Math 69: Logic

Abelian Groups

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Abstract

This paper describes a set of axioms and a language for abelian groups and explores what the theory of abelian groups can tell us about completeness.

Contents

1. Preliminary Questions

For this problem, we will start with the following definitions. An element g of a group G has order n if n is the smallest positive natural number such that

$$ng = \underbrace{g + \ldots + g}_{n \text{ times}} = 0.$$

For example, g has order 2 if $g \neq 0$ and $g + g = 0$.
An element is said to be *torsion free* if it does not have order n for any $n \in \mathbb{N}$ with $n > 0$. A group is said to be *torsion free* if each of its elements, other than the identity, is torsion free.
Lastly, we say that a group G is divisible if for each $g \in G$ and $n \in \mathbb{N}$ with $n > 0$, there exists $h \in G$ such that

$$nh = \underbrace{h + \ldots + h}_{n \text{ times}} = g.$$

1.1. Definitions.

For convenience, we define the following shorthands used in the rest of the writing:

(i) $\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\}$.
(ii) $(x \neq y)$ is shorthand for $\neg(x = y)$.

1.2. Axiomatizing Abelian Groups.

Define a language $\mathcal{L}$ and a set of axioms Σ such that any model that satisfies Σ is an abelian group. Next, define a set of axioms T such that any model which satisfies $\Sigma \cup T$ is a divisible torsion free abelian group.

Using additive notation for groups, we define $\mathcal{L}$ to specify group operation $(+)$ and the group identity, 0. We also define element equality in the group as a two-place predicate. Precisely, two elements g and h in the group are considered equal if and only if $g + x = h + x$ for every other element x in the group.
Thus, if $\mathfrak{A}$ is a model for $\mathcal{L}$, then

$$(=^{\mathfrak{A}}) = \{(g, h) \mid \forall x(g + x = h + x)\}.$$

We specify $\mathcal{L}$ as follows:

$$\mathcal{L} = \langle 0, +, = \rangle \tag{1.1}$$

For any structure to be abelian, it needs to satisfy the group axioms (closure, associativity, identity, and inversion) and commutativity Σ to contain the following axioms:

$$\forall g\, \forall h\, \exists i\, ((g+h) = i) \qquad \text{(closure)} \qquad (1.2)$$

$$\forall g\, \forall h\, \forall i\, ((g+(h+i)) = ((g+h)+i)) \qquad \text{(associativity)} \qquad (1.3)$$

$$\exists h\, \forall g((g+h) = h) \qquad \text{(group identity element } h\text{, equivalent to } 0) \qquad (1.4)$$

$$\forall g\, \exists h\, ((g+h) = 0) \qquad \text{(existence of inverses)} \qquad (1.5)$$

$$\forall g\, \forall h\, (((g+h) = i) \;\leftrightarrow\; ((h+g) = i)) \qquad \text{(commutativity)} \qquad (1.6)$$

We define an extra set of conditions, T, to contain the following axioms, which must be satisfied (in addition to those in Σ) for a structure to be a divisible torsion-free abelian group. T contains the following axioms:

$$\forall g\, \forall n\, (((g \neq 0) \wedge (0 < n)) \rightarrow (ng \neq 0)) \qquad \text{(torsion-free)} \qquad (1.7)$$

$$\forall g\, \forall n\, \exists h, ((n \neq 0) \rightarrow (g = nh)) \qquad \text{(divisibility)} \qquad (1.8)$$

1.3. **Existence of $\mathbb{Q}$-vector space structures.**

Show that any divisible torsion free abelian group has a $\mathbb{Q}$-vector space structure.

Hint: Show that if G is such a group, $n \in \mathbb{N}$ with $n > 0$ and $g \in G$ then there is a unique $h \in G$ such that $nh = g$. Note that to show there is a $\mathbb{Q}$-vector space structure, you must define scalar multiplication (and prove it is well-defined).

1.3.1. *Scalar Multiplication (Definition).*

For any group element $h \in G$ and a scalar $n \in \mathbb{N}_{>0}$, we define the scalar multiplication of h by n to be the unique element $g \in G$ such that

$$g = \underbrace{h + \ldots + h}_{n \text{ times}}.$$

Let f be the function that maps every such element $h \in G$ and scalar $n \in \mathbb{N}$ to the scalar product g. That is;

$$\begin{aligned} f\colon G \times \mathbb{N}_{>0} &\rightarrow G \\ (h,n) &\mapsto g \quad \text{where} \quad g = \underbrace{h + \ldots + h}_{n \text{ times}} \end{aligned}$$

Claim 1.9. *f is well-defined.*

Proof.

(i) Suppose $h \in G$ and $m, n \in \mathbb{N}$ such that $f(h, m) = g$ and $f(h, n) = g$. Then

$$\underbrace{h + \ldots + h}_{m \text{ times}} = \underbrace{h + \ldots + h}_{n \text{ times}} = g.$$

Let $k = \mathbf{max}(m, n)$ and $l = \mathbf{min}(m, n)$, then $k - l \geq 0$, and:

$$\begin{aligned} kh - lh &= g - g = 0 \\ \therefore (k - l)\, h &= 0 \\ \therefore k - l &= 0 && \text{(since } G \text{ is torsion-free)} \\ \therefore n &= m \end{aligned}$$

(ii) Suppose $h_1, h_2 \in G$ and $n \in \mathbb{N}_{>0}$ such that $f(h_1, n) = g$ and $f(h_2, n) = g$. Then

$$\begin{aligned} \underbrace{h_1 + \ldots + h_1}_{n \text{ times}} &= \underbrace{h_2 + \ldots + h_2}_{n \text{ times}} = g \\ \therefore \left(\underbrace{h_1 + \ldots + h_1}_{n \text{ times}}\right) - \left(\underbrace{h_2 + \ldots + h_2}_{n \text{ times}}\right) &= g - g = 0 \\ \therefore \underbrace{(h_1 - h_2) + \ldots + (h_1 - h_2)}_{n \text{ times}} &= 0 \end{aligned}$$

Since G is torsion-free, $ng \neq 0$ for any $g \in G$ given $g \neq 0$. Therefore, if $n(h_1 - h_2) = 0$, it follows that $h_1 - h_2 = 0$, so $h_1 = h_2$.

Therefore, scalar multiplication is well-defined. □

1.4. **Any divisible torsion-free abelian group has a $\mathbb{Q}$-vector space structure.**

As seen in the proof to 1.9, if a group G is divisible and torsion-free, then scalar multiplication is well-defined. Since G is divisible, then for every $g \in G$ and $n \in \mathbb{N}_{>0}$ there is a unique $h \in G$ such that $nh = g$. As seen in the proof to 1.9, for fixed n, any such h must be unique (likewise, for fixed h, any such n must be unique). Therefore, G has a $\mathbb{Q}$-vector space structure.

1.5. **Axiomatizing Abelian Groups Wherein Each Element Other Than the Identity Has Order 2.**

Define a set of axioms S such that any model that satisfies $\Sigma \cup S$ is an abelian group in which each element other than the identity has order two. Can we give a model for $\Sigma \cup S$ a vector space structure?
Hint: Be creative in your choice of the scalar field.

We define S to contain the following axioms:

$$\forall g\left((g \neq 0) \rightarrow ((g+g)=0)\right) \qquad \text{(every element has order 2)} \qquad (1.10)$$

For examples of models of $\Sigma \cup S$; first, let's consider finite abelian groups in which every non-identity element has order 2. The smallest such group is the cyclic group of order 2:

$$C_2 = (\{0,1\}, +)$$

To generate larger groups where every non-identity element has order 2, we can take the direct product of C_2 with itself. For instance, the Klein-Four group V_4 is isomorphic to $C_2 \times C_2$. For an infinite such group, take an infinite sequence of direct products of C_2:

$$C_2^\infty = C_2 \times C_2 \times C_2 \times \ldots$$

Take $g \in C_2^\infty$ such that g is not the identity. For C_2^∞ to have a $\mathbb{Q}$-vector space structure,

Since g has order 2, we have that $3g = 2g + g = 0 + g = g$, and, more generally, for all $n \in \mathbb{N}_{>0}$, $ng = kg$ where $n \equiv k \mod 2$. Therefore, G cannot have a $\mathbb{Q}$-vector space structure under scalar multiplication with elements of $\mathbb{N}_{>0}$.

To specify a model for $\Sigma \cup S$, such that the group is a scalar field, we have to limit our choice of scalars to the Galois field of 2 elements

$$\mathbb{F}_2 = (\{0,1\}, \times).$$

Therefore, by defining $G = C_2^\infty$ and limiting the scalar multiplication function $f : G \times \mathbb{F}_2 \rightarrow G$, we can construct a model for $\Sigma \cup S$ that has a $\mathbb{Q}$-vector space structure.