Homework3

Qinglin Li, 5110309074

Problem 1

$\forall a \nmid b, \mu(a,b) = 0$
$\forall a \mid b, \mu(a,b) = \mu(1, \frac{b}{a})$
$\mu(a,b) = -1 \iff \frac{b}{a}$ is the product of odd number of distinct prime numbers
Calculated by MATLAB, there are 179 pairs
So the probability is $\frac{179}{10000}$ $\square$

Problem 2

$s(n,k) = (-1)^{n-k}c(n,k) \Rightarrow s(n,k)$ is even iff $c(n,k)$ is even
By induction.

1. $n = 1, k = 0, c(n,k) = 0$ is even
2. Suppose $\forall n' < n, 2k' < n', c(n', k')$ is even
   $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$
   If n is even, $2k < n \Rightarrow 2k < n-1$. $c(n,k)$ is even by hypothesis.
   If n is odd, $n-1$ is even and $2k < n \Rightarrow 2(k-1) < n-1$, $c(n,k)$ is even

So $s(n,k)$ is even $\square$

Problem 3

T is the coefficient of x^{50} in $x(x-1)\cdots(x-99)$
So

$$T = \sum_{X \in \binom{[99]}{50}} \prod_{x \in X} -x$$

If a multiple is in X, $\prod_{x \in X} -x = 0$
Since there are 33 numbers equals 1 mod 3 and 33 numbers equals 2 mod 3

$$T = \sum_{i=0}^{33} \binom{33}{i}\binom{33}{50-i} 2^i = \sum_{i=17}^{25} \binom{33}{i}\binom{33}{50-i} 2^i + \sum_{i=17}^{24} \binom{33}{i}\binom{33}{50-i} 2^{50-i} \mod 3$$

By calculating, it is found that $17 \leq i \leq 25$, $\binom{33}{i} = 0 \mod 3$
Thus $T = 0 \mod 3$

Problem 4

Let $S = \{p | p \text{ is a permutation of } [2n]\}$
Let $S_i = \{p | p \text{ is a permutation of } [2n] \text{ and } x_i + x_{i+1} = 2n + 1\}$
$S_i \cap S_{i+1} = \emptyset$ because each element can appear only once
There are $\binom{2n-i}{i}$ ways not to choose i sets consecutively

$$|S - \cup_{i=0}^{i \leq 2n-1} S_i| = \sum_{i=1}^{n} (-1)^i \sum_{X \in \binom{[2n-1]}{i}} |\cap_{j \in X} S_j| = \sum_{i=0}^{n} (-1)^i \binom{2n-i}{i}\binom{n}{k}(2n-2k)!k!2^k$$

Problem 5

The coefficient of x^{n-k} is $\sum_{t_1+\cdots+t_k=n-k} \prod_{i=1}^{k} i^{t_i}$
Consider a process of inserting 1 to n into k sets $S_1 \cdots S_k$ in k rounds
Initially, all sets are empty
In round i, insert $t_i + 1$ numbers. The first number is inserted into S_i. And the other t_i numbers can be inserted into any of $S_1 \cdots S_i$. So there are i^{t_i} ways to insert them
Thus, $\sum_{t_1+\cdots+t_k=n-k} \prod_{i=1}^{k} i^{t_i} = S(n, k)$