Homework 1

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Problem 1

(1) Toss the coin twice, then we can generate a pair (x, y), where $x, y \in$ {HEAD,TAIL}
If $x \neq y$, use x as the result.
If $x = y$, toss the coin twice again, until $x \neq y$.
$\because \texttt{Pr}(x \neq y) = 2p(1-p)$
$\therefore$ expected number of pairs is $\dfrac{1}{2p(1-p)}$
$\therefore$ expected number of flips is $\dfrac{1}{p(1-p)}$ $\square$

(2) Present the sequence as pairs $(x_1, y_1), (x_2, y_2), \cdots .(x_n, y_n)$
$\forall x_i \neq y_i$, let the result be x_i
And then, generate two sequence from the original sequence:

(a) Remove all the pairs with different elements.
For all remaining pairs (x_i, y_i), let $x_i' = x_i$

(b) $\forall (x_i, y_i)$, let $x_i' = \begin{cases} HEAD & (x_i = y_i) \\ TAIL & (x_i \neq y_i) \end{cases}$
And we can do the same thing on the two new generated sequence. $\square$

Problem 2

(1) Let N be the smallest power of 2 greater or equal than n
Generate random bits for $\log_2 N$ times, and these random bits makes up a number m.
If $m < n$, let m be the sample, else regenerate m until $m < n$.
Let X be the time of generating m.
Thus, $\Pr(m < n) = \dfrac{n}{N}$

$E[X] = \dfrac{N}{n}$

The expected time to generate bits is $\dfrac{N \log_2 N}{n}$ $\square$

(2) If n is a power of 2, in any case, we need to generate $\lceil \log_2 n \rceil$ bits.
If n is not a power of 2, in the worst case, we always need to generate random bits. $\square$

(3) Let N be the smallest power of p greater or equal than n
Generate random numbers for $\log_p N$ times, and these random numbers makes up a number m under the base of p.
If $m < n$, let m be the sample, else regenerate m until $m < n$.

If n is a power of p, in any case, we need to generate $\lceil \log_p n \rceil$ bits.
If n is not a power of p, in the worst case, we always need to generate random bits. $\square$

Problem 3

(1) $\Pr(R \cap S = \emptyset) = (1-p)^n$
$\Pr(R \cap S \neq \emptyset) = 1 - (1-p)^n$
$\Pr(R \cap S = \emptyset \wedge R \cap S \neq \emptyset) = (1 - \frac{1}{n})^n [1 - (1 - \frac{1}{n})^n]$
$\lim_{n \to +\infty} (1 - \frac{1}{n})^n [1 - (1 - \frac{1}{n})^n] = (1 - \frac{1}{e}) \frac{1}{e}$

Arbitrarily choose $\varepsilon_0 \in \mathbb{R} \qquad$ s.t. $\varepsilon_0 < (1 - \frac{1}{e}) \frac{1}{e}$
$\exists N, \forall n > N$
$|(1 - \frac{1}{n})^n [1 - (1 - \frac{1}{n})^n] - (1 - \frac{1}{e}) \frac{1}{e}| < \varepsilon_0 \Rightarrow (1 - \frac{1}{n})^n [1 - (1 - \frac{1}{n})^n] > (1 - \frac{1}{e}) \frac{1}{e} - \varepsilon_0 > 0$
Let $M = \min\{\Pr|_{n=1}, \Pr|_{n=2}, \cdots, \Pr|_{n=N-1}, (1 - \frac{1}{e}) \frac{1}{e} - \varepsilon_0\}$

$\forall n, \Pr \geq M$ $\square$

(2) Let X_i indicate whether the i-th element of S is choosen.
Let Y_i indicate whether the i-th element of T is choosen.
Let X be the number of choosen elements of S, and Y be the choosen elements of T.
Thus, $X = \sum_i X_i \; , \; Y = \sum_i Y_i$

$$
\begin{aligned}
&\Pr(R \cap S = \emptyset \wedge R \cap S \neq \emptyset) \\
=& \Pr(X = 0 \wedge Y > 0) \\
=& \Pr(Y > 0) - \Pr(X > 0 \wedge Y > 0) \\
=& \Pr(Y > 0) - \Pr(XY \geq 1)
\end{aligned}
$$

By Markov's inequality, $\Pr(XY \geq 1) \leq \dfrac{E[XY]}{1} = E[XY]$

For an arbitrary positive number λ

$$
\begin{aligned}
E[Y] &= \sum_{0<x<\lambda} x \Pr(Y = x) + \sum_{x \geq \lambda} x \Pr(Y = x) \\
&\leq \lambda \Pr(Y > 0) + \sum_{x \geq \lambda} \frac{x^2}{\lambda} \Pr(Y = x) \\
&\leq \lambda \Pr(Y > 0) + \frac{1}{\lambda} E[X^2]
\end{aligned}
$$

$\therefore \Pr(Y > 0) \geq \dfrac{1}{\lambda} E[Y] - \dfrac{1}{\lambda^2} E[Y^2]$

$\therefore \Pr(Y > 0) - \Pr(XY \geq 1) \geq \dfrac{1}{\lambda} E[Y] - \dfrac{1}{\lambda^2} E[Y^2] - E[XY]$

Let $\alpha = E[Y]$

$E[Y^2] = E[\sum_i Y_i Y_i + \sum_{i \neq j} Y_i Y_j] = E[Y] + \sum_{i \neq j} E[Y_i Y_j]$

By pairwise independent, $E[Y^2] = \alpha + \dfrac{n-1}{n} \alpha, E[XY] = E[X]E[Y] = \alpha^2$

$\therefore \Pr(R \cap S = \emptyset \wedge R \cap S \neq \emptyset) \geq \dfrac{1}{\lambda} \alpha - \dfrac{1}{\lambda^2} \left(\alpha + \dfrac{n-1}{n} \alpha^2 \right) - \alpha^2$ $\square$

Problem 4

If currently k balls have been choosen from the bin.
Generate a random number r less than n.
If $r < k$ randomly pick one of the k balls, else pick a ball from the bin.
Let b be one of the k balls.

$\Pr(\text{pick} b) = \dfrac{k}{n} * \dfrac{1}{k} = \dfrac{1}{n}$

Let b' be a ball in the bin.
Since all balls in the bin are randomly choosen from all balls.

$\Pr(\text{pick} b') = \dfrac{n-k}{n} * \dfrac{1}{n-k} = \dfrac{1}{n}$ $\square$