Homework4

Qinglin Li, 5110309074

Problem 1

$\forall \lambda > 0$

$$\Pr\left(X > (1+\delta)(2n)\right) = \Pr\left(e^{\lambda X} > e^{\lambda(1+\delta)(2n)}\right) \leq \frac{E\left[e^{\lambda X}\right]}{e^{\lambda(1+\delta)(2n)}} = \frac{\prod_{i=1}^{n} E\left[e^{\lambda X_i}\right]}{e^{\lambda(1+\delta)(2n)}} \tag{1}$$

$\because E[e^{\lambda X_i}] = \sum_{k=1}^{\infty} e^{\lambda k}\Pr(X_i = k) = \sum_{k=1}^{\infty} e^{\lambda k}\left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty}\left(\frac{e^{\lambda}}{2}\right)^k = \dfrac{e^{\lambda}}{2-e^{\lambda}}$

$\therefore (1) = \dfrac{\left(\frac{e^{\lambda}}{2-e^{\lambda}}\right)^n}{e^{\lambda(1+\delta)(2n)}} = \left[\dfrac{1}{e^{\lambda(2\delta+1)}\left(2-e^{\lambda}\right)}\right]^n$

let $\lambda = \ln\dfrac{4\delta+2}{2\delta+2}, (1) \leq \left[\dfrac{(\delta+1)^{2\delta+2}}{(2\delta+1)^{2\delta+1}}\right]^n$

Problem 2

1. $\forall x \in \{0,1\}^k$, random choose $y \in \{0,1\}^n$, let $C(x) = y$
   number all messages as $0 \cdots 2^k - 1$
   for some constant λ randomly choose $y1, y2 \in \{0,1\}^n$
   let $X = \sum_{i=1}^{n} |y1(i) - y2(i)|, \quad E[X] = \dfrac{n}{2}$
   by chernoff bound

   $$\Pr\left(X \leq (1-\delta)\frac{n}{2}\right) \leq e^{-\frac{n\delta^2}{4}}$$

   let $\delta = 2\lambda\sqrt{r}\;\; (0 < 2\lambda\sqrt{r} < 1)$

   $$\Pr\left(X \leq \frac{n}{2} - \lambda n\sqrt{r}\right) \leq e^{-\lambda^2 nr} = e^{-\lambda^2 k}$$

let $X_{i,j}$ be the event $d(C(x_i), C(x_j)) \leq \frac{n}{2} - \lambda n\sqrt{r}$
the max degree of the dependency graph is $2 \cdot 2^k$
if $\forall k, 2^{k+1}e^{1-\lambda^2 k} \leq 1$, $\Pr(\cap\overline{X_{i,j}})$, which means $d \geq \frac{n}{2} - \lambda n\sqrt{r}$
let $k = 1$, $\lambda = \sqrt{2\ln 2 + 1}$, $2\lambda\sqrt{r} < 1$ is satisfied.

2. randomly assign 0 and 1 for elements in $A_{n\times k}$

$$\begin{aligned}
&\Pr(\exists x_1, x_2 \in \{0,1\}^k, x_1 \neq x_2 \wedge d(Ax_1, Ax_2) < t) \\
=&\Pr(\exists x \in \{0,1\}^k \setminus \{\mathbf{0}\}, x_1 \neq x_2 \wedge d(Ax, 0) < t) \\
\leq&\sum_{x\in\{0,1\}^k\setminus\{\mathbf{0}\}} \Pr(d(Ax, 0) < t) \\
\leq&2^k \sum_{\substack{y \in \{0,1\}^n \\ d(y,0) < t}} \Pr(Ax = y) \\
=&2^k \sum_{\substack{y \in \{0,1\}^n \\ d(y,0) < t}} 2^{-n}
\end{aligned} \tag{2}$$

let $Vol(t, n)$ be the volumn of n-dim 2-ary Hamming ball with radius t

$$(2) \leq 2^{k-n}Vol(t, n)$$

let $t = (\frac{1}{2} - \sqrt{r})n$

$$(2) \leq 2^{nr-n(1-H(\frac{1}{2}-\sqrt{r}))}$$

$H(x) = -x\log_2 x - (1-x)\log_2 x$
let $2^{nr-n(1-H(\frac{1}{2}-\sqrt{r}))} < 1$
verify with MATLAB, the proposition is true

Problem 3

1. let $X_i = \begin{cases} 1 & \text{i-th machine fails} \\ 0 & \text{otherwise} \end{cases}$

   let $X = \sum_{i=1}^n X_i$
   $\because E[X] = \sum_{i=1}^n E[X_i] \leq 0.05n$

$\therefore \Pr(X \geq 0.08n) = \Pr(X - 0.05n \geq 0.03n) \leq \Pr(X - E[X] \geq 0.03n)$

$\leq \Pr(|X - E[X]| \geq 0.03n) < \exp(-\dfrac{9 \times 10^4 n}{2})$

So the probability is exponentially small as the size of the system grows

2. suppose the # virtual machin is m, # real mathine is n

   let $X_i = \begin{cases} 1 & \text{i-th real machine fails} \\ 0 & \text{otherwise} \end{cases}$

   let $\mathbf{X} = (X_1, \cdots, X_n)$

   let $f(\mathbf{X})$ indicates the number of failed virtual machine when real mathine fails with $\mathbf{X}$

   $|f(x_1, \cdots, x_i, \cdots, x_n) - f(x_1, \cdots, x_i', \cdots, x_n)| \leq d$

   $\Pr\left(f(\mathbf{X}) \geq \frac{m}{100000}\right) \leq \Pr(f(\mathbf{X}) - E[f(\mathbf{X})] \geq 3.75 \times 10^{-6})$

   $\leq \exp\left(\dfrac{2 \times 10^{-11} m^2}{nd^2}\right) \leq \exp\left(\dfrac{2 \times 10^{-11} m}{d^2}\right)$

   So the probability that the system fails is exponentially small as n grows

Problem 4

1. let $Y_k = \#$ runs fo length k

   $Z_i = \begin{cases} 1 & S[i..(i+k-1)] is a run \\ 0 & \text{otherwise} \end{cases}$

   $E[Z_i] = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2}$

   $E[Y_k] = \sum_{i=1}^{n-k+1} E[Z_i] = \left(\frac{1}{2}\right)^{k+1} + (n-k-1)\left(\frac{1}{2}\right)^{k+2}$

   $E[X_k] = \sum_{j=k}^{n} E[Y_j] = \sum_{j=k}^{n} (n-j-1)\left(\frac{1}{2}\right)^{j+2} + \left(\frac{1}{2}\right)^j = (n-k)2^{-k-1} + 2^{-n}$

2. $\because E[X_k] = \sum_{j=k}^{n} E[Y_j], Y_j \in \left[0, (n-k+1)\left(\frac{1}{2}\right)^{k+2}\right]$

   $\therefore \Pr(|X - E[X]| \geq t) \leq 2\exp\left(\dfrac{t^2}{2(n-k+1)^2\left(\frac{1}{2}\right)^{k+2}}\right)$