Homework5

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Problem 1

1. Let $Z_v[t] = \begin{cases} 1 & \text{vertex } v \text{ changes color at step } t \\ 0 & \text{otherwise} \end{cases}$

   Let $C_t[v]$ be the color of vertex v at step t and d_v be the degree of the vertex v

   Let $\begin{aligned} W_t &= \{v \in V | C_t[v] = \texttt{white}\} \\ B_t &= \{v \in V | C_t[v] = \texttt{black}\} \end{aligned}$

   Let $D_t[v] = \Big| \{u \in V | (u, v) \in E \land C_t[u] \neq C_t[v]\} \Big|$, thus

   $$Y_{t+1} - Y_t = \sum_{v \in B_t} d_v Z_t[v] - \sum_{v \in W_t} d_v Z_t[v] \tag{1}$$

   $$E\left[Z_t[v]\right] = \frac{D_t[v]}{2d_v} \tag{2}$$

   Combine (1) with (2)

   $$E\big[Y_{t+1} - Y_t | X_1, \cdots, X_t\big] = \sum_{v \in B_t} d_v \frac{D_t[v]}{2d_v} - \sum_{v \in W_t} d_v \frac{D_t[v]}{2d_v} = \frac{1}{2}\left(\sum_{v \in B_t} D_t[v] - \sum_{v \in W_t} D_t[v]\right)$$

   So

   $$E\big[Y_{t+1} - Y_t | X_1, \cdots, X_t\big] = 0$$

   Then Y_i is a martingale of $\{X_i\}$

2. $\because |Y_{t+1} - Y_t| \leq 2m$ (the total degree)
   $\therefore E\left[|Y_{t+1} - Y_t| \big| X_1, \cdots X_{t-1}\right] \leq 2m$
   Let T be the stopping time, since there exists a system of linear equations of all stopping of all initial states, $E[T] < \infty$

By OST, $E[Y_T] = E[Y_0]$
Let p be the probability that the process terminates in the all-white configuration

$$E[Y_T] = 2mp \Rightarrow p = \frac{Y_0}{2m}$$

Problem 2

1. Let d_v be the degree of vertex v
   Define a random walk, $\forall u \in V$

$$\begin{cases} P(u,v) = \dfrac{1}{\Delta+1} & (u,v) \in E \\ P(u,u) = 1 - \dfrac{d_u}{\Delta+1} \end{cases}$$

   Since $P(u,u) > 0$, the random walk is a aperiodic, then ergodic.

   For uniform distribution π, $\forall u,v \in V \land (u,v) \in E, \pi_u P(u,v) = \pi_v P(v,u)$, So the random walk is time reversible.

   $\forall (u,v) \in E, P(u,v) > 0 \Rightarrow$ the random walk is irreducible
   By fundamental theorem of markov chain
   irreducible + ergodic $\Rightarrow$ stationary distribution is unique $\Rightarrow$ stationary distribution is the uniform distribution

2. $\forall u \in V$, let $S_u = \sum_{(u,v)\in E} \pi_v$

$$\begin{cases} P(u,v) = \dfrac{\pi_v}{2} & (u,v) \in E \\ P(u,u) = 1 - \dfrac{S_u}{2} \end{cases}$$

   By similar process, it can be proved that

Problem 3

The original process is equal to choosing a number i uniformly at random from $\{1, 2, \cdots, n+1\}$ and then changing b_i if $i \neq n+1$

For two process X_t, Y_t, at each step, define the following coupling

1. If $i \neq n+1 \wedge X_t[i] = Y_t[i]$, flip $X_t[i]$ and $Y_t[i]$
2. If $i = n+1 \vee X_t[i] \neq Y_t[i]$, let $I = \{i | X_t[i] \neq Y_t[i]\} \cup \{n+1\} = \{i_1, i_2, \cdots, i_m = n+1\}$
   Define $f : I \to I$
   $$f(i_x) = i_{(x \mod m)+1}$$
   Then flip $X_t[i]$ and $Y_t[f(i)]$

Every time (2) happens, then hamming distance of X_t and Y_t decreases by at least 1
So this coupling is equal to an coupon collector model
The mixing time should be $O(n \ln n)$

Problem 4

1. **Irreducible**

   It is supposed to prove that it is reachable from any coloring C to any coloring D
   Induction on n

   If $n = 1$, then $\Delta = 0$ irreducible is satisfied by two colors
   If $n > 1$, suppose $\forall G$ with $n' < n, \Delta' < \Delta$ is irreducible

   (a) If there exists an vertex v with color $\Delta + 2$, choose arbitary $c \in \{1, 2, \cdots \Delta + 1\} \setminus \{\text{color of } v\text{'s neighbor}\}$ and change the color of v to c
   (c always exists since the maximum degree of v is Δ)

   Repeat this process until there exists no vertex with color $\Delta + 2$ and the whole process can be finished in finite steps
   Apply the whole process on C and D, then get C' and D'
   Since the process is reversible, the reachability of C and D cannot change

(b) Let $V_\Delta \subset V$ be all verteces with degree Δ
If $V_\Delta \neq \emptyset$, let T be a maximum independent set on V_Δ, then $G - T$ has no vertex with degree of Δ
(Otherwise some vertex with degree Δ in $G - T$ has no neighbor with degree Δ in G implying that T is not a maximum indepentdent set)
If $V_\Delta = \emptyset$ let T be any nonempty independent set of G

(c) Since $G - T$ has $n' < n, \Delta' < D$, it is reachable from C' to D'

(d) change the color on T

Thus it is reachable from C to D

Aperiodic

For any state, because it is non-zero probability doing nothing, aperiodic holds

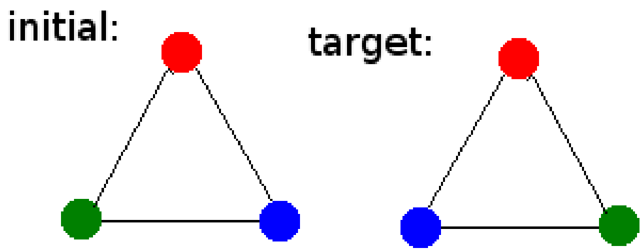
Time reversible

$\forall x, y \in \Omega, P(x, y) > 0$ iff at most one vertex has different color
If $x = y$, $P(x, y) = P(y, x)$
If one vertex has different color, since the probability of choosing that vertex is equal in both states, $P(x, y) = P(y, x)$
$\therefore \forall x, y \in \Omega, \pi_x P(x, y) = \pi_y P(y, x)$
$\therefore$Time reversible

The stationary distribution is the uniform distribution

$\because \pi$ in time reversible proof is stationary distribution
$\therefore$ By fundamental teorem of markov chain the stationary distribution is unique
$\therefore$ The stationary distribution is the uniform distribution

2. Consider $\Delta = 2, q = 3$



Problem 5

1. **Ergodic**

   $\forall 0 < p < 1$, because the probability of doing nothing is nonzero, it is aperiodic then is ergodic

   Time reversible

   $\forall x, y \in \Omega, P(x, y) > 0$ iff at most one element of x and y is different
   If $x = y$, $P(x, y) = P(y, x)$
   If one element of x and y is different, since the probability of choosing that element is equal in both x and y, $P(x, y) = P(y, x)$
   Therefore, $\forall x, y \in \Omega, \pi_x P(x, y) = \pi_y P(y, x)$. So it is time reversible

   The stationary distribution is the uniform distribution

   $\because$ π in time reversible proof is stationary distribution
   $\therefore$ By fundamental teorem of markov chain the stationary distribution is unique
   $\therefore$ The stationary distribution is the uniform distribution

2. For two markov chains X_t and Y_t, define the following coupling
   At step t, X_t do nothing with probability p

(a) If X_t do nothing, Y_t do nothing as well

(b) Otherwise choose a', b' for Y_t as follow

   i. If $a \in Y_t$, $a' = a$ else randomly choose a' in $Y_t \setminus X_t$

   ii. If $b \in [n] - S_2$, $b' = b$ else randomly choose b' in $\overline{Y_t} \cap X_t$

$X_{t+1} = X_t - a + b$
$Y_{t+1} = Y_t - a' + b'$
Let $D_t = \left| X_t \setminus Y_t \right|$
At each step, if $a' \neq a$ or $b' \neq b$, D_t decreases by at least 1

$$
\begin{aligned}
&\Pr(D_t \text{ decreases by at least } 1) \\
=&(1-p)\left(1 - \Pr(a \in X_t \cup Y_t)\Pr(b \in \overline{X_t} \cup \overline{Y_t})\right) \\
=&(1-p)\left(1 - \frac{k - D_t}{k} \cdot \frac{n-k-D_t}{n-k}\right) \\
\leq&(1-p)\frac{D_t}{k}
\end{aligned}
$$

Thus the upperbound of mixing time is

$$\sum_{i=1}^{k} \frac{k}{1-p}\frac{1}{i} = \frac{1}{p} k \ln k = O(n \ln k)$$

So mixing time is asymptotically $O(n \ln k)$ or less