

Homework2

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Problem 1

Since every person have only two hands, the game continue n rounds. The final state must be some cycles.

Before the n -th round, the state must be some lines(a single without self-loop is also considered to be a line) with some cycles.

In every round, we either change a line into a cycle or combine two lines into one line.

Let X_n be the number of cycles when there are n lines remaining.

Let A_n be the event changing a line into cycle when there are n lines remaining.

$$\Pr(A_n) = \frac{n}{\binom{2n}{2}} = \frac{1}{2n-1}$$

$$X_n = \begin{cases} X_{n-1} + 1 & \Pr = \frac{1}{2n-1} \\ X_{n-1} & \Pr = \frac{2n-2}{2n-1} \end{cases} \quad (1)$$

$$E[X_n] = \frac{1}{2n-1} (1 + E[n-1]) + \frac{2n-2}{2n-1} E[n-1] = E[n-1] + \frac{1}{2n-1}$$

$$E[X_n] = \sum_{i=1}^n \frac{1}{2i-1}$$

Problem 2

Problem 3

Let H_i be Number of Heads - Number of Tails

$$E[|H_i|] = \sum_{x>0} x \Pr(H_i = x) - \sum_{x<0} x \Pr(H_i = x) = 2 \sum_{x>0} x \Pr(H_i = x)$$

Let $i = 2n$

$$E[|H_i|] = 4 \sum_{y=0}^n y \frac{\binom{2n}{n+y}}{2^{2n}} = 2^{2-2n} \sum_{y=0}^n y \binom{2n}{n+y}$$

$$\text{Let } S_1 = \sum_{y=0}^n y \binom{2n}{n+y} = 0 \cdot \binom{2n}{n} + \dots + n \cdot \binom{2n}{2n}$$

$$\text{Let } S_2 = \sum_{y=0}^n y \binom{2n}{y} = n \cdot \binom{2n}{n} + \dots + 0 \cdot \binom{2n}{0} = n \cdot \binom{2n}{n} + \dots + 0 \cdot \binom{2n}{2n}$$

$$S_1 + S_2 = n \sum_{y=n}^{2n} \binom{2n}{y} = \frac{n}{2} \left(2^{2n} + \binom{2n}{n} \right)$$

$$S_2 = \sum_{y=0}^n y \binom{2n}{y} = \sum_{y=0}^n 2n \binom{2n-1}{y-1} = 2n \sum_{y=0}^{n-1} \binom{2n-1}{y} = n \cdot 2^{2n-1}$$

$$S_1 = (S_1 + S_2) - S_2 = \frac{n}{2} \binom{2n}{n}$$

$$E[|H_i|] = 2^{2-2n} S_1 = 2^{2-2n} \frac{(2n)!n}{n!n!}$$

By Stirling's approximation

$$E[|H_i|] \approx 2^{2-2n} \frac{\sqrt{4\pi n} \left(\frac{2n}{e}\right)^{2n} n}{2\pi n \left(\frac{n}{e}\right)^{2n}} = 4\sqrt{\frac{n}{\pi}} = \Theta(\sqrt{n})$$

$$\begin{aligned} & \Pr(H \geq k) \\ &= \Pr(H \geq k | H_n \geq k) \cdot \Pr(H_n \geq k) + \Pr(H \geq k | H_n < k) \cdot \Pr(H_n < k) \\ &= \Pr(H_n \geq k) + \Pr(H \geq k | H_n < k) \cdot \Pr(H_n < k) \end{aligned}$$

Since H_n is either greater or less than H_{n-1} by 1, by reflection principle

$$\Pr(H \geq k | H_n < k) \cdot \Pr(H_n < k) = \Pr(H_n \geq k+1)$$

Thus

$$\Pr(H \geq k) = \Pr(H_n \geq k) + \Pr(H_n \geq k+1)$$

$$\begin{aligned}
& \Pr(H = k) \\
&= \Pr(H \geq k) - \Pr(H \geq k + 1) \\
&= \Pr(H_n \geq k) + \Pr(H_n \geq k + 1) - \Pr(H_n \geq k + 1) - \Pr(H_n \geq k + 2) \\
&= \Pr(H_n = k) + \Pr(H_n = k + 1)
\end{aligned}$$

$$\begin{aligned}
E[H_i] &= \sum_{k \geq 0} k(\Pr[H_n = k] + \Pr[H_n = k + 1]) \\
&= \sum_{k \geq 1} (2k - 1) \Pr[H_n = k] \\
&= \Theta(E[H_n]) \\
&= \Theta(\sqrt{n})
\end{aligned}$$

Problem 4

$$1. \text{ Let } Z = \frac{X - \mu_X}{\sigma_X}$$

$$\Pr(X - \mu_X \geq t\sigma_X) = \Pr(Z \geq t) = \Pr\left(Z + \frac{1}{t} \geq t + \frac{1}{t}\right) \leq \Pr\left(\left(Z + \frac{1}{t}\right)^2 \geq \left(t + \frac{1}{t}\right)^2\right)$$

By Markov's inequality

$$\Pr(X - \mu_X \geq t\sigma_X) \leq \frac{E\left[\left(Z + \frac{1}{t}\right)^2\right]}{\left(t + \frac{1}{t}\right)^2}$$

$$E[Z^2] = \frac{E[(X - \mu_X)^2]}{E[Var[X]]} = 1$$

$$E[Z] = \frac{E[X - \mu_X]}{E[Var[X]]} = 0$$

Thus

$$\Pr(X - \mu_X \geq t\sigma_X) \leq \frac{1 + \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2} = \frac{1}{1 + t^2}$$

2.

$$\Pr(|X - \mu_X| \geq t\sigma_X) = \Pr((X - \mu_X)^2 \geq t^2 \text{Var}[X]) = \Pr\left(\frac{(X - \mu_X)^2}{\text{Var}[X]} + 1 \geq t^2 + 1\right)$$

By Markov's inequality

$$\Pr(|X - \mu_X| \geq t\sigma_X) \leq \frac{E\left[\frac{(X - \mu_X)^2}{\text{Var}[X]} + 1\right]}{t^2 + 1} = \frac{2}{t^2 + 1}$$