Homework2

Qinglin Li, 5110309074

Problem 1

Since every person have only two hands, the game continue n rounds. The final state must be some cycles.
Before the n-th round, the state must be some lines(a single without self-loop is also considered to be a line) with some cycles.
In every round, we either change a line into a cycle or combine two lines into one line.
Let X_n be the number of cycles when there are n lines remaining.
Let A_n be the event changing a line into cycle when there are n lines remaining.
$\Pr(A_n) = \frac{n}{\binom{2n}{2}} = \frac{1}{2n-1}$

$$X_n = \begin{cases} X_{n-1} + 1 & \Pr = \dfrac{1}{2n-1} \\ \\ X_{n-1} & \Pr = \dfrac{2n-2}{2n-1} \end{cases} \tag{1}$$

$E[X_n] = \frac{1}{2n-1}(1 + E[n-1]) + \frac{2n-2}{2n-1}E[n-1] = E[n-1] + \frac{1}{2n-1}$
$E[X_n] = \sum_{i=1}^{n} \frac{1}{2i-1}$

Problem 2

Let X_i be the number of balls in each bins in uniformly random case
Let Y_i be the number of balls in each bins in 2-choice paradigm case
Obviously, in paradigm 1–3, the answer is $E[\max_i(X_i + Y_i)]$

$$E[\max_i(X_i + Y_i)] \leq E[\max_i X_i] + E[\max_i Y_i]$$

Thus

$$E[\max_i(X_i + Y_i)] = O\left(\frac{\ln \frac{n}{2}}{\ln\ln \frac{n}{2}} + \ln\ln \frac{n}{2}\right) = O\left(\frac{\ln n}{\ln\ln n}\right)$$

$$E[\max_i(X_i + Y_i)] \geq E[\max_i X_i]$$

Thus

$$E[\max_i(X_i + Y_i)] = \Omega\left(\frac{\ln \frac{n}{2}}{\ln\ln \frac{n}{2}}\right) = \Omega\left(\frac{\ln n}{\ln\ln n}\right)$$

So the answer is $\Theta\left(\dfrac{\ln n}{\ln\ln n}\right)$

Problem 3

Let H_i be `Number of Heads - Number of Tails`

$$E[|H_i|] = \sum_{x>0} x\Pr(H_i = x) - \sum_{x<0} x\Pr(H_i = x) = 2\sum_{x>0} x\Pr(H_i = x)$$

Let $i = 2n$

$$E[|H_i|] = 4\sum_{y=0}^{n} y\frac{\binom{2n}{n+y}}{2^{2n}} = 2^{2-2n}\sum_{y=0}^{n} y\binom{2n}{n+y}$$

Let $S_1 = \sum_{y=0}^{n} y\binom{2n}{n+y} = 0\cdot\binom{2n}{n} + \cdots + n\cdot\binom{2n}{2n}$
Let $S_2 = \sum_{y=0}^{n} y\binom{2n}{y} = n\cdot\binom{2n}{n} + \cdots + 0\cdot\binom{2n}{0} = n\cdot\binom{2n}{n} + \cdots + 0\cdot\binom{2n}{2n}$

$$S_1 + S_2 = n\sum_{y=n}^{2n}\binom{2n}{y} = \frac{n}{2}\left(2^{2n} + \binom{2n}{n}\right)$$

$$S_2 = \sum_{y=0}^{n} y\binom{2n}{y} = \sum_{y=0}^{n} 2n\binom{2n-1}{y-1} = 2n\sum_{y=0}^{n-1}\binom{2n-1}{y} = n\cdot 2^{2n-1}$$

$$S_1 = (S_1 + S_2) - S_2 = \frac{n}{2}\binom{2n}{n}$$

$$E[|H_i|] = 2^{2-2n} S_1 = 2^{2-2n} \frac{(2n)!n}{n!n!}$$

By Stirling's approximation

$$E[|H_i|] \approx 2^{2-2n} \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n} n}{2\pi n \left(\frac{n}{e}\right)^{2n}} = 4\sqrt{\frac{n}{\pi}} = \Theta(\sqrt{i})$$

$$\begin{aligned}
&\Pr(H \geq k) \\
=&\Pr(H \geq k | H_n \geq k) \cdot \Pr(H_n \geq k) + \Pr(H \geq k | H_n < k) \cdot \Pr(H_n < k) \\
=&\Pr(H_n \geq k) + \Pr(H \geq k | H_n < k) \cdot \Pr(H_n < k)
\end{aligned}$$

Since H_n is either greater or less than H_{n-1} by 1, by reflection principle

$$\Pr(H \geq k | H_n < k) \cdot \Pr(H_n < k) = \Pr(H_n \geq k+1)$$

Thus

$$\Pr(H \geq k) = \Pr(H_n \geq k) + \Pr(H_n \geq k+1)$$

$$\begin{aligned}
&\Pr(H = k) \\
=&\Pr(H \geq k) - \Pr(H \geq k+1) \\
=&\Pr(H_n \geq k) + \Pr(H_n \geq k+1) - \Pr(H_n \geq k+1) - \Pr(H_n \geq k+2) \\
=&\Pr(H_n = k) + \Pr(H_n = k+1)
\end{aligned}$$

$$\begin{aligned}
E[H_i] =& \sum_{k \geq 0} k(\Pr[H_n = k] + \Pr[H_n = k+1]) \\
=& \sum_{k \geq 1} (2k-1) \Pr[H_n = k] \\
=& \Theta(E[H_n]) \\
=& \Theta(\sqrt{n})
\end{aligned}$$

Problem 4

1. Let $Z = \dfrac{X-\mu_X}{\sigma_X}$

   $$\Pr(X - \mu_X \geq t\sigma_X) = \Pr(Z \geq t) = \Pr\left(Z + \frac{1}{t} \geq t + \frac{1}{t}\right) \leq \Pr\left(\left(Z + \frac{1}{t}\right)^2 \geq \left(t + \frac{1}{t}\right)^2\right)$$

   By Markov's inequallity

   $$\Pr(X - \mu_X \geq t\sigma_X) \leq \frac{E\left[\left(Z + \frac{1}{t}\right)^2\right]}{\left(t + \frac{1}{t}\right)^2}$$

   $$E[Z^2] = \frac{E\left[(X-\mu_X)^2\right]}{E\left[Var[X]\right]} = 1$$

   $$E[Z] = \frac{E[X - mu_X]}{E[Var[X]]} = 0$$

   Thus

   $$\Pr(X - \mu_X \geq t\sigma_X) \leq \frac{1 + \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2} = \frac{1}{1+t^2}$$

2. $$\Pr(|X - \mu_X| \geq t\sigma_X) = \Pr\left((X-\mu_X)^2 \geq t^2 Var[X]\right) = \Pr\left(\frac{(X-\mu_X)^2}{Var[X]} + 1 \geq t^2 + 1\right)$$

   By Markov's inequality

   $$\Pr(|X - \mu_X| \geq t\sigma_X) \leq \frac{E\left[\frac{(X-\mu_X)^2}{Var[X]} + 1\right]}{t^2+1} = \frac{2}{t^2+1}$$