Math 69: Logic **Winter '23**

Homework assigned February 01, 2023

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Problem 3.

(a) Let $\mathfrak{A}$ be a structure and let $s : V \to |\mathfrak{A}|$. Define a truth assignment on the set of prime formulas by

$$v(\alpha) = T \quad \text{iff} \quad \vDash_{\mathfrak{A}} \alpha[s].$$

Show that for any formula (prime or not),

$$\overline{v}(\alpha) = T \quad \text{iff} \quad \vDash_{\mathfrak{A}} \alpha[s].$$

Remark: This result reflects the fact that ¬ and → were treated in Chapter 2 the same was as in Chapter 1.

Since the set $\{\neg, \to\}$ is complete we can construct any formula α from prime formulas $\alpha_1, \ldots, \alpha_n$ using some combination of ¬ and → connectives. Suppose $\overline{v}(\alpha) = T$. We prove by induction on the form of α that $\vDash_{\mathfrak{A}} \alpha[s]$.

Base Case: α is prime.

Then $\overline{v}(\alpha) = v(\alpha)$, so $\overline{v}(\alpha) = T$ iff $v(\alpha) = T$, and $v(\alpha) = T$ iff $\vDash_{\mathfrak{A}} \alpha[s]$.
Therefore, $\overline{v}(\alpha) = T$ iff $\vDash_{\mathfrak{A}} \alpha[s]$

Inductive Step 1: Suppose $\alpha = \neg\alpha_1$ for some formula α_1. Then:

$$\overline{v}(\alpha) = \overline{v}(\neg\alpha_1) = \neg\overline{v}(\alpha_1)$$

$$\overline{v}(\alpha) = T \iff \overline{v}\alpha_1 = F$$

$$\overline{v}(\alpha) = T \iff \nvDash_{\mathfrak{A}} \alpha_1[s]$$

$$\overline{v}(\alpha) = T \iff \vDash_{\mathfrak{A}} \neg\alpha_1[s]$$

$$\overline{v}(\alpha) = T \iff \vDash_{\mathfrak{A}} \alpha[s]$$

Inductive Step 2: Suppose $\alpha = \alpha_1 \to \alpha_2$ for some formulas α_1, α_2. Then:

$$\overline{v}(\alpha) = \overline{v}(\alpha_1 \to \alpha_2) = \overline{v}(\alpha_1) \to \overline{v}(\alpha_2)$$

$$\overline{v}(\alpha) = T \iff \overline{v}(\alpha_1) = F \text{ or } \overline{v}(\alpha_2) = T$$

$$\overline{v}(\alpha) = T \iff \nvDash_{\mathfrak{A}} \alpha_1[s] \text{ or } \vDash_{\mathfrak{A}} \alpha_2[s]$$

$$\overline{v}(\alpha) = T \iff \vDash_{\mathfrak{A}} (\alpha_1 \to \alpha_2)[s]$$

$$\overline{v}(\alpha) = T \iff \vDash_{\mathfrak{A}} \alpha[s]$$

Therefore, for all formulas α, $\overline{v}(\alpha) = T$ iff $\vDash_{\mathfrak{A}} \alpha[s]$.

(b) Conclude that if Γ tautologically implies φ, then Γ logically implies φ.

Let $\mathfrak{A}$, s, and v be as defined above. Suppose $\mathfrak{A}$ satisfies all members of Γ with s, then $\overline{v}(\gamma) = T$ for all $\gamma \in \Gamma$.

Since Γ tautologically implies φ, and s satisfies all members of Γ, we have that $\Gamma \vDash_{\mathfrak{A}} \varphi[s]$, so $\overline{v}(\varphi) = T$.

$\mathfrak{A}$ and s are arbitrary, so the same condition holds for any other structure $\mathfrak{A}$ and assignment function s given $\mathfrak{A}$ satisfies all members of Γ with s. Therefore, Γ logically implies φ.

Problem 4.

Give a deduction (from $\varnothing$) of $\forall x\varphi \to \exists x\varphi$.

Note: You should not merely prove that such a deduction exists; write out the entire deduction.

(i) $(\forall x\neg\varphi \to \neg\varphi) \to (\varphi \to \neg\forall x\neg\varphi)$ (Tautology)

(ii) $(\forall x\neg\varphi \to \neg\varphi)$ (Axiom 2)

(iii) $(\varphi \to \neg\forall x\neg\varphi)$ (Modus Ponens on (i), (ii))

(iv) $(\varphi \to \neg\forall x\neg\varphi) \to ((\neg\forall x\neg\varphi \to \exists x\varphi) \to (\varphi \to \exists x\varphi))$ (Tautology)

(v) $(\neg\forall x\neg\varphi \to \exists x\varphi) \to (\varphi \to \exists x\varphi)$ (Modus Ponens on (iii), (iv))

(vi) $(\neg\forall x\neg\varphi \;\leftrightarrow\; \exists x\varphi)$ (Axiom 5)

(vii) $(\varphi \to \exists x\varphi)$ (Modus Ponens on (v), (vi))

(viii) $(\forall x\varphi \to \varphi) \to ((\varphi \to \exists x\varphi) \to (\forall x\varphi \to \exists x\varphi))$ (Tautology)

(ix) $\forall x\varphi \to \varphi$ (Axiom 2)

(x) $(\varphi \to \exists x\varphi) \to (\forall x\varphi \to \exists x\varphi)$ (Modus Ponens on (viii), (ix))

(xi) $\forall x\varphi \to \exists x\varphi$ (Modus Ponens on (vii), (x))

Problem 9.

(Re-replacement lemma)

(a) Show by example that $(\varphi_y^x)_x^y$ is not in general equal to φ. And that it is possible for both for x to occur in $(\varphi_y^x)_x^y$ at a place it did not occur in φ, and for x to occur in φ at a place it does not occur in $(\varphi_y^x)_x^y$.

Let P be a one-place predicate symbol and Q be a two-place predicate symbol.

Let $\varphi = \forall y Px \to Qxy$, then $\varphi_y^x = \forall y Py \to Pyy$ and $(\varphi_y^x)_x^y = \forall y Py \to Qxx$

In the example above, we see one instance where x occurs in $(\varphi_y^x)_x^y$ at a position where it does not occur at in φ (at Qxx vs. Qxy) and one instance where x occurs in φ at a position where it does not occur in $(\varphi_y^x)_x^y$ (at $\forall y Px$ vs. $\forall y Py$).

(b) Show that if y does not occur at all in φ then x is substitutable for y in φ_y^x and that $(\varphi_y^x)_x^y$ is equal to φ.

Suggestion: Use induction on φ.

Base Case 1: $\varphi = \alpha$ for some variable or constant α.

Then $\varphi_y^x = \alpha_y^x = y$ if $\alpha = x$, or else $\alpha_y^x = \alpha$. Therefore, $(\varphi_y^x)_x^y = x$ iff $\alpha = x$ else α, meaning $(\varphi_y^x)_x^y = \varphi$ irrespective of whether $\alpha = x$.

Base Case 2: $\varphi = Px_1x_2 \ldots x_n$ for some n-place predicate symbol p.

Then $\varphi_y^x = (Px_1x_2 \ldots x_n)_y^x = P(x_1)_y^x(x_2)_y^x \ldots (x_n)_y^x$.

Therefore, $(\varphi_y^x)_x^y = P((x_1)_y^x)_x^y((x_2)_y^x)_x^y \ldots ((x_n)_y^x)_x^y$.

Since $((x_k)_y^x)_x^y = x_k$ for all $k \in \{1, \ldots, n\}$ $(\varphi_y^x)_x^y = Px_1x_2 \ldots x_n = \varphi$, so $(\varphi_y^x)_x^y = \varphi$.

Inductive Step: Let $\varphi = (\beta * \gamma)$ for some β, γ, and a connective $*$, and that y does not occur in φ. Then y must not occur in β or γ.

By definition, $\varphi_y^x = (\beta_y^x * \gamma_y^x)$ and $(\varphi_y^x)_x^y = (\beta_y^x * \gamma_y^x)_x^y = (\beta_y^x)_x^y * (\gamma_y^x)_x^y$.

By the inductive hypothesis, assume $(\beta_y^x)_x^y = \beta$ and $(\gamma_y^x)_x^y = \gamma$, then $(\varphi_y^x)_x^y = (\beta_y^x)_x^y * (\gamma_y^x)_x^y = \beta * \gamma = \varphi$.