1}{q} \cdot h)$. p is an integer, so the multiplication by p is well-defined as above. For multiplication by $\frac{1}{q}$, we define $\frac{1}{q} \cdot g = h$ such that $g = qh$. For multiplication to be well-defined, we must show that any such element h is unique, i.e. if $qh_1 = qh_2$

then we necessarily have that $h_1 = h_2$.

$$\begin{aligned} qh_1 = qh_2 &= g \\ \underbrace{h_1 + \ldots + h_1}_{q \text{ times}} = \underbrace{h_2 + \ldots + h_2}_{q \text{ times}} &= g \\ \therefore \left(\underbrace{h_1 + \ldots + h_1}_{q \text{ times}}\right) - \left(\underbrace{h_2 + \ldots + h_2}_{q \text{ times}}\right) = g - g &= 0 \\ \therefore \underbrace{(h_1 - h_2) + \ldots + (h_1 - h_2)}_{q \text{ times}} &= 0 \\ \therefore h_1 - h_2 &= 0 \qquad \text{(since } G \text{ is torsion free)} \\ \therefore h_1 &= h_2 \end{aligned}$$

□

Corollary 3.2. *Any divisible, torsion free abelian group has a vector space structure over* $\mathbb{Q}$*.*

Proof. This follows from the proof to claim 3.1 □

3.2. **Abelian Groups Wherein Each Non-Identity Element Has Order 2.**

Claim 3.3. *Any two abelian groups of the same finite order* n *in which all non-identity elements have order* 2 *are isomorphic.*

Proof. Let G and H be two such groups. Since G and H have the same order n, they each have exactly $n-1$ elements of order 2. To construct an isomorphism between G and H, map the identity element in G to the identity element in H, and map each non-identity element $g \in G$ to some nonidentity element $h \in H$ such that no two elements are mapped to the same element. This isomorphism is always constructible since there's exactly $n - 1$ such elements in each of G and H.

If every element in a group has order 2, and the group has finite order, then the group must have an order that is a power of 2 (Lagrange/Sylow theorems in [1]) However, any such group or order 2^k is isomorphic to the direct product of k copies of C_2, where C_2 is the cyclic group of order 2,

$$C_2 = (\{0, 1\}, 0, +)$$

. For instance, the Klein-Four group, V_4, is an abelian group of order 4 where every non-identity element has order 2. We can construct an isomorphism from V_4 to $C_2 \times C_2$ as follows:

$$V_4 = (\{1, a, b, ab\}, 1, \cdot)$$
$$C_2 \times c_2 = (\{(0,0), (0,1), (1,0), (1,1)\}, (0,0), +)$$

$$\begin{aligned}\psi : V_4 &\to C_2 \times C_2 \\ 1 &\mapsto (0,0) \\ a &\mapsto (0,1) \\ b &\mapsto (1,0) \\ ab &\mapsto (1,1)\end{aligned}$$

Therefore, since both G and H are isomorphic to a sequence of direct products of C_2 with itself, so they are isomorphic to each other. □

Claim 3.4. *Any two groups G and H of infinite order such that every non-identity element has order* 2 *are isomorphic.*

Proof. First, note that G and H are each countably infinite sets since every element has order 2. Construct a countably infinite group where every non-identity element has order 2 by taking the direct product of C_2 with itself an infinite number of times,

$$C_2^\infty = C_2 \times C_2 \times C_2 \times \dots$$

We can then construct isomorphisms from each of G and H to C_2^∞, so G and H are also isomorphic to each other. □

Let G be an abelian group in which every non-identity element has order 2. This means that $g = g = 0$, or equivalently $2g = 0$ for all $g \in G$. As a result, $ng = 0$ for all $n \in 2\mathbb{Z}$ (multiples of 2), so G cannot have a vector space structure over Q. To define a vector space structure for G, we must limit the scalars to a two-element field such as $\mathbb{F}_2$.

$$\mathbb{F}_2 = (\{0,1\}, +, \times).$$

Claim 3.5. *Any group G in which every non-identity element has a vector space structure over the Galois field of two elements,* $\mathbb{F}_2$.

Proof. Define multiplication as follows:

$$ng = \begin{cases} 0 & \text{if } n = 0 \\ g & \text{if } n = 1 \end{cases}$$

Then all teh vector space axioms are satisfied. Particularly;

1. $0g = 0$ for all $g \in G$.

2. $1g = g$ for all $g \in G$.

3. $a(bx) = (ab)x$ for any vector $x \in V$ and any two scalars $a, b \in F$.

(i) $0(0g) = (0 \cdot 0)g = 0g = 0.$

(ii) $1(0g) = (1 \cdot 0)g = 0g = 0.$

(iii) $0(1g) = (0 \cdot 1)g = 0g = 0.$

(iv) $1(1g) = (1 \cdot 1)g = 1g = g.$

4. $(a + b)g = ag + bg$ for any vector $xg \in G$ and any two scalars $a, b \in F$.

(i) $(0 + 0)g = 0g + 0g = 0.$

(ii) $(0 + 1)g = 0g + 1g = 0 + g = g.$

(iii) $(1 + 0)g = 1g + 0g = g.$

(iv) $(1 + 1)g = 1g + 1g = 1g + 1g = 2g = 0.$

5. $a(g + h) = ag + ah$ for any two vectors $x, y \in V$ and any scalar $a \in F$.

(i) $1(g + h) = 1g + 1h = g + h.$

(ii) $0(g + h) = 0g + 0h = 0.$

□

4. Axiomatizing Groups

In the previous section, we derived two vector space structures over two distinct fields, $\mathbb{Q}$ and $\mathbb{F}_2$. In this section, we will officially model those two structures using first-order logic. First, we specify the following language:

$$\mathcal{L} = \langle 0, +, -, = \rangle \tag{4.1}$$

The language dictates which symbols are permitted in well-formed axioms for our structures.

4.1. **Axioms Defining Abelian Groups.**

For any structure to be an abelian group, it needs to satisfy the group axioms (closure, associativity, identity, and inversion) and commutativity (see section 2.1 for reference). We define the set Σ to contain the following abelian group axioms:

$$\forall g\, \forall h\, (g + h = h + g) \qquad \text{(commutativity)} \tag{4.2}$$

$$\forall g\, \forall h\, \forall i\, ((g + (h + i)) = ((g + h) + i)) \qquad \text{(associativity)} \tag{4.3}$$

$$\forall g (g + 0 = g) \qquad \text{(0 is the group identity element)} \tag{4.4}$$

$$\forall g\, (g + (-g) = 0) \qquad \text{(group inversion)} \tag{4.5}$$

In first-order logic, n-ary functions are always defined to map elements from $|\mathfrak{A}|^n$ back to $|\mathfrak{A}|$, so we do not need an explicit axiom for closure.

Any structure satisfying Σ is an abelian group.

4.2. **Axioms Defining Divisible, Torsion Free Abelian Groups.**

Any abelian group needs additional restrictions to be divisible and torsion free.

1. Recall that a group G is *divisible* if for any $g \in G$ and any $n \in \mathbb{N}$ with $n > 0$, there exists an element $h \in G$ such that $nh = g$. We represent this condition using an infinite set of axioms specifying that, for each possible positive integer, there exists such sn element h for each element g in the structure:

$$T_1 = \left\{ \forall g\, \exists h,\ (g = \underbrace{h + \ldots + h}_{n \text{ times}}) \mid n = 1, 2, 3, \ldots \right\}.$$

2. Recall that a group G is *torsion free* if every non-identity element is torsion free. That is, if $g \neq 0$ then $ng \neq 0$ for any positive integer $n > 0$. We can represent this condition using an infinite set of elements specifying that, for each possible positive integer n, the product ng is non-zero for each non-zero element in G:

$$T_2 = \left\{ \forall g\, ((g \neq 0) \rightarrow (\underbrace{g + \ldots + g}_{n \text{ times}} \neq 0)) \mid n = 1, 2, 3, \ldots \right\}.$$

Let $T = T_1 \cup T_2$, then any model satisfying $\Sigma \cup T$ is a divisible, torsion free abelian group.

4.3. **Axioms Defining Abelian Groups Wherein Each Element Other Than the Identity Has Order 2.** To specify that each element in an abelian group has order 2, define the set

$$S = \{\forall g\,(g \neq 0 \to (g + g = 0))\},$$

then any model satisfying $\Sigma \cup S$ is an abelian group wherein each element other than the identity has order 2.

5. Categorical Analysis of the Structures

In this section, we will define some properties of the specific sets of axioms based on what we have established about the corresponding structures.

Claim 5.1. $(\Sigma \cup T)$ *is not countably categorical.*

Proof. By definition of Σ and T, any structure satisfying $\Sigma \cup T$ is a divisible, torsion free abelian group. Let $\mathfrak{A}$ be such a structure. As we established in the previous section, $\mathfrak{A}$ has a vector space structure over $\mathbb{Q}$. Since all finite-dimensional vector spaces over $\mathbb{Q}$ are countable and no two vector spaces of different dimensions can be isomorphic, there are multiple countable models of $\Sigma \cup T$ so $\Sigma \cup T$ is not countably categorical. □

Claim 5.2. $(\Sigma \cup T)$ *is categorical in the cardinality of* $\mathbb{R}$*.*

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two models of $\Sigma \cup T$, each having the cardinality of $\mathbb{R}$. By considering these two models as vector spaces over $\mathbb{F}_2$ with multiplication defined by $0g = 0$ and $1g = g$, then every basis of $\mathfrak{A}$ must contain all non-zero vectors in $\mathfrak{A}$, and every basis of $\mathfrak{B}$ must contain all non-zero vectors in $\mathfrak{B}$. Therefore,

$$\mathbf{dim}(\mathfrak{A}) = \mathbf{dim}(\mathfrak{B}) = |\mathbb{R}|$$

. Since any two vector spaces of the same dimension are isomorphic, $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic as vector spaces hence isomorphic as groups. Therefore, $\Sigma \cup T$ has a single model in the cardinality $\mathbb{R}$ up to isomorphism and is therefore categorical in the cardinality of $\mathbb{R}$. □

Claim 5.3. $(\Sigma \cup S)$ *is countably categorical.*

Proof. By definition of Σ and S, any structure satisfying $\Sigma \cup S$ is an abelian group wherein each element other than the identity has order 2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two such structures with the same cardinality n. As we saw earlier, $\mathfrak{A}$ and $\mathfrak{B}$ can only have a vector space structure over $\mathbb{F}_2$, with multiplication defined by $0g = 0$ and $1g = g$. Therefore, any basis for $\mathfrak{A}$ must contain all non-zero elements in $\mathfrak{A}$, and any basis for $\mathfrak{B}$ must contain all non-zero elements in $\mathfrak{B}$. Therefore,

$$\mathbf{dim}(\mathfrak{A}) = \mathbf{dim}(\mathfrak{B}),$$

hence $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic as vector spaces. $\square$

6. Conclusions

Claim 6.1. *Cn Σ is not complete.*

Proof. For an easy example, consider the fact that any model for $\Sigma \cup T$ is a model for Σ (or any divisible, torsion free abelian group is an abelian group). Now, there are sentences in $\mathcal{L}$ that are true in Mod $(\Sigma \cup T)$ but are not in Cn Σ. For instance, take the sentence saying that no element in a model of $\Sigma \cup T$ has order 2:

$$\forall g\,(g \neq 0 \to (g + g \neq 0)).$$

This sentence is not deducible from Σ — and neither is its negation. In fact, it is entirely okay for elements in a model of Σ to have order 2! Take C_2, the cyclic group of order 2, for example, wherein the element 1 has order 2. □

Claim 6.2. *Cn $(\Sigma \cup T)$ is complete.*

Proof. As we established in the previous section, $\Sigma \cup T$ has no finite models except the trivial vector space. Additionally, $\Sigma \cup T$ is categorical in the cardinality of the reals but not countably categorical. First, note that Th Mod $(\Sigma \cup T)$ = Cn $(\Sigma \cup T)$, since Th Mod $(\Sigma \cup T)$ is the set of all sentences true in all models of $\Sigma \cup T$, and each sentence true in all models of $\Sigma \cup T$ is in Cn $(\Sigma \cup T)$.

By the Los-Vaught Test [2], any theory with no finite models that is κ-categorical for some infinite cardinal κ is complete. □

Corollary 6.3. *Cn $(\Sigma \cup T)$ is not decidable.*

Proof. We've seen that Cn $(\Sigma \cup T)$ = Th Mod $(\Sigma \cup T)$. This means that for any sentence σ definable in $\mathcal{L}$, if Cn $(\Sigma \cup T) \vDash \sigma$ then $\sigma \in$ Cn $(\Sigma \cup T)$.

Furthermore, Cn $(\Sigma \cup T)$ is complete, so for any sentence σ definable in $\mathcal{L}$, either Cn $(\Sigma \cup T) \vDash \sigma$ or Cn $(\Sigma \cup T) \vDash \neg\sigma$.

Given an arbitrary sentence φ, one thought would be to determine if $\varphi \in$ Cn $(\Sigma \cup T)$ by checking if Cn $(\Sigma \cup T) \vDash \varphi$ or Cn $(\Sigma \cup T) \vDash \neg\varphi$. In the first case, $\varphi \in$ Cn $(\Sigma \cup T)$, and in the second case, $\varphi \notin$ Cn $(\Sigma \cup T)$. However, this is not necessarily possible since Cn $(\Sigma \cup T)$ can be infinite especially when the model has an infinite cardinal. □

Claim 6.4. *Cn $\Sigma \cup S$ is not complete.*

Proof. As shown in claim 3.3, every finite model of order 2^k for some integer $k > 0$ is isomorphic to the direct product of k copies of C_2. Now, take any two finite models $\mathfrak{A}$ and $\mathfrak{B}$ of different orders, say 2^a and 2^b. Define the sentence σ_a to say that the structure has exactly 2^a elements, and the sentence σ_b to say that the structure has exactly 2^b elements.

$$\sigma_a = \forall g_1 \exists g_2 \exists \ldots \exists g_{2^a} ((g_1 \neq g_2) \wedge (g_1 \neq g_3) \wedge \ldots \wedge (g_2 \neq g_3) \wedge (g_2 \neq g_4) \wedge \ldots \wedge g_{2^a-1} \neq (g_{2^a}))$$

$$\sigma_b = \forall g_1 \exists g_2 \exists \ldots \exists g_{2^b} ((g_1 \neq g_2) \wedge (g_1 \neq g_3) \wedge \ldots \wedge (g_2 \neq g_3) \wedge (g_2 \neq g_4) \wedge \ldots \wedge g_{2^b-1} \neq (g_{2^b}))$$

Then, σ_a is true in $\mathfrak{A}$ but not in $\mathfrak{B}$, and σ_b is true in $\mathfrak{B}$ but not in $\mathfrak{A}$. However;

$$\text{Cn}\,(\Sigma \cup S) \nvDash \sigma_a$$

$$\text{Cn}\,(\Sigma \cup S) \nvDash (\neg \sigma_a)$$

$$\text{Cn}\,(\Sigma \cup S) \nvDash \sigma_b$$

$$\text{Cn}\,(\Sigma \cup S) \nvDash (\neg \sigma_b)$$

Consequently, Cn $\Sigma \cup S$ is not complete. □

For Cn $\Sigma \cup S$, we must specify that no model is finite, thus limiting the cardinality to infinite models. Define a new set R specifying that, for each possible finite cardinality n, there exists more than n elements in the structure.

$$R = \left\{ \exists g_1 \, \exists g_2 \, \exists \ldots \exists g_n \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n+1} (g_i \neq g_j) \mid n = 1, 2, 3, \ldots \right\}$$

References

[1] David S. Dummit and Richard M. Foote. *Abstract Algebra*. 3rd ed. John Wiley & Sons, 2003.

[2] David Marker. *Model Theory: An Introduction*. Springer, 2002.