Homework3

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Problem 1

1. Let X be a subset of V. $\forall v \in V$, with probability p, $v \in X$

   $$E[|X|] = np$$

   Let Y be the set of all vertices in $V - X$ that have no neighbours in X

   $$E[|Y|] = n(1-p)^{d+1}$$

   $X \cup Y$ is a dominating set

   $$\begin{aligned} E[|X \cup Y|] &\le E[|X|] + E[|Y|] = np + n(1-p)^{d+1} \\ &= n\left[p + (1-p)^{d+1}\right] \le n\left[p + e^{-p(d+1)}\right] \end{aligned}$$

   let $p = \dfrac{\ln(d+1)}{d+1}$

   $$E[|X \cup Y|] \le n\left[\frac{\ln(d+1)}{d+1} + \frac{1}{d+1}\right] = \frac{n(1+\ln(d+1))}{d+1}$$

2. Let X be a subset of V. $\forall v \in V$, with probability p, $v \in X$
   Let A_i be the event that neither the i-th vertex nor its neighbors are in X
   By Lovasz local lemma, if $e(1-p)^{d+1}n \le 1$, with nonzero probability none of these events happens .
   $\because e(1-p)^{d+1}n \le e^{1-p(d+1)}n$
   $\therefore e^{1-p(d+1)}n \le 1 \Rightarrow e(1-p)^{d+1}n \le 1$
   $\therefore p \ge \dfrac{\ln n + 1}{d+1}$
   $\therefore$ the bound is $n\dfrac{\ln n + 1}{d+1}$
   So the bound is worse.

Problem 2

Suppose $G(V, E)$ dosen't contain H

Covering K_n with k graphs isomorph of G is corresponding to color K_n with k colors.

If an edge is covered more than once, let it be of the first color.

Since H is not a subgraph of the k graphs, if every edge is covered, a k-coloring meeting the conditions exists.

let $X_i = \begin{cases} 1 & \text{the i-th edge is not covered} \\ 0 & \text{otherwise} \end{cases}$

let $X = \sum X_i$

$$E[X] = \sum E[X_i] = \binom{n}{2}\left(1 - \frac{m}{\binom{n}{2}}\right)^k \leq \frac{n^2}{2}e^{\frac{-2mk}{n^2}}$$

let $k = \dfrac{n^2 \ln n}{m}$, $E[X] < 1$

So there is an edge k-coloring for K_n that K_n contains no monochromatic H.

Problem 3

Lemma:

Let $G(V, E)$ be a graph with degree at most 1 for each vertex and have $2n$ vertices.

Let $V = V_1 \cup V_2 \cup \cdots \cup V_n$ be a partition s.t. $\forall V_i, |V_i| = 2$

Then there exists an independent set of G containing precisely one vertex from each V_i

The independent can be find in the following way:

1. Arbitrarily choose a vertex v_1 for an arbitrary set V_1
2. For each i, suppose $v \in V_i$ is choosen.
   If $\exists w\ s.t.\ (v, w) \in E \wedge \forall j \leq i, w \notin V_j$, let the set containing w be V_{i+1}. Else let any remaining set be V_{i+1}.
   Find $v_{i+1} \in V_{i+1}\ s.t.\ (v_i, v_{i+1}) \notin E$

Let $G = (V, E)$ be a cycle of length $4n$. Reorder the vertices of G such that $E = \{(i, (i+1)\text{mod n})\,|i \leq 4n\}$

Let $G' = (V, E')$ with $E' = \{(2i-1, 2i) | i \le 2n\} \subset E$.
Each vertex in G' has exact degree 1
Split V_i of G into 2 sets V_{i1} and V_{i2}
So there must be an independent set S of G' contains exact 1 vertex of each V_{ij} $(j \in \{1, 2\})$
Then an independent set S' of $V \setminus S$ with $V_i \setminus S$ can be found
And S' is obviously an independent set of G containing precisely one vertex from each V_i

Problem 4

Let $V \subseteq U$, each element in U present in V with probability p

$$\begin{aligned} q &\triangleq \Pr(V \text{is isolating set}) \\ &= \Pr(V \cup S = \emptyset) \left[1 - \Pr(T_i \cap V = \emptyset \vee \cdots \vee T_m \cap V = \emptyset)\right] \\ &\geq (1-p)^{|S|} \left[1 - m(1-p)^{|S|}\right] \end{aligned}$$

Let $p = \dfrac{\ln 2m}{|S|}$, $\Pr(V \text{is isolating set}) \geq \dfrac{1}{2m} \cdot \dfrac{1}{2} = \dfrac{1}{4m}$
Let $N = |\mathcal{F}|$
$\forall$ safe set instance I,let $X_I = \begin{cases} 1 & \text{no isolating set in } \mathcal{F} \\ 0 & \text{at least one isolating set in } \mathcal{F} \end{cases}$
Let $X = \sum_I X_I$, # of distinct $I = \dbinom{n}{|S|}\dbinom{n-|S|}{|S|}^m \leq 2^{n(m+1)}$

$$E[X] = \sum_I E[X_i] \leq 2^{n(m+1)}(1-q)^N$$

$$\Pr(X = 0) = 1 - \Pr(X \geq 1) \geq 1 - E[X]$$

Let $N > 4mn(m+1)$

$$\Pr(X = 0) > 2^{n(m+1)} \left(1 - \frac{1}{4m}\right)^{4mn(m+1)} \approx \left(\frac{2}{e}\right)^{n(m+1)} > 0$$

Problem 5

Let $Y = \left|\frac{X}{\delta}\right|$

$$\frac{E\left[e^{t|X|}\right]}{e^{t\delta}} = \frac{E\left[e^{t\delta Z}\right]}{e^{t\delta}} = \frac{E\left[\sum_{i=0}^{\infty} \frac{(t\delta)^i}{i!} Y^i\right]}{e^{t\delta}} = \sum_{i=0}^{\infty} \frac{(t\delta)^i}{i! e^{t\delta}} E[Y^i]$$

$\because \displaystyle\sum_{i=0}^{\infty} \frac{(t\delta)^i}{i! e^{t\delta}} = 1$

$\therefore$ let $k = \mathrm{argmin_i} E[Y^i], E[Y^k] = \dfrac{E\left[|X|^k\right]}{\delta^k} \leq \dfrac{E\left[e^{t|X|}\right]}{e^{t\delta}}$

We prefer Chernoff bound because it's easy to use and calculate.