Homework 1

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Problem 1

1. $\binom{10}{8} = \binom{10}{2} = \frac{10\times 9}{2\times 1} = 45$

2. $1^{-1} = 1 \mod 7$
   $2^{-1} = 4 \mod 7$
   $3^{-1} = 5 \mod 7$
   $4^{-1} = 2 \mod 7$
   $5^{-1} = 3 \mod 7$
   $6^{-1} = 6 \mod 7$

3. $449 = 2^0 + 2^6 + 2^7 + 2^8 = 4 \times 5^0 + 4 \times 5^1 + 2 \times 5^2 + 3 \times 5^3$
   $137 = 2^0 + 2^3 + 2^7 = 2 \times 5^0 + 2 \times 5^1 + 1 \times 5^3$
   $\binom{449}{137} = \binom{1}{1}\binom{0}{1}\binom{1}{0}\binom{1}{1}\binom{1}{0} = 0 \mod 2$
   $\binom{449}{137} = \binom{4}{2}\binom{4}{2}\binom{2}{0}\binom{3}{1} = 3 \mod 5$

   By Chinese Remainder Theorem
   $\binom{449}{137} = 8 \mod 10$

Problem 2

Firstly, choose $k (0 \leq k \leq n)$ elements from $[n]$ as A. And then the remained $n-k$ elements can be either in B or not.

$$\#\text{oredered pairs} = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^{n-k} = (1+2)^n = 3^n$$

Problem 3

For any 9×9 array, consider generating it by filling 1 to 81 into the units one by one.
Suppose every line or every column is filled for the first time after the number k is filled.
WLOG, suppose every line is filled.
Because not all columns are filled before k is filled, there must be at least one blank in all rows after k is filled. In other word, there must be at least 9 blanks adjacent to filled units.
Then the maxmium number filled into these blanks must be at least $k+9$ because 1 to k are used.
So the difference is at least $(k+9) - k = 9$

Problem 4

Suppose $\exists a, \exists b \neq c$, $\overline{ab} = \overline{ac}$, clearly abc is a fun triple.
$\forall d \neq a, b, c$, either both abd and acd are fun triples, or none of abd and acd is fun triple.
Since the number of fun triples in $\{a, b, c, d\}$ is even, bcd must be a fun triple.
So $\overline{bc}$ is of size n.

Otherwise $\forall a, \forall b \neq c, \overline{ab} \neq \overline{ac}$
Then there must exist a club C such that $\exists a \notin C$
For other people $b_1, b_2, \cdots, b_{n-1}, \forall i \neq j, \overline{ab_i} \neq \overline{ab_j}$
So n clubs $\overline{ab_1}, \cdots, \overline{ab_{n-1}}, C$ are different.