

Shortest non-separating st-path on chordal graphs

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Abstract

Many NP-Hard problems on general graphs, such as maximum independence set, maximal cliques and graph coloring can be solved efficiently on chordal graphs. In this paper, we explore the problem of non-separating st-paths defined on edges: for a connected undirected graph and two vertices, a non-separating path is a path between the two vertices such that if we remove all the edges on the path, the graph remains connected. We show that on general graphs, checking the existence of non-separating st-paths is NP-Hard, but the same problem can be solved in linear time on chordal graphs. In the case that such path exists, we introduce an algorithm that finds the shortest non-separating st-path on a connected chordal graph of n vertices and m edges with positive edge lengths that runs in $O(n \log n + m)$ time.

1 Introduction

1.1 Overview

1.1.1 Non-separating defined on edges vs. vertices

In this paper, we study non-separating paths defined in the following way:

Definition 1.1 (Non-separating path (edge)). Let $G = \langle V, E \rangle$ be an undirected graph. A non-separating path is a path p in G such that after removing all edges on p . The remaining graph remains connected.

Traditionally, there have been studies on non-separating paths defined with vertex removal instead of edge removal.

Definition 1.2 (Non-separating path (vertex)). Let $G = \langle V, E \rangle$ be an undirected graph. A non-separating path is a path p in G such that after removing all *vertices* on p . The remaining graph remains connected.

For this definition, many studies on their relation to graph connectivity have been done [1, 2, 3, 4, 5] since it first gained major academic interest in 1975, when Lovász made a famous conjecture relating path removal to k -connectivity [6]. The first major work on related optimization problem was published in 2009 by B. Y. Wu and S. C. Chen, who showed that the optimization problem on general graphs is NP-hard in the strong sense [7]. Later in 2014, Wu showed an efficient $O(N \log N)$ time algorithm for the optimization problem on grid graphs [8].

In this paper we study the version defined on edges. We are only interested in the optimization problem. Like Wu's work in 2009 and 2014 which proved NP-hardness in general and showed an efficient algorithm on a special type of graphs, we will obtain an efficient algorithm for both the decision and the optimization problem on connected *chordal graphs* with positive edge lengths, and show that the decision problem with non-separating paths defined with edge removal is NP-hard on general graphs. For the rest of the paper, the term "non-separating path" refers to the edge version.

1.1.2 Chordal Graphs

Definition 1.3 (Chordal Graph). A chordal graph is a simple, undirected graph where all cycles with four or more vertices have at least one chords. A chord is an edge that is not part of the cycle but connects two vertices of the cycle.

Given a connected chordal graph $G = \langle V, E \rangle$ with positive lengths on edges and a pair of distinct vertices S and T , the decision problem is to decide whether a non-separating path exists between S and T , and the optimization problem, which is the main topic of this paper, is to calculate the shortest non-separating path between S and T if such paths exist.

Chordal graphs are a wide class of graphs with many subclasses, including interval graphs, split graphs, Ptolemaic graphs, and K-trees. Many studies have been done on chordal graphs in the past. The most famous technique for these graphs is the computation of perfect elimination ordering, which can be done in linear time by the famous algorithm by Rose, Leuker and Tarjan [9]. It can be used to compute the clique tree of a chordal graph [10]. The tree decomposition technique is another famous technique, which is due to Gavril's work showing that chordal graphs are intersection graphs of subtrees, and a representation of a chordal graph as an intersection of subtrees forms a tree decomposition of the graph [11].

A significance of chordal graphs is that on these types of graphs, many generally NP-hard problems can be efficiently solved. For example, the perfect elimination ordering can be used to compute the maximum

independent set, maximal cliques, graph coloring, etc., which are all examples of NP-hard problems on general graphs. The tree decomposition technique can be used to solve the routing algorithm for chordal graphs [12]. **As our work shows, the problem we study is another example of a generally NP-hard problem that can be efficiently solved on chordal graphs.** Our algorithm is not based on famous techniques such as perfect elimination ordering, clique tree or tree decomposition. Instead, it is based on a number of new observations and original sub-procedures related to our problem.

1.2 Our Results

Assume that G is a connected chordal graph, and S, T are distinct vertices in G . An edge is called a *bridge* if after removing such edge, G becomes disconnected.

Our main result shows that the optimization problem can be solved efficiently, and the majority of this paper deals with this optimization problem.

Theorem 1.4. *For a connected chordal graph $G = \langle V, E \rangle$ with non-negative weights on edges and a pair of vertices S and T , the shortest non-separating path between the S and T can be calculated in $O(T_{SPTD}(2|V|, 8|E|))$ time, where $T_{SPTD}(2|V|, 8|E|)$ is the asymptotic time to compute a single source shortest path tree in a **directed** graph with no more than $2|V|$ vertices and $8|E|$ edges.*

Since the single source shortest path tree in a directed graph with n vertices and m edges can be calculated in $O(n \log n + m)$ time using the famous Dijkstra's algorithm equipped with a Fibonacci heap [13, 14], our problem can be solved in $O(|V| \log |V| + |E|)$ time.

For the decision problem, we have the following result:

Theorem 1.5. *There exists a non-separating path from S to T if and only if S and T are not separated by a bridge. When such path exists, the path from S to T that contains the minimum number of edges is a non-separating path.*

This implies an $O(|V| + |E|)$ time algorithm for checking the existence of non-separating path by doing a simple breadth first graph traversal. Proof for Theorem 1.5 will be given in section 5.

We also show that the problem of deciding the existence of non-separating st-paths is NP-hard on general graphs.

Theorem 1.6. *On general graphs, deciding if non-separating st-paths exist is NP-Hard.*

1.3 Paper Organization

In this paper, we will first give our algorithm for the optimization problem on chordal graphs, where we will first give an overview of the algorithm, and then prove important key lemma and theorems that we use. With these we can complete the proof for Theorem 1.5. After that, we give details on the components of the main algorithm, and then give our final correctness proof. Finally we will show the NP-hardness of the existence problem on general graphs.

2 Definitions and Notations

Let $G = \langle V, E \rangle$ be a simple undirected chordal graph with edge lengths. For two vertices $u, v \in V$, u and v are *adjacent* if $u \neq v$ and u and v are connected by an edge. If u and v are adjacent, the length of the edge between them is denoted by $L(u, v)$.

Throughout the paper, we use 0-based indexing. A *path* p is defined to be a sequence of vertices $p_0 \rightarrow p_1 \rightarrow \dots \rightarrow p_{|p|-1}$ such that for $0 \leq i < |p| - 1$, the vertices p_i and p_{i+1} are adjacent in G . A vertex u is *on* p if $u = p_i$ for $0 \leq i < |p|$, and we say p *contains* u or *visits* u , and is denoted by $u \in p$. An edge e is *on* p if the edge is equal to the edge between p_i and p_{i+1} for $0 \leq i < |p| - 1$, and we say p *contains* e or *visits* e , and is denoted by $e \in p$. The *length* of p is the sum of the length of the edges on p .

The vertices p_0 and $p_{|p|-1}$ are called the *endpoints* of G , and we say p is a path *between* p_0 and $p_{|p|-1}$. A vertex on p other than an endpoints is called an *inner vertex* of p . For $0 \leq i < |p|$, define $\text{INDEX}(p, p_i) = i$. A path p is called *simple* if for $0 \leq i < j < |p|$, the vertices p_i and p_j are different. The *length* of p denoted by $\|p\|$ is defined by $\|p\| = \sum_{0 \leq i < |p|-1} L(p_i, p_{i+1})$. For $0 \leq i \leq j < |p|$, $p_{i,j}$ denotes the path $p_i \rightarrow p_{i+1} \rightarrow \dots \rightarrow p_j$. Two paths u and v are *different* if $|u| \neq |v|$ or $\exists 0 \leq i < |u|, u_i \neq v_i$. Let p be a path. The *reverse* of p is the path $p_{|p|-1} \rightarrow p_{|p|-2} \rightarrow \dots \rightarrow p_1 \rightarrow p_0$. Two paths u and v are called *distinct* if u and both v and the reverse of v are different. For two paths p_0 and p_1 , p_0 *contains* p_1 if and only if $\exists 0 \leq i \leq j < |p_0|, p_1 = (p_0)_{i,j}$, and we denote this by $p_1 \subset p_0$. Note that this is different from containment relationship between the sets of edges on these paths.

Let D be a set of edges in G . $G \setminus D$ denote the graph we get by removing all edges in D from G . If G is connected, a *separating* set of edges is a set of edges D such that $G \setminus D$ is disconnected. A *separating path* is a path such that the set of edges on the path is separating.

We will assume that there is no tie between lengths of paths, and at the end of the paper we will talk about how to break ties in practice.

3 Main Algorithm

Let $G = \langle V, E \rangle$ be a connected chordal graph, and let S, T be distinct vertices in V . Due to Theorem 1.5, we can remove all vertices that are un-reachable from S without visiting a bridge without affecting the answer to our problem. It is easy to see that the graph remains chordal and connected. Therefore, from now on we will suppose the graph does not contain any bridges.

A simple path r is called a *separator path* if and only if the path is separating and does not properly contain a separating path. For separator paths, we have the following theorem.

Theorem 3.1. *A simple path p is not separating if and only if the path does not contain a separator path.*

Since the answer must be a simple path, the answer is not separating if and only if it does not contain a separator path. A separator path r is called *traversable* if there exists a simple path from S to T that contains r . Obviously, we do not need to worry about non-traversable separator paths. We will be dealing with traversable separator path of length two separately. A vertex v is called a *bad vertex* if there exists a separator path r with length two such that $r_1 = v$ (i.e. v is the middle vertex). For bad vertices we have the following theorem.

Theorem 3.2. *If u is a bad vertex, then no non-separating simple path from S to T visits u .*

With this theorem, if we want to make sure the path does not contain any separator paths of length two. We only need to ensure we don't visit any bad vertices. What remains are separator paths with length more than two. We design a sub-procedure, $\text{AVOID}(X)$, where X is a set of separator paths of length more than two, that finds the shortest path from S to T in G that does not contain any element of X , **and does not contain a bad vertex**. Obviously, this shortest path must also be simple. As a shorthand, we say a path *avoids* X if the path does not contain any element of X , and does not contain a bad vertex. The sub-procedure runs

in $O(T_{\text{SPTD}}(2|V|, 8|E|))$ time. There is an important pre-requisite to this sub-procedure: any two separator paths in X must not share a common edge or a common inner vertex.

For separator paths, we have the following theorem:

Theorem 3.3. *Let r be a separator path. For any $0 \leq i < |r| - 2$, the vertex r_i and r_{i+2} are adjacent.*

Let r be a separator path. If for some $0 \leq i < |r| - 2$, $\|r_{i,i+2}\| > L(r_i, r_{i+2})$, then intuitively, one can simply get around r by going through the edge between r_i and r_{i+2} instead of going through $r_{i,i+2}$, and such separator path intuitively should never become an issue. r is called *useful* if for any $0 \leq i < |r| - 2$, $\|r_{i,i+2}\| < L(r_i, r_{i+2})$. A separator path r is called *normal* if it is both traversable and useful and contains more than two edges. We will formally prove in sub-section 7.1, that any separator path contained in a path that can possibly be produced by $\text{AVOID}(X)$ for some set of normal separator paths X is indeed useful, and therefore normal. Normal separator paths have two very important properties and the pre-requisite to the sub-procedure will be met:

Theorem 3.4. *Two different normal separator paths do not share a common edge.*

Theorem 3.5. *Two different normal separator paths do not share a common inner vertex.*

The framework of the algorithm is as follows: we first find a set X of normal separator paths such that one can guarantee that the shortest path that avoids X does not contain a normal separator path not in X . Thus $\text{AVOID}(X)$ will give us the answer directly. An obvious way is to let X be the set of all normal separator paths in G . However, such X will be very hard to compute. A good approach would be to gradually increment X : start from $X = \emptyset$ and keep calling $\text{AVOID}(X)$ and expanding X to include new unseen separator paths until we find the answer. This gives a simple quadratic time algorithm. However, we will introduce a more efficient approach. We will introduce some special types of normal separator paths that are easy to compute, and we claim that a X that can give us the answer directly can be found doing an extra-sub-procedure based on these separator paths.

Let P be the the shortest path from S to T that does not visit a bad vertex. Let r be a normal separator path r such that for any two vertices u and v shared by r and P , $\text{INDEX}(r, u) < \text{INDEX}(r, v)$ if and only if $\text{INDEX}(P, u) < \text{INDEX}(P, v)$. r is called an *S-separator path* if $r_{2,|r|-1}$ lies entirely on P , and r is called a *T-separator path* if $r_{0,|r|-3}$ lies entirely on P . Consider laying P flat with S on one side and T on the other. The edges on an *S-path* seperator can only be off P on the *S-side*, and those on a *T-separator path* can only be off P on the *T-side*. If a separator path is on P , it is both an *S-separator path* and a *T-separator path* per definition.

The framework of the computation for X is as follows: We first the set of bad vertices. We then find the shortest path P from S to T that does not contain any bad vertices. Then, we find the set X_{ST} consisting of the *S-separator paths* and the *T-separator paths*. We will show, in sub-section 6.2, that the bad vertices, the *S-separator paths* and *T-separator paths* can all be found in $O(|V| + |E|)$ time, by methods based on biconnected components. Now we will compute the set of extra separator paths needed in X . Let r be a separator path. The edge between r_0 and r_1 is called the *head* of r , denoted by $\text{HEAD}(r)$, and the edge between $r_{|r|-2}$ and $r_{|r|-1}$ is called the *tail* of r , denoted by $\text{TAIL}(r)$. We build a graph G_0 which is the maximal subgraph of G that does not contain either of the following:

- A bad vertex
- The tail of an *S-separator path*
- The head of a *T-separator path*

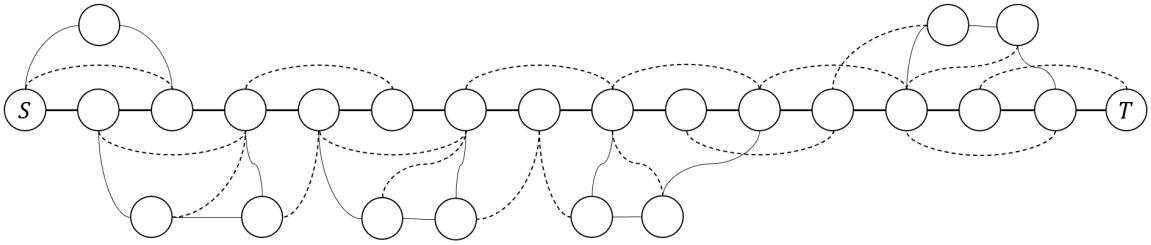


Figure 1: The chordal graph G .

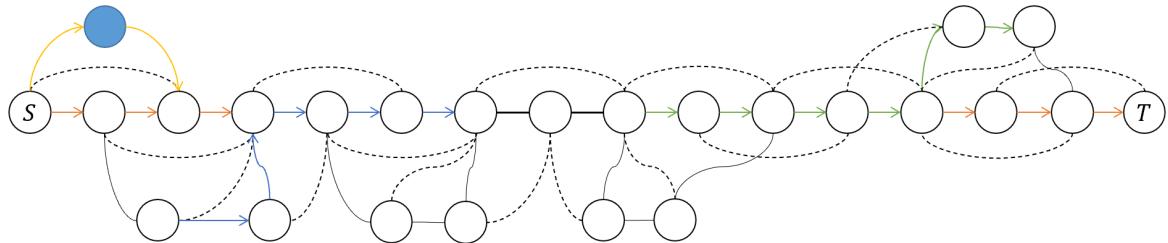


Figure 2: Bad vertices, path P and X_{ST} .

Let the shortest path from S to T in G_0 be P_0 . Then P_0 corresponds to a simple path in G . Let X_{EXTRA} be the set of normal separator paths in G that P_0 contains, which as we will show in sub-section 6.3 can be computed in $O(|V| + |E|)$ time. We set X to $X_{ST} \cup X_{\text{EXTRA}}$. We will prove, in sub-section 7.3, that the set X calculated in this way is such that the shortest path that avoids X does not contain a normal separator path not in X .

Here is an example that shows how the algorithm works:

Figure 2 shows a chordal graph G , where dashed edges have length 100 and other edges have length 1. The path P is the horizontal chain in bold from S to T in the middle.

As shown in Figure 2, the only bad vertex in the graph is filled in light blue, since it is the middle vertex of the separator path in yellow which is a traversable separator path of length 2. The path P is the horizontal chain from S to T in the middle. The two separator paths in orange are both S -separator paths and T -separator paths. The separator path in light blue is an S -separator path and the separator path in green is a T -separator path. X_{ST} contains the set of S -separator paths and T -separator paths (i.e. all separator paths shown except the one in yellow).

Figure 3 shows the graph G_0 , the maximal subgraph of G that does not contain a bad vertex, the tail of an S -separator path or the head of a T -separator path, and P_0 , the shortest path from S to T in G_0 in red.

Figure 4 shows the path P_0 in G in red, and it contains an extra normal separator path in yellow.

Figure 5 shows graph G , the bad vertex in light blue and the set X , the only separator path in X_{EXTRA} is the one in yellow and the other separator paths are in X_{ST} . The algorithm then proceeds to calculate $\text{AVOID}(X)$ which will give us the answer.

The pseudocode is given below, where $E(P)$ denotes the set of edges on P :

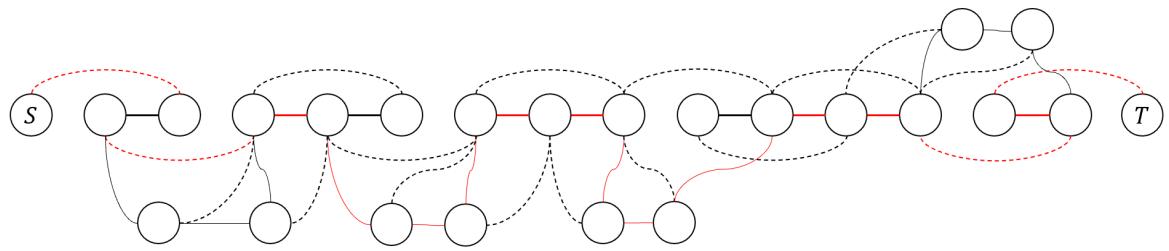


Figure 3: The graph G_0 and the path P_0 .

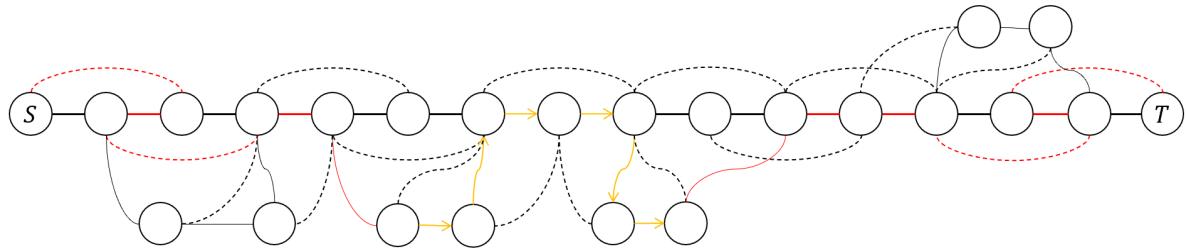


Figure 4: P_0 shown in G and X_{EXTRA} .

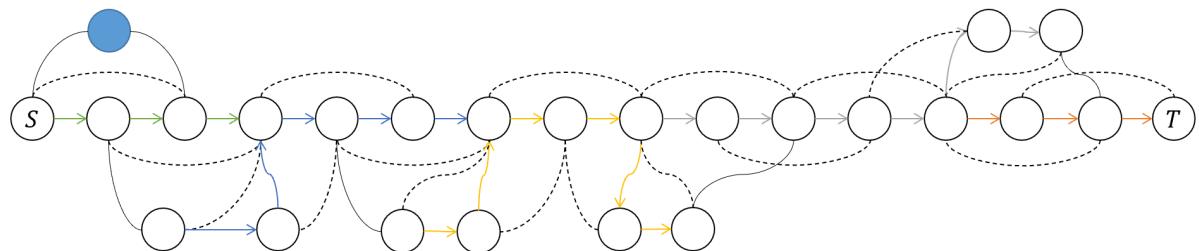


Figure 5: The graph G , bad vertices and the set X .

Algorithm 1 Main Algorithm

Require: Chordal Graph $G = \langle V, E \rangle$, $S, T \in V (S \neq T)$

Ensure: Shortest Non-Separating Path Between S and T

- 1: Calculate BAD_G , the set of bad vertices in G
 - 2: $G' = \langle V', E' \rangle \leftarrow$ the subgraph of G induced by $V \setminus \text{BAD}_G$
 - 3: $P \leftarrow$ the shortest path from S to T in G'
 - 4: $X_{ST} \leftarrow$ the set of S -separator paths and T -separator paths
 - 5: Let G_0 be the induced subgraph of $E' \setminus (\{\text{TAIL}(r) \mid r \text{ is an } S\text{-separator path}\}) \setminus (\{\text{HEAD}(r) \mid r \text{ is an } T\text{-separator path}\})$ of G' .
 - 6: $P_0 \leftarrow$ the shortest path from S to T in G_0
 - 7: $X_{\text{EXTRA}} \leftarrow$ the set of normal separator paths in G that P_0 contains
 - 8: $X \leftarrow X_{ST} \cup X_{\text{EXTRA}}$
 - 9: **return** $\text{AVOID}(X)$.
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4 Properties of Separator Paths and Proof

4.1 An Important Lemma and an Important Theorem

We have already seen Theorem 3.3, which states that for a separator path r , for any $0 \leq i < |r| - 2$, the vertex r_i and r_{i+2} are adjacent. In fact, more properties of separator paths exist.

Let p be a simple path. Two vertices u and v are called weakly p -connected if there exists a path between u and v that does not share a common edge with p . u and v are called strongly p -connected if there exists a path p' between u and v such that except at the endpoints, p' does not share a common vertex with p , and p' does not share a common edge with p . We have the following important lemma and theorem:

Lemma 4.1. *Let r be a separator path. For all $0 \leq i < |p| - 2$, let j be the minimum index such that $i < j < |p|$ and r_i and r_j are weakly r -connected, then $j = i + 2$. For $i \geq |p| - 2$, no such index exists.*

Theorem 4.2. *Let r be a separator path. For all $0 \leq i < j < |r|$, r_i and r_j are strongly r -connected if and only if $j - i = 2$.*

To help understand these properties of separator paths, we have a visualization for a separator path r with 6 vertices in Figure 6. Recall that due to Theorem 3.3, There is an edge between r_i and r_{i+2} for $0 \leq i < |r| - 2$. we can see that the relation of r to the graph is as follows. There are five components $C_{02}, C_{13}, C_{24}, C_{35}$. For component C_{uv} , all vertices in C_{uv} can not be strongly r -connected to any vertices on r except r_u and r_v , and. If a vertex is only strongly r -connected to a single r_x on r , add the vertex to an arbitrary component involving r_x . For example if y is strongly r -connected to r_2 only, y can be in either C_{02} or C_{24} . C_{uv} also include edges from vertices inside it to r_u and r_v , and the edge between r_u and r_v . Moreover, it is not hard to see that if r is traversable, $S \in C_{02}$ and $T \in C_{35}$. Moreover, there exists a path from S to r_0 which is entirely inside C_{02} and a path from r_5 to T which is entirely inside C_{35} . Note that S may or may not be strongly r -connected to r_2 and the same goes for T and r_3 . For any separator path in a connected chordal graph, one can draw a diagram like this.

Theorem 4.1 has many corollaries.

Corollary 4.3. *If r is a normal separator path, and p is a path from S to T . At least one inner vertex of r is on p*

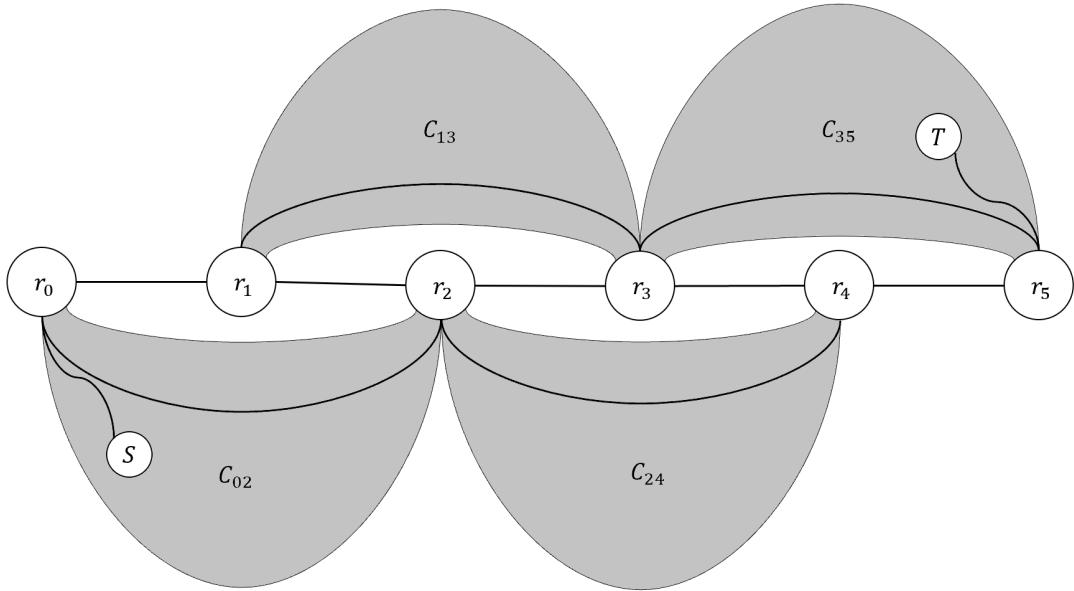


Figure 6: Separator path visualization.

Proof. if p does not visit an inner of r , then r_0 and $r_{|r|-1}$ are strongly r -connected. Since $|r| > 3$, this violates Theorem 4.2. \square

Corollary 4.4. Let r be a normal separator path, then r_0 and T are not strongly r -connected. $r_{|r|-1}$ and S are not strongly r -connected. Consequently S and T are not strongly r -connected. Consequently the reverse of r is not a normal separator path.

Proof. We know $r_{|r|-1}$ and T are strongly r -connected and $|r| > 3$, if r_0 and T are strongly r -connected then r_0 and $r_{|r|-1}$ are strongly T connected, which violates Theorem 4.2.

By the same reasoning, $r_{|r|-1}$ and S are not strongly r -connected. \square

Corollary 4.5. If r is a normal separator path, for all $2 \leq i < |r| - 1$, all paths from S to T must visit at least one of r_{i-1} and r_i .

Proof. Let p be a path from S to T . From Corollary 4.3 we know p must visit an inner vertex on r . Suppose neither r_{i-1} nor r_i is visited. If no inner vertex that p visits precedes r_{i-1} on r , then the first inner vertex that p visits is r_k for $k > 2$, but r_k and r_0 will be strongly r -connected and this violates Theorem 4.2. Similarly p must visit an inner vertex after r_i . Find the last r_u ($u < i - 1$) that p visits and the first r_v ($v > i$) that p visits, and r_u and r_v are strongly r -connected. Since $v - u > 2$, this violates Theorem 4.2. \square

Corollary 4.6. If r is a normal separator path, and p is a path from S to T . The first vertex on r that p visits is either r_0 or r_2 , and the last vertex on r that p visits is either $r_{|r|-3}$ or $r_{|r|-1}$.

Proof. If the first vertex on r that p visits is r_k where $k \neq 0$, then r_k and r_0 are strongly r -connected. From Theorem 4.2, this is only possible if $k = 2$. The other direction is similar. \square

Corollary 4.7. Let p be a simple path from S to T and let r be a normal separator path. For $0 < i < |r| - 1$ ($i \neq 2$), if $p_j = r_i$ ($j < |p| - 1$) and $p_{j+1} \notin \{r_{i-1}, r_{i+1}\}$, then for all $k > j$ and $l \leq i$, $p_k \neq r_l$.

Proof. If $p_{j+1} \notin \{r_{i-1}, r_{i+1}\}$, then the next visit of a vertex on r after r_i on p , if it exists, must be either r_{i-2} or r_{i+2} . In the former case, p hasn't visited r_{i-2} before visiting r_i . We have $i \geq 2$ and $i \neq 2$ and therefore $i > 2$. Since r_{i-2} is not visited before r_i , one can see that p must visit r_{i-1} before r_i . Now that we are currently at r_{i-2} and both r_i and r_{i-1} are already visited, it will be impossible for us to reach T again while the path remains simple. In the latter case, we have $i < |r| - 2$, and one can see that with r_i already visited, in order to visit some r_l ($l < i$) again, we need to visit r_{i+1} first. Now that we are at r_l , with r_i and r_{i+1} both visited and $i < |r| - 2$, it will be impossible for us to reach T again while the path remains simple. \square

4.2 Proof of Key Theorems

First of all, we want to prove Theorem 3.1, which is dependent on the following lemma.

Lemma 4.8. If D is a minimal separating set of edges, then the induced subgraph of D is connected.

Proof of Lemma 4.8. If $G \setminus D$ contains more than two connected components, we can take two connected components and the edges between them in G will be a separating proper subset of D , which is a contradiction. Therefore $G \setminus D$ contains exactly two connected components.

Let the two connected components of $G \setminus D$ be C_A and C_B . If D is disconnected, then we can find $u, x \in C_A$ ($u \neq x$) and $v, y \in C_B$ ($v \neq y$) such that u and v are adjacent in G , and so are v and y . Let p_{ux} be the path between u and x with the minimum amount of edges, and p_{vy} be such path between v and y . The cycle formed by p_{ux} , p_{vy} , the edge between u and v and the edge between x and y will form a simple cycle of length at least four without a chord, which contradicts the fact that G is chordal. \square

Proof of Theorem 3.1. If p contains a separator path then p is obviously separating.

If p is separating, let D be a minimally separating subset of the edges in p . If D corresponds to $p_{i,j}$ for some $0 \leq i < j < |p|$, then $p_{i,j}$ is a separator path. Otherwise D is disconnected, which is impossible due to Lemma 4.8. \square

Proof of Theorem 3.2. If r is a traversable separator of length 2 and a path p between S and T visits r_1 , we can show that p must contain the edge between r_0 and r_1 . Suppose it does not. If r_1 is the first or the last vertex on r that p visits, then r_1 and either S or T are strongly r -connected, and therefore r_1 and either r_0 or r_2 are strongly r -connected, violating Theorem 4.2. Otherwise, r_1 must be either the last vertex on r that p visits before r_1 or the first vertex on r that p visits after r_1 , and r_0 and r_2 are strongly r -connected, violating Theorem 4.2. By the same reasoning p must contain the edge between r_1 and r_2 . Therefore both edges on r will be contained in the path p and the path will not be non-separating. \square

Proof of Lemma 4.1. If $i = |p| - 1$ then no index greater than i exists.

If $i = |p| - 2$ and r_i and r_{i+1} are weakly r -connected, then since r is separating, $r_{0,|r|-2}$ is separating. Therefore r isn't minimal, which is a contradiction. Similarly, we can conclude that r_0 and r_1 are not weakly r -connected.

If $j - i > 2$ and r_i and r_j are weakly r connected but for all $i < k < j$, r_i and r_k are not weakly r -connected, r_i and r_j are strongly $r_{i,j}$ -connected. Consider the simple cycle consisting of $r_{i,j}$ and the path p with the minimum amount of edges between r_i and r_j that do not share a common vertex with $r_{i,j}$ except for p 's endpoints. This cycle is simple and has a length greater than three and therefore must have a chord.

By the way the path p is chosen, the chord can't have both endpoints on p , and therefore an endpoint of the chord must be r_k for some $i < k < j$. This makes r_i and r_k weakly r -connected, which is a contradiction.

If r_i and r_{i+1} are weakly r -connected for some $1 \leq i < |p| - 1$, find the maximum $a < i$ such that r_a and r_i are not weakly r -connected and the minimum $b > i$ such that r_b and r_i are not weakly r -connected. Since r_0 and r_1 , $r_{|r|-2}$ and $r_{|r|-1}$ are not weakly r -connected, both a and b exist. Let R be the set of edges on r . From the fact that R is a minimum separating set of edges, and from Lemma 4.8 we know $G \setminus R$ contains exactly two connected components. Therefore r_a and r_b are weakly r -connected, but since $b - a > 2$, this has been shown to be impossible. \square

Proof of Theorem 3.3. For $0 \leq i < |r| - 2$, from Lemma 4.1, r_i and r_{i+2} are strongly $r_{i,i+2}$ -connected. If r_i and r_{i+2} are not adjacent, consider the simple cycle consisting of $r_{i,i+2}$ and the path p with the minimum amount of edges between r_i and r_{i+2} that don't share a common vertex with $r_{i,i+2}$ except the endpoints. The cycle will have at least four edges. Using a similar reasoning as the one in the proof of Lemma 4.1, an endpoint of the chord is r_{i+1} . It is easy to see that the other endpoint will be on p and the chord makes r_{i+1} and r_i weakly r -connected, which is impossible from Lemma 4.1. \square

Proof of Theorem 4.2. For $0 \leq i < j < |r|$, if $j - i = 2$, then from Theorem 3.3, r_i and r_j are strongly r -connected.

Reversely, if r_i and r_j are strongly r -connected and therefore weakly r -connected, from Lemma 4.1 we can easily see that $j - i$ is even. Let i and j be such that there exists no $i \leq k < l \leq j$ such that $(i, j) \neq (k, l)$ and r_k and r_l are strongly r -connected.

If $j - i > 4$, consider the cycle consisting of the edge between r_i and r_{i+2} , the edge between r_{i+2} and r_{i+4}, \dots , the edge between r_{j-2} and r_j , and the path p with the minimum amount of edges between r_i and r_j that don't share a common vertex with r except for p 's endpoints. The cycle contains more than three edges and it is straightforward, from the similar reasoning we had in the proof of Lemma 4.1, that any chord of the cycle will imply the existence of $i \leq k < l \leq j$ such that $(i, j) \neq (k, l)$ and r_k and r_l are strongly r -connected.

If $j - i = 4$, consider the cycle consisting of the edge between r_i and r_{i+1} , the edge between r_{i+1} and r_{i+3} , the edge between r_{i+3} and r_{i+4} , and the path p with the minimum amount of edges between r_i and r_j that don't share a common vertex with r except for p 's endpoints. The cycle has more than three edges and we can see that any chord of the cycle will imply that either r_i and r_{i+3} are strongly r -connected or r_{i+1} and r_{i+4} are strongly r -connected, neither of which is impossible since both pairs of indices have different parities. \square

Proof of Theorem 3.4. Due to Corollary 4.4 it suffices to show that two distinct (disregarding direction) normal separators don't share a common edge. In fact, the theorem applies to two distinct useful separators.

Let u and v be distinct useful separator paths and u and v share a common edge. Let $0 \leq i < j < |u|$ be such that $u_{i,j}$ is entirely on v and $j - i$ is maximal. Since the set of edges u is not contained in that of v , either $i > 0$ or $j < |p| - 1$. Without loss of generality, assume $i > 0$ and that $\text{INDEX}(v, u_{i+1}) = \text{INDEX}(v, u_i) + 1$. Let $x = \text{INDEX}(v, u_i)$.

If $x = 0$, then $v_0 = u_i$ and $v_1 = u_{i+1}$. From Theorem 3.3 we know u_{i-1} and u_{i+1} are adjacent. If the edge is on v , then we violate Lemma 6.2; otherwise since the edge between u_{i-1} and u_i is not on v , v_0 and v_1 are strongly v -connected, which is a contradiction of Theorem 4.2.

If $x > 0$, then $v_x = u_i$ and $v_{x+1} = u_{i+1}$. If $v_{x-1} \notin u$, then v_{x-1} and v_{x+1} , v_{x-1} and v_x are both connected by an edge not on u , and therefore u_i and u_{i+1} are strongly u -connected, which is a contradiction of Theorem 4.2. \square

Proof of Theorem 3.5. Due to Corollary 4.4 it suffices to show this for two distinct normal separators. Let u and v be two distinct normal separators.

Let $u_i (1 \leq i < |u| - 1)$ and $v_j (1 \leq j < |v| - 1)$ be the same vertex. We argue that for some u_a and u_b such that $a \leq i - 2$ and $b \geq i + 2$, u_a and v_{j-1} are strongly u -connected and u_b and v_{j+1} are strongly u -connected. We know v_{j-1} and v_{j+1} are adjacent, and since u and v don't share a common edge due to Theorem 3.4, neither of v_{j-1} and v_{j+1} is on u (otherwise one can see that v_{j-1} or v_{j+1} and v_j are weakly v -connected, violating Lemma 4.1). Therefore, since u_a and v_{j-1} are strongly u -connected and u_b and v_{j+1} are strongly u -connected, u_a and u_b are strongly u -connected, but since $b - a \geq 4$, this violates Theorem 4.2.

By symmetry we will only show the existence of a . Let p be a path from S to T that contains v . If the part of p before v_j does not contain a vertex on u , then S and u_0 are strongly u -connected and S and v_{j-1} are strongly u -connected, and u_0 and v_{j-1} are strongly u -connected. Therefore, we can let $a = 0$. If $i = 1$, that would imply u_0 and u_1 are strongly u -connected, which is impossible from Theorem 4.2. Therefore $i > 1$ and $a \leq i - 2$.

If the last vertex on u that p visits before v_j is u_j , since from Theorem 3.4 u and v don't share a common edge, u_j and u_i are strongly u -connected. If $j = 0$, we can use the same reasoning above to deduce $i > 1$ and we can set $a = 0$. If $j > 0$, from Corollary 4.7 we know $i - j$ is -1, or, 1 or ≥ 2 . Since u_j and u_i are strongly u -connected, from Theorem 4.2 we know $|j - i| \neq 1$ and therefore $j = i - 2$. Therefore we can set $a = i - 2$.

□

5 Proof of Theorem 1.5

This short section completes the proof of Theorem 1.5, our result pertaining to the decision problem.

It suffices to show that the separator paths on the path that visits the minimum amount of edges have length one (i.e. bridges).

Suppose there is a separator path on this path with length more than one, then Theorem 3.3 shows that two non-adjacent vertices on the path are adjacent, this contradicts the fact that the path visits the minimum amount of edges. Therefore, all separator paths on the path that visits the minimum amount of edge are bridges.

6 Sub-procedures

In this section, we will detail on the sub-procedures required in the main algorithm.

6.1 Shortest Path That Avoids a Set of Normal Separator Paths

6.1.1 The Sub-procedure

The sub-procedure AVOID(X) takes a set X of normal separator paths in the chordal graph G , and computes the shortest path from S to T in G that avoids X . In order to do this, we build an auxiliary **directed** graph $G_1(X) = \langle V_1, E_1 \rangle$ that contains no more than two copies of each vertex in V . Specifically, both S and T only have one copy. Of course, we want to ensure that we don't visit any bad vertices. Therefore, G_1 does not contain any copies of bad vertices. It also contains no more than four copies of each edge in G , for either direction, yielding a total of no more than $2|V|$ vertices and $8|E|$ edges. The shortest path from the vertex

corresponding to S in G_1 to the vertex corresponding to T in G_1 will correspond to a shortest path from S to T in G that avoids X .

The motivation behind G_1 is as follows: suppose we are halfway on a path from S to T and we are at a vertex that is an inner vertex r_i of some normal separator path r . There are two possibilities: the part of the path before might already contain the entirety of $r_{0,i}$, which means that we do not want to cover the entirety of $r_{i,|r|-1}$, and we need to get off r . Or the path before might not contain the entirety of $r_{0,i}$. In this case if we assume our path is simple and will remain so, it would mean that we are “safe” and do not need to worry about containing r . Therefore for each inner vertex u of a normal separator path, the graph G_1 contains a *high vertex* $(u, 1)$, which indicates the first, unsafe case, and a *low vertex* $(u, 0)$, which indicates the second, safe case. Neither S nor T is an inner vertex of a normal separator path, so only low vertices of them exist.

In order to make this work, for any separator path r , the edge from $(r_0, 0)$ to $(r_1, 0)$ must not exist, so that we can’t get on r without entering the high vertex. The edge between $(r_{|r|-2}, 1)$ and $(r_{|r|-1}, 1)$ must not exist, since going through this edge would imply containing the entirety of r . Moreover, we are forbidden to go from $(r_i, 1)$ to $(r_{i+1}, 0)$ so that we can’t get off the high vertices without getting off r . To help simplify the formal proof in sub-section 7.3, we forbid the edges to the corresponding high vertices as well.

However, if we simply keep all the other edges, we still have a problem: since we now have two copies of a vertex. A simple path in the new graph does not necessarily correspond to a simple path in the original graph. For an inner vertex r_i on separator path r , one can go from the high vertex $(r_i, 1)$ to $(x, 0)$ for some $x \notin r$, and then go to the low vertex $(r_i, 0)$ from $(x, 0)$. We will introduce a fix. For $v \notin p$. Define $L(p, v)$ to be the p_i with the smallest index i such that p_i is strongly p -connected to v , and define $R(p, v)$ to be the p_i with the largest index i such that p_i is strongly p -connected to v . From Theorem 4.2, if r is a separator path, then for any r_i and $v \notin r$, at least one of $r_i = L(r, v)$ and $r_i = R(r, v)$ is true. In section 6.1.2 we will show that the values of $L(r, u)$ and $R(r, v)$ required in the procedure can be efficiently computed. For the moment, assume that these values are accessible. Consider Corollary 4.7, for most r_i , if we go from an inner vertex r_i to a vertex $x \notin r$ and want to keep our path (in the original graph) simple, then we can not ever visit a vertex on $r_{0,i}$. If $R(r, x) = r_i$ and we are at the high vertex for r_i , then we can not go to either copy of x and keep our path simple, and therefore we forbid both edges. Similarly, if $L(r, x) = r_i$, suppose we are at a copy of vertex x , then we must have either visited r_i before, or one can verify that we have visited r_{i+1} and r_{i+2} before. In the former case the path will not be simple, and in the latter case the path will not be able to reach T while the path remains simple. Therefore we forbid the edge from $(x, 0)$ or $(x, 1)$ (if it exists) to $(r_i, 0)$. This will forbid the scenario where one leaves the high vertex $(r_i, 1)$ and goes around a cycle that does not contain another vertex on r back to $(r_i, 0)$. Recall that for each vertex $u \neq r$, at least one of $L(r, u) = r_i$ and $R(r, u) = r_i$ is true. Therefore either the last edge on the cycle back to the low vertex r_i is forbidden, or the first edge on the cycle leaving the high vertex r_i is forbidden. To prevent similar exploits, for $i > 0$, we are also forbidden to go from $(r_{i+1}, 1)$ to $(r_i, 0)$ or $(r_{i-1}, 0)$.

There are two caveats. Firstly, it is possible to leave r_i and then immediately visit the head of another separator path. In this case, if we are going from r_i to $u = r'_1$ for another separator path r' , and $R(r, u) \neq r_i$, we go from $(r_i, 1)$ to $(u, 1)$ instead of $(u, 0)$. Secondly if $i = |r| - 3$ and $L(r, x) = r_i$, we might have gotten off r from $r_{|r|-1}$, which is not an inner vertex and Corollary 4.7 no longer applies. If this is the case, since we can not visit $r_{|r|-1}$ again, we will have to get off r again, and therefore we allow the edge from $(x, 0)$ to the high vertex $(r_i, 1)$. One can show that even with this edge, whenever we are at $(r_i, 1)$, all vertices on $r_{0,i}$ have been visited. A similar issue exists for $i = 2$. Fortunately if we get off r from r_2 and visits a vertex on $r_{0,2}$, the first vertex we visit will be r_0 . This implies that we must currently be on the low vertex of r_2 since otherwise r_0 is already visited. Our graph does not forbid going (directly or indirectly) from the low vertex of r_2 to r_0 and therefore our auxiliary graph still admits such paths. Currently, one might have concerns about

the correctness of this sub-procedure, especially because we can have multiple separator paths. Fortunately, Theorem 3.4 and Theorem 3.5 state that these separator paths do not share an edge of an inner vertex, and a formal proof of the sub-procedure will be given in sub-section 7.3 which makes use of these two theorems.

The pseudocode is given below. For a separator path r , the set of inner vertices is denoted by $\text{INNER}(r)$.

Algorithm 2 Computation of $\text{AVOID}(X)$, The Shortest Path Avoiding X

```

1: procedure  $\text{AVOID}(X)$ 
2:    $V_1 \leftarrow ((V \setminus \text{BAD}_G) \times \{0\}) \cup ((\cup_{r \in X} \text{INNER}(r)) \setminus \text{BAD}_G) \times \{1\})$ 
3:    $E_1 \leftarrow \emptyset$ 
4:   for ordered pair  $(u, v) \in V$  where  $u$  and  $v$  are connected by an edge  $e$  of length  $w$  in  $G$  do
5:     if  $u \notin \text{BAD}_G$  AND  $v \notin \text{BAD}_G$  then
6:       if  $\exists r \in X, ((u \notin r \text{ AND } v = r_{|r|-3}) \text{ OR } (u = r_0 \text{ AND } v = r_1))$  then
7:         Add an edge from  $(u, 0)$  to  $(v, 1)$  with length  $w$  to  $E_1$ 
8:       end if
9:       if NOT  $(\exists r \in X, (u, v) = (r_0, r_1)) \text{ OR } (\exists r \in X, u \notin r \text{ AND } v \in \text{INNER}(r) \text{ AND } L(r, u) = v)$  then
10:        Add an edge from  $(u, 0)$  to  $(v, 0)$  with length  $w$  to  $E_1$ 
11:       end if
12:       if  $(u, 1) \in V_1 \text{ AND } (v, 1) \in V_1 \text{ AND } \text{NOT } ((\exists r \in X, (u, v) = (r_{|r|-2}, r_{|r|-1})) \text{ OR } (\exists r \in X, 2 \leq i < |r|-1, (u, v) = (r_i, r_{i-2}) \text{ OR } (u, v) = (r_i, r_{i-1})) \text{ OR } (\exists r \in X, v \notin r \text{ AND } u \in \text{INNER}(r) \text{ AND } R(r, v) = u))$  then
13:         Add an edge from  $(u, 1)$  to  $(v, 1)$  with length  $w$  to  $E_1$ 
14:       end if
15:       if  $(u, 1) \in V_1 \text{ AND } \text{NOT } ((\exists r \in X, e \in r) \text{ OR } (\exists r \in X, 2 \leq i < |r|-1, (u, v) = (r_i, r_{i-2})) \text{ OR } (\exists r \in X, ((u \notin r \text{ AND } v \in \text{INNER}(r) \text{ AND } L(r, u) = v) \text{ OR } (v \notin r \text{ AND } u \in \text{INNER}(r) \text{ AND } R(r, v) = u)))$  then
16:         Add an edge from  $(u, 1)$  to  $(v, 0)$  with length  $w$  to  $E_1$ 
17:       end if
18:     end if
19:   end for
20:    $p \leftarrow$  the shortest path from  $(S, 0)$  to  $(T, 0)$  in  $G_1(X) = \langle V_1, E_1 \rangle$ 
21:   for  $0 \leq i < |p|$  do
22:     Replace  $p_i$  with the vertex in  $G$  that  $p_i$  corresponds to
23:   end for
24:   return  $p$ 
25: end procedure

```

6.1.2 Calculation of $L(r, u)$ and $R(r, v)$

We show a way to calculate $L(r, u)$ for all queries involved in the algorithm. The queries for $R(r, v)$ can be calculated similarly.

We notice that every time we want to calculate $L(r, u)$, there must be a vertex $v \in \text{INNER}(r)$ that is adjacent to u . Let $v = r_i$. From Theorem 4.2, the answer can only be r_i or r_{i-2} . All that we need to do is to check if r_{i-2} is strongly r -connected to u . Our problem becomes, given r_i , a vertex u adjacent to r_i , decide whether u is strongly r -connected to r_{i-2} .

Firstly, if $|X| = 1$ and the only element in X is the separator path r . Then let G' be the graph after we removing all edges on r from G . One can see, from Theorem 4.2 and Theorem 3.3, that vertices r_k and r_l share a biconnected component if and only if $|k - l| = 2$. From this we can see that given there is an edge between u and r_i , u is strongly r -connected to r_{i-2} if and only if u share a biconnected component with both r_i and r_{i-2} . This can be done by checking if the biconnected component containing the edge between r_i and u contains r_{i-2} .

For the case where $|X| > 1$, calling this sub-procedure for all $r \in X$ will be too costly. We want to calculate the values for all $r \in X$ at once. We let G' be the graph we get after removing from G all edges on some separator path in X . Still for $r \in X$ vertices r_k and r_l share a biconnected component if and only if $|k - l| = 2$. Our problem is that if u is r strongly-connected to r_{i-2} , it is no longer obvious whether u share a biconnected component with both r_i and r_{i-2} in G' , since some edges not on r are removed. Fortunately, in this case, we have the following Theorem:

Theorem 6.1. *If u is strongly r -connected to r_{i-2} and there exists an edge between u and r_i in G' , then u , r_i and r_{i-2} share a biconnected component in G' .*

To prove Theorem 6.1, we first introduce the following lemma:

Lemma 6.2. *If r and q are useful separator paths, then for $0 \leq i < |r| - 2$, the edge between r_i and r_{i+2} (which exists due to Theorem 3.3) is not on q .*

Proof of Lemma 6.2. If the edge between r_i and r_{i+2} belongs to another useful separator path q . Then q must contain either the edge between r_i and r_{i+1} or the edge between r_{i+1} and r_{i+2} . Otherwise two adjacent vertices on q are strongly q -connected, violating Theorem 4.2. Without loss of generality, suppose q contains the edge between r_i and r_{i+1} and $r_{i+1} = q_j$, $r_i = q_{j+1}$ and $r_{i+2} = q_{j+2}$, then since r is useful:

$$\begin{aligned} L(q_j, q_{j+2}) &= L(r_{i+1}, r_{i+2}) \\ &< L(r_{i+1}, r_{i+2}) + L(r_{i+1}, r_i) \\ &< L(r_i, r_{i+2}) \\ &< L(q_{j+1}, q_{j+2}) \\ &< L(q_{j+1}, q_{j+2}) + L(q_{j+1}, q_j) \end{aligned}$$

which contradicts the fact that q is useful. \square

Proof of Theorem 6.1. Since r_i and r_{i-2} are adjacent due to Theorem 3.3, it suffices if we can show there is a simple path between u and r_{i-2} that does not visit another vertex in r . Consider a simple path p in G (not G') between u and r_{i-2} that does not visit another vertex in r . We show that we can adjust p so that p does not visit any edge on a separator path in X , and then p will exist in G' . For $r' \in X$, if p does not visit any

vertex on r' , p obviously does not visit an edge on r' . Otherwise, let the first vertex on r' that p visits be $(r')_u$ and the last vertex be $(r')_v$. Since the edge between u and r_i is not on r' and from Theorem 3.4, no edges on $r_{i-2,i}$ are on r' , $(r')_u$ and $(r')_v$ are weakly (r') -connected. It follows from Lemma 4.1 that $|u - v|$ is even. From Theorem 3.3 and Lemma 6.2 we can get a path between $(r')_u$ and $(r')_v$ that is not on any separator path in X using the edges between $(r')_k$ and $(r')_{k+2}$ ($\min\{u, v\} \leq k < \max\{u, v\}$ and $k - u$ is even). Therefore we can replace the part of p from $(r')_i$ to $(r')_j$ with this path. It suffices to do the adjustment for all $r' \in X$. \square

Now, if the edge between u and r_i exists in G' , we can readily check if u and r_{i-2} are strongly r -connected. Otherwise, there must be some $r' \in X$ where $r_i = (r')_j$ and $u = (r')_k$ where $|k - j| = 1$. Due to theorem 3.5, $j = 0$ or $j = |r'| - 1$. Without loss of generality let $j = 0$, and then $k = 1$. Consider $(r')_2$. If $(r')_2 \notin r$, then u and r_{i-2} are strongly r -connected if and only if $(r')_2$ and r_{i-2} are strongly r -connected. Since $r_i = (r')_0$ and $(r')_2$ are adjacent and by Lemma 6.2 the edge between them is in G' , we can apply 6.1. If $(r')_2 \in r$, then u and r_{i-2} are strongly r -connected if and only if $(r')_2$ and both r_i and r_{i-2} are strongly connected, which is only possible if $(r')_2 = r_{i-2}$.

6.2 Computation of Bad Vertices, S -Separators Paths and T -Separators Paths

In this sub-section, we will introduce procedures that calculate all the bad vertices, and the S -separator paths and T -separator paths given P , the shortest path between S and T in G' . Both procedures will be based on the calculation of the *block-cut tree*, a data structure based on biconnected components of undirected graphs, which can be done in $O(n + m)$ time for a graph with n vertices and m edges due to the famous algorithm by John Hopcroft and Robert Tarjan [15].

6.2.1 Computation of Bad Vertices

Per definition, a bad vertex is the middle point of a traversable separator path of length 2. If r is a separator path of length 2. Then from Theorem 4.2 one can see that r_1 is an articulation vertex of G , and r_0, r_1 and r_2 share the same biconnected component. Moreover, if x is a vertex in a biconnected component C such that the degree of x within C is two. Then the two edges associated with x in C constitute a separator path of length 2. x will be bad vertex as long as r is traversable.

Consider the block-cut tree τ of G . Let S_τ be the vertex corresponding S itself if S is an articulation point and to the biconnected component S is in otherwise, and let T_τ be that for T . Then one can easily verify that r is traversable if and only if C is on the path from S_τ to T_τ on τ . Therefore, BAD_G can be computed using the following sub-procedure:

Algorithm 3 Computation of BAD_G

Require: Chordal Graph $G = \langle V, E \rangle$, $S, T \in V (S \neq T)$

Ensure: BAD_G , the set of bad vertices

```
1: Build the block-cut tree  $\tau$  of  $G$ .
2: if  $S$  is an articulation vertex then
3:    $S_\tau \leftarrow$  the vertex corresponding to  $S$  on  $\tau$ 
4: else
5:    $S_\tau \leftarrow$  the vertex corresponding to the biconnected component  $S$  is in on  $\tau$ 
6: end if
7: if  $T$  is an articulation vertex then
8:    $T_\tau \leftarrow$  the vertex corresponding to  $T$  on  $\tau$ 
9: else
10:   $T_\tau \leftarrow$  the vertex corresponding to the biconnected component  $T$  is in on  $\tau$ 
11: end if
12:  $\text{BAD}_G \leftarrow \emptyset$ 
13: for Vertex  $v$  between  $S_\tau$  and  $T_\tau$  on  $\tau$  that corresponds to a biconnected component do
14:    $C \leftarrow$  the biconnected component  $v$  corresponds to
15:   for  $u \in C$  do
16:     if  $u$  has a degree of exactly 2 in  $C$  then
17:        $\text{BAD}_G \leftarrow \text{BAD}_G \cap \{u\}$ 
18:     end if
19:   end for
20: end for
21: return  $\text{BAD}_G$ 
```

6.2.2 Computation of S -Separator Paths and T -Separator Paths on P

Since S -separator paths and T -separator paths are symmetric, it suffices to design an algorithm that calculates S -separator paths. We can split S -separator paths into two categories by the parity of the index of r_2 on P . An S -separator path is called *even* if $\text{INDEX}(P, r_2)$ is even, and *odd* if $\text{INDEX}(P, r_2)$ is odd. We will calculate all the odd S -separator paths, and the even ones can be calculated similarly.

In the previous section, we were able to pick out a length-2 separator path since all edges on it are associated with its midpoint. An odd S -separator path can contain more than 3 vertices and no single vertex is associated with all the edges on the path. Fortunately, we can design a way to merge r_1, r_3, \dots into a fat vertex without including any of r_0, r_2, \dots , so that all edges on the path will be associated with the fat vertex. Here by merging u, v into a fat vertex we mean replacing u and v with a new vertex w such that for every edge between u and x or v and x ($x \notin \{u, v\}$) before merging, there is an edge between w and x with the same length as that edge after merging. We also use a shorthand: if we merge a and b into a fat vertex x first and merge b and c thereafter, we actually merge x and c .

To merge r_1, r_3, \dots into a fat vertex, we first go through $0 \leq i < |P| - 2$ where i is even, and merge P_i with P_{i+2} if they are adjacent in G . This will make sure that r_3, r_5, \dots are merged into a fat vertex, since all these vertices are on P with even indices and there are edges inter-connecting them due to Theorem 3.3, and none of r_0, r_2, \dots will be merged into this fat vertex due to Theorem 4.2.

It remains to merge r_1 and r_3 . If $r_1 \in P$, then since $\text{INDEX}(P, r_1) < \text{INDEX}(P, r_2)$ per definition, the only way to make sure that r_1 and r_2 are not strongly r connected is for $r_{1,2}$ to be on P and therefore $\text{INDEX}(P, r_1) = \text{INDEX}(P, r_2) - 1$ which is even. This would mean that r_1 is already merged into the fat vertex. If $r_1 \notin P$, since r_1 and r_3 are adjacent and $\text{INDEX}(r_3)$ is odd, a way we can do this is to merge all $u \notin P$ with $v \in P$ where u and v are adjacent and $\text{INDEX}(v)$ is odd. This will not merge r_1 with r_2 since $\text{INDEX}(r_2)$ is even. However, this method can merge r_1 with r_0 when $r_0 \in P$ and $\text{INDEX}(P, r_0)$ is odd. To prevent this, note that if $r_0 \in P$ and we let $j = \text{INDEX}(r_0)$, then for no $k < j$ can P_k and r_1 be adjacent — otherwise r_1 and r_0 will be strongly r -connected. Therefore, we do not merge u with v if v has the smallest index among all $v \in P$ adjacent to u . This will not prevent r_1 from merging with r_3 since $\text{INDEX}(P, r_2) < \text{INDEX}(P, r_3)$ and r_2 is adjacent to r_1 .

Now that the edges on the odd S -separator path are associated with a single fat vertex, we can pick out these separator paths by checking for each biconnected component, if the set of edges associated with each vertex inside that biconnected component corresponds to a normal separator path. Our sub-procedure will be as follows:

Algorithm 4 Computation of Odd S -Separators Paths

Require: Chordal Graph $G = \langle V, E \rangle$, $S, T \in V (S \neq T)$, path P from S to T

Ensure: X_{SODD} , the set of odd S -separator paths

```

1: TO_MERGE  $\leftarrow \emptyset$ 
2: for  $0 \leq i < |P| - 2$  do
3:   if  $i$  is odd then
4:     if  $P_i$  and  $P_{i+2}$  are adjacent then
5:       Add  $(P_i, P_{i+2})$  to TO_MERGE.
6:     end if
7:   end if
8: end for
9: for Vertex  $u \notin P$  do
10:   MIN_INDEX  $\leftarrow \infty$ 
11:   for Vertex  $v$  such that  $u$  and  $v$  are adjacent do
12:     if  $v \in P$  then
13:       MIN_INDEX  $\leftarrow \min\{\text{MIN\_INDEX}, \text{INDEX}(v, P)\}$ 
14:     end if
15:   end for
16:   for Vertex  $v$  such that  $u$  and  $v$  are adjacent do
17:     if  $v \in P$  AND  $\text{INDEX}(P, v) \neq \text{MIN\_INDEX}$  then
18:       Add  $(u, v)$  to TO_MERGE.
19:     end if
20:   end for
21: end for
22:  $G_{\text{MERGE}} \leftarrow G$ 
23: for  $(u, v) \in \text{TO\_MERGE}$  do
24:   Merge  $u$  and  $v$  in  $G_{\text{MERGE}}$ .
25: end for
26: Build the block-cut tree  $\tau$  of  $G_{\text{MERGE}}$ .
27:  $X_{\text{SODD}} \leftarrow \emptyset$ 
28: for Vertex  $v$  on  $\tau$  that corresponds to a biconnected component do
29:    $C \leftarrow$  the biconnected component  $v$  corresponds to
30:   for  $u \in C$  do
31:      $R \leftarrow$  edges associated with  $u$  inside  $C$ 
32:     if  $R$  corresponds to an odd  $S$ -separator path  $r$  in  $G$  then
33:        $X_{\text{SODD}} \leftarrow X_{\text{SODD}} \cap \{r\}$ 
34:     end if
35:   end for
36: end for
37: return  $X_{\text{SODD}}$ 

```

▷ Including cut vertices

We still need to check if a set of edges R in G corresponds to an odd S -separator path in $O(|R|)$ time. Firstly, we check if R corresponds to a path. It is not hard to see that R is a minimal separating set of edges, so if R corresponds to a path, the path is to a separator path r . Secondly, one can determine the direction of the path based on the indices of the vertices r shared with p — recall that for any two vertices u and v shared by r and P , $\text{INDEX}(r, u) < \text{INDEX}(r, v)$ if and only if $\text{INDEX}(P, u) < \text{INDEX}(P, v)$. If the constraint on indices can't be satisfied in either direction, then by definition r is not an S -separator path. After r has been fully determined, we can then check whether $r_{2,|r|-1}$ is on P . Thirdly, we check for the usefulness constraint per definition, and finally we need to check for the traversability constraint, we have the following theorem.

Theorem 6.3. *If r is a useful separator path such that $r_{2,|r|-1}$ is on P and for any two vertices u and v shared by r and P , $\text{INDEX}(r, u) < \text{INDEX}(r, v)$ if and only if $\text{INDEX}(P, u) < \text{INDEX}(P, v)$. Then let $j = \text{INDEX}(P, r_2)$ and r is traversable if and only if r_0 is equal to or adjacent to P_k for some $k < j$.*

Proof of Theorem 6.3. If r_0 is equal to or adjacent to P_k for some $k < j$, we can obviously find a path from S to T that contains r , and therefore r is traversable.

If r is traversable and therefore normal, then if $r_0 \in P$ the theorem is obviously true. If $r_0 \notin P$, then from Corollary 4.6 r_2 is the first vertex on r on P and $r_1 \notin P$ from the constraint on indices, and the indices of the vertices of r on P form an interval. Since r is traversable, there exists a path from S to r_0 that does not visit a vertex on r before r_0 . If the path last leaves P at P_l , then $l < j (= \text{INDEX}(P, r_2))$ since otherwise $l > \text{INDEX}(P, r_{|r|-1})$ and r_0 and $r_{|r|-1}$ will be strongly r -connected, contradicting $|r| \geq 3$. Now, let k be the largest index such that $k < j$ and there exists a path between P_k and r_0 that does not visit a vertex on r . Find the shortest of such path p . If p contains at least two edges, then consider the simple cycle consisting of p , $P_{k,j}$ and the edge between r_2 and r_0 . This cycle has a length of at least four edges and one can verify that it is impossible for any chord to exist, which contradicts the fact that G is chordal. If p contains only one edge, then P_k is adjacent to r_0 . \square

With Theorem 6.3, we can pre-compute for each $u \notin P$, the minimum k such that P_k is adjacent to u . With this we can check in $O(|R|)$ time whether R corresponds to an odd S -separator path. Since the total number of edges ever involved in R in the sub-procedure is $O(|E|)$. The entire sub-procedure runs in $O(|V| + |E|)$ time.

However, there are still two issues left to address. Firstly, if we merge vertices using typical disjoint-set data structures, it would take $O(\alpha(|V|))$ time asymptotically per action, yielding a total time complexity of $O(|V| + |E|\alpha(|V|))$. We can improve this to $O(|V| + |E|)$: we can move all the actions offline by creating a graph representing all the merges, and a fat vertex will correspond to a connected component of the graph.

Secondly, checking whether a separator path is useful involves looking up the length of edges between some given pairs of vertices. Although these look-ups can easily be done in $O(1)$ per query if the graph is stored with an adjacency matrix, for many other ways one stores a graph (e.g. linked lists), the most obvious way of doing these operations in $O(1)$ time would be to use a hash table¹, which is somewhat awkward since hash tables inherently introduce randomness. An alternative way to hashing is by doing the look-ups offline. We can use a bucket for each vertex and store the queries into the bucket for either of the associated vertex. After that we can deal with each bucket alone. The look-ups for edges associated with a given vertex can be done in $O(1)$ time per query by using a 1-D array of size $|V|$. All the look-ups can therefore be done in $O(|V| + Q)$ time offline where Q is the number of queries, without the need for a hash table.

¹Assuming the RAM model.

6.3 Computation of Normal Separator Paths Contained in a Simple Path

During the main procedure we find a path P_0 from S to T in G_0 and we want to find all separator paths contained in P_0 .

The sub-procedure will work as follows: firstly, let G_3 be the graph with all the edges on P_0 removed. We then find all the connected components in G_3 . Then we have the following theorem:

Theorem 6.4. *If $(P_0)_{i,j}$ ($0 \leq i < j < |P_0|$) is a separator path, then in G_3 all the vertices with odd indices on $(P_0)_{i,j}$ belong to a connected component C_o , and all the vertices with even indices on $(P_0)_{i,j}$ belong to a separate connected component C_e . Reversely, for $0 \leq i < j < |P_0|$, if in G_3 all the vertices with odd indices on $(P_0)_{i,j}$ belong to a connected component C_o , and all the vertices with even indices on $(P_0)_{i,j}$ belong to a separate connected component C_e , and to no $(P_0)_{x,y}$ ($0 \leq x < y < |P_0|$) where $(P_0)_{i,j} \subset (P_0)_{x,y}$ such condition applies (i.e. $(P_0)_{i,j}$ is maximal), then $(P_0)_{i,j}$ is a separator path.*

Proof of Theorem 6.4. Firstly, if $(P_0)_{i,j}$ is a separator path, then straightforwardly from Theorem 4.2 we know all the vertices with odd indices belong to the same connected component in G_3 and so do the ones with even indices. The two connected components are different since otherwise $(P_0)_{i,j}$ is not separating.

Secondly, if in G_3 all the vertices with odd indices on $(P_0)_{i,j}$ belong to a connected component C_o , and all the vertices with even indices on $(P_0)_{i,j}$ belong to a separate connected component C_e , and $(P_0)_{i,j}$ is not a separator path. Then consider a minimal subset D of edges on P_0 such that after removing D from G , vertices in C_o and vertices in C_e are disconnected. From Lemma 4.8 we know there must be $0 \leq k < l < |P_0|$ such that D is the set of edges on $(P_0)_{k,l}$. If $(P_0)_{i,j} \not\subset (P_0)_{k,l}$, then a edge not in D will be between C_o and C_e and removing D from G will not disconnect C_o and C_e , which is a contradiction. Therefore $(P_0)_{i,j} \subset (P_0)_{k,l}$. \square

With Theorem 6.4 we can design an algorithm based on the *two-pointer* technique. We iterate through the index j in the increasing order and maintain the minimum i where $(P_0)_{i,j}$ satisfies the condition in the theorem. Every time we need to change i , $(P_0)_{i,j-1}$ is maximal and is a separator path. If the minimum i for $j = |P_0| - 1$ is less than j , $(P_0)_{i,j}$ is maximal and is a separator path. To make sure the separator path is normal, we check if it is useful. The pseudocode is as follows:

Algorithm 5 Computation Normal Separator Paths on a Simple Path

Require: Chordal Graph $G = \langle V, E \rangle$, a simple path P_0

Ensure: X_{EXTRA} , the set of separator paths contained in P_0 .

```
1:  $i \leftarrow 0$ 
2:  $G_3 \leftarrow G$  with edges on  $P_0$  removed.
3: for  $0 \leq i < |P_0|$  do
4:    $\text{BEL}_i \leftarrow$  the connected component  $(P_0)_i$  belongs to in  $G_3$ 
5: end for
6:  $X_{\text{EXTRA}} \leftarrow \emptyset$ 
7: for  $j$  from 1 to  $|P_0| - 1$  do
8:   if  $j > 1$  AND  $\text{BEL}_j \neq \text{BEL}_{j-2}$  then
9:     if  $(i < j - 1)$  AND  $((P_0)_{i,j-1}$  is useful) then
10:       $X_{\text{EXTRA}} \leftarrow X_{\text{EXTRA}} \cup \{(P_0)_{i,j-1}\}$ 
11:    end if
12:     $i \leftarrow j - 1$ 
13:  end if
14:  if  $\text{BEL}_j = \text{BEL}_{j-1}$  then
15:     $i \leftarrow j$      $\triangleright$  It is not hard to see that  $i$  always equals  $j - 1$  before this line, and no new separator
       path is found
16:  end if
17: end for
18: if  $i < |P_0| - 1$  then
19:    $X_{\text{EXTRA}} \leftarrow X_{\text{EXTRA}} \cup \{(P_0)_{i,|P_0|-1}\}$ 
20: end if
21: return  $X_{\text{EXTRA}}$ 
```

7 Correctness

7.1 Correctness of Normal Separator Paths

We will show that any separator paths contained in a path P that can be produced by $\text{AVOID}(X)$ for some X is indeed useful, and therefore normal.

Lemma 7.1. *Let p be a simple path and let $0 \leq i < j < |p|$ be such that $j - i > 1$ and p_i and p_j are adjacent, let p' be the path we get from p after replacing $p_{i,j}$ by the edge between p_i and p_j . For all $u, v \in V$, if u and v are weakly p -connected, then u and v are weakly p' -connected.*

Proof. For $u, v \in V$ such that u and v are weakly p -connected., for any path between u and v that do not share an edge with p , if the path goes through the edge between p_i and p_j , replace that edge with $p_{i,j}$ and we find a path between u and v that do not share an edge with p' . Therefore u and v are weakly p' -connected. \square

Theorem 7.2. *If r is a separator path that is not useful, then for all simple path p such that $r \subset p$, there exists another simple path p' of shorter length such that for all separator path $r' \subset p'$, $r' \subset p$.*

Proof. If $\|r_{i,i+2}\| > L(r_i, r_{i+2})$ for some $0 \leq i < |r| - 2$, let p' be p after replacing the $r_{i,i+2}$ with the edge between r_i and r_{i+2} . p' will have a shorter length. For separator path $r' \subset p'$, if $r' \notin p$, suppose the edge between $(r')_i$ and $(r')_{i+1}$ is not on p . $(r')_i$ and $(r')_{i+1}$ are not weakly r' -connected according to Lemma 4.1, and therefore not weakly p' connected, but are weakly p -connected. This is a contradiction of Lemma 7.1. \square

Therefore, if P contains a separator path that is not useful, from Theorem 7.2 we know we can make P shorter while still avoiding X , which contradicts the fact that P is the shortest.

7.2 Correctness of the Set X

In this sub-section, we show that the set X computed in the main algorithm is indeed such that the shortest path that avoids X does not contain a new normal separator path not in X .

Let P_{AVOID} be the path produced by $\text{AVOID}(X)$. Let X_n be the set of normal separator paths that P_{AVOID} contains. Obviously, $X \cap X_n = \emptyset$. Our goal is to show that $X_n = \emptyset$. Now, suppose we have a hypothetical $n \in X_n$, and we want to prove such n actually does not exist and therefore conclude $X_n = \emptyset$. If P_i is an inner vertex of n , we show that we have two vertices $a = P_j$ and $b = P_k$, where $j < i < k$, such that we have the following four conditions:

1. For any $i < x < j$, if P_x is the inner vertex of an S -separator path s' , then $(s')_{|s'|-2} \notin P_{\text{AVOID}}$; if P_x is the inner vertex of an T -separator path t' , then $t'_1 \notin P_{\text{AVOID}}$
2. No inner vertex of a separator path $r \in X_{\text{EXTRA}}$ is contained in $P_{i,j}$
3. $a = s_{|s|-3}$ for an S -separator path s , and $b = t_2$ for a T -separator path t .
4. $a, b \in P_{\text{AVOID}}$, a is the last inner vertex of s that P_{AVOID} visits, and b is the first inner vertex of t that P_{AVOID} visits.

We show that the indices of the inner vertices of a S -separator path or a T -separator path on P must be an interval. Note that if r is an S -separator path and $r_1 \in P$, r_1 must be adjacent to r_2 on P since otherwise r_1 and r_2 are strongly r -connected. Similar reasoning applies for T -separator paths. We can define an order

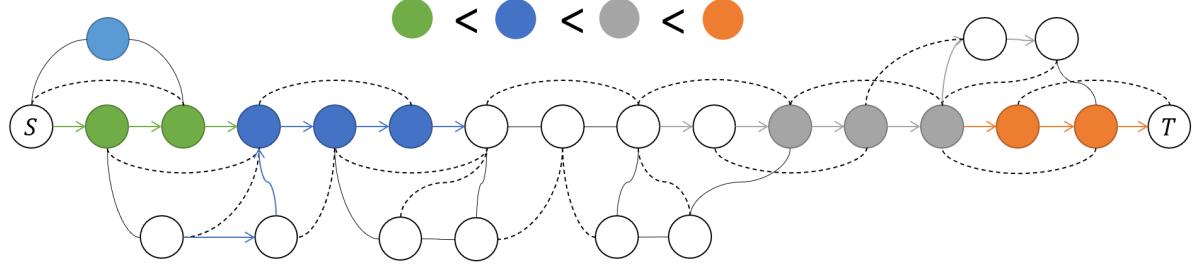


Figure 7: Intervals for X_{ST}

on the set X_{ST} according to the order their corresponding intervals appear on P . For distinct $x, y \in X_{ST}$, $x < y$ if the interval for x appears closer to the S -side than the interval for y , that is, $\forall P_u \in \text{INNER}(x)$ and $\forall P_v \in \text{INNER}(y)$, $u < v$. Figure 7 shows the intervals and the ordering in the example graph we have shown before.

We have the following lemma:

Lemma 7.3. *Let p be a simple path from S to T .*

If r is a normal separator path such that the indices of the set of inner vertices of r on P is an interval, and P_u is an inner vertex of r on P , then:

- After p visits the an inner vertex of r for the last time, p does not visit an inner vertex of any normal separator path x such that there exists $P_k \in \text{INNER}(x)$ (i.e. an inner vertex of x on P) such that $k < u$.
- Before p visits an inner vertex of r for the first time, p does not visit an inner vertex of any normal separator path x such that there exists $P_k \in \text{INNER}(x)$ such that $k > u$.

Consequently, if $r \in X_{ST}$:

- After p visits an inner vertex of r for the last time, p does not visit an inner vertex of any separator path $x \in X_{ST}$ where $x < r$.
- Before p visits an inner vertex of r for the first time, p does not visit an inner vertex of any separator path $x \in X_{ST}$ where $r < x$.

Proof. Suppose after p visits the an inner vertex of r for the last time, p visits an inner vertex v of any normal separator path x such that there exists $P_k \in \text{INNER}(x)$ such that $k < u$. Then v are S are strongly r -connected. Since S and r_0 are strongly r -connected, v and r_0 are strongly r -connected. Since p does not visit any inner vertex of r after v , v and $r_{|r|-1}$ are strongly r -connected and therefore r_0 and $r_{|r|-1}$ are strongly r -connected, which is impossible since $|r| > 3$.

The other direction is similar. \square

We argue that both P_{AVOID} and P_0 must visit a and b , and a before b . Moreover, let p_{AVOID} be the segment of P_{AVOID} from a to b , and p_0 be the segment of P_0 from a to b , and we argue that $p_{\text{AVOID}} = p_0$. We will show it by contradiction both when $\|p_{\text{AVOID}}\| < \|p_0\|$ and $\|p_{\text{AVOID}}\| > \|p_0\|$. To do this, we will argue that p_{AVOID} is also a path in G_0 . Therefore if $\|p_{\text{AVOID}}\| < \|p_0\|$ one can replace, in P_0 , the segment p_0 by p_{AVOID} and find

a shorter path than P_0 between S and T in G_0 , which is a contradiction. We also argue that in P_{AVOID} , if we replace p_{AVOID} by p_0 and get P_{NEW} , P_{NEW} still avoids X , and therefore if $\|p_{\text{AVOID}}\| < \|p_0\|$, P_{NEW} becomes a shorter X -avoiding path in G between S and T in G than P_{AVOID} , which is a contradiction.

We show that P_0 , the shortest path between S and T in G_0 , must visit both a and b , and a before b . Recall that $a = s_{|s|-3}$ and $b = t_2$. Note that in G_0 , t_2 is an articulation vertex. To show this, it suffices to show that any simple path between S and T visits t_2 . Consider Corollary 4.6. If the first vertex of t that P_0 visits is not t_2 , that vertex is t_0 . It follows from Corollary 4.4 that the path visits some other vertex of t after visiting t_0 . However, since the edge between t_0 and t_1 does not exist in G_0 and the only other vertex of t that is strongly t -connected to t_0 is t_2 , the path must also visit t_2 . Note that this reasoning also shows that $b(t_2)$ is the first inner vertex of t that P_0 visits. Similarly, $a(s_{|s|-3})$ is an articulation vertex and is the last inner vertex that P_0 visits. Therefore P_0 must visit both a and b and since $s < t$, if b is visited before a , we immediately violate Lemma 7.3. Therefore, a is visited before b .

By definition P_{AVOID} visits both a and b . It remains to show that a is visited before b . Suppose b is visited before a . Let p_{ba} be the segment on P_{AVOID} from b to a , and p_{ab} be the reverse of p_{ba} . Since p_{ba} contains n , p_{ab} contains the reverse of n . Consider the path concatenated by $P_{0,i}$ (the segment from S to a), p_{ab} and $P_{j,|P|-1}$ (the segment from b to T). This is a path that contains p_{ab} and therefore the reverse of n . The reverse of n is also traversable and both n and its reverse are normal separator paths. This violates Corollary 4.4.

Now we show that p_{AVOID} , the segment of P_{AVOID} from a to b , is a path in G_0 . It suffices to show that it does not contain the tail of an S -separator path and the head of a T -separator path. Let x be an S -separator path and we want to show that its tail is not on p_{AVOID} . The case where $x \in \{s, t\}$ is trivial. The case where $s < x < t$ is straightforward from the first condition. The case where $x < s$ or $x > t$ is straightforward from Lemma 7.3. The case for T -separator paths is similar.

Now we show that P_{NEW} , the path we get after replacing p_{AVOID} by p_0 in P_{AVOID} , still avoids X . Consider $x \in X$. The first case is if $x \in X_{ST}$. In this case if $x < s$ or $x > t$, then from Lemma 7.3 no inner vertices of x and therefore no edges on x are on p_{AVOID} . Since P_{AVOID} does not contain x , neither does P_{NEW} . If $x \in \{s, t\}$, p_{AVOID} also does not contain an edge on x due to the fourth condition. If $s < x < t$, without loss of generality suppose x is an S -separator path. From the first condition, $x_{|x|-2} \notin P_{\text{AVOID}}$ and therefore $x_{|x|-2,|x|-1} \notin P_{\text{AVOID}}$. Since p_0 is a path in G_0 , $x_{|x|-2,|x|-1} \notin p_0$. Therefore $x_{|x|-2,|x|-1} \notin P_{\text{NEW}}$ and P_{NEW} does not contain x . If $x \in X_{\text{EXTRA}}$, we argue that no inner vertex of x and therefore no edge on x is on p_0 , which will finish the proof. Suppose an inner vertex u of x is on p_0 . From Corollary 4.3, we know an inner vertex v of x is on P , and from the second condition either $v < s$ or $v > t$, both would violate Lemma 7.3.

It remains to show that such a and b exist. For a separator path r , a path p is called r -free if p does not visit a bad vertex and, except at its endpoints, p does not visit a vertex of r . A normal separator r is said to be *truly useful* if, in addition to being useful, for all $1 \leq i < |r| - 3$, the shortest r -free path between r_i and r_{i+2} has a length greater than $\|r_{i,i+2}\|$. Let $X^+ = X \cup X_n$. We have the following:

Lemma 7.4. *All separator paths in $X^+ \setminus X_{ST}$ are truly useful.*

To prove Lemma 7.4, we introduce the following lemma, to be proven at the end of this sub-section.

Lemma 7.5. *Let r be a normal separator. For all $1 \leq i < |r| - 3$, any r -free path between r_i and r_{i+2} does not visit an inner vertex of another normal separator path.*

Proof of Lemma 7.4. A separator path $r \in X^+ \setminus X_{ST}$ is on a path p which is either the shortest X_{ST} -avoiding path or the shortest X -avoiding path. If r is not truly useful, then there exists $i \leq |r| - 3$ such that in p we can replace $r_{i,i+2}$ with the shortest r -free path between r_i and r_{i+2} and make p shorter. Due to Lemma 7.5, no r -free path between r_i and r_{i+2} can visit an inner vertex of a different normal separator path r' , and thus

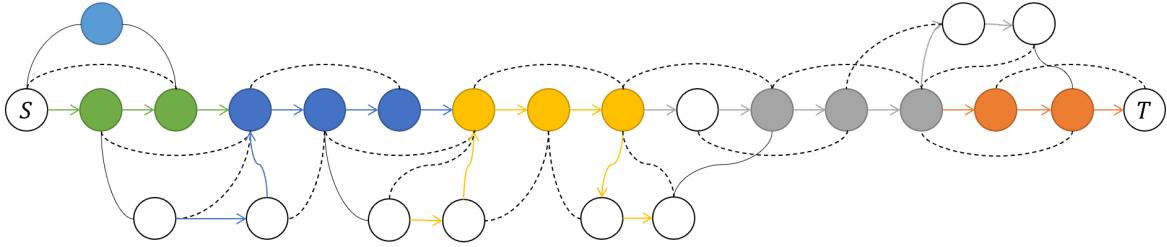


Figure 8: Intervals for X

after such replacement the path still avoids X_{ST} or X . Therefore p is not the shortest X_{ST} -avoiding path or the shortest X -avoiding path, which is a contradiction. \square

We have already shown that X_{ST} has the *interval property*: indices of inner vertices of separator paths in X_{ST} on P are intervals on P . With Lemma 7.4, the same interval property also applies to X^+ . Figure 7 shows the intervals for X (which in reality is equal to X_n) in the example graph we have shown before. To show this, let r be a separator path in $X^+ \setminus X_{ST}$, and P_x, P_y are inner vertices of r such that $|x - y|$ and no $P_z (x < z < y)$ is an inner vertex of r . Then P_x and P_y are strongly r -connected. Since P is the shortest path, the edge between P_x and P_y is longer than $P_{x,y}$ and therefore r is not truly useful, violating Lemma 7.4.

Recall that $n \in X_n$ is a hypothetical normal separator path $n \in X_n$ and P_i is an inner vertex of n . We can let j be the maximum $j < i$ such that one of the following is true:

1. P_j is an inner vertex of an S -separator path α
2. $P_j \in P_{\text{AVOID}}$ and P_j is not an inner vertex of n . (Note that $j = 0$ satisfies this condition and therefore j is always well-defined)
3. P_j is an inner vertex of a separator path $\alpha \in X_{\text{EXTRA}}$

We argue that all but the first case are impossible. In the first case, and we let $s = \alpha$ and $a = \alpha_{|\alpha|-3}$. We will show that a is the last inner vertex of α that P_{AVOID} visits.

Similarly we let k be the minimum $k > i$ such that one of the following is true:

1. P_k is an inner vertex of a T -separator path β
2. $P_k \in P_{\text{AVOID}}$ and P_k is not an inner vertex of n .
3. P_k is an inner vertex of a separator path $\beta \in X_{\text{EXTRA}}$

And we also argue that all but the first case are impossible. In the first case, let $t = \beta$ and $b = \beta_2$. If for both directions the first case is true, it is straightforward to verify that such a and b satisfies all four conditions stated at the beginning of this proof.

Due to symmetry, we will only need to examine the direction for α . To proceed, we first introduce two lemmas:

Lemma 7.6. For $r \in X^+$:

If the vertex $u \in r$ on P with the smallest $\text{INDEX}(u, P)$ is an endpoint of r , and $r_{0,1}$ is on P , then r is an T -separator path.

Similarly, if the vertex $u \in r$ on P with the largest $\text{INDEX}(u, P)$ is an endpoint of r , and $r_{|r|-2,|r|-1}$ is on P , then r is an S -separator path.

Proof. We prove the first half of the lemma.

Suppose r is not a T -separator path. Then r is not an S -separator path, and therefore from Lemma 7.4 r is truly useful, let u be the vertex of r on P with the smallest $\text{INDEX}(u, P)$. Due to Corollary 4.4, if u is an endpoint of r , u can not be $r_{|r|-1}$. Therefore $u = r_0$. If x is the smallest index such that $r_{x,x+1}$ is not on P . If for some $y > x$, r_y is back on P again for the first time. If $\text{INDEX}(P, r_y) > \text{INDEX}(P, r_x)$, then r_x and r_y are strongly r -connected and $y = x + 2$. Since P is the shortest path between S and T , by the definition of truly usefulness this is only possible when $x = |r| - 3$ and $y = |r| - 1$. Thus r is a T -separator path. If $\text{INDEX}(P, r_y) < \text{INDEX}(P, r_x)$, $\text{INDEX}(P, r_y) < \text{INDEX}(P, r_0)$, which violates the fact that r_0 has the smallest index. Then If such y does not exist, then r_x and T are strongly r -connected. Therefore $x = |r| - 3$ or $x = |r| - 1$, in both cases r is a T -separator path.

The reasoning for the second half is similar. \square

Lemma 7.7. If $x \in X$ is not an S -separator path and P_i is an inner vertex of x . For any $j > i$ such that P_j is an inner vertex of another normal separator path $y \in X^+$, no simple path from S to T containing $P_{i,j}$ or $P_{j,i}$ can contain x .

If $x \in X$ is not a T -separator path and P_i is an inner vertex of x . Then for any $j < i$ such that P_j is an inner vertex of another normal separator path $y \in X^+$, no simple path from S to T containing $P_{j,i}$ or $P_{i,j}$ can contain x .

Proof. We prove the first half of the lemma.

Suppose $x \in X$ is not an S -separator path and P_i is an inner vertex of x . It suffices to prove this for the case where for no $i < i' < j$, $P_{i'}$ is an inner vertex of x . If a simple path contains $P_{i,j}$ or $P_{j,i}$ and contain x . If $P_i = x_k$, consider the edge between x_k and x_{k+1} , if $x_{k,k+1} \neq P_{i,i+1}$, then no simple path between S and T can contain $x_{k,k+1}$ and $P_{i,i+1}$ at the same time (note that this is not the same as $x_{k+1,k}$ and $P_{i,i+1}$), and no path can contain $P_{i+1,i}$ without visiting x_k twice. Therefore, x_{k+1} must be an endpoint and therefore $k+1 = |x| - 1$. From the interval property we have shown before, $x_{|x|-1}$ must be the vertex of x on P with the largest index. From Lemma 7.6, x is an S -separator path, which is a contradiction.

The other direction is similar. \square

Since P_{AVOID} contains n , it visits P_i . Let i' be the smallest index such that $P_{i'}$ is also inner vertex of n . Due to the interval property $P_{i',i}$ consists entirely of inner vertices of n and therefore $i' > j$. If P_{AVOID} also visits P_j , consider replacing the part of P_{AVOID} between P_j and $P_{i'}$ by $P_{i',j}$ or $P_{j,i'}$ depending on the order of the visits. Here we assume P_j is visited before $P_{i'}$. The reasoning for the other direction is similar. To force a similar contradiction as we have seen before. We argue that the some edge on $P_{j,i'} \not\subseteq P_{\text{AVOID}}$ — therefore the replacement changes the path, and that the new path after the replacement still avoids X unless P_j is an inner vertex of some an S -separator path α . This will preclude the second case.

Firstly, we argue that $P_{j,i'} \not\subseteq P_{\text{AVOID}}$. Suppose $P_{j,i'} \subset P_{\text{AVOID}}$. Suppose P_j is the inner vertex of some normal separator path α . If $n_{0,1}$ is not on $P_{j,i'}$, then since $P_{i'}$ is an inner vertex of n and P_j is not an inner vertex of n , the path P_{AVOID} , which contains $P_{j,i'}$, can not contain n . If $n_{0,i}$ is on $P_{j,i'}$, we can apply Lemma 7.6 and argue that $n \in X_{ST}$, which contradicts $X \cap X_n = \emptyset$.

Obviously, the new path is still simple, if P_j is the inner vertex of some separator path α , from lemma 7.7, unless α is an S -separator path, the new path does not contain α . Since none of the other elements of X has an inner vertex on $P_{i,j}$, the new path still avoids X .

The only case left is if P_j is the inner vertex of $\alpha \in X_{\text{EXTRA}}$ and P_{AVOID} does not visit P_j . Suppose $P_j = \alpha_k$. From Corollary 4.5 we have P_{AVOID} must visit α_{k-1} and α_{k+1} , unless they are endpoints. If α_{k-1} is not an endpoint and is on P , then $P_{j-1} = \alpha_{k-1}$ since otherwise α_{k-1} and α_k become strongly α -connected. Since α_{k-1} is also on P_{AVOID} , one can let $j = k - 1$ and one can apply the same reasoning as when P_{AVOID} visits P_j to argue that α is an S -separator path, with minor modification, which contradicts $\alpha \in X_{\text{EXTRA}}$. Then if α_{k-1} is an inner vertex, $\alpha_{k-1} \notin P$. By the way we find j , if α_{k+1} is an inner vertex, $\alpha_{k+1} \notin P$.

If both α_{k-1} and α_{k+1} are inner vertices, since α is truly useful according to Lemma 7.4, we can replace the segment of P_{AVOID} from α_{k-1} to α_{k+1} by $\alpha_{k-1,k+1}$ and make the path P_{AVOID} shorter. Note that the new path might not be X -avoiding since it might contain α , but it visits $P_j (= \alpha_k)$, and now we can use the reasoning before using replacement to find a even shorter path that does not contain α and is X -avoiding, which is a contradiction.

If one of α_{k-1} and α_{k+1} is an endpoint. Suppose $k - 1 = 0$. Then α_{k+1} is an inner vertex since $|\alpha| > 3$, and therefore $\alpha_{k+1} \notin P$. From Corollary 4.6, since $\alpha_{k+1} \notin P$, $\alpha_{k-1}(\alpha_0)$ is the first vertex on α that P visits, from Lemma 7.6 α is a T -separator path, which contradicts $\alpha \in X_{\text{EXTRA}}$. Since our reasoning is symmetric, we also know that $k + 1$ is not an endpoint.

To show that $\alpha_{|\alpha|-3}$ is the last inner vertex of α that P_{AVOID} visits, we show that P_{AVOID} does not visit the tail of α . Suppose it does. Then if $\alpha_{|\alpha|-1}$ is not an inner vertex of n , then P_j should have been $\alpha_{|\alpha|-1}$; if $\alpha_{|\alpha|-1}$ is an inner vertex of n , one can see that P_{AVOID} can not contain n .

Proof of Lemma 7.5. Let p be an r -free path between r_i and r_{i+2} . Suppose p an inner vertex of another normal separator path r' . We note the following fact due to Theorem 4.2: no other vertex on r is r -connected to both r_i and r_{i+2} . Let p' be any path between S and T that contains r' . Let u be the last vertex of r that p' visits before r' , and if such vertex does not exist, let $u = S$. Let v be the first vertex of r that p' visits after r' , and if such vertex does not exist, let $v = T$. If $u = r_i$ and $v = r_{i+2}$, since r_i and r_{i+2} are adjacent due to Lemma 6.3, $(r')_0$ and $(r')_{|r'|-1}$ are strongly r' -connected. Otherwise without loss of generality suppose $u \notin \{r_i, r_{i+2}\}$. If $u = S$, both r_i and r_{i+2} are r -connected to r_0 , and otherwise both r_i and r_{i+2} are r -connected to u . Neither is possible. \square

7.3 Correctness of AVOID(X)

In this section, we will show the correctness of AVOID(X) based only on the fact that X is a set of normal separator paths. We will ignore all the extra properties of the set X computed in the main procedure we have seen in the previous section. We believe this makes the sub-procedure more general. We will do this in two steps: first, we show that the shortest X -avoiding path p from S to T in G corresponds to a path from $(S, 0)$ to $(T, 0)$ in G_1 . Second, we show that the shortest path p' from $(S, 0)$ to $(T, 0)$ in G_1 corresponds to an X -avoiding path in G . Obviously these will show the correctness of the path we compute.

7.3.1 p to p'

If p is the shortest X -avoiding path from S to T in G , we can construct p' in this way: start from $(S, 0)$. We move along p and construct p' alongside. Imagine the two paths being two pointers moving in sync, one in G and the other in G_1 . Whenever p visit a new vertex, p' will try to visit the corresponding low vertex, and

if an edge to the low vertex does not exist, we visit the high vertex. We show that there will never be a case when neither edge exists. Note that p is a simple path.

We first argue that, for $r \in X$, suppose we are currently at index $i + 1$. $(p')_{i+1}$ is the high vertex of an inner vertex r_j of r but p_i is not an inner vertex of r , then either $p_i = r_0$, $p_{i+1} = r_1$, or $p_{i+1} = r_{|r|-3}$ and both $r_{|r|-2}$ and $r_{|r|-4}$ are visited before p_i .

They are only three ways this can happen. Only two of them are possible.

The first way is that $p'_{i+1} = (r_1, 1)$ and $p_i = r_0$. Therefore $p_i = r_0$ and $p_{i+1} = r_1$.

The second way is that $p'_{i+1} = (r_{|r|-3}, 1)$ and $p_i = u$ where $L(r, u) = r_{|r|-3}$. In this case, p hasn't visited $r_{|r|-3}$ but has visited $r_{|r|-1}$. From Corollary 4.5 we can see that both $r_{|r|-4}$ and $r_{|r|-2}$ are visited.

The third is that $(p')_i = (v, 1)$ where $(v \notin r)$ and $(p')_{i+1} = (r_j, 1)$ where $0 < j < |r| - 1$. From the way we built G_1 , this will imply the edge e between v and r_i is on some separator path $r' \in X$ and Theorem 3.4 implies that such r' is unique. From Theorem 3.5 we know r_i is an endpoint of r' . From the way G_1 is built we know e is not the tail of r' . Therefore $v = (r')_0$ and the edge e is from $(r')_1$ to $(r')_0$, which means that $(r')_2$ is visited by p . With $(r')_1$ and $(r')_2$ both visited before and p currently at $(r')_0$, p can not get to T while being simple.

Therefore, if p' visits $(r_i, 1)$ ($0 < i < |r| - 1$) and was not on the high vertex of an inner vertex of r before this visit, then r_{i-1} has been visited by p . If after the current visit, p' immediately visits another high vertex of an inner vertex of r , it will be $(r_{i+1}, 1)$, while r_i has been visited by p . With this induction we can further deduce that whenever p' visits a high vertex $(r_k, 1)$, regardless of what we visited before this visit, all vertices on $r_{0,k-1}$ have been visited by p .

If p visits a vertex and p' is not able to reach any copy of the vertex, the possibilities are: it is trying to visit (some copy of) r_{i-1} or r_{i-2} from $(r_i, 1)$; it is trying to visit r_i from $(u, 0)$ where $u \notin r$, $L(u, r_i) = r_i$ and $0 < i < |r| - 1$ and $i \neq |r| - 3$; it is trying to visit v from $(r_i, 1)$ where $0 < i < |r| - 1$, $v \notin u$ and $R(r, v) = r_i$. It is trying to visit $r_{|r|-1}$ from $(r_{|r|-2}, 1)$.

If p' is trying to visit a copy of r_{i-1} or r_{i-2} from $(r_i, 1)$, since r_{i-1} and r_{i-2} have already been visited by p , p will not be simple.

If p' is trying to visit v from $(r_i, 1)$, where $0 < i < |r| - 1$, $v \notin u$ and $R(r, v) = r_i$. Since p' is at $(r_i, 1)$, r_{i-1} has been visited by p . If we visit v the next inner vertex of r that we visit on p will be r_{i-2} , p will not be able to reach T without visiting r_{i-1} or r_i again, but it can not visit do either of them while being simple, which leads to a contradiction.

If p' is trying to visit $r_{|r|-1}$ from $(r_{|r|-2}, 1)$. Consider the last time it visits a high vertex of an inner vertex of r . As we showed there are only two ways this can happen. If it got from u to $(r_{|r|-3}, 1)$ where $L(r, u) = r_{|r|-3}$ then $r_{|r|-1}$ has already been visited by p , so it will not visit $r_{|r|-1}$ again. If it visited $(r_1, 1)$ from r_0 , then by visiting $r_{|r|-1}$, p will contain r , which violates the fact that p avoids X .

If p' is trying to visit r_i from $(u, 0)$ where $u \notin r$, $L(u, r_i) = r_i$ and $0 < i < |r| - 1$ and $i \neq |r| - 3$. From Theorem 4.2, suppose the last time p gets off r it was at r_j . If $0 < i < |r| - 3$, then $j = i$ or $i + 2$. If $i = |r| - 2$, $j = i$. In both cases we violate Corollary 4.7.

7.3.2 p' to p

If p' is the shortest X -avoiding path from $(S, 0)$ to $(T, 0)$ in G_1 . We can see that p' will not visit any $(u, 0)$ before $(u, 1)$ since all paths starting from $(u, 1)$ can be replaced by an path starting from $(u, 0)$, by replacing some high vertices with the corresponding low vertices. For the same reason, we can assume that p' will not visit $(u, 1)$ from a vertex when it can visit $(u, 0)$ from that vertex, and similar to what we had in the previous part, we can deduce that whenever p' visits the high vertex of an inner vertex r_i of $r \in X$ from a vertex that

does not correspond to an inner vertex of r , either it is from r_0 to r_1 or $i = |r| - 3$ or $i = |r| - 3$ and we visit r_i from $u \notin r$ where $L(r, u) = r_i$.

Let p be the path from S to T in G corresponding to p' in G_1 . p obviously does not contain a bad vertex. It suffices to prove that the set of edges on p does not contain the set of edges on any element of X (Note that we are not sure if p is simple yet). Suppose it does for r in X . We first show that p' can not visit any low vertex $(u, 0)$ of $u \notin r$ from any inner vertex $(r_i, 1)$. Suppose it does. Note that since $(r_i, 0)$ hasn't been visited before. One of the edge associated with r_i on r hasn't be visited by the corresponding path p . So p' will have to visit $(r_i, 0)$ again in order to visit the other edge. By the way we built G_1 , $L(r, u) = r_i$ and the next vertex on r that p visits must be r_{i+2} , and the corresponding visit on p' is either $(r_{i+2}, 0)$ or $(r_{i+2}, 1)$. Between them we can rule out $(r_{i+2}, 1)$ since p' visits $r(r_{i+2}, 1)$ from a vertex that does not correspond to an inner vertex of r , and we can not be in either of the two cases this can happen. Since p' visited $(r_{i+2}, 0)$ before visiting $(r_i, 0)$, p will not visit r_{i+2} again after visiting r_i . Thus one can see that after visiting r_i , there will be a segment of p corresponding to a path from r_{i+1} to r_{i+3} (or to T if $i + 2 = |r| - 1$) such that no vertices on the segment except its endpoints are on r . The suffix of p after the start of this segment corresponds to a path from $(r_{i+1}, 1)$ in G_1 , and we can replace the part of p' from $(r_{i+1}, 0)$ with this suffix from $(r_{i+1}, 1)$. This replacement preserves p' 's length as well as its correspondence with p . Note that to contain all the edges on r , p needs to visit at least three edges associated with r_{i+1} . Therefore p visits r_{i+1} at least twice. Therefore p' after the length-preserving replacement must visit either $(r_{i+1}, 1)$ or $(r_{i+1}, 0)$ again after visiting $(r_{i+1}, 1)$, and in either case we can do a second replacement and make p' a shorter path between S and T in G_1 , which is impossible since p' is the shortest.

Now that we have shown that p' can not visit any low vertex $(u, 0)$ of $u \notin r$ from any inner vertex $(r_i, 1)$. Since p contains all the edges on r , p must visit the edge between r_0 and r_1 . If p never visits r_1 from r_0 but has visited r_0 from r_1 , then by the time p visits r_1 , p' must have visited $(r_1, 0)$ and $(r_2, 0)$ before. Since p' can not visit any copy of r_1 or r_2 again, p' can not reach $(T, 0)$. Suppose p visits r_1 from r_0 . After visiting the edge p' will be at $(r_1, 1)$. Since the edge from $(r_{i+1}, 1)$ to $(r_i, 0)$ for $i > 0$ or $(r_{i-2}, 0)$ for $i > 1$ does not exist, we will have to take the route $(r_1, 1) \rightarrow (r_2, 1) \rightarrow (r_3, 1) \dots$. Eventually, we will be forced to go from $(r_{|r|-2}, 1)$ to $(r_{|r|-1}, 1)$, which is an edge that does not exist in G_1 , and we have a contradiction.

8 Tie Breaking

Throughout this paper we have assumed that any two different paths between the same pair of vertices (ordered pair) in the input graph have different lengths. However, in reality, this is not already true. There can be ties where two different paths between some pair of vertices have the same lengths. One can see that if we break ties arbitrarily, the final path we get might be a separating path equal in length to the real answer. In Figure 9, numbers beside the edges are their lengths. If both the path P and the path P_0 go through the blue part. The final path we compute may go through the green part, which means that the path is separating. Fortunately, there are ways we can break ties.

A tie-breaking scheme is called *consistent* if for any two paths connecting the same pair of vertices with the same length, the scheme always favors one after another. Many implementations of the Dijkstra's Algorithm achieves uses this time breaking scheme since the order in which we iterate through all the edges associated with a particular vertex is usually fixed. If we use Dijkstra's Algorithm in our main algorithm, for most of the common ways one can store an undirected graph on a machine using the RAM model, a consistent tie-breaking scheme is easy to achieve. For example, if the graph is stored by using linked lists, for all the auxiliary graphs involved in our algorithm, it suffices to store the linked lists in a way such that if at a particular vertex the edge e_0 before the edge e_1 , then at the corresponding vertex in the auxiliary graph,

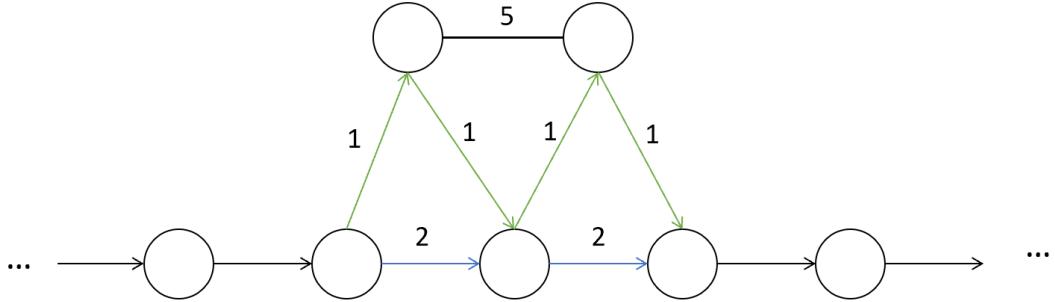


Figure 9: Tie breaking issues

all edges corresponding to e_0 comes before all edges corresponding to e_1 .

9 NP-hardness proof for general graphs

We will show that the problem of deciding the existence of non-separating st-paths is NP-hard on general graphs.

We will reduce the famous 3-SAT problem to the problem of deciding the existence of non-separating st-paths. Here we adopt the notational framework in [16]. In the 3-SAT problem, we are given a 3-CNF formula, which is a conjunction of n clauses and m variables, each with exactly 3 literals, and we want to decide whether a satisfying assignment exists. The well-known Cook-Levin Theorem [17, 18] states that this problem is NP-Hard. We show that we can solve 3-SAT by deciding the existence of non-separating st-paths on a graph with $O(n)$ (assuming that $m = O(n)$) vertices and edges, which will then prove Theorem 1.6.

Let the i -th variable be x_i . Let k_i be the amount of times its complement \bar{x}_i appears in the formula. Let \bar{k}_i be the amount of times x_i appears in the formula. To build the graph, we first create the vertex S . For variable x_i in order, we create $k_i + \bar{k}_i + 1$ nodes: $a_{i,0 \dots k_i-1}, \bar{a}_{i,0 \dots \bar{k}_i-1}, b_i$. Let $t = b_{i-1}$ ($t = S$ if $i = 0$). Make edges between t and $a_{i,0}$ (if it exists), between t and $\bar{a}_{i,0}$ (if it exists), between $a_{i,j}$ and $a_{i,j+1}$ (for all j such that both vertices exist), between $\bar{a}_{i,j}$ and $\bar{a}_{i,j+1}$ (for all j such that both vertices exist), between a_{i,k_i-1} (if it exists) and b_i , between \bar{a}_{i,\bar{k}_i-1} and b_i (if it exists). If $k_i = 0$ or $\bar{k}_i = 0$, make an edge between t and b_i . Finally, let $T = b_{m-1}$.

We use the following shorthand: To make a *fat edge* between u and v , we make a dummy node w , and then make edges between u and w and between w and v . A fat edge can be treated as a single edge that can not be traversed by the path, since any path that goes through $u \rightarrow w \rightarrow v$ makes w disconnected from the rest of the graph. For the k -th clause, create a vertex c_k . If the clause contains literal x_i , and is the j -th clause that contains that literal, make a fat edge between $\bar{a}_{i,j}$ and c_k . If the clause contains literal \bar{x}_i , and is the j -th clause that contains that literal, make a fat edge between $a_{i,j}$ and c_k .

One can see that the 3-CNF formula is satisfiable if and only if the there exists a non-separating path from S to T in the graph. A solution of the formula corresponds to the path in this way: for the i -th variable, if x_i is true, the path goes through $a_{i,*}$. Otherwise, the path goes through $\bar{a}_{i,*}$. (The path goes through the edge between b_{i-1} (or S) and b_i if the corresponding k_i or \bar{k}_i is 0) In this way, it is easy to see that for a non-satisfying assignment, any clause that is violated will be disconnected from the rest of the graph.

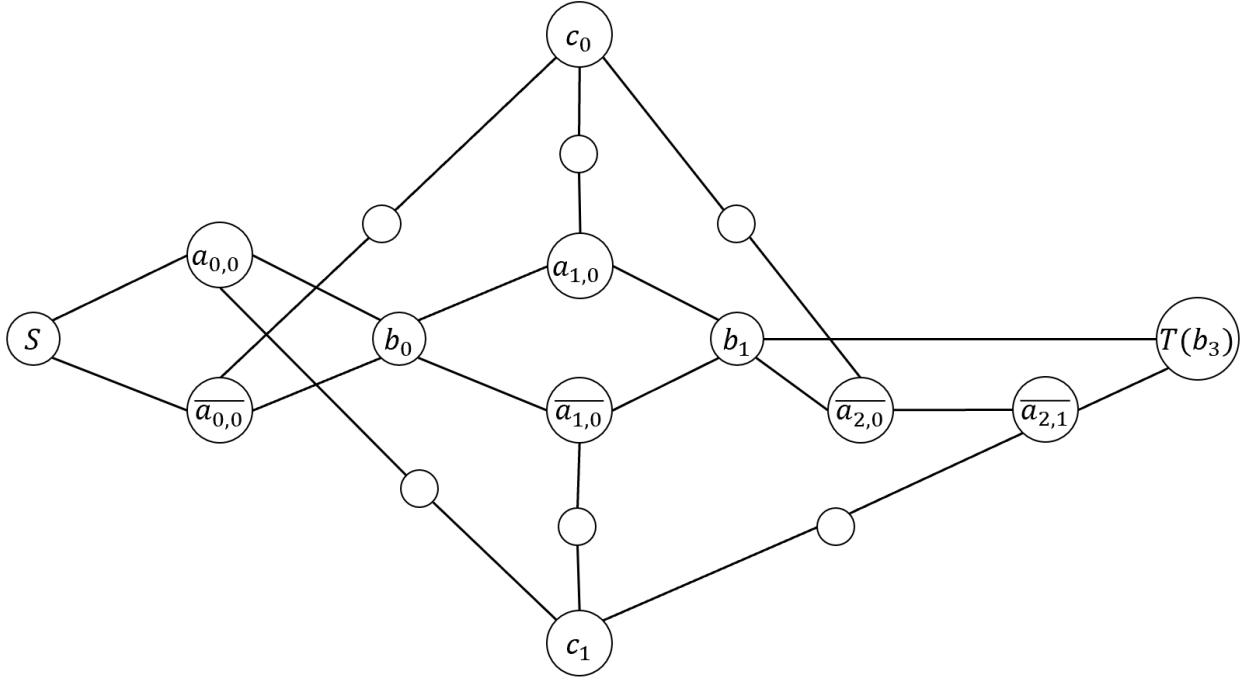


Figure 10: The graph.

The following is an example for the formula: $(x_0 \vee \bar{x}_1 \vee x_2) \wedge (\bar{x}_0 \vee x_1 \vee x_2)$:

Figure 10 shows the graph we build for the formula.

Figure 11 shows the non-separating path corresponding to assignment $\{x_0 = \text{TRUE}, x_1 = \text{TRUE}, x_2 = \text{TRUE}\}$, which is a satisfying assignment.

Figure 12 shows the path corresponding to assignment $\{x_0 = \text{FALSE}, x_1 = \text{TRUE}, x_2 = \text{FALSE}\}$ in green, which is not a satisfying assignment as clause $(x_0 \vee \bar{x}_1 \vee x_2)$ is violated. The path is not a non-separating path. After removing all the edges on the path, the connected component for the violated clause c_0 in orange is disconnected from the rest of the graph in light blue.

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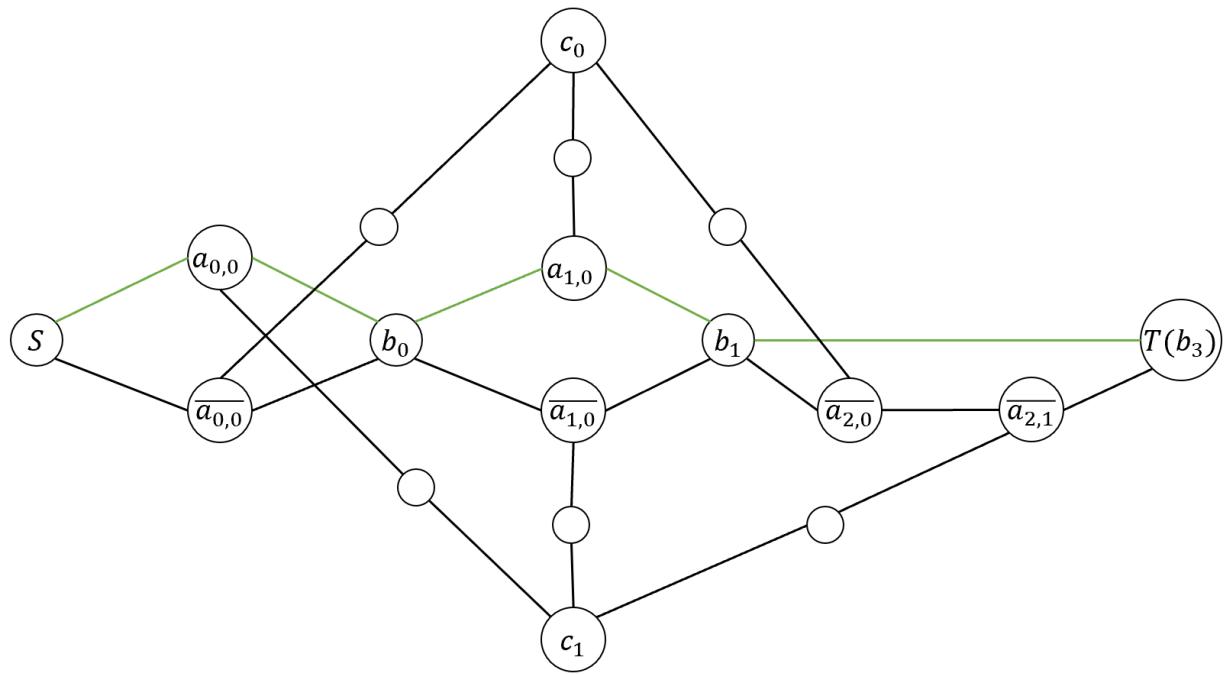


Figure 11: A non-separating path

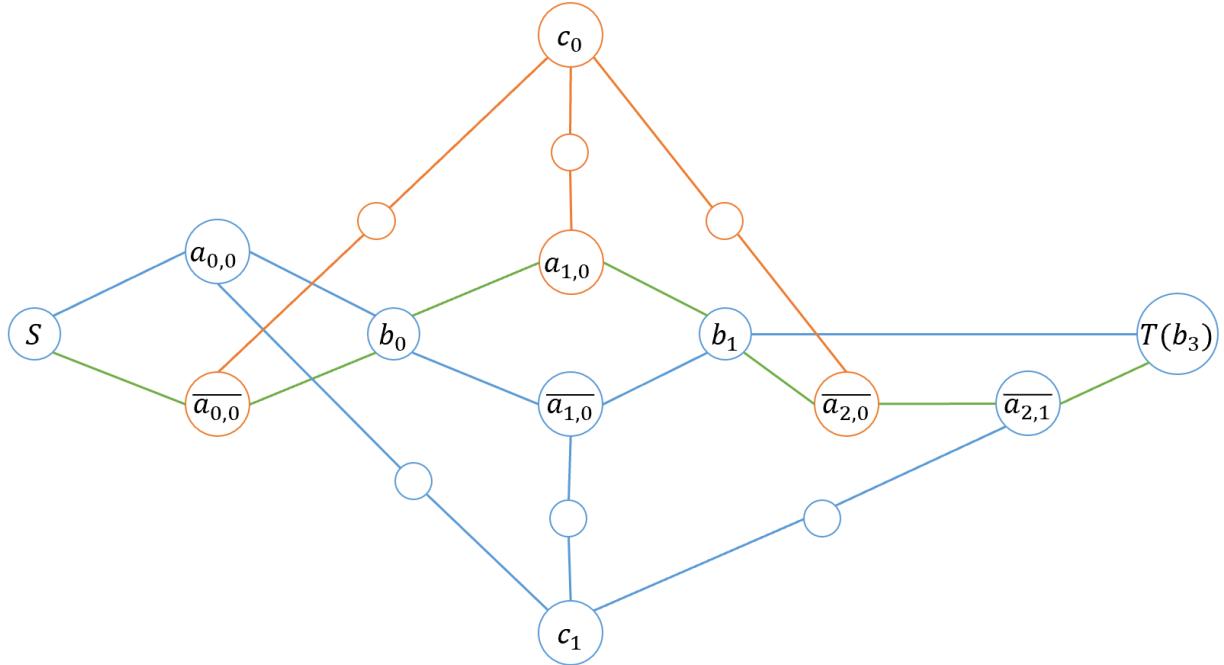


Figure 12: Not a non-separating path

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