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Math 69: Logic **Winter '23**

Homework assigned January 4, 2023

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Credit Statement

I worked on these problems alone, with reference to class notes and the following books:

(a) ***A Mathematical Introduction to Logic*** by **Herbert Enderton**.

Problems

1. Give three sentences in English together with translations into our formal language. The sentences should be chosen so as to have an interesting structure, and the translations should each contain 15 or more symbols.

Building Blocks

(i) S: "It is snowing."

(ii) C: "I am wearing a coat."

(iii) O: "I am outdoors."

(iv) F: "I am freezing."

(v) H: "I run into the house."

(vi) G: "I am going to get sick."

Sentences

(i) "It is snowing and I am outdoors; if I am not wearing a coat then I am freezing and I must run into the house or I am going to get sick."

$$((S \wedge O) \rightarrow ((\neg C) \rightarrow (F \wedge (H \vee G))))$$

(ii) "If it is snowing and I am outdoors and I am wearing a coat, I am not going to get sick if I do not run into the house."

$$(((S \wedge O) \wedge C) \ \rightarrow \ ((\neg H) \rightarrow (\neg G)))$$

(iii) "If I am outdoors and it is snowing, I can run into the house only if I am not sick."

$$((O \wedge S) \rightarrow (H \leftarrow (\neg G)))$$

5. Suppose that α is a wff not containing the negation symbol $\neg$. Show that the length of α (i.e., the number of symbols in the string) is odd.

We can prove that the length of α is odd by induction on the number of sentence symbols in α.

Base Case

Suppose α has a single sentence symbol. Recall that α may not have the negation symbol $\neg$. For α to contain a single sentence symbol and be a wff, it may not have any of the binary connectives $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$, as those would necessitate the inclusion of a second sentence symbol to retain the well-formulation. Therefore, the wff must have a length of 1 (since all connectives are disallowed).

Inductive Step

Let α_n be a wff containing n sentence symbols, and having arbitrary length l without loss of generality. Consider the addition of a new sentence symbol β to α_n to yield a new wff, α_{n+1}, containing $n+1$ sentence symbols.

(i) In the simplest case, β is adjoined to the end (or beginning) of α_n using a connective $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, and the resulting phrase is wrapped in parentheses, resulting in the wff $\alpha_{n+1} = (\alpha_n * \beta)$ which has two additional parentheses, one additional connective, and one additional sentence symbol. Therefore, α_{n+1} has length $l+4$.

(ii) In the more complicated case, β is embedded into a subwff γ of α_n. However, the length of γ changes by 4 as in case (a) above.

Since the length of α_1 is known (as shown in the base case), we can recursively define the length of α_n for any number of sentence symbols n.

$$\mathcal{L}(n) = \begin{cases} 1 & \text{if } n = 1 \\ \mathcal{L}(n-1) + 4 & \text{otherwise} \end{cases}$$

In unrolling the recursion, the length function becomes $\mathcal{L}(n) = 4(n-1)+1$ (or $\mathcal{L}(n+1) = 4n+1$). We see that:

(a) $\mathcal{L}(n)$ is odd for all $n \geq 1$, since we have the addition of 1 to $4(n-1)$ which is always even or zero.

(b) The ratio of sentence symbols in a wff $(n+1)$ to its length $(4n+1)$ is always greater than $1/4$, since

$$\frac{n+1}{4n+1} > \frac{n+1}{4n+4} = 1/4$$