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REGIONAL STABILITY AND STABILIZATION OF TIME-DELAY SYSTEMS WITH ACTUATOR SATURATION AND DELAY

Yan-Ming Fu, Bin Zhou, and Guang-Ren Duan

ABSTRACT

This paper is concerned with regional stability analysis and regional stabilization of time-delay systems subject to both actuator saturation and delay. By using a novel Lyapunov-Krasovskii functional, the corresponding conditions for regional stability and stabilization are derived. The free parameters in the solutions are optimized to maximize the domain of attraction of the closed-loop system. It is shown that, even in the absence of actuator delay, the obtained result is less conservative than the existing results. Two numerical examples are presented to illustrate the effectiveness of the proposed approach.

Key Words: Time-delay systems, actuator saturation, actuator delay, regional stabilization.

I. INTRODUCTION

Time-delay is frequently encountered in many engineering applications such as analog circuits, digital systems, and neural networks. Control of time-delay systems is a challenging task because it is of infinite dimension [9,18]. Over the past several decades, many approaches have been developed to solve various control problems for time-delay systems [4,16,14,19]. Since explicit conditions and solutions for stability and stabilization are generally hard to obtain, many numerical conditions and solutions based on linear matrix inequalities (LMIs) have been proposed [3,10,20,25,26] and the references therein.

On the other hand, saturation nonlinearity commonly exists in various practical systems like analog circuits, digital filters and control systems. Saturation nonlinearity is not only the cause of performance degeneration, but also the source of instability of practical systems [12]. During the last twenty years, considerable research has been directed towards this topic, yielding many analysis and synthesis approaches. For example, the semi-global stabilization problem is considered in [15], while problems of estimation of the domain of attraction and local stabilization are studied in [12] and [21], respectively. For more related works, see [2,5,11–13,23] and the references therein.

It is thus natural and very important to consider control problems that involve both time delay and actuator saturation. Indeed, many results have been proposed for such problems in recent years. Investigating time-delay systems subject to actuator saturation appears even more difficult. The estimation of the domain of attraction and the design of controllers that maximize it were considered in [21]. Anti-windup design for linear time-delay systems with input saturation was considered in [7] and [22] where the methods can be easily adapted to state feedback synthesis problems. In [6], a constraint on the derivative of the initial conditions was introduced in characterizing the invariant set of linear time-delay systems with input saturation. In [1], the problem of estimation of the domain of attraction and controller design for time-delay systems subject to actuator saturation was considered and a series of delay-independent and delay-dependent solutions were proposed by using LMIs. The global stabilization of a chain of integrators by delayed and bounded feedback were considered in [17]. Recently, by using the parametric Lyapunov equation based approach, we have solved the semi-global stabilization problem for a class of linear systems with arbitrarily large delay and saturation nonlinearity in the input in [28–30].

The aim of this paper is to consider local stability and stabilization of linear systems subject to both state delay and actuator saturation and delay. This problem, however, to the best of our knowledge, has not been well studied in the literature. A novel Lyapunov-Krasovskii functional is introduced to establish the corresponding stability condition and existence condition for stabilization. The free parameters introduced by using the free weighting matrix technique (e.g. [10]) in such LMI-type conditions can be optimized so as to

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Bin Zhou is the corresponding author (e-mail: binzhou@hit.edu.cn; binzhoulee@163.com).

The authors are with Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin, China.

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enlarge the domain of attraction of the resulting closed-loop system. Although there are plenty of papers that are devoted to dealing with different problems for systems with actuator saturation, the main difference and difficulty lie in their treatment of saturation nonlinearity. In this paper, we have used the polytopic representation of saturation nonlinearity that was initially proposed in [21] and [12] (see also [1]), which can considerably reduce the conservatism as has been noticed in the literature. Such a technique can also reduce the number of slack variables compared with the method in [27]. Even for the degenerated case that there is no actuator delay, it will be seen from numerical examples that, compared with the existing results, the proposed conditions can lead to a lower conservatism. The applicability of the proposed method is exemplified by numerical results.

The remainder of this paper is organized as follows. The problem formulation is given in Section II and the main results are presented in Section III, which contains two subsections dealing with systems with both state and actuator delays and systems with only state delay, respectively. Numerical examples are given in Section IV and Section V concludes the paper.

II. PROBLEM FORMULATION

In this paper we are interested in the following linear time-delay system subject to actuator saturation and delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_r x(t-r) + B\mathrm{sat}(u(t-\tau)) \\ x(t) = \phi(t), \forall t \in [-d, 0], d = \max\{r, \tau\}, \end{cases} \tag{1}$$

where $A, A_r \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are given constant matrices, r and τ are two nonnegative scalars, and sat(u): $\mathbf{R}^m \to \mathbf{R}^m$ is a vector valued standard saturation function, *i.e.*,

$$\mathrm{sat}(u) = [\mathrm{sat}(u_1) \quad \cdots \quad \mathrm{sat}(u_m)]^{\mathrm{T}}$$

with $\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\min\{1, |u_i|\}$. Here we have assumed, without loss of generality, the unity saturation level [1]. When a linear state feedback

$$u(t) = F^{\mathrm{T}} x(t), F \in \mathbf{R}^{n\times m}$$

is applied on to system (1), the resulting closed-loop system reads

$$\begin{cases} \dot{x}(t) = Ax(t) + A_r x(t-r) + B\mathrm{sat}(F^{\mathrm{T}} x(t-\tau)) \\ x(t) = \phi(t), \forall t \in [-d, 0]. \end{cases} \tag{2}$$

Let $\mathcal{C}_{n,\tau} = \mathcal{C}([-\tau, 0], \mathbf{R}^n)$ denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^n$ with the topology of uniform convergence, and $x_t \in \mathcal{C}_{n,\tau}$ denote the restriction of $x(t)$ to the interval $[t-\tau, t]$ translated to $[-\tau, 0]$. For $x_0 \in \mathcal{C}_{n,d}$, the solution of system (2) is denoted by $x(x_0, t)$. If the trivial solution $x(t) \equiv 0$ is asymptotically stable for system (2), then the domain of attraction is defined as

$$\mathcal{D} = \left\{ x_0 \in \mathcal{C}_{n,d} : \lim_{t\to\infty} \|x(x_0, t)\| = 0 \right\}.$$

A set $\mathcal{I} \subset \mathcal{C}_{n,d}$ is called invariant for system (2) if $x_0 \in \mathcal{I} \Rightarrow x(x_0, t) \in \mathcal{I}$, $\forall t \geq 0$. Moreover, if a set $\mathcal{I}$ is an invariant set for system (2) and is in the domain of attraction, then it is called a contractively invariant set.

The problems under investigation in this paper are stated as follows.

Problem 1. Consider the linear time-delay system (1) subject to actuator saturation and delay.

(i) For a given F, how to check whether the closed-loop system (2) is locally stable?
(ii) For a given F, if system (2) is locally stable, how to find a set $\mathcal{M} \subset \mathcal{C}_{n,d}$ such that $\mathcal{M}$ is an estimate of the domain of attraction for the system, and is as large as possible?
(iii) How to design a matrix F such that an estimate of the domain of attraction for the closed-loop system (2) is maximized?

Let $\mathcal{V}$ be the set of all $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0, so there are 2^m elements in $\mathcal{V}$. Assume that each element in $\mathcal{V}$ is labeled as D_i, $i \in \mathcal{V}_m \triangleq \{1, 2^m\}$. Moreover, let $D_i^- = I - D_i, i \in \mathcal{V}_m$. Then we can introduce the following lemma for later use.

Lemma 1 [12]**.** For two vectors $u, v \in \mathbf{R}^m$. If $\|v\|_\infty \leq 1$, then

$$\mathrm{sat}(u) \in \mathrm{co}\{D_i u + D_i^- v, i \in \mathcal{V}_m\},$$

where $\mathrm{co}\{\cdot\}$ denotes the convex hull of a set.

Finally, $\mathrm{He}(A) = A^{\mathrm{T}} + A$ and ★ in a matrix denotes the term that can be induced by symmetry. For example,

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix},$$

where X and Z are symmetric matrices.

III. MAIN RESULTS

In this section, we first present in Subsection 3.1 the solutions to Problem 1 in the general case that both r and τ are nonzero. To show that the proposed approach is less conservative than the existing results, the special case that $\tau = 0$, namely, there is no delay in the actuator, will be presented in Subsection 3.2.

3.1 Input saturated systems with both state and actuator delays

For a given $\phi \in \mathcal{C}_{n,d}$ and a series of given matrices $P_i > 0$, $i \in \{1, 7\}$, we define

$$\begin{aligned} v(\phi) &= \phi^{\mathrm{T}}(0)P_1\phi(0) + \int_{-\tau}^{-r}\phi^{\mathrm{T}}(s)(\tau - r)P_4\phi(s)\mathrm{d}s \\ &\quad + \int_{-\tau}^{0}\phi^{\mathrm{T}}(s)P_3\phi(s)\mathrm{d}s + \int_{-r}^{0}\phi^{\mathrm{T}}(s)P_2\phi(s)\mathrm{d}s \\ &\quad + \int_{-r}^{0}\int_{s}^{0}\dot{\phi}^{\mathrm{T}}(\theta)P_5\dot{\phi}(\theta)\mathrm{d}\theta\mathrm{d}s \\ &\quad + \int_{-\tau}^{0}\int_{s}^{0}\dot{\phi}^{\mathrm{T}}(\theta)P_6\dot{\phi}(\theta)\mathrm{d}\theta\mathrm{d}s \\ &\quad + \int_{-\tau}^{-r}\int_{s}^{0}\dot{\phi}^{\mathrm{T}}(\theta)(\tau - r)P_7\dot{\phi}(\theta)\mathrm{d}\theta\mathrm{d}s. \end{aligned} \tag{3}$$

We first present the following theorem.

Theorem 1. Consider the linear time-delay system (2). Assume that $F \in \mathbf{R}^{n\times m}$ is given, $\tau \neq 0$, $r \neq 0$ and $r \neq \tau$. If there exist matrices $0 < P_i \in \mathbf{R}^{n\times n}$, $i \in \{1, 7\}$, X_r, X_τ, $X_{r\tau} \in \mathbf{R}^{4n\times n}$, $Y \in \mathbf{R}^{n\times n}$, $H \in \mathbf{R}^{n\times m}$, and scalars ψ_0, ψ_τ, ψ_r such that the following inequalities

$$\begin{bmatrix} \Omega(i) & X_r & X_\tau & X_{r\tau} \\ \star & -\frac{1}{r}P_5 & 0 & 0 \\ \star & \star & -\frac{1}{\tau}P_6 & 0 \\ \star & \star & \star & -P_7 \end{bmatrix} < 0, i \in \mathcal{V}_m, \tag{4}$$

and the relations

$$\begin{aligned} &\left|H_i^{\mathrm{T}}x(t)\right| \leq 1, i \in \{1, m\}, \\ &H = \begin{bmatrix} H_1 & H_2 & \cdots & H_m \end{bmatrix} \end{aligned} \tag{5}$$

are satisfied for all $x_t \in \mathcal{M}_v$, where

$$\mathcal{M}_v = \{\phi \in \mathcal{C}_{n,d} : v(\phi) \leq 1\}, \tag{6}$$

then the solution $x(t) \equiv 0$ is asymptotically stable for the closed-loop system (2) with the set $\mathcal{M}_v$ contained in the domain of attraction. In (4), the matrices X_r, X_τ, $X_{r\tau}$ are related with

$$\begin{bmatrix} X_r^{\mathrm{T}} \\ X_\tau^{\mathrm{T}} \\ X_{r\tau}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} X_{r1} & X_{\tau1} & X_{r\tau1} \\ X_{r2} & X_{\tau2} & X_{r\tau2} \\ X_{r3} & X_{\tau3} & X_{r\tau3} \\ X_{r4} & X_{\tau4} & X_{r\tau4} \end{bmatrix}, \tag{7}$$

and $\Omega(i) = [\Omega_{jk}]$ in which Ω_{jk}, $j, k \in \{1, 4\}$ are given by

$$\Omega_{11} = \mathrm{He}\left(X_{\tau1} + X_{r1} + \psi_0 YA + \psi_0 YBD_iH^{\mathrm{T}}\right) + P_2 + P_3,$$

$$\begin{aligned} \Omega_{12} &= X_{r2}^{\mathrm{T}} - X_{r1} - X_{r\tau1} + X_{\tau2}^{\mathrm{T}} + \psi_0 YA_r \\ &\quad + \psi_r A^{\mathrm{T}}Y^{\mathrm{T}} + \psi_r HD_iB^{\mathrm{T}}Y^{\mathrm{T}}, \end{aligned}$$

$$\begin{aligned} \Omega_{13} &= X_{r\tau1} + X_{\tau3}^{\mathrm{T}} - X_{\tau1} + X_{r3}^{\mathrm{T}} + \psi_\tau A^{\mathrm{T}}Y^{\mathrm{T}} \\ &\quad + \psi_0 YBD_i^- F^{\mathrm{T}} + \psi_\tau HD_iB^{\mathrm{T}}Y^{\mathrm{T}}, \end{aligned}$$

$$\Omega_{14} = X_{\tau4}^{\mathrm{T}} + X_{r4}^{\mathrm{T}} + A^{\mathrm{T}}Y^{\mathrm{T}} - \psi_0 Y + HD_iB^{\mathrm{T}}Y^{\mathrm{T}} + P_1,$$

$$\Omega_{22} = \mathrm{He}\left(\psi_r YA_r - X_{r\tau2} - X_{r2}\right) - P_2 + (\tau - r)P_4,$$

$$\Omega_{23} = X_{r\tau2} - X_{r\tau3}^{\mathrm{T}} - X_{\tau2} - X_{r3}^{\mathrm{T}} + \psi_\tau A_r^{\mathrm{T}}Y^{\mathrm{T}} + \psi_r YBD_i^- F^{\mathrm{T}},$$

$$\Omega_{24} = -X_{r\tau4}^{\mathrm{T}} - X_{r4}^{\mathrm{T}} + A_r^{\mathrm{T}}Y^{\mathrm{T}} - \psi_r Y,$$

$$\Omega_{33} = \mathrm{He}\left(X_{r\tau3} - X_{\tau3} + \psi_\tau YBD_i^- F^{\mathrm{T}}\right) - P_3 - (\tau - r)P_4,$$

$$\Omega_{34} = X_{r\tau4}^{\mathrm{T}} - X_{\tau4}^{\mathrm{T}} - \psi_\tau Y + FD_i^- B^{\mathrm{T}}Y^{\mathrm{T}},$$

$$\Omega_{44} = -Y - Y^{\mathrm{T}} + rP_5 + \tau P_6 + (\tau - r)^2 P_7.$$

Proof. Choose the Lyapunov-Krasovskii functional $V(x_t) = v(x_t)$, where $v(x_t)$ is defined in (3). Then by denoting $\eta^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t)x^{\mathrm{T}}(t-r)x^{\mathrm{T}}(t-\tau)\dot{x}^{\mathrm{T}}(t)\right]$, we have

$$\begin{aligned} \dot{V}(x_t) &= \dot{x}^{\mathrm{T}}(t)P_1x(t) + x^{\mathrm{T}}(t)P_1\dot{x}(t) \\ &\quad + x^{\mathrm{T}}(t)P_2x(t) - x^{\mathrm{T}}(t-r)P_2x(t-r) \\ &\quad - x^{\mathrm{T}}(t-\tau)P_3x(t-\tau) + x^{\mathrm{T}}(t)P_3x(t) \\ &\quad + x^{\mathrm{T}}(t-r)(\tau - r)P_4x(t-r) \\ &\quad - x^{\mathrm{T}}(t-\tau)(\tau - r)P_4x(t-\tau) \\ &\quad - \int_{t-r}^{t}\dot{x}^{\mathrm{T}}(\theta)P_5\dot{x}(\theta)\mathrm{d}\theta + r\dot{x}^{\mathrm{T}}(t)P_5\dot{x}(t) \\ &\quad - \int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(\theta)P_6\dot{x}(\theta)\mathrm{d}\theta + \tau\dot{x}^{\mathrm{T}}(t)P_6\dot{x}(t) \\ &\quad - \int_{t-\tau}^{t-r}\dot{x}^{\mathrm{T}}(\theta)(\tau - r)P_7\dot{x}(\theta)\mathrm{d}\theta \\ &\quad + \dot{x}^{\mathrm{T}}(t)(\tau - r)^2P_7\dot{x}(t) \\ &= \eta^{\mathrm{T}}(t)T_1\eta(t) - \int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(\theta)P_6\dot{x}(\theta)\mathrm{d}\theta \\ &\quad - \int_{t-\tau}^{t-r}\dot{x}^{\mathrm{T}}(\theta)(\tau - r)P_7\dot{x}(\theta)\mathrm{d}\theta \\ &\quad - \int_{t-r}^{t}\dot{x}^{\mathrm{T}}(\theta)P_5\dot{x}(\theta)\mathrm{d}\theta, \end{aligned} \tag{8}$$

where T_1 is defined as

$$T_1=\begin{bmatrix} T_{11} & 0 & 0 & P_1 \\ 0 & T_{22} & 0 & 0 \\ 0 & 0 & T_{33} & 0 \\ P_1 & 0 & 0 & T_{44} \end{bmatrix}, \tag{9}$$

with $T_{11} = P_2 + P_3$, $T_{22} = (\tau - r)P_4 - P_2$, $T_{33} = -P_3 - (\tau - r)P_4$ and $T_{44} = rP_5 + \tau P_6 + (\tau - r)^2P_7$. By using the Newton-Leibniz formula and the system (2), we have the following four identities:

$$\begin{aligned} 0 &= 2\eta^{\mathrm{T}}(t)X_{\mathrm{r}}\left(x(t)-x(t-r)-\int_{t-r}^{t}\dot{x}(\theta)\mathrm{d}\theta\right) \\ 0 &= 2\eta^{\mathrm{T}}(t)X_{\tau}\left(x(t)-x(t-\tau)-\int_{t-\tau}^{t}\dot{x}(\theta)\mathrm{d}\theta\right) \\ 0 &= 2\eta^{\mathrm{T}}X_{\mathrm{r}\tau}\left(x(t-\tau)-x(t-r)-\int_{t-r}^{t-\tau}\dot{x}(\theta)\mathrm{d}\theta\right) \\ 0 &= 2\eta^{\mathrm{T}}(t)Y_{\eta}\left(Ax(t)+A_{\mathrm{r}}x(t-r)+B\mathrm{sat}\left(F^{\mathrm{T}}x(t-\tau)\right)-\dot{x}(t)\right), \end{aligned} \tag{10}$$

where X_{r}, X_{τ} and $X_{r\tau}$ are in the form of (7) and

$$Y_{\eta}^{\mathrm{T}}=\left[\psi_0 Y^{\mathrm{T}}\ \psi_{\mathrm{r}}Y^{\mathrm{T}}\ \psi_{\tau}Y^{\mathrm{T}}\ Y^{\mathrm{T}}\right]. \tag{11}$$

Inserting the above four identities into (8) gives

$$\begin{aligned} \dot{V}(x_t) &= \eta^{\mathrm{T}}T_3\eta+2\eta(t)^{\mathrm{T}}Y_{\eta}B\mathrm{sat}\left(F^{\mathrm{T}}x(t-\tau)\right) \\ &\quad -\int_{t-\tau}^{t}z_2^{\mathrm{T}}(\theta)P_6^{-1}z_2(\theta)\mathrm{d}\theta-\int_{t-r}^{t}z_1^{\mathrm{T}}(\theta)P_5^{-1}z_1(\theta)\mathrm{d}\theta \\ &\quad -\int_{t-\tau}^{t-r}z_3^{\mathrm{T}}(\theta)\left((\tau-r)P_7\right)^{-1}z_3(\theta)\mathrm{d}\theta \\ &\le \eta^{\mathrm{T}}(t)T_3\eta(t)+2\eta(t)^{\mathrm{T}}Y_{\eta}B\mathrm{sat}\left(F^{\mathrm{T}}x(t-\tau)\right), \end{aligned} \tag{12}$$

where

$$z_1(\theta)=X_{\mathrm{r}}^{\mathrm{T}}\eta(t)+P_5\dot{x}(\theta),$$

$$z_2(\theta)=X_{\tau}^{\mathrm{T}}\eta(t)+P_6\dot{x}(\theta),$$

$$z_3(\theta)=X_{\mathrm{r}\tau}^{\mathrm{T}}\eta(t)+(\tau-r)P_7\dot{x}(\theta),$$

and

$$T_3=T_1+T_2+rX_{\mathrm{r}}P_5^{-1}X_{\mathrm{r}}^{\mathrm{T}}+\tau X_{\tau}P_6^{-1}X_{\tau}^{\mathrm{T}}+X_{\mathrm{r}\tau}P_7^{-1}X_{\mathrm{r}\tau}^{\mathrm{T}}, \tag{13}$$

with T_2 defined as

$$\begin{aligned} T_2 &= \mathrm{He}\left(X_{\mathrm{r}}[I\ \ -I\ \ 0\ \ 0]\right)+\mathrm{He}\left(X_{\tau}[I\ \ 0\ \ -I\ \ 0]\right) \\ &\quad +\mathrm{He}\left(X_{\mathrm{r}\tau}[0\ \ -I\ \ I\ \ 0]\right)+\mathrm{He}\left(Y_{\eta}[A\ \ A_{\mathrm{r}}\ \ 0\ \ -I]\right). \end{aligned} \tag{14}$$

By Lemma 1, we conclude from (5) that

$$2\eta^{\mathrm{T}}(t)Y_{\eta}B\mathrm{sat}\left(F^{\mathrm{T}}x(t-\tau)\right)\le \eta^{\mathrm{T}}(t)\max_{i\in\mathcal{V}_m}\{T_4(i)\}\eta(t),$$

where

$$T_4(i)=\mathrm{He}\left(\left[Y_{\eta}BD_iH^{\mathrm{T}}\ \ 0\ \ Y_{\eta}BD_i^{-}F^{\mathrm{T}}\ \ 0\right]\right).$$

Thus the inequality in (12) can be continued as

$$\dot{V}(x_t)\le \max_{i\in\mathcal{V}_m}\eta^{\mathrm{T}}(t)\Upsilon(i)\eta(t),$$

where $\Upsilon(i) = T_3 + T_4(i)$. This implies that $\dot{V}(x_t)<0$ if $\Upsilon(i) < 0$, $\forall i\in\mathcal{V}_m$, which, via a Schur complement procedure and noting that $T_1 + T_2 + T_4(i) = \Omega(i)$, is equivalent to (4). Consequently, the set $\mathcal{M}_v$ is a contractively invariant set, and the proof is completed.

The contractively invariant set $\mathcal{M}_v$ is complicated as integrals are involved. To simplify this set, as done in [27], we can consider a subset $\mathcal{B}(\vartheta_1,\vartheta_2)\subset\mathcal{C}_{n,d}$, where

$$\mathcal{B}(\vartheta_1,\vartheta_2)=\left\{\phi\in\mathcal{C}_{n,d}:\|\phi\|_{\mathrm{c}}\le\vartheta_1,\|\dot{\phi}\|_{\mathrm{c}}\le\vartheta_2\right\}, \tag{15}$$

in which $\|\mu\|_{\mathrm{c}} = \max_{\theta\in[-d,0]}\|\mu(\theta)\|$, $\mu\in\mathcal{C}_{n,d}$, where $\|\cdot\|$ denotes the usual Euclidean norm. Note that

$$v(\phi)\le p_1\|\phi\|_{\mathrm{c}}^2+p_2\|\dot{\phi}\|_{\mathrm{c}}^2, \tag{16}$$

where

$$p_1=\|P_1\|+r\|P_2\|+\tau\|P_3\|+(\tau-r)^2\|P_4\|,$$

$$p_2=\frac{r^2}{2}\|P_5\|+\frac{\tau^2}{2}\|P_6\|+\frac{(\tau-r)^2(\tau+r)}{2}\|P_7\|.$$

Then we conclude from (6) that

$$\mathcal{B}_p=\mathcal{B}\left(\sqrt{\frac{1}{2p_1}},\sqrt{\frac{1}{2p_2}}\right)\subseteq\mathcal{M}_v,$$

that is to say, the set $\mathcal{B}_p$ is an estimate of the domain of attraction for the nonlinear time-delay system (2). In particular, if we set $\vartheta_1 = \vartheta_2$ in (15) and introduce

$$\mathcal{B}(\vartheta)=\left\{\phi\in\mathcal{C}_{n,d}:\|\phi\|_{\mathrm{c}}\le\vartheta,\|\dot{\phi}\|_{\mathrm{c}}\le\vartheta\right\}, \tag{17}$$

then it follows from (16) that $\mathcal{B}(\rho)$ with

$$\rho=\sqrt{\frac{1}{p_1+p_2}}, \tag{18}$$

is an estimate of the domain of attraction for system (2).

Remark 1. If we assume that the initial condition ϕ satisfies

$$\dot{\phi}(\theta)=0,\theta\in[-d,0], \tag{19}$$

then it follows from (3) and (6) that the set $\mathcal{M}_v$

$$\mathcal{M}_v = \{\phi \in \mathcal{C}_{n,d} : v(\phi) \le 1\},$$

with

$$\begin{aligned} v(\phi) &= \phi^{\mathrm{T}}(0)\left(P_1 + rP_2 + \tau P_3 + (\tau - r)^2 P_4\right)\phi(0) \\ &\triangleq \phi^{\mathrm{T}}(0) P_0 \phi(0), \end{aligned}$$

where we have noticed that $\phi(s) = \phi(0)$, $\forall s \in [-d, 0]$ by assumption, namely, the "ellipsoid"

$$\mathcal{CE}(P_0) = \{\phi \in \mathcal{C}_{n,d} : \phi^{\mathrm{T}}(0) P_0 \phi(0) \le 1\},$$

is a contractively invariant set for the nonlinear time-delay system (2).

Remark 2. In the area of controlling delay systems by using LMIs based techniques, there are many methods for introducing slack matrices so as to reduce the conservatism, for example, the descriptor system approach [6] and the Jensen inequality based approach [8]. In the present paper, we have adopted the free weighting matrix approach [10]. However, as studied in [24], more free parameters does not necessarily imply less conservatism as many results having different number of free parameters are equivalent. But in our situation, these slack matrices can indeed lead to less conservative results as illustrated in the numerical example in Section IV.

In the following, we will transform (4) into a linear form. Let

$$\mathcal{Y} = \mathrm{diag}\{Y^{-1}, Y^{-1}, Y^{-1}, Y^{-1}, Y^{-1}, Y^{-1}, Y^{-1}\}. \tag{20}$$

Then performing a congruence transformation by $\mathcal{Y}$ on the left-hand side and $\mathcal{Y}^{\mathrm{T}}$ on the right-hand side of (4), we get

$$\begin{bmatrix} \Omega^{\mathrm{y}}(i) & \mathscr{X}_{\mathrm{r}} & \mathscr{X}_{\tau} & \mathscr{X}_{\mathrm{r}\tau} \\ \star & -\frac{1}{r}\mathscr{P}_5 & 0 & 0 \\ \star & \star & -\frac{1}{\tau}\mathscr{P}_6 & 0 \\ \star & \star & \star & -\mathscr{P}_7 \end{bmatrix} < 0 \tag{21}$$

where $i \in \mathcal{V}_m$, $\mathscr{X}_{\mathrm{r}}$, $\mathscr{X}_{\tau}$ and $\mathscr{X}_{\mathrm{r}\tau}$ are related with

$$\mathscr{Y} = Y^{-1}, \mathscr{X}_{\mathrm{r}j} = \mathscr{Y} X_{\mathrm{r}j} \mathscr{Y}^{\mathrm{T}},$$

$$\mathscr{X}_{\tau j} = \mathscr{Y} X_{\tau j} \mathscr{Y}^{\mathrm{T}}, \mathscr{X}_{\mathrm{r}\tau j} = \mathscr{Y} X_{\mathrm{r}\tau j} \mathscr{Y}^{\mathrm{T}}, j = 1, 2, 3,$$

and $\Omega^{\mathrm{y}}(i) = \left[\Omega^{\mathrm{y}}_{jk}\right]$, $j, k \in \{1, 4\}$ are given by

$$\begin{aligned} \Omega^{\mathrm{y}}_{11} &= \mathrm{He}\left(\mathscr{X}_{\tau 1} + \mathscr{X}_{\mathrm{r}1} + \psi_0 A \mathscr{Y}^{\mathrm{T}} + \psi_0 BD_i \mathscr{H}^{\mathrm{T}}\right) + \mathscr{P}_2 + \mathscr{P}_3, \\ \Omega^{\mathrm{y}}_{12} &= \mathscr{X}^{\mathrm{T}}_{\mathrm{r}2} - \mathscr{X}_{\mathrm{r}1} - \mathscr{X}_{\mathrm{r}\tau 1} + \mathscr{X}^{\mathrm{T}}_{\tau 2} \\ &\quad + \psi_0 A_{\mathrm{r}} \mathscr{Y}^{\mathrm{T}} + \psi_{\mathrm{r}} \mathscr{Y} A^{\mathrm{T}} + \psi_{\mathrm{r}} \mathscr{H} D_i B^{\mathrm{T}}, \\ \Omega^{\mathrm{y}}_{13} &= \mathscr{X}_{\mathrm{r}\tau 1} + \mathscr{X}^{\mathrm{T}}_{\tau 3} - \mathscr{X}_{\tau 1} + \mathscr{X}^{\mathrm{T}}_{\mathrm{r}3} \\ &\quad + \psi_{\tau} \mathscr{Y} A^{\mathrm{T}} + \psi_0 BD_i^{-} F^{\mathrm{T}} \mathscr{Y}^{\mathrm{T}} + \psi_{\tau} \mathscr{H} D_i B^{\mathrm{T}}, \\ \Omega^{\mathrm{y}}_{14} &= \mathscr{X}^{\mathrm{T}}_{\tau 4} + \mathscr{X}^{\mathrm{T}}_{\mathrm{r}4} + \mathscr{Y} A^{\mathrm{T}} - \psi_0 \mathscr{Y}^{\mathrm{T}} + \mathscr{H} D_i B^{\mathrm{T}} + \mathscr{P}_1, \\ \Omega^{\mathrm{y}}_{22} &= \mathrm{He}\left(\psi_{\mathrm{r}} A_{\mathrm{r}} \mathscr{Y}^{\mathrm{T}} - \mathscr{X}_{\mathrm{r}\tau 2} - \mathscr{X}_{\mathrm{r}2}\right) - \mathscr{P}_2 + (\tau - r)\mathscr{P}_4, \\ \Omega^{\mathrm{y}}_{23} &= \mathscr{X}_{\mathrm{r}\tau 2} - \mathscr{X}^{\mathrm{T}}_{\mathrm{r}\tau 3} - \mathscr{X}_{\tau 2} - \mathscr{X}^{\mathrm{T}}_{\mathrm{r}3} + \psi_{\tau} \mathscr{Y} A^{\mathrm{T}}_{\mathrm{r}} + \psi_{\mathrm{r}} BD_i^{-} F^{\mathrm{T}} \mathscr{Y}^{\mathrm{T}}, \\ \Omega^{\mathrm{y}}_{24} &= -\mathscr{X}^{\mathrm{T}}_{\mathrm{r}\tau 4} - \mathscr{X}^{\mathrm{T}}_{\mathrm{r}4} + \mathscr{Y} A^{\mathrm{T}}_{\mathrm{r}} - \psi_{\mathrm{r}} \mathscr{Y}^{\mathrm{T}}, \\ \Omega^{\mathrm{y}}_{33} &= \mathrm{He}\left(\mathscr{X}_{\mathrm{r}\tau 3} - \mathscr{X}_{\tau 3} + \psi_{\tau} BD_i^{-} F^{\mathrm{T}} \mathscr{Y}^{\mathrm{T}}\right) - \mathscr{P}_3 - (\tau - r)\mathscr{P}_4, \\ \Omega^{\mathrm{y}}_{34} &= \mathscr{X}^{\mathrm{T}}_{\mathrm{r}\tau 4} - \mathscr{X}^{\mathrm{T}}_{\tau 4} - \psi_{\tau} \mathscr{Y}^{\mathrm{T}} + \mathscr{Y} FD_i^{-} B^{\mathrm{T}}, \\ \Omega^{\mathrm{y}}_{44} &= -\mathscr{Y} - \mathscr{Y}^{\mathrm{T}} + r\mathscr{P}_5 + \tau \mathscr{P}_6 + (\tau - r)^2 \mathscr{P}_7, \end{aligned} \tag{22}$$

with $\mathscr{H}^{\mathrm{T}} = H^{\mathrm{T}} \mathscr{Y}^{\mathrm{T}}$, $\mathscr{P}_j = \mathscr{Y} P_j \mathscr{Y}^{\mathrm{T}}$, $j \in \{1, 7\}$. Notice that the inequality (21) is still nonlinear because of the scalars ψ_0, ψ_τ, and ψ_{r}. These three parameters should be prescribed before using LMI techniques.

Remark 3. Here we give an explanation why we impose the special structure (11) on the weighting matrix Y_η. Though this special structure will introduce limitations for Theorem 1, this special structure is necessary for transforming the inequality in (4) into a linear one. This can be observed from

$$T_4(i) = \mathrm{He}\left(\begin{bmatrix} Y_\eta BD_i H^{\mathrm{T}} & 0 & Y_\eta BD_i^{-} F^{\mathrm{T}} & 0 \end{bmatrix}\right)$$

in which both Y_η and H are unknown matrices. Hence, to ensure that the congruence transformation (20) can transform the inequality in (4) into a linear one, we have to impose the special structures on Y_η as (11). We should point out that an even more restrictive structure, say, in the form of

$$Y^{\mathrm{T}}_\eta = \left[Y^{\mathrm{T}}\ 0\ 0\ Y^{\mathrm{T}}\right],$$

which corresponds to $\psi_0 = 1$, $\psi_\tau = 0$ and $\psi_{\mathrm{r}} = 0$, has been used in [27].

We next transform the condition in (5) into LMI. Since $\mathcal{M}_v$ is an invariant set, we have $\phi \in \mathcal{M}_v \Rightarrow x_t \in \mathcal{M}_v$, $\forall t \ge 0$. Let $\mathcal{E}(P_1) = \{x \in \mathbf{R}^n : x^{\mathrm{T}} P_1 x \le 1\}$. So, if

$$\mathcal{E}(P_1) \subseteq \mathcal{L}(H_i) \triangleq \{x \in \mathbf{R}^n : |H_i^{\mathrm{T}} x| \le 1\}, i \in \{1, m\},$$

it can be seen from (3) and (6) that $\phi \in \mathcal{M}_v$ implies (5). Note that for $i \in \{1, m\}$ [12]

$$\mathcal{E}(P_1) \subseteq \mathcal{L}(H_i) \Leftrightarrow \begin{bmatrix} 1 & \mathscr{H}_i^{\mathrm{T}} \\ \star & \mathscr{P}_1 \end{bmatrix} \ge 0. \tag{23}$$

Thus Problem 1 (i) is solved if a feasibility problem with LMI constraints (21) and (23) is solved.

It is natural to ask the question that, if the closed-loop system (2) is guaranteed for stability with $\tau = \tau_{\max}$ and $r = r_{\max}$ (namely, Problem 1 (i) is solvable with $\tau = \tau_{\max}$ and $r = r_{\max}$), can the stability be maintained for all $\tau \le \tau_{\max}$ and $r < r_{\max}$ (namely, Problem 1 (i) is solvable with $\tau \le \tau_{\max}$ and $r < r_{\max}$). Motivated by this question, we present the following proposition:

Proposition 1. Assume that Problem (i) is solvable with $\tau = \tau_{\max}$ and $r = r_{\max}$. Then Problem (i) is solvable with

$$(\tau, r) \in \left\{ (\tau, r) : \begin{cases} \tau \le \tau_{\max} \\ r \le r_{\max} \\ \operatorname{sign}(r-\tau) = \\ \operatorname{sign}(r_{\max} - \tau_{\max}) \\ |r-\tau| \le |r_{\max} - \tau_{\max}| \end{cases} \right\}. \tag{24}$$

Proof. By defining $\mathscr{R}_5 = r\mathscr{R}_5'$, $\mathscr{R}_6 = \tau\mathscr{R}_6'$, the LMI in (21) becomes

$$\Upsilon(\tau, r) = \begin{bmatrix} \Omega^{y}(i) & \mathscr{X}_{r} & \mathscr{X}_{\tau} & \mathscr{X}_{r\tau} \\ \star & -\mathscr{R}_5' & 0 & 0 \\ \star & \star & -\mathscr{R}_6' & 0 \\ \star & \star & \star & -\mathscr{R}_7 \end{bmatrix} < 0, \tag{25}$$

where $i \in \mathcal{V}_m$, $\Omega^{y}(i) = \left[\Omega^{y}_{jk}\right]$, with $j \ne k$ are given by (22), and with $j = k \in \{1, 2, 3, 4\}$ are given by

$$\Omega^{y}_{11} = \mathrm{He}\left(\mathscr{X}_{\tau 1} + \mathscr{X}_{r1} + \psi_0 A\mathscr{Y}^{\mathrm{T}} + \psi_0 BD_i\mathscr{H}^{\mathrm{T}}\right) + \mathscr{R}_2 + \mathscr{R}_3,$$

$$\Omega^{y}_{22} = \mathrm{He}\left(\psi_r A_r \mathscr{Y}^{\mathrm{T}} - \mathscr{X}_{r\tau 2} - \mathscr{X}_{r2}\right) - \mathscr{R}_2 + (\tau - r)\mathscr{R}_4,$$

$$\Omega^{y}_{33} = \mathrm{He}\left(\mathscr{X}_{r\tau 3} - \mathscr{X}_{\tau 3} + \psi_\tau BD_i^{-} F^{\mathrm{T}}\mathscr{Y}^{\mathrm{T}}\right) - \mathscr{R}_3 - (\tau - r)\mathscr{R}_4,$$

$$\Omega^{y}_{44} = -\mathscr{Y} - \mathscr{Y}^{\mathrm{T}} + r^2\mathscr{R}_5' + \tau^2\mathscr{R}_6' + (\tau - r)^2\mathscr{R}_7.$$

Assume that the LMIs in (25) and (23) are feasible with $\tau = \tau_{\max}$ and $r = r_{\max}$, and the associated feasible solutions are

$$\mathscr{Y}_{\max}, \mathscr{H}_{\max}, \mathscr{R}_{1\max}, \mathscr{R}_{2\max}, \mathscr{R}_{3\max}, \mathscr{R}_{4\max},$$

$$\mathscr{R}_{5\max}', \mathscr{R}_{6\max}', \mathscr{R}_{7\max}, \mathscr{X}_{r\max}, \mathscr{X}_{\tau\max}, \mathscr{X}_{r\tau\max}.$$

For any (τ, r) belong to the set defined in (24), we claim that the LMIs in (25) and (23) have feasible solutions

$$\mathscr{Y}_{\max}, \mathscr{H}_{\max}, \mathscr{R}_{1\max}, \mathscr{R}_{2\max}, \mathscr{R}_{3\max}, \frac{\tau_{\max} - r_{\max}}{\tau - r}\mathscr{R}_{4\max}, \mathscr{R}_{5\max}',$$
$$\mathscr{R}_{6\max}', \mathscr{R}_{7\max}, \mathscr{X}_{r\max}, \mathscr{X}_{\tau\max}, \mathscr{X}_{r\tau\max}.$$

Notice that we need only to verify LMI (25). In fact, direct manipulation shows that

$$\Upsilon(\tau, r) - \Upsilon(\tau_{\max}, r_{\max}) = \begin{bmatrix} \Delta\Omega^{y}(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\Delta\Omega^{y}(i) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta\Omega^{y}_{44} \end{bmatrix}$$

in which

$$\begin{aligned} \Delta\Omega^{y}_{44} &= \left(-\mathscr{Y}_{\max} - \mathscr{Y}_{\max}^{\mathrm{T}} + r^2\mathscr{R}_{5\max}' + \tau^2\mathscr{R}_{6\max}' + (\tau - r)^2\mathscr{R}_{7\max}\right) \\ &\quad - \left(-\mathscr{Y}_{\max} - \mathscr{Y}_{\max}^{\mathrm{T}} + r_{\max}^2\mathscr{R}_{5\max}' + \tau_{\max}^2\mathscr{R}_6' \right. \\ &\quad \left. + (\tau_{\max} - r_{\max})^2\mathscr{R}_{7\max}\right) \\ &= \left(r^2 - r_{\max}^2\right)\mathscr{R}_{5\max}' + \left(\tau^2 - \tau_{\max}^2\right)\mathscr{R}_6' \\ &\quad + \left((\tau - r)^2 - (\tau_{\max} - r_{\max})^2\right)\mathscr{R}_{7\max} \\ &\le 0. \end{aligned}$$

That is to say, $\Upsilon(\tau, r) \le \Upsilon(\tau_{\max}, r_{\max}) < 0$. The proof is completed.

To tackle Problem 1 (ii), we need to find a way to quantify the "size" of the contractively invariant set $\mathcal{M}_v$. Because the "size" of $\mathcal{M}_v$ is proportional to the "sizes" of the ellipsoids $\mathcal{E}(P_i)$, $i \in \{1, 7\}$, an alternative way to maximize $\mathcal{M}_v$ is to minimize the "sizes" of these seven matrices. Take P_1 for example. Notice that

$$\begin{aligned} \|P_1\| &= \lambda_{\max}\left(\mathscr{Y}^{-1}\mathscr{R}_1\mathscr{Y}^{-\mathrm{T}}\right) \\ &\le \lambda_{\max}(\mathscr{R}_1)\,\lambda_{\max}\left(\mathscr{Y}^{-\mathrm{T}}\mathscr{Y}^{-1}\right). \end{aligned}$$

Let $\mathscr{Y}^{-\mathrm{T}}\mathscr{Y}^{-1} \le \chi_y I$ and $\mathscr{R}_1 < \chi_1 I$. Then, to minimize P_1, we can alternatively minimize χ_y and χ_1. Note that

$$\begin{aligned} \chi_y I \ge \mathscr{Y}^{-\mathrm{T}}\mathscr{Y}^{-1} &\Leftrightarrow \mathscr{Y}^{\mathrm{T}}\mathscr{Y} \ge \chi_y^{-1} I \\ &\Leftarrow -\alpha_1^2 I + \alpha_1\left(\mathscr{Y} + \mathscr{Y}^{\mathrm{T}}\right) \ge \chi_y^{-1} I, \forall \alpha_1 > 0 \\ &\Leftrightarrow \begin{bmatrix} \chi_y I & \varepsilon_1 I \\ \star & -I + \varepsilon_1\left(\mathscr{Y} + \mathscr{Y}^{\mathrm{T}}\right) \end{bmatrix} \ge 0, \forall \varepsilon_1 > 0, \end{aligned} \tag{26}$$

where $\varepsilon_1 = \alpha_1^{-1}$. The other matrices P_i, $i \in \{2, 7\}$ can be handled similarly. The scalars ε_i, $i \in \{1, 7\}$ should also be prescribed before utilizing an LMI solver.

With this, we can state the following optimization based solution to Problem 1 (ii):

$$\begin{aligned}&\inf_{\mathscr{R}_i>0,\chi_i>0,i\in\{1,7\},\chi_y>0,\mathscr{X}_r,\mathscr{X}_\tau,\mathscr{X}_{r\tau},\mathscr{Y},\mathscr{H},} \chi \\ &\text{s.t.} \begin{cases}(\mathbf{a})\ \text{LMIs (21), (23) and (26)},\\ (\mathbf{b})\ \mathscr{R}_i < \chi_i I, i\in\{1,7\},\end{cases}\end{aligned} \tag{27}$$

where $\chi = \sum_{i=1}^{7}\omega_i\chi_i + \omega_y\chi_y$ with ω_i, $i \in \{1, 7\}$, and ω_y being some prescribed positive weighting factors. As a result, the approximate maximal contractively invariant set is given by $\mathcal{M}_v$ defined in (3) and (6) where $P_i = \mathscr{Y}^{-1}\mathscr{R}_i\mathscr{Y}^{-T}$, $i \in \{1, 7\}$.

Remark 4. To solve Problem (iii), we need only to replace $\mathscr{Y}F$ in (21) with $\mathscr{F}$, which is a free variable. Then the resulting feedback gain is given by $F^{\mathrm{T}} = \mathscr{F}^{\mathrm{T}}\mathscr{Y}^{-\mathrm{T}}$.

3.2 Input saturated delay systems without actuator delay

If the actuator in (2) is only subjected to input saturation, namely, the system in (2) becomes

$$\dot{x}(t) = Ax(t) + A_r x(t-r) + B\mathrm{sat}\left(F^{\mathrm{T}}x(t)\right), \tag{28}$$

where $x_0 = \phi \in \mathcal{C}_{n,r}$, then solutions to Problems 1 (i)–(iii) associated with system (28) can be easily obtained by using the same method for system (2). For simplicity, we only give solutions to Problems 1 (ii)–(iii) associated with system (28).

Corollary 1. Consider the nonlinear time-delay system (28). Assume that $F \in \mathbf{R}^{n\times m}$ is given. If there exist matrices $0 < \mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_5 \in \mathbf{R}^{n\times n}$, $\mathscr{X}_r \in \mathbf{R}^{3n\times n}$, $\mathscr{Y} \in \mathbf{R}^{n\times n}$ and $\mathscr{H} \in \mathbf{R}^{n\times m}$ and two scalars ψ_0 and ψ_r such that the following inequalities

$$\begin{bmatrix}\Omega^{0y}(i) & \mathscr{X}_r \\ \star & -\frac{1}{r}\mathscr{R}_5\end{bmatrix} < 0, i \in \mathcal{V}_m, \tag{29}$$

and the conditions in (5) are satisfied for all $x_t \in \mathcal{M}_{v_r}$, where

$$\mathcal{M}_{v_r} = \{\phi \in \mathcal{C}_{n,r} : v_r(\phi) \le 1\}, \tag{30}$$

with

$$\begin{aligned}v_r(\phi) = {}& \phi^{\mathrm{T}}(0)P_1\phi(0) + \int_{-r}^{0}\phi^{\mathrm{T}}(s)P_2\phi(s)\mathrm{d}s \\ &+ \int_{-r}^{0}\int_{s}^{0}\dot{\phi}^{\mathrm{T}}(\theta)P_5\dot{\phi}(\theta)\mathrm{d}\theta\mathrm{d}s,\end{aligned} \tag{31}$$

and $P_i = \mathscr{Y}^{-1}\mathscr{R}_i\mathscr{Y}^{-\mathrm{T}}$, $i \in \{1, 2, 5\}$, then the solution $x(t) \equiv 0$ is asymptotically stable for system (28) with the set $\mathcal{M}_{v_r}$ contained in the domain of attraction. In (29),

$$\mathscr{H}_r^{\mathrm{T}} = \begin{bmatrix}\mathscr{X}_{r1}^{\mathrm{T}} & \mathscr{X}_{r2}^{\mathrm{T}} & \mathscr{X}_{r3}^{\mathrm{T}}\end{bmatrix}$$

and $\Omega^{0y}(i) = \left[\Omega_{jk}^{0y}\right]$ in which Ω_{jk}^{0}, j, $k = 1, 2, 3$ are given by

$$\begin{cases}\Omega_{11}^{0y} = \mathrm{He}\left(\mathscr{X}_{r1} + \psi_0 A\mathscr{Y}^{\mathrm{T}}\right) + \mathscr{R}_2 \\ \qquad + \mathrm{He}\left(\psi_0 BD_i\mathscr{H}^{\mathrm{T}} + \psi_0 BD_i^{-}F^{\mathrm{T}}\mathscr{Y}^{\mathrm{T}}\right) \\ \Omega_{12}^{0y} = \mathscr{X}_{r2}^{\mathrm{T}} - \mathscr{X}_{r1} + \psi_0 A_r\mathscr{Y}^{\mathrm{T}} + \psi_r\mathscr{Y}A^{\mathrm{T}} \\ \qquad + \psi_r\mathscr{H}D_iB^{\mathrm{T}} + \psi_r\mathscr{Y}FD_i^{-}B^{\mathrm{T}}, \\ \Omega_{13}^{0y} = \mathscr{X}_{r3}^{\mathrm{T}} + \mathscr{Y}A^{\mathrm{T}} - \psi_0\mathscr{Y}^{\mathrm{T}} + \mathscr{H}D_iB^{\mathrm{T}} \\ \qquad + \mathscr{Y}FD_i^{-}B^{\mathrm{T}} + \mathscr{R}_1, \\ \Omega_{22}^{0y} = \mathrm{He}\left(\psi_r A_r\mathscr{Y}^{\mathrm{T}} - \mathscr{X}_{r2}\right) - \mathscr{R}_2 \\ \Omega_{23}^{0y} = -\mathscr{X}_{r3}^{\mathrm{T}} + \mathscr{Y}A_r^{\mathrm{T}} - \psi_r\mathscr{Y}^{\mathrm{T}}, \\ \Omega_{33}^{0y} = -\mathrm{He}\left(\mathscr{Y}\right) + r\mathscr{R}_5.\end{cases} \tag{32}$$

Thus, by specifying the values of ψ_0 and ψ_r, and solving the following optimization problem

$$\begin{aligned}&\inf_{\mathscr{R}_i>0,\chi_i>0,i\in\{1,2,5\},\chi_y>0,\mathscr{X}_r,\mathscr{Y},\mathscr{H},} \chi \\ &\text{s.t.} \begin{cases}(\mathbf{a}).\ \text{LMIs (29), (23) and (26)},\\ (\mathbf{b}).\ \mathscr{R}_i < \chi_i I, i\in\{1,2,5\},\end{cases}\end{aligned} \tag{33}$$

where $\chi = \omega_1\chi_1 + \omega_2\chi_2 + \omega_5\chi_5 + \omega_y\chi_y$ with χ_i, $i \in \{1, 2, 5\}$ and χ_y being some prescribed positive weighting factors, the approximate maximal contractively invariant set for system (28) is given by $\mathcal{M}_{v_r}$ defined in (30)–(31). Similarly, if we set $\mathscr{Y}F$ in (29) with $\mathscr{F}$, which is treated as a decision variable, then solutions to Problem 1 (iii) associated with system (28) can also be obtained.

Remark 5. Note that there are 2^m constraints in (29) while there are $(2^m)^2$ constraints in [27] for solving a similar problem. Evidently, if m is large, the number of the constraints in the optimization problem (33) is significantly less than that in [27], which indicates that Corollary 1 requires less computation burden than the results in [27]. Moreover, Corollary 1 contains more auxiliary (freedom) variables, namely, ψ_0 and ψ_r, than that in [27] where ψ_0 is fixed and ψ_r vanishes. These variables can be well selected to reduce conservatism.

Remark 6. According to Proposition 1, in the case $\tau = 0$ or the case $\tau = r$, if Problems (ii)-(iii) are solvable with $r = r_{\max}$ and the associated domain of attraction is $\mathcal{M}_{\max}$ then they are also solvable with $r < r_{\max}$ and the resulting domain of attraction $\mathcal{M}$ contains $\mathcal{M}_{\max}$.

IV. NUMERICAL EXAMPLES

Example 1. Consider the nonlinear delay system (2) with

$$A = \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.2 \end{bmatrix},\ A_r = \begin{bmatrix} -0.3 & 0 \\ -0.2 & 0.6 \end{bmatrix},$$

$$B = \begin{bmatrix} -\frac{1}{2} \\ 2 \end{bmatrix}.$$

First, for different values of τ, we find the maximal value of r such that system (2) can be stabilized. Here we choose $\psi_0 = 0.01, \psi_r = 0$ and $\psi_\tau = 0$. The results are shown in Fig. 1, from which it can be seen that the system cannot be stabilized for arbitrary delay r if $\tau \geq 1.32$.

The selections of ψ_0, ψ_r and ψ_τ will certainly influence the maximal values of r and τ such that the system can be locally stabilized. To see this, we first let ψ_r and ψ_τ be fixed and let ψ_0 change. We also let τ be prescribed as $\tau = 1$. For different values of ψ_0, the maximal value of r (denoted by $r_{\max}$) such that the system can be locally stabilized are recorded at the top of Fig. 2 from which we can see that there exists an optimal value of ψ_0 such that $r_{\max}$ is maximized. Such value is about 0.02 observed from the figure. Second, we let ψ_0 be fixed as $\psi_0 = 0.01$ and let ψ_r and ψ_τ change. For different values of ψ_r and ψ_τ, the maximal value of r (also denoted by $r_{\max}$) such that the system can be locally stabilized are again computed and recorded at the bottom of Fig. 2. From the figure we can see that $r_{\max}$ is maximized with $\psi_r = \psi_\tau = 0$, which is the just pair of selection we have used before. Notice that $(\psi_0, \psi_r, \psi_\tau) = (0.01, 0, 0)$ is not a globally optimal selection which is however hard to obtain in practice.

For a pair of fixed delays $(r, \tau) = (1.1, 1)$, according to Remark 4, by solving the corresponding optimization problem with $\omega_y = 10^3$, $\omega_i = 1, i \in \{1, 7\}$ and $\varepsilon_i = 15$ (here ψ_0, ψ_r and ψ_τ are the same chosen as before), we get the optimal feedback gain

$$F^{\mathrm{T}} = [-0.2444 \quad -0.7358],$$

and $\rho = 0.3092$. Then the set $\mathcal{B}(\rho)$ defined in (17) is an estimate of the domain of attraction for system (2). For a set of initial conditions on the boundary of $\mathcal{B}(\rho)$, the trajectories of the system are recorded in Fig. 3, which clearly indicates that the closed-loop system is asymptotically stable. If it is assumed that the initial condition ϕ satisfies (19), then according to Remark 1, a contractively invariant set is given by $\mathcal{CE}(P_0)$ with

$$P_0 = \begin{bmatrix} 2.8810 & 3.0473 \\ 3.0473 & 3.4615 \end{bmatrix}.$$

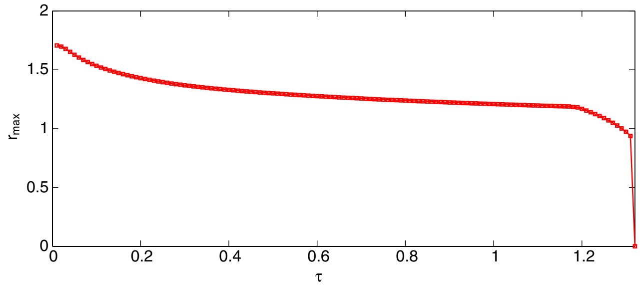

Fig. 1. The maximal value of r for different τ.

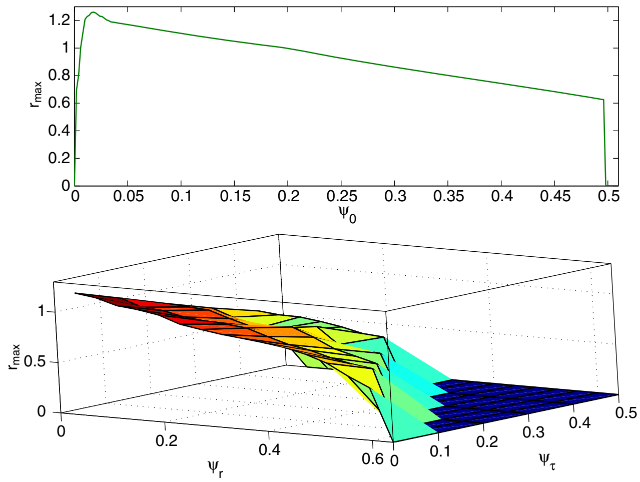

Fig. 2. Top: The maximal value of r for different values of ψ_0 with $\psi_r = \psi_\tau = 0$ and $\tau = 1$. Bottom: The maximal value of r for different values of ψ_r and ψ_τ with $\psi_0 = 0.01$ and $\tau = 1$.

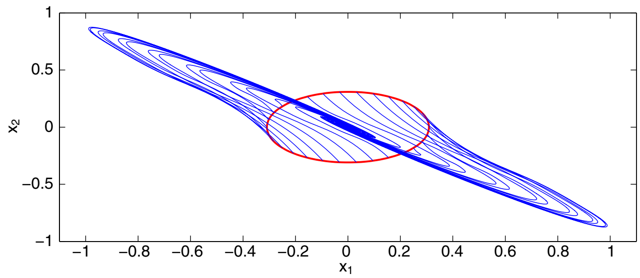

Fig. 3. The set $\mathcal{B}(\rho)$ is an estimate of the domain of attraction for the nonlinear system (2).

For a set of initial conditions on the boundary of $\mathcal{CE}(P_0)$, the trajectories of the closed-loop system (2) are plotted in Fig. 4. It is clearly seen that $\mathcal{CE}(P_0)$ is a contractively invariant set.

Example 2. Consider the nonlinear delay system (28) with

$$A = \begin{bmatrix} 1 & 0.6 \\ -1 & 0.8 \end{bmatrix},\ A_r = \begin{bmatrix} 1 & 0.5 \\ -0.5 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Table I. Comparison of Different Approaches.

	Approaches	Theorem 5 in [1]	Theorem 1 in [6]	Corollary 1 in [27]	Corollary 1
	r_{max}	0.0915	1.6540	1.8346	2.8985
ρ_{max}	$r = 0.09$	0.1759	0.1300	0.1694	0.2057
	$r = 1.20$	Infeasible	0.0742	0.0772	0.0806
	$r = 1.80$	Infeasible	Infeasible	0.0066	0.0419
	$r = 2.00$	Infeasible	Infeasible	Infeasible	0.0327
	$r = 2.80$	Infeasible	Infeasible	Infeasible	0.0018

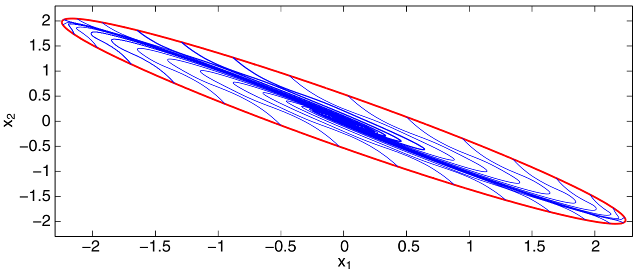

Fig. 4. The set $\mathcal{CE}(P_0)$ is a contractively invariant set for the nonlinear system (2) if the initial condition ϕ satisfies (19).

We intend to compare our method in Corollary 1 with some existing results. First, we find the maximal delay r_{max} such that the system in (16) can be stabilized locally by using different approaches. By setting $\omega_y = 10^3$, $\omega_1 = \omega_2 = \omega_5 = 1$, $\psi_0 = 0.01$, $\psi_r = 0$ and $\varepsilon_i = 15$ in (33), we get the maximal delay $r_{max} = 2.8985$. The comparisons with the methods given in [1], [6] and [27] are shown in Table I, from which it can be seen that our methods are able to give much less conservative estimates of the maximal delay. Notice that, though reference [27] considered a delay-range-dependent approach, its corollary 1 is concerned with constant delays, which is the topic of this paper. Second, we make a comparison of the estimate of the domain of attraction for system (28) by using different approaches. Here we have used the set in (17) while the maximal value of ρ (denoted by ρ_{max}) is used to measure the size of the domain of attraction. It can be observed that our method can also lead to a larger domain of attraction. Moreover, the results obtained by Corollary 1 reveal that there is a trade-off between maximizing the size of the delay and the "size" of the domain of attraction.

The main reasons that Corollary 1 is more effective than that in [27] are that the number of the constraints in Corollary 1 is significantly less than that in [27] (see Remark 5 for details), and Corollary 1 contains more auxiliary (freedom) variables, namely, ψ_0 and ψ_r, than that in [27] where ψ_0 is fixed as 1 and ψ_r vanishes. On the other hand, our result in Corollary 1 is more effective than that in [1] not only because Corollary 1 contains more auxiliary (freedom) variables, but also has taken the derivative of the initial condition into consideration, which is not the case in [1]. Finally, the number of free parameters in Corollary 1 is $N_1 = 4n^2 + \frac{3n(n+1)}{2} + 2nm + 2 = \frac{11}{2}n^2 + \frac{3}{2}n + 2mn + 2$, while the number of free parameters in Theorem 1 in [6] is $N_2 = 5n^2 + 2n + 2mn + 2$, from which it follows that $N_1 - N_2 = \frac{1}{2}n(n-1) > 0, n \geq 2$, namely, Corollary 1 contains more free parameters than that in [6]. This is the main reason why Corollary 1 is less conservative than theorem 1 in [6].

V. CONCLUSION

This paper has investigated the regional stability and regional stabilization problem for linear time-delay systems in the presence of both actuator saturation and delay. The main theoretical results are the LMI-type conditions for regional stability and stabilization established by using a new Lyapunov-Krasovskii functional. The maximization of the estimated domain of attraction has been made by solving an LMIs based optimization problem. The usefulness of the obtained results has been demonstrated by two numerical examples.

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Yan-Ming Fu received his B.S. degree in Automatic Control from Northeast Institute of Electric Power, China, in 2001, and his Ph.D. degree in Control Theory and Control Engineering from Harbin Institute of Technology, China, in 2006. He is currently an Associate Professor in Harbin

Institute of Technology, China. His research interests include robust control and filtering, jumping systems, and time-delay systems.

Bin Zhou was born in Luotian County, Huanggang, Hubei Province, China on July 28, 1981. He received Bachelor's degree, Master's Degree and Ph.D. degree from Department of Control Science and Engineering at Harbin Institute of Technology, Harbin, China in 2004, 2006 and 2010, respectively. He was Research Associate at Department of Mechanical Engineering, University of Hong Kong from December 2007 to March 2008, Visiting Fellow at School of Computing and Mathematics, University of Western Sydney from May 2009 to August 2009, and a Visiting Research Scientist at Department of Electrical and Computer Engineering, University of Virginia from July 2012 to August 2013. In February 2009, he joined School of Astronautics, Harbin Institute of Technology, where he has been Professor since December 2012.

His current research interests include constrained control, time-delay systems, nonlinear control, and control applications in astronautic engineering. In these areas, he has published about 100 papers, over 70 of which are in archival journals.

He is a reviewer for American Mathematical Review and is an active reviewer for a number of journals and conferences. He was selected as "New Century Excellent Talents in University", the Ministry of Education of China in 2011. He received "National Excellent Doctoral Dissertation Award" in 2012 from the Academic Degrees Committee of the State Council and the Ministry of Education of China.

Guang-Ren Duan was born in Heilongjiang Province, China, on April 5, 1962. He received his B.Sc. Degree in Applied Mathematics, and both his M.Sc. and Ph.D. Degrees in Control Systems Theory. From 1989 to 1991, he was a postdoctoral researcher at Harbin Institute of Technology, Harbin, China, where he became a professor of control systems theory in 1991. He is currently Director of the Centre for Control Systems and Guidance Technology at Harbin Institute of Technology. He is the author and co-author of over 300 publications. His main research interests include robust control, eigenstructure assignment, descriptor systems, missile autopilot control and magnetic bearing control.