Stability of Time-Delay Systems

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ABSTRACT abstract

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Preface

This book is devoted to the study of stability of time-delay systems, a subject that has long been vigorously pursued by such learned societies as much diversely represented as in mathematics, science, engineering, and economics. One may wonder, given the richness of its history and its maturity, what may be new, and what the authors have to say, beyond the extensive body of knowledge acquired in a long, fruitful past which commands both depth and breadth of enormity. It is comforting, and so is the reader assured, that the subject is more enduring than transient, and indeed has sustained a surprising degree of vitality. In particular, for the last decade or so, it has received visible research attention and the advance has been notable. It appears warranted to assert, due to this period of creative work, that the subject has undergone a significant leap conceptually and on practical measures; both its nature and scope have been dramatically advanced and broadened. Our aim in the present book is to present some of the highlights resulted from this advance, and to further develop the techniques and tools along the recent trends.

What defines a time-delay system is the feature that the system's future evolution depends not only on its present state, but also on a period of its history. This particular cause-effect relationship can be most succinctly captured, and indeed has been traditionally so modeled by differential-difference equations, or more generally, by functional differential equations. While in practice many dynamical systems may be satisfactorily described by ordinary differential equations alone, for which the system's future evolution depends solely on its current state, there are times when delay effect cannot be neglected, or it will be more beneficial for it to be accounted for. In a word–we reckon that many will agree–time delay is by no means a matter of rarity; in fact, it is more prevalent than uncommon; numerous examples presented in this book and elsewhere serve to solidify this standpoint. It is thus unsurprising, due to their omnipresence, and for their intrinsic scientific interest and practical implication, that time-delay systems have been studied long and well. It has for decades been an active area of scientific research in mathematics, biology, ecology, economics, and in engineering, under such terms as hereditary systems, systems with aftereffect, or systems with time-lag, and more generally as a subclass of functional differential equations and infinite dimensional systems. The field of time-delay systems as a whole has its beginning dated back to the eighteenth century, and it received substantial attention in the early twentieth century in works devoted to the modeling of biological, ecological, as well as engineering systems.

Stability of time-delay systems became a formal subject of study in the 1940's, with contributions from such towering figures as Pontryagin and Bellman. Over the years it enjoyed a steady growth in interest and popularity. In particular, in the last ten to fifteen years there has been a surge of research activities and henceforth a proliferation of new techniques and results, dramatically uplifting the area to a new, more exciting stage; a cursory glance of the large numbers of articles published during this period, in international conferences, organized workshops and archive journals leave one the strong impression of the scale and magnitude of this progress. It is clear that advances in numerical methods and control theory, especially the theory of robust stability and control, have contributed much to make the progress possible. Indeed, not only did the concepts of robust stability inject new life into an otherwise old topic, but the techniques and methods found in robust control and numerical optimization have afforded tools necessary for the new development. Of these, we come to note such notions as robust stability, quadratic stability, small gain theorem, structured singular value, linear matrix inequalities (LMI), and advances in the development of efficient convex optimization algorithms, notably that of interior point algorithms. To give the reader a glimpse into the power of these tools at the outset, the small gain theorem constitutes the very catalyst that inspired stability problem formulations for time-delay systems as one of robust stability and of the structured singular values. Techniques in robust stability analysis of uncertain polynomials find generalizations to uncertain quasipolynomials, which serve as models for uncertain time-delay systems. Additionally, efficient numerical algorithms for solving LMI programs, which form a special category of convex optimization problems, have become an enabling tool that led the stability problems to be posed as LMI conditions and thus rendered solvable computationally. It is fair to conclude that these concepts and techniques have helped revitalize the field and by now have been routinely used in the study of time-delay systems.

In recognition of these advances, the authors came to the conclusion that a book project will be a most natural outcome, one that is borne out of more than a decade's creative work and that will, as the authors so hoped, contribute to the future studies of the subject. Our primary objective then is to present a sufficiently thorough, yet reasonably focused development of the key techniques and results that were developed in the past decade or so. This will serve a humble yet useful purpose; it will allow us to organize and develop, in a systematic manner, in a coherent way, and in sufficient technical depth, many of the results that are otherwise scattered in various journals and conference proceedings; the sheer volume of these publications can be rather overwhelming. This, we hope, will consequently offer a convenient source of reference to readers who are interested in the subject, and to furnish a quick access to some of the important issues and results.

We attempt to provide a particular viewpoint, which is centered on computability and is approached from a perspective distinctively that of robust stability and robust control. We believe these lie at the very essence and indeed have been the driving force responsible for the latest development. In a bolder attempt, we emphasize conceptual significance over technical details, while without sacrificing mathematical rigor. At times we labor to articulate on fundamental though intuitive observations behind methods and results, and to build links and relationships that may seem nonexistent; in other words, we strive to provide an "insider's story" based on our experiences with the subject. This perspective has guided us in the selection of the materials and in our presentation. Thus, while we expect that the facts and results presented in this book be worthy materials, we too hope that the ideas and intuitive thoughts may serve as a useful guide in understanding often the intricate, nitty-gritty technical details, especially to a reader new to the area. On the other hand, we face the unpleasant yet inevitable task of limiting ourselves to a few selected topics, while excluding materials that to some may be equally or even more important; examples include classical results, recent developments on stabilization, control, and filtering of time-delay systems, and those on stochastic time-delay systems, to name just a few. We assure, nevertheless, that this merely reflects the authors' taste and viewpoints rather than the value of particular issues. Fortunately, we have the luxury to resort to the rescue of several classical treaties and several others that were published recently.

It is uncommon to find in an educational institution time-delay systems adopted as a regular course of learning, despite the widespread research interest; for many, the first and perhaps the only encounter with the subject is likely to take place in an undergraduate system modeling or automatic control course. For this reason, our book has been conceived and written as a research monograph, with a readership aimed at researchers, practicing engineers, and graduate students. We do not purport, though used judiciously, that it can serve as a textbook for an advanced graduate course. It nevertheless is appropriate as a self-study text; the majority of the contents are easily accessible to a typical second year graduate student in engineering, who is presumed to have been exposed to state-space and transfer function descriptions and stability concepts of dynamical systems. A few more specialized topics will require more advanced a mathematical background. A reader with less preparation may opt to skip proofs of such results, but quickly move on to the theorem statements. The book can be read in an order at the reader's interest. For those who are mainly interested in frequency domain results, the first four chapters can be read in order. For time domain techniques, the reader can read Chapter 1 and then move directly to Chapters 5 to 7. Most of the materials in Chapter 8 can be understood as well after reading Chapters 5 and 6. The book has been made sufficiently self-contained, with necessary technical preliminaries integrated into the respective chapters and included in two appendices.

The contents of this book are centered at the theme of stability and robust stability of time-delay systems. Chapter 1 begins with a number of practical examples in which time delays are seen to play an important role. It then continues to provide an introductory exposition of some basic concepts and results keen to stability analysis, such as functional differential equation representation, characteristic quasipolynomial, Lyapunov-Krasovskii theorem, and Razumikhin theorem. It ends with a brief outline of the subsequent development.

The next three chapters develop frequency domain criteria for stability and robust stability of linear time-invariant systems. Chapter 2 focuses on systems with commensurate delays only. It presents both frequency-sweeping and constant matrix tests, which are both necessary and sufficient conditions for delay-dependent and delay-independent stability, and both require computing matrix pencils. Chapter 3 studies systems with incommensurate delays. The development is built upon a small gain approach, and the necessary and sufficient condition for stability independent of delay is shown to be equivalent to a problem of computing the structured singular value. The chapter opens with a brief exposure of key concepts found in robust stability analysis, such as the small gain theorem and the structured singular value, and ends with a formal analysis of the computational complexity inherent in the stability problem. Both Chapter 2 and Chapter 3 treat systems modeled either by state-space descriptions or quasipolynomials, but with no consideration of system uncertainty. Uncertain time-delay systems are addressed in Chapter 4. More specifically, it studies uncertain quasipolynomials of systems with incommensurate delays, that is, families of multivariate polynomials whose coefficients are permitted to vary in a prescribed set; notable examples in this class include interval and diamond quasipolynomials. In much the same spirit as in robust control, this chapter develops robust stability conditions that insure the stability of the entire family of quasipolynomials, by checking the so-called edge, and in some instances only the vertices of the quasipolynomial family.

The rest of the main book proper, Chapter 5 to Chapter 7, is devoted to time domain methods. It is in this setting that such fundamental tools as Lyapunov-Krasovskii theorem and Razumikhin theorem play a critical role. Chapter 5 concentrates on systems with a single delay. While providing various stability conditions, this chapter also introduces a number of techniques prevailing in the analysis of time-delay systems, including model transformation, discretized Lyapunov functional method, and LMI conditions. This paves the way for extensions to uncertain systems **with memoryless uncertainty** and systems with multiple incommensurate delays, pursued in Chapter 6 and Chapter 7, respectively. It is worth noting that unlike in frequency domain approaches, time domain methods are more advantageous in accommodating nonlinear, time-varying systems. Furthermore, by the nature of time domain techniques, these chapters deal exclusively with state-space descriptions, and the stability tests amount to sufficient conditions that are posed as solutions of LMI programs.

The final chapter, Chapter 8, **discusses the robust stability under dynamic uncertainty. By modeling time delay as dynamic uncertainty, some stability conditions derived earlier are shown to arise naturally from small gain condition. In more general case, the nominal system can be time delay system. {previous description deleted}** The book ends with two appendices; Appendix A summarizes the key matrix facts and identities used throughout the book, and Appendix B provides basic concepts and techniques of LMI's.

The writing of a book is often a painstakingly long and drawn-out process of hard labor and agonizing deliberations, which invariably involves the participation of many. We begin our acknowledgement, in a superficially unorthodox fashion, of the contributions of our families. KG would like to express his heartfelt thanks to Xinxin, Siyao and Patrick; VK to Olga; and JC to Yan, Christopher, and Stephen; in a way, this book is their triumph, who had to tolerate the countless hours which otherwise would have been better spent with them. Indeed, had it not been the families' support and understanding, we would have long abandoned the project. Furthermore, KG dedicates the book to his parents Lijian Gu and Jieping Jiang for their selfless love and support, VK to his teacher the late Professor Vladimir Ivanovich Zubov (1930-2000) for his genuine friendship and mentorship, and JC to his wife Yan and sons Christopher and Stephen for their unfading trust. The authors would also like to thank their personal friends, who might have unknowingly become yet will never cease to be a source of inspiration. It is the caring eyes of these fine people, far or near, that have often urged us to clear the hurdles hidden here and there during the writing. We hope we have carried out their vision and expectation.

Of the friends and colleagues whom we have had the fortune to work with on problems directly pertinent to this book, it is a pleasure to acknowledge: Guoxiang Gu, Qing-Long Han, Silviu-**Iulian** Niculescu, Bahram Shafai, André Tits, Demin Xu, and Alexey Zhabko. Perhaps one of the more heartening aspects in writing the book has been to revisit some of our collaborative work with these individuals, which occupies a significant portion of the book. We are equally grateful to Chaouki Abdallah, Jean-Michel Dion, Luc Dugard, Michael Fan, Didier Georges, Jacob Kogan, Brad Lehman, James Louisell, Olivier Sename, Gilead Tadmor, Sophie Tarboriech, Onur Toker, Li Qiu, Erik Verriest, and Kemin Zhou; we have benefited from their ideas and wisdom through numerous discussions. These friends contributed either directly to the book via joint work, or indirectly by imparting on us their discerning taste, helping uphold a standard we have strived to achieve. We thank Springer for granting us the opportunity to publish the book, and its fine editorial staff for their patience and technical support; they have leniently granted us several "delays" which we hope have turned out to be positive and paid off. We thank Rex Pierce and Gang Chen, who each helped prepare a number of computer-generated graphes. Finally, we gratefully acknowledge the US National Science Foundation for its financial support during the period of this project, under the direction of Kishan Baheti and Rose Gombay, respectively.

In closing we ask the reader to grant us the privilege for a moment of reflection, for us to indulge in the fond memory of our collaborative process. It may seem somewhat odd that the authors have never had a record of collaboration before initiating such a major, time-consuming project. The idea was first due to KG, which came up in the 1997 American Control Conference held at Albuquerque, New Mexico, during a casual conversation between KG and JC. It seemed at first rather remote a possibility. It remained so until the summer of 1998, in a robust control workshop held in Sienna, Italy, where VK and JC rekindled the idea. The picturesque setting of Toscanny's rolling hills and stunning sunsets seemed to promise to bear fruit and foretell a happy ending. The project thus took off. Discussions were soon made between the three authors via emails and phone calls, with contents conceived and plans carved out. It would take another four years nonetheless, to implement the plans, and after numerous drafts and iterations, for the project to be finally brought to its present form, in which we have attempted with good faith to communicate our views and passion to the subject. The book, in retrospect, is a triumph of will and friendship. We thus thank each other as well.

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Notations

R, R^n, $\mathsf{R}^{n\times n}$	The sets of real numbers, n component real vectors, and n by n real matrices
C, C^n, $\mathsf{C}^{n\times n}$	The sets of complex numbers, n component complex vectors, and n by n complex matrices
D	The open unit disk $\{s \mid s \in \mathsf{C}, \ \lvert s\rvert < 1\}$
R_+	The set of positive real numbers
T	The unit circle $\{s \mid s \in \mathsf{C}, \ \lvert s\rvert = 1\}$
$\bar{\mathsf{S}}$, S°, $\partial\mathsf{S}$	The closure, interior and boundary of S, where S is any set. For example, $\bar{\mathsf{D}}$ is the closed unit disk
S^n	$\{(s_1, s_2, ..., s_n) \mid s_i \in \mathsf{S}, \ i = 1, 2, ..., n\}$, S any set
C_+	The set $\{w \in \mathsf{C} \mid \mathrm{Re}(w) > 0\}$
$\mathrm{Re}(w)$, $\mathrm{Im}(w)$	The real and imaginary parts of $w \in \mathsf{C}$
w^*	The complex conjugate of w
$\lVert\cdot\rVert$	Vector or matrix norm.
$\mathcal{C}[a, b]$	The set of R^n valued continuous functions on $[a, b]$
$\mathcal{C}$	$\mathcal{C}[-r, 0]$
$\lVert\phi\rVert_c$	The continuous norm $\max\limits_{a\le\xi\le b} \lVert\phi(\xi)\rVert$ for $\phi \in \mathcal{C}[a, b]$
$\dot{x}$	Derivative of x with respect to time t, $\frac{dx}{dt}$
$\mathcal{L}(\cdot), \mathcal{L}^{-1}(\cdot)$	Laplace transform and the inverse Laplace transform
A^T, A^*, A^H	Transpose, component-wise complex conjugate, and Hermitian transpose of matrix A.
$\lvert A\rvert$	Matrix formed by taking absolute value of each entry of A
$A > 0$, $A \ge 0$	The matrix A is positive (semi)definite ($<$ or $\le$ similar)
$A > B$, $A \ge B$	$A - B > 0$, $A - B \ge 0$
$\lambda(A), \lambda_i(A)$	An eigenvalue and the ith eigenvalue of matrix A
$\lambda_{\max}(A), \lambda_{\min}(A)$	The maximum and the minimum eigenvalue of A
$\rho(A)$	Spectrum radius of matrix A, $\max\limits_i \lvert\lambda_i(A)\rvert$
$\sigma(A)$	Spectrum (the set of all the eigenvalues) of matrix A
$\sigma_i(A)$	The ith singular value of A
$\sigma_{\max}(A)$ or $\bar{\sigma}(A)$	The maximum singular value of A
$\sigma_{\min}(A)$ or $\underline{\sigma}(A)$	The minimum singular value of A
$A \otimes B$	Kronecker product of A and B
$A \oplus B$	Kronecker sum of A and B, $A \otimes I + I \otimes B$

Additional notations are defined in the text. Please check the index for the locations of definitions.

1

Introduction to Time-Delay Systems

1.1 Introduction

We begin this chapter by first presenting a number of practical examples of time-delay systems. This should give the reader a glimpse in how widely time delays may occur in practice. For it to be more accessible, we then follow to analyze a simple time-delay system, which serves to illustrate some of the important concepts in a concrete manner. This sets the stage for a formal introduction in much generality, introducing functional differential equations as representations of time-delay systems, and such stability analysis tools as Lyapunov-Krasovskii theorem and Razumikhin theorem. Linear delay systems are discussed in greater length with frequency domain descriptions. The chapter is concluded with a brief outline of the subsequent contents.

Ordinary differential equations in the form of

$$\dot{x}(t) = f(t, x(t)) \tag{1.1}$$

have been a prevalent model description for dynamical systems. In this description, the variables $x(t) \in \mathsf{R}^n$ are known as the *state variables*, and the differential equations characterize the evolution of the state variables with respect to time. A fundamental presumtion on a system modelled **modeled {both are correct, but needs to be consistent throughout the book, my fault}** as such is that the future evolution of the system is completely determined by the current value of the state variables. In other words, the value of the state variables $x(t)$, $t_0 \leq t < \infty$, for any t_0, can be found once the initial condition

$$x(t_0) = x_0 \tag{1.2}$$

is known. Needless to say, ordinary differential equations in general, and stability and control of dynamical systems so modelled in particular, have been an extensively developed subject of scientific learning.

In practice, however, many dynamical systems cannot be satisfactorily modeled by an ordinary **differential** equation. In particular, for a particular class of **many {two “particular” in a row sounds bad}** systems, the future evolution of the state variables $x(t)$ not only depends on their current value $x(t_0)$, but also on their past values, say $x(\xi)$,

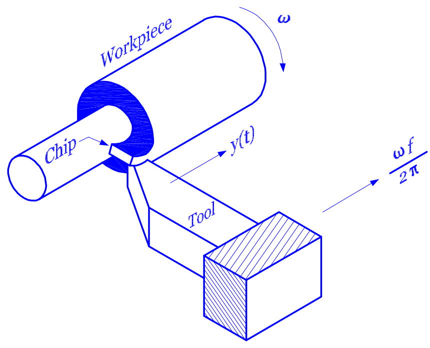

FIGURE 1.1. Geometry of Turning

$t_0 - r \leq \xi \leq t_0$. Such a system is called a *time-delay system*. Time-delay systems may arise in practice for a variety of reasons.

Example 1.1 Regenerative chatter in metal cutting.

Shown in Figure 1.1 is the metal cutting process in a typical machine tool such as a lathe. A cylindrical workpiece rotates with constant angular velocity ω and the cutting tool translates along the axis of the workpiece with constant linear velocity $\omega f/2\pi$, where f is the feed rate in length per revolution corresponding to the normal thickness of the chip removed. The tool generates a surface as the material is removed, shown as shaded, and any vibration of the tool is reflected on this surface. In regenerative chatter, the surface generated by the previous pass becomes the upper surface of the chip on the subsequent pass. A model often used in studying such a process is shown in Figure 1.2. This time-delay system can be described by the equation

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = -F_t(f + y(t) - y(t-\tau)), \tag{1.3}$$

where m, c and k reflect the inertia, damping and stiffness characteristics of the machine tool, the delay time $\tau = 2\pi/\omega$ corresponds to the time for the workpiece to complete one revolution, and $F_t(\cdot)$ is the thrust force depending on the instantaneous chip thickness $f + y(t) - y(t-\tau)$. It is often sufficient to consider $F_t(\cdot)$ to be linear, and techniques for linear time-delay systems are often used.

Notice that in the above example the system is autonomous (without feedback control), and the delay is the result of an *intrinsic property of the system*. In response to recent demands on high speed machining, the study on the time-delay and nonlinear nature of the system is receiving renewed interest.

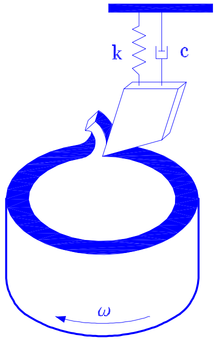

FIGURE 1.2. Model of Regenerative Chatter

Example 1.2 Internal combustion engine. *In the control of internal combustion engines, the* Mean Torque Production Model *is often used. In this model, the crankshaft rotation is modeled by the motion equation,*

$$J\dot{\omega}(t) = T_i(t-\tau_i) - T_f(t) - T_{load}(t), \tag{1.4}$$

where T_i is the indicated torque generated by the engine, which is delayed by τ_i seconds due to engine cycle delays, resulted from, e.g., fuel-air mixing, ignition delay, and cylinder pressure force propagation; T_f represents the friction, T_{load} the load, J the moment of inertia, and ω the angular velocity of the crankshaft. Feedback control action is applied to manipulate the indicated torque T_i, through the controller dynamics governed by

$$\dot{x}(t) = f(x(t), \omega(t)), \tag{1.5}$$
$$T_i(t) = h(x(t), \omega(t)). \tag{1.6}$$

Using (1.4) to (1.6), the closed-loop system becomes

$$\begin{aligned} \dot{\omega}(t) &= \frac{1}{J}[h(x(t-\tau_i), \omega(t-\tau_i)) - T_f(t) - T_{load}(t)], \\ \dot{x}(t) &= f(x(t), \omega(t)). \end{aligned}$$

Note that unlike in Example 1.1, herein the delay results from *feedback control action*, instead from the system itself; that is, due to the nature of the system, the feedback is delayed. Another type of delays of similar effects may be incurred due to *delayed measurements*. In both delayed control and delayed measurement, the delay is usually considered undesirable, which has the tendency to deteriorate the system performance or even destabilize the system.

Example 1.3 Delayed resonator.

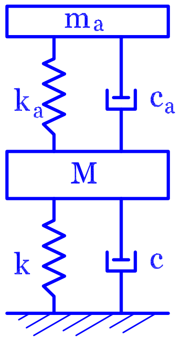

FIGURE 1.3. Classical Vibration Absorber

A delay resonator is a modification of the classical vibration absorber shown in Figure 1.3, which consists of an object with mass m, a spring with constant k, and a damper with damping factor c, subject to a harmonic excitation $f(t)$. With the parameters designed appropriately, the The *classical vibration absorber consisting of m_a, c_a and k_a is attached to a main structure with mass m.* With the parameters designed appropriately, *this can substantially reduce the amplitude of vibration in the structure. However, the classical absorber suffers from a number of shortcomings such as its sensitivity to the excitation frequency. To enhance the performance, an additional force proportional to the delayed* distance between the two masses displacement of *M is introduced. Such a structure is known as a delayed resonator, and is shown in Figure 1.4. The equation of motion for the entire system can be written as* {equation corrected}

$$\begin{aligned} &m_a\ddot{x}_a(t) + c_a(\dot{x}_a(t) - \dot{x}(t)) + k_a(x_a(t) - x(t)) \\ &\qquad -gx(t-\tau) = 0, \\ &M\ddot{x}(t) + (c + c_a)\dot{x}(t) - c_a\dot{x}_a(t) + (k + k_a)x(t) \\ &\qquad -k_a x_a(t) + gx(t-\tau) = f(t). \end{aligned}$$

It is known that with a proper design, the delayed resonator can significantly enhance the system performance, in reducing sensitivity to the excitation frequency, eliminating the main structure's vibration with the presence of damping, and simultaneously suppressing two excitation frequencies.

In this example, the time delay also takes place by way of feedback control action. However, in contrast to the delay effect in Example 1.2, the delay at present is *intentionally introduced* to enhance the system performance.

Time-delay systems may also arise from simplification of some partial differential equations. Time-delay systems are used to

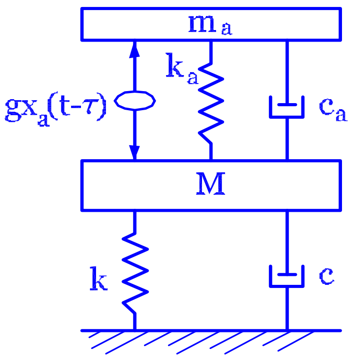

FIGURE 1.4. Delayed Resonator

model dynamical systems in many other scientific disciplines including engineering, biology, ecology, and economics. The *Notes* **Section at the end of the chapter will give a very brief historic overview to discuss some related literature.**

1.2 A simple time-delay system

We now attempt to acquaint ourselves with some of the basic concepts of time-delay systems, by analyzing the following simple system:

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-r) + h(t). \tag{1.7}$$

Here a_0, a_1 and r are real numbers, and $r > 0$ is the delay time. We want to solve the equation from the time instant $t = 0$. In order to calculate $\dot{x}(0)$, so as to advance the solution beyond the time instant $t = 0$, we need the values of $x(0)$ and $x(-r)$. Similarly, in order to calculate $\dot{x}$ at the instant $t = \xi$, $0 \leq \xi < r$, we need $x(\xi)$ and $x(\xi - r)$. For such a ξ, since $-r \leq \xi - r < 0$, $x(\xi - r)$ cannot be generated in the solution process. It is therefore clear that in order for the solution to be uniquely defined, it is necessary to specify the value of $x(t)$ for the entire interval $-r \leq t \leq 0$; in other words, we need to specify the *initial condition*

$$x(t) = \phi(t), \qquad t \in [-r, 0], \tag{1.8}$$

with some given continuous function $\phi : [-r, 0] \to \mathsf{R}$.

Once the initial condition (1.8) is given, the solution is well defined. Indeed, for $t \in [0, r]$, since $x(t-r)$ is already known in this interval, we can treat (1.7) as an ordinary differential equation, so as to obtain

$$x(t) = e^{a_0 t}x(0) + \int_0^t e^{a_0(t-u)}[a_1 x(u-r) + h(u)]du.$$

Upon obtaining $x(t)$ for $t \in [0, r]$, we can calculate $x(t)$ for $t \in [r, 2r]$ similarly, that is,

$$x(t) = e^{a_0(t-r)}x(r) + \int_r^t e^{a_0(t-u)}[a_1 x(u-r) + h(u)]du.$$

Continuing this process will allow us to obtain the solution $x(t)$, $-r \leq t < \infty$. Such a process is known as *the method of steps.*

In the following development, we shall assume that the forcing function $h(t)$ is exponentially bounded, so that

$$|h(t)| \leq Ke^{ct} \text{ for some } K > 0,\ c \in \mathsf{R},$$

and that its Laplace transform

$$H(s) = \mathcal{L}[h(t)]$$

exists. It can be shown that the solution of (1.7) satisfies the exponential bound

$$|x(t)| \leq ae^{bt}(||\phi||_c + \int_0^t |h(u)|du)$$

for some $a > 0$ and $b > 0$, where

$$||\phi||_c = \max_{-r \leq \theta \leq 0} |\phi(\theta)|$$

is the continuous norm of ϕ. This implies that the Laplace transform of $x(t)$ exists as well, and as such it can be used to solve the equation. Indeed, if we take the Laplace transform of (1.7), together with the initial condition (1.8), we obtain

$$sX(s) - \phi(0) = a_0 X(s) + a_1[e^{-sr}X(s) + \int_{-r}^0 e^{-s(u+r)}\phi(u)du] + H(s),$$

where $X(s)$ is the Laplace transform of $x(t)$. Solving for $X(s)$, we obtain

$$X(s) = \frac{1}{\Delta(s)}[\phi(0) + a_1 \int_{-r}^0 e^{-s(u+r)}\phi(u)du + H(s)], \tag{1.9}$$

where

$$\Delta(s) = s - a_0 - a_1 e^{-rs} \tag{1.10}$$

is the *characteristic quasipolynomial* of the system. Let $\Phi(t)$ denote the inverse Laplace transform of $1/\Delta(s)$,

$$\Phi(t) = \mathcal{L}^{-1}[1/\Delta(s)]. \tag{1.11}$$

Then $\Phi(t)$ is known as the *fundamental solution* of (1.7). It can be seen from (1.9) that $\Phi(t)$ is the solution of (1.7) with

$$h(t) = 0$$

and the initial condition

$$x(\theta) = \begin{cases} 1 & \theta = 0 \\ 0 & -r \leq \theta < 0. \end{cases}$$

Given the fundamental solution (1.11), it is easy to see, using the convolution theorem, that the solution of (1.7) under an arbitrary initial condition $\phi(t)$ and forcing function $h(t)$ can be expressed as

$$x(t) = \Phi(t)\phi(0) + a_1 \int_{-r}^{0} \Phi(t-u-r)\phi(u)du + \int_{0}^{t} \Phi(t-u)h(u)du \quad (1.12)$$

Equation (1.12) is called the Cauchy formula (also known as the variation-of-constant formula) for the system, which explicitly expresses the solution in terms of the initial condition ϕ and the forcing function h.

From (1.12), it is clear that the growth of the general solution for any bounded h is determined by the growth rate of **intimately related to the rate of exponential growth of** the fundamental solution Φ, and thus in term **which is determined** by the *poles* (**also known as the** characteristic roots) of the system, *i.e.*, the solutions of the *characteristic equation*

$$\Delta(s) = 0. \quad (1.13)$$

alternatively known as the characteristic roots.**{delete}** Unlike for systems without delay, generally (1.13) has an infinite number of solutions. According to complex variables theory, however, since $\Delta(s)$ is an entire function, (1.13) cannot have an infinite number of zeros within any compact set $|s| \leq R$, for any finite $R > 0$. Therefore, "most" of the system's poles go to infinity.

To understand the distribution of the poles, notice that (1.10) and (1.13) imply that

$$|s| \leq |a_0| + |a_1| e^{-r\,\mathrm{Re}(s)}. \quad (1.14)$$

When $s \to \infty$, the left hand side of (1.14) approaches infinity, and therefore the right hand side must also approach infinity. But this means that

$$\lim_{s \to \infty} \mathrm{Re}(s) = -\infty$$

Therefore, for any given real scalar α, there are only a finite number of poles of the system with real parts greater than α. With this in mind, let s_i, $i = 1, 2, \cdots$ be the poles of the system, and any α such that

$$\mathrm{Re}(s_i) \neq \alpha, \qquad i = 1, 2, \cdots$$

Using the *residual theorem* from complex variables theory, we may write (1.11) as

$$\Phi(t) = \int_{\alpha - j\infty}^{\alpha + j\infty} e^{-st}\Delta^{-1}(s)ds + \sum_{\mathrm{Re}(s_i)>\alpha} \mathrm{Res}[e^{s_i t}\Delta^{-1}(s_i)]. \tag{1.15}$$

It can be shown that the first term satisfies the condition **{"$t \to \infty$" is corrected, "$t > 0$" is added}**

$$\lim_{\alpha \to \infty} e^{-\alpha t} \int_{\alpha - j\infty}^{\alpha + j\infty} e^{-st}\Delta^{-1}(s)ds = 0, \qquad \text{for } t > 0.$$

For the particular time-delay system (1.7), we may now summarize the above discussion in the following proposition.

Proposition 1.1 *For any $\alpha \in \mathsf{R}$, there are only a finite number of poles with real parts greater than α. Let $h(t) \equiv 0$ and s_i, $i = 1, 2, \cdots$, be the poles of system (1.7), and let*

$$\alpha_0 = \max_i \mathrm{Re}(s_i).$$

Then for any $\alpha > \alpha_0$, there exists a $K > 1$ such that the solution of (1.7) with the initial condition (1.8) satisfies the inequality

$$|x(t)| \le K e^{\alpha t} ||\phi||_c.$$

Thus, in order for the solution to approach zero under an arbitrary initial condition, it is sufficient that all the poles of the system have negative real parts; in fact, this is also neccesary. This conclusion is the same as that for systems without delay.

1.3 Functional differential equations

We can use *functional differential equations* to describe time-delay systems. To formally introduce the concept of functional differential equations, let $\mathcal{C}([a,b], \mathsf{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to R^n. In many situations, one may wish to identify a *maximum time-delay* r of a system. In this case, we are often interested in the set of continuous functions mapping $[-r, 0]$ to R^n, for which we simplify the notation to $\mathcal{C} = \mathcal{C}([-r, 0], \mathsf{R}^n)$. For any $A > 0$ and any continuous function of time $\psi \in \mathcal{C}([t_0 - r, t_0 + A], \mathsf{R}^n)$, and $t_0 \le t \le t_0 + A$, let $\psi_t \in \mathcal{C}$ be a segment of the function ψ defined as $\psi_t(\theta) = \psi(t + \theta)$, $-r \le \theta \le 0$. The general form of a *retarded functional differential equation* (RFDE) (or functional differential equation of *retarded type*) is

$$\dot{x}(t) = f(t, x_t), \tag{1.16}$$

where $x(t) \in \mathsf{R}^n$ and $f : \mathsf{R}\times\mathcal{C} \to \mathsf{R}^n$. Equation (1.16) indicates that the derivative of the state variables x at time t depends on t and $x(\xi)$ for $t - r \leq \xi \leq t$. As such, to determine the future evolution of the state, it is necessary to specify the initial state variables $x(t)$ in a time interval of length r, say, from $t_0 - r$ to t_0, *i.e.*,

$$x_{t_0} = \phi, \tag{1.17}$$

where $\phi \in \mathcal{C}$ is given. In other words, $x(t_0 + \theta) = \phi(\theta)$, $-r \leq \theta \leq 0$.

For examples of retarded functional differential equations, consider

$$\begin{aligned}
\dot{x}(t) &= ax(t) + b(x - r) + \cos\omega t, && (1.18)\\
\dot{x}(t) &= (2 + \sin\omega t)x(t), && (1.19)\\
\dot{x}(t) &= \int_{-r}^{0} c(\theta)x(t+\theta)d\theta, && (1.20)\\
\dot{x}(t) &= ax(t)x(t-r). && (1.21)
\end{aligned}$$

Here Equation (1.18) represents a nonhomogeneous linear time-invariant system with a single delay. Equation (1.19) is a linear time-varying ordinary differential equation, which is a special functional differential equation. Equation (1.20) represents a system with distributed delay. Equation (1.21) represents a nonlinear system. A retarded functional differential equation may also involve higher order derivatives, which is known as a higher order RFDE. As with differential equations without delay, we may introduce additional variables to transform a higher order RFDE to a standard first order functional differential equation of the form (1.16). For example,

$$\ddot{y}(t) + 2\dot{y}(t-1) + y(t) = 0$$

can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x_1(t-1) \\ x_2(t-1) \end{pmatrix}$$

with the variables

$$\begin{aligned}
x_1(t) &= y(t),\\
x_2(t) &= \dot{y}(t).
\end{aligned}$$

It is important to point out that in any a RFDE, the highest order derivative does not contain any delayed variables. When such a term does occur **appear**, then we have a functional differential equation of *neutral* type. For example,

$$5\dot{x}(t) + 2\dot{x}(t-r) + x(t) - x(t-r) = 0$$

is a *neutral functional differential equation* (NFDE). In this book, we will mainly focus on retarded functional differential equations, and thus for simplicity will often refer to a RFDE as a functional differential equation, unless otherwise confusion may be incurred. We will also use the terms *time-delay system* and *functional differential equation* interchangeably. Delay systems of neutral type, i.e., systems described by NFDE's, will only be discussed briefly, and will be spelled out where they appear.

For an $A > 0$, a function x is said to be a solution of Equation (1.16) on the interval $[t_0 - r, t_0 + A)$ if within this interval x is continuous and satisfies the RFDE (1.16). Here the derivative with respect to time t should be interpreted as one-sided derivative in the forward direction. Of course, a solution also implies that (t, x_t) is within the domain of definition of f. If the solution also satisfies the initial condition (1.17), we say that it is a solution of the equation with the initial condition (1.17), or simply a solution with **through** (t_0, ϕ). We denote it as $x(t_0, \phi, f)$ when it is important to distinguish the specific RFDE and the **specify the particular RFDE with the given** initial condition. The value of $x(t_0, \phi, f)$ at t is denoted as $x(t; t_0, \phi, f)$. We will also omit f to write $x(t_0, \phi)$ or $x(t; t_0, \phi)$ when f is obvious from the context.

An issue of foremost importance **A fundamental issue** in the study of differential equations, be it an **both** ordinary differential equations or a **and** functional differential equations, is the existence and uniqueness of solution. It is not our aim in the present book, however, to delve into the existence and uniqueness **this** issue for RFDE's. Instead, we state the following theorem without proof.

Theorem 1.2 *Suppose that Ω is an open set in $\mathsf{R} \times \mathcal{C}$, $f : \Omega \to \mathsf{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in Ω, that is, for each given compact set $\Omega_0 \subset \Omega$, there exists a constant L, such that*

$$||f(t, \phi_1) - f(t, \phi_2)|| \leq L||\phi_1 - \phi_2||$$

for any $(t, \phi_1) \in \Omega_0$ and $(t, \phi_2) \in \Omega_0$. If $(t_0, \phi) \in \Omega$, then there exists a unique solution of Equation (1.16) with through *(t_0, ϕ).*

Nearly in all cases the systems treated in this book satisfy the conditions in this theorem, and hence the existence and uniqueness of the solutions is insured. Exceptions will be spelled **pointed {spell out usually refer to details, and is more oral in flavor}** out explicitly.

1.4 Stability of time-delay systems

1.4.1 Stability concept

Let $y(t)$ be a solution of the RFDE (1.16). The stability of the solution concerns the system's behavior when the system trajectory $x(t)$ deviates from $y(t)$. Throughout this book, without loss of generality, we will assume that the functional differential equation (1.16) admits the solution $x(t) = 0$, which will be referred to as the *trivial solution*. Indeed, if it is desirable to study the stability of a non-trivial solution $y(t)$, then we may resort to the variable transformation $z(t) = x(t) - y(t)$, so that the new system

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t) \tag{1.22}$$

has the trivial solution $z(t) = 0$.

For a function $\phi \in \mathcal{C}([a, b], \mathsf{R}^n)$, define the **continuout** norm $\|\cdot\|_c$ by

$$||\phi||_c = \max_{a \le \theta \le b} ||\phi(\theta)||$$

In this definition, the vector norm $\|\cdot\|$ represents the 2-norm $\|\cdot\|_2$. It should be pointed out that other norms such as 1-norm and ∞-norm can also be used in most of our subsequent development, as long as the same norm is used consistently.

Definition 1.1 *For the system (1.16), the trivial solution $x(t) = 0$ is said to be* stable *if for any $t_0 \in \mathsf{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $||x_{t_0}||_c < \delta$ implies $||x(t)|| < \epsilon$ for $t \ge t_0$. It is said to be* asymptotically stable *if* it is stable, and *for any $t_0 \in \mathsf{R}$ and any $\epsilon > 0$, there exists a* {use δ_a *instead of* δ in asymptotic stability context} $\delta_a = \delta_a(t_0, \epsilon) > 0$ *such that $||x_{t_0}||_c < \delta_a$ implies $\lim_{t\to\infty} x(t) = 0$* and $||x(t)|| < \epsilon$ for $t \ge t_0$. *It is said to be* uniformly stable *if it is stable and $\delta(t_0, \varepsilon)$ can be chosen independently of t_0. It is* uniformly asymptotically stable *if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that $||x_{t_0}||_c < \delta$ implies $||x(t)|| < \eta$ for $t \ge t_0 + T$ and $t_0 \in \mathsf{R}$. It is globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and δ_a can be an arbitrarily large, finite number.*

One should note that the stability notions herein are not at all different from their counterparts for systems without delay, modulo to the different assumptions on the initial conditions. Note also that in this book, we shall only be concerned with asymptotic stability. Thus, while the Lyapunov-Krasovskii theorem and Razumikhin theorem will be stated in completeness in the next two sections, inclusive of both stability and asymptotic stability, elsewhere a stable system will be equated to an asymptotically stable system **is often simply said to be "stable"**. Correspondingly, we will sometimes refer to a system which is not asymptotically stable as "unstable".

1.4.2 Lyapunov-Krasovskii theorem

As in the study of systems without delay, an effective method for determining the stability of a time-delay system is Lyapunov method. For a system without delay, this requires the construction of a Lyapunov function $V(t, x(t))$, which in some sense is a potential measure quantifying the deviation of the state $x(t)$ from the trivial solution 0. Since for a delay-free system $x(t)$ is needed to specify the system's future evolution beyond t, and since in a time-delay system the "state" at time t required for the same purpose is the value of $x(t)$ in the interval $[t-r, t]$, i.e., x_t, it is natural to expect that for a time-delay system, the corresponding Lyapunov function be a *functional* $V(t, x_t)$ depending on x_t, which also should measure the deviation of x_t from the trivial solution 0. Such a functional is known as a Lyapunov-Krasovskii functional.

More specifically, let $V(t, \phi)$ be differential, and let $x_t(\tau, \phi)$ be the solution of (1.16) at time t with the initial condition $x_\tau = \phi$. We may calculate the derivative of $V(t, x_t)$ with respect to t and evaluate it at $t = \tau$. This gives rise to

$$\begin{aligned}\dot{V}(\tau, \phi) &= \left.\frac{d}{dt}V(t, x_t)\right|_{t=\tau,\ x_t=\phi} \\ &= \limsup_{\Delta t \to 0} \frac{1}{\Delta t}[V(\tau + \Delta t, x_{\tau+\Delta t}(\tau, \phi)) - V(\tau, \phi)]\end{aligned}$$

Intuitively, a nonpositive $\dot{V}$ indicates that x_t does not grow with t, which in term means that the system under consideration is stable in light of Definition 1.1. The more precise statement of this observation is the following theorem.

Theorem 1.3 (Lyapunov-Krasovskii Stability Theorem) *Suppose that* $f : \mathsf{R}\times\mathcal{C} \to \mathsf{R}^n$ given {delete} *in (1.16), maps* every{delete, add $\mathsf{R}\times(\ldots)$} $\mathsf{R} \times$ (*bounded set in* $\mathcal{C}$) *into a bounded set in* R^n, *and that* $u, v, w : \bar{\mathsf{R}}_+ \to \bar{\mathsf{R}}_+$ *are continuous nondecreasing functions, where additionally* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$, *and* $u(0) = v(0) = 0$. *If there exists a continuous differentiable functional* $V : \mathsf{R}\times\mathcal{C} \to \mathsf{R}$ *such that*

$$u(||\phi(0)||) \le V(t, \phi) \le v(||\phi||_c)$$

and

$$\dot{V}(t, \phi) \le -w(||\phi(0)||),$$

then the trivial solution of (1.16) is uniformly stable. If $w(s) > 0$ *for* $s > 0$, *then it is uniformly asymptotically stable.* In addition, if If, in addition, $\lim_{s\to\infty} u(s) = \infty$, *then it is globally uniformly asymptotically stable.*

Proof. For any $\varepsilon > 0$, we can find a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that $v(\delta) < u(\varepsilon)$. Hence, for any initial time t_0 and any initial condition $x_{t_0} = \phi$ with $||\phi||_c < \delta$, we have $\dot{V}(t, x_t) \le 0$, and therefore $V(t, x_t) \le V(t, \phi)$, for any $t \ge t_0$. This implies that

$$u(||x(t)||) \le V(t, x_t) \le V(t_0, \phi) \le v(||\phi||_c) \le v(\delta) < u(\varepsilon),$$

which in turn implies that $||x(t)|| < \varepsilon$ for $t \geq t_0$. This proves the uniform stability.

To prove uniform asymptotic stability, let $\varepsilon > 0$ and $\delta_a > 0$ such that $v(\delta_a) < u(\varepsilon)$. Then $||x_{t_0}||_c \leq \delta_a$ implies that $||x(t)|| < \varepsilon$ for $t \geq t_0$. For this δ_a and any $\eta > 0$, we need to show that there exists a $T = T(\delta_a, \eta)$, such that $||x(t)|| < \eta$ for $t \geq t_0 + T$. Let $\delta_b > 0$ satisfy $v(\delta_b) < u(\eta)$. Then, it suffices to show that $||x_{t_0+T}||_c < \delta_b$, which implies $u(||x(t)||) \leq V(t, x_t) < v(\delta_b) < u(\eta)$ for $t \geq t_0 + T$ and therefore $||x(t)|| < \eta$. We establish this fact by contradiction. Suppose that this is not true; in other words, no such T exists. It follows that $||x_t||_c \geq \delta_b$ for all $t \geq t_0$. This means that a sequence $\{t_k\}$ exists such that

$$t_0 + (2k-1)r \leq t_k \leq t_0 + 2kr, \qquad k = 1, 2, ...,$$

and

$$||x(t_k)|| \geq \delta_b.$$

For a sufficiently large L, due to the assumption on f, we have $||\dot{x}(t)|| = ||f(t, x_t)|| < L$ for all $t \geq t_0$. Consider any $t \in I_k = [\underline{t}_k, \overline{t}_k]$, where $\underline{t}_k = t_k - \frac{\delta_b}{2L}$, and $\overline{t}_k = t_k + \frac{\delta_b}{2L}$. Without loss of generality, assume that $t \in [t_k - \frac{\delta_b}{2L}, t_k]$; the case for $t \in [t_k, t_k + \frac{\delta_b}{2L}]$ can be dealt with in the same manner. Since

$$x(t) = x(\underline{t}_k) + \dot{x}(t + \theta(t_k - \underline{t}_k)(t - \underline{t}_k), \qquad 0 \leq \theta \leq 1,$$

it follows that

$$\|x(t)\| \geq \|x(\underline{t}_k)\| - \|\dot{x}(t + \theta(t_k - \underline{t}_k)\| \frac{\delta_b}{2L} > \frac{\delta_b}{2}.$$

{The above are replaced by the following} Therefore, for $t \in I_k = [t_k - \frac{\delta_b}{2L}, t_k + \frac{\delta_b}{2L}]$**, due to the Mean Value Theorem, for some** $\theta \in [0, 1]$**,**

$$\begin{aligned} ||x(t)|| &= ||x(t_k)|| + \dot{x}(t_k + \theta(t - t_k))(t - t_k)|| \\ &\geq ||x(t_k)|| - ||\dot{x}(t_k + \theta(t - t_k))|| \cdot |(t - t_k)| \\ &\geq \delta_b - L \cdot \frac{\delta_b}{2L} \\ &= \frac{\delta_b}{2} \end{aligned}$$

{end of replacement} Therefore,

$$\dot{V}(t, x_t) \leq -w(\delta_b/2), \text{ for } t \in I_k$$

and $\dot{V}(t, x_t) \leq 0$ otherwise. By increasing L if necessary, we can assume that these intervals do not overlap, and hence

$$V(t_k, x_{t_k}) \leq V(t_0, \phi) - w(\delta_b/2) \frac{\delta_b}{L}(k-1).$$

But this implies that $V(t_k, x_{t_k}) < 0$ for a sufficiently large k, which is a contradiction. This proves the uniform asymptotic stability.

Finally, if $\lim_{s\to\infty} u(s) = \infty$, then δ_a above may be arbitrarily large, and ε can be chosen after δ_a is given to satisfy $v(\delta_a) < u(\varepsilon)$, and therefore global uniform asymptotic stability can be concluded. ∎

It is obvious from the above proof that u, v, w and V need only be defined in a neighborhood of 0 except for the case of global stability. Notice also that the lower bound of V need only to be a positive function of $\phi(0)$.

1.4.3 Razumikhin theorem

That the Lyapunov-Krasovskii functional requires the state variable $x(t)$ in the interval $[t-r, t]$ necessitates the manipulation of functionals, which consequently makes the application of the Lyapunov-Krasovskii theorem rather difficult. This difficulty may sometimes be circumvent using the Razumikhin theorem, an alternative result involving essentially only functions rather than functionals, made available by Razumikhin.

The key idea behind the Razumikhin theorem also dwells on a function $V(x)$ representative of the size of $x(t)$. For such a function,

$$\bar{V}(x_t) = \max_{\theta\in[-r,0]} V(x(t+\theta))$$

serves to measure the size of x_t. If $V(x(t)) < \bar{V}(x_t)$, then $\dot{V}(x) > 0$ does not make $\bar{V}(x_t)$ grow. Indeed, for $\bar{V}(x_t)$ not to grow, it is only necessary that $\dot{V}(x(t))$ is not positive whenever $V(x(t)) = \bar{V}(x_t)$. The precise statement is as follows.

Theorem 1.4 (Razumikhin Theorem) *Suppose $f : \mathsf{R}\times\mathcal{C} \to \mathsf{R}^n$ in (1.16) takes $\mathsf{R}\times$ (bounded sets of $\mathcal{C}$) into bounded sets of R^n, and $u, v, w : \bar{\mathsf{R}}_+ \to \bar{\mathsf{R}}_+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$, v strictly increasing. If there exists a continuous functional $V : \mathsf{R}\times\mathsf{R}^n \to \mathsf{R}$ such that*

$$u(||x||) \le V(t,x) \le v(||x||), \text{ for } t \in \mathsf{R} \text{ and } x \in \mathsf{R}^n \tag{1.23}$$

and the derivative of V along the solution $x(t)$ of (1.16) satisfies

$$\dot{V}(t,x(t)) \le -w(||x(t)||) \text{ whenever } V(t+\theta, x(t+\theta)) \le V(t,x(t)) \tag{1.24}$$

for $\theta \in [-r, 0]$, then system (1.16) is uniformly stable.

If, in addition, $w(s) > 0$ for $s > 0$, and there exist a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that condition (1.24) is strengthened to

$$\dot{V}(t,x(t)) \le -w(||x(t)||) \text{ if } V(t+\theta, x(t+\theta)) \le p(V(t,x(t))) \tag{1.25}$$

for $\theta \in [-r, 0]$, then system (1.16) is uniformly asymptotically stable.

If in addition $\lim_{s\longrightarrow\infty} u(s) = \infty$*, then system (1.16) is globally uniformly asymptotically stable.*

Proof. To prove uniform stability, for any given $\varepsilon > 0$, let $0 < \delta < v^{-1}(u(\varepsilon))$. Then for any given t_0 and ϕ, $||\phi|| < \delta$, we have $V(t_0+\theta, \phi(\theta)) \leq v(\delta) < u(\varepsilon)$ for $\theta \in [-r, 0]$. Let x be the solution of (1.16) with initial condition $x_{t_0} = \phi$. According to (1.24), as t increases, whenever $V(t, x(t)) = v(\delta)$ and $V(t+\theta, x(t+\theta)) \leq v(\delta)$ for $\theta \in [-r, 0]$, $\dot{V}(t, x(t)) \leq 0$. Due to continuity of $V(t, x(t))$, it is therefore impossible for $V(t, x(t))$ to exceed $v(\delta)$. In other words, we have $V(t, x(t)) \leq v(\delta) < u(\varepsilon)$ for $t \geq t_0 - r$. But this implies $||x(t)|| \leq \varepsilon$ for $t \geq t_0 - r$.

To prove uniform asymptotic stability under the strengthened condition (1.25), we need to show that for any sufficiently small δ_a (such that there exists an $\varepsilon > 0$ to satisfy $v(\delta_a) = u(\varepsilon)$), and any arbitrarily given η, $0 < \eta < \varepsilon$, there exists a $T = T(\delta_a, \eta)$, such that $||x_t|| < \eta$ for $t \geq t_0 + T$, $||\phi|| < \delta_a$ and $t_0 \in \mathsf{R}$.

Let V_l satisfy $0 < V_l < u(\eta)$ (say, $V_l = 0.9u(\eta)$). The continuity of function p implies that there exists an $a > 0$, such that $a < p(s) - s$ for $V_l \leq s < v(\delta_a)$. Let N be the smallest positive integer to satisfy $V_l + Na \geq v(\delta_a)$, and let $\gamma = \min_{v^{-1}(V_l)\leq s\leq\varepsilon} w(s)$. Let $t_k = t_0 + k(a/\gamma + r)$. We will show that

$$V(t, x(t)) \leq \min\{V_l + (N-k)a, v(\delta_a)\} \text{ for } t_k - r \leq t \leq t_k \tag{1.26}$$

implies

$$V(t, x(t)) \leq V_l + (N-k-1)a \text{ for } t \geq t_k + a/\gamma \tag{1.27}$$

especially, it implies (1.27) for $t_{k+1} - r \leq t \leq t_{k+1}$. An induction then shows that $V(t, x(t)) \leq V_l < u(\eta)$ for $t \geq t_N = t_0 + N(a/\gamma + r)$. This implies $||x(t)|| \leq \eta$ for $t \geq t_0 + T$ for $T = N(a/\gamma + r)$, which will complete the proof.

Assume that $V(t_k, x(t_k)) > V_l+(N-k-1)a$. Then, according to condition (1.25), there exists a $d > 0$ such that $\dot{V}(t, x(t)) \leq -\gamma$ and $V(t_k, x(t_k)) \geq V(t, x(t)) > V(t_k+d, x(t_k+d)) = V_l + (N-k-1)a$ for $t_k \leq t < t_k + d$ for some $d > 0$. Clearly, $d \leq \frac{V(t_k,x(t_k))-V(t_k+d,x(t_k+d))}{\gamma} \leq a/\gamma$. Also, it is easy to see that the inequality in (1.27) holds for $t \geq t_k+d$ (and therefore also holds for $t \geq t_k+a/\gamma$) since $\dot{V}(t, x(t)) \leq 0$ whenever $V(t, x(t)) = V_l+(N-k-1)a$. Thus the uniform asymptotic stability is proven.

If $\lim_{s\longrightarrow\infty} u(s) = \infty$, then for arbitrary large $\delta_a > 0$ there exists an $\varepsilon > 0$ to satisfy $v(\delta_a) = u(\varepsilon)$. This allows us to conclude global uniform asymptotic stability. ∎

1.5 Linear systems

Among RFDE's, of particular interest is the instance that f is linear with respect to x_t. Systems in this category are linear retarded delay systems. Linear delay systems avail more analysis tools and hence enable more in-depth studies. We shall first give a brief review of general linear time-varying delay systems and next focus on linear time-invariant systems.

A general linear time-delay system can be described by the RFDE

$$\dot{x}(t) = L(t)x_t + h(t), \tag{1.28}$$

where $L(t)$ is a time-varying linear operator acting on x_t. In this case, it is always possible to find a matrix function $F : \mathsf{R} \times [-r, 0] \to \mathsf{R}^{n\times n}$ of bounded variation, such that

$$F(t, 0) = 0$$

and

$$L(t)\phi = \int_{-r}^{0} d_\theta[F(t, \theta)]\phi(\theta). \tag{1.29}$$

Here in (1.29), the Stieltjes integral is required in general, and the subscript θ is used to indicate that θ (rather than t) is the integration variable. As such, a general linear RFDE can be represented as

$$\dot{x}(t) = \int_{-r}^{0} d_\theta[F(t, \theta)]x(t + \theta) + h(t). \tag{1.30}$$

In particular, many linear RFDE's can be further specialized to

$$\dot{x}(t) = \sum_{k=0}^{K} A_k(t)x(t - r_k(t)) + \int_{-r}^{0} A(t, \theta)x(t + \theta)d\theta + h(t), \tag{1.31}$$

where $A_k(t)$ and $A(t, \theta)$ are given $n \times n$ real continuous matrix functions, and $r_k(t)$ are given continuous functions representing time-varying delays, which can be ordered with no loss of generality, so that

$$0 = r_0 < r_1 < ... < r_K = r.$$

Under such circumstances,

$$F(t, \theta) = -\int_{\max(\theta,-r)}^{0} A(t, \tau)d\tau - \sum_{-r_k>\theta} A_k(t)$$

The fundamental solution $\Phi(t, t_0)$ of the RFDE (1.30) is the $n\times n$ matrix function satisfying the homogeneous equation

$$\dot{\Phi}(t, t_0) = \int_{-r}^{0} d_\theta[F(t, \theta)]\Phi(t + \theta, t_0)$$

together with the initial condition

$$\Phi(t_0+\theta,t_0)=\begin{cases} I & \theta=0 \\ 0 & -r\le\theta<0 \end{cases}. \tag{1.32}$$

Note that although this initial condition does not satisfy the existence condition in Theorem 1.2, the existence and uniqueness of the solution can nevertheless be established. In fact, with the fundamental solution so given, the solution of the linear RFDE (1.30) with the initial condition $x_{t_0}=\phi$ can be expressed as

$$x(t;t_0,\phi)=\Phi(t,t_0)\phi(0)+\int_{t_0}^{t}\Phi(t,\alpha)d_\alpha[G(\alpha;t_0,\phi,h)], \tag{1.33}$$

where

$$G(t;t_0,\phi,h)=\phi(0)+\int_{t_0}^{t}\int_{-r}^{t_0-s}d_\theta[F(s,\theta)]\phi(s-t_0+\theta)ds+\int_{t_0}^{t}h(\alpha)d\alpha.$$

Note that herein we have extended the domain of definition of $F(t,\theta)$ to $-\infty<\theta<+\infty$, with the understanding that

$$\begin{array}{ll} F(s,\theta)=0 & \theta>0 \\ F(s,\theta)=F(s,-r) & \theta<-r. \end{array}$$

It is clear that for a linear system, and for any of its solutions $y(t)$, $z(t)=x(t)-y(t)$ satisfies the equation

$$\dot{z}(t)=\int_{-r}^{0}d_\theta[F(t,\theta)]z(t+\theta). \tag{1.34}$$

Thus, for a linear RFDE, the stability of any solution is equivalent to the stability of its trivial solution. For this reason, we will simply say that a system is stable, instead that any particular solution is stable. It should also be rather clear that a linear system is stable if and only if it is globally stable.

1.6 Linear time-invariant systems

If the function F in (1.30) is independent of time t, then the system described by (1.30) is linear time-invariant (LTI). An LTI RFDE can be written as

$$\dot{x}(t)=\int_{-r}^{0}dF(\theta)x(t+\theta)+h(t) \tag{1.35}$$

In this case, the fundamental solution $\Phi(t,t_0)$ only depends on $t-t_0$. As a result, without loss of generality, we only need to consider $t_0=0$. We write $\Phi(t)$ for $\Phi(t,0)$.

Like their counterparts without delay, LTI time-delay systems can be studied effectively using frequency domain methods. This distinguishing advantage is made available by such frequency domain analysis tools as Laplace transforms. The case study given in Section 1.2 has sheded light in how this may proceed, and is now extended below to general LTI systems. Let

$$x_0 = \phi \tag{1.36}$$

be the initial condition. Taking the Laplace transform of (1.35) with the initial condition (1.36), we obtain

$$sX(s) - \phi(0) = \int_{-r}^{0} e^{\theta s} dF(\theta) X(s) + \int_{-r}^{0} e^{\theta s} dF(\theta) \int_{\theta}^{0} e^{-\alpha s} \phi(\alpha) d\alpha + H(s)$$

where $X(s)$ and $H(s)$ are the Laplace transform of $x(t)$ and $h(t)$, respectively,

$$\begin{aligned} X(s) &= \mathcal{L}[x(t)] = \int_{0}^{\infty} x(t) e^{-st} dt \\ H(s) &= \mathcal{L}[x(t)] = \int_{0}^{\infty} h(t) e^{-st} dt \end{aligned}$$

Solving for $X(s)$ yields

$$X(s) = \Delta^{-1}(s) \left[\phi(0) + \int_{-r}^{0} e^{\theta s} dF(\theta) \int_{\theta}^{0} e^{-\alpha s} \phi(\alpha) d\alpha + H(s) \right] \tag{1.37}$$

where

$$\Delta(s) = sI - \int_{-r}^{0} e^{\theta s} dF(\theta) \tag{1.38}$$

is called the *characteristic matrix*. The equation

$$\det[\Delta(s)] = 0 \tag{1.39}$$

is called the *characteristic equation*, and the expression $\det[\Delta(s)]$ the *characteristic function*, or alternatively, the *characteristic quasipolynomial*. The solutions to (1.39) are called the *characteristic roots* or *poles* of the system.

Let $h(t) = 0$ and choose the initial condition (1.32). Then the right hand side of (1.37) becomes $\Delta^{-1}(s)$. From this and the definition of fundamental solution, it follows that

$$\mathcal{L}[\Phi] = \Delta^{-1}(s),$$

i.e., $\Delta^{-1}(s)$ is the Laplace transform of the fundamental solution. This shows that the fundamental solution also satisfies the equation

$$\dot{\Phi}(t) = \int_{-r}^{0} \Phi(t+\theta) dF(\theta),$$

together with the same initial condition (1.32).

The key result that enables frequency domain analysis of stability for time-delay systems is the following theorem, which generalizes Proposition 1.1. This result is given herein without proof. One may, however, go through an analysis analogous to that in Section 1.2 for an intuitive understanding.

Theorem 1.5 *For any real scalar γ, the number of the solutions, counting their multiplicities, to the characteristic equation (1.39) with real parts greater than γ is finite. Define*

$$\alpha_0 = \max\{\mathrm{Re}(s) |\ \det[\Delta(s)] = 0\} \tag{1.40}$$

The following statements are true.

(i) The LTI delay system (1.35) is stable if and only if $\alpha_0 < 0$.

(ii) For any $\alpha > \alpha_0$, there exists a $K > 0$ such that any solution $x(t)$ of (1.35) with $h(t) = 0$ and the initial condition (1.36) is bounded as

$$||x(t)|| \leq Ke^{\alpha t}||\phi||_c. \tag{1.41}$$

(iii) α_0 is continuous with respect to r_k, for all $r_k \geq 0$, $k = 1,\ 2,\ \cdots,\ K$.

The number α_0 is known as the *stability exponent* of the system. The above theorem states that an LTI delay system is stable if and only if its stability exponent is strictly negative. This is equivalent to that all the poles of the system have negative real parts, a fact that forms the very basis of frequency domain stability analysis, in the same spirit as for delay-free LTI systems; indeed, it renders the study of the stability of an LTI delay system into one on the zeros of the system's characteristic quasipolynomial. Moreover, as a fundamental fact, while unlike polynomials the zeros of a quasipolynomial are not necessarily continuous with respect to the delays r_k, $k = 1,\ 2,\ \cdots,\ K$ as $r_k \downarrow 0$, the theorem makes it clear that for an LTI RFDE, the stability exponent α_0 defines a continuous function of r_k, for all $r_k \geq 0$, $k = 1, 2, ..., K$. This continuity property is reassuring and will play a crucial role in our subsequent stability studies.

One particular class of LTI delay systems under (1.35) are those with pointwise **(or concentrated)** delays, which can be further simplified to the description

$$\dot{x}(t) = \sum_{k=0}^{K} A_k x(t - r_k).$$

Here A_k are given $n \times n$ real constant matrices, and r_k are given real constants, ordered such that

$$0 = r_0 < r_1 < \cdots < r_K = r.$$

For such systems, the characteristic quasipolynomial $\Delta(s)$ takes the form

$$p\left(s;\ e^{-r_1 s},\ \cdots,\ e^{-r_K s}\right) = \left(sI - \sum_{k=0}^{K} e^{-r_k s} A_k\right) = p_0(s) + \sum_{k=1}^{m} p_k(s) e^{-h_k s},$$

where $p_k(s)$, $k = 0,\ 1,\ \cdots,\ m$ are polynomials, and h_k, $k = 1,\ 2,\ \cdots,\ m$ are sums of the delay parameters r_k. It is evident that this class of systems can also be described by the scalar differential-difference equation

$$y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{k=0}^{m} p_{ki} y^{(i)}(t - h_k) = 0,$$

which admits the same quasipolynomial with

$$\begin{aligned} p_0(s) &= s^n + \sum_{i=0}^{n-1} p_{0i} s^i, \\ p_k(s) &= \sum_{i=0}^{n-1} p_{ki} s^i, \qquad k = 1,\ 2,\ \cdots,\ m. \end{aligned}$$

Thus, both the state-space and differential-difference equations can be used to describe the same system, and that the two representations are mutually exchangeable. In general, the ratios between the delays, r_i/r_j, may be irrational numbers, in which case, the delays are said to be *incommensurate*. When all such ratios are rational numbers, on the other hand, we say that the delays are *commensurate*. In the latter case, the delays r_k (and hence h_i) become integer multiples of a certain positive τ. The characteristic quasipolynomial $\Delta(s)$ of systems with commensurate delays can then be written as

$$a(s,\ e^{-\tau}) = \sum_{k=0}^{q} a_k(s) e^{-k\tau s}.$$

It is worth noting that for systems with incommensurate delays the characteristic quasipolynomial is in effect a *multivariate polynomial* of several variables, while for those with commensurate delays, it may be viewed as a bivariate polynomial, or even further, a polynomial with coefficients themselves as polynomials. This will prove to be a fundamental distinction. Indeed, it will be seen in the subsequent chapters that for systems with commensurate delays, the characteristic quasipolynomial is much easier to analyze.

1.7 Neutral time delay systems

{This is a new section to support some contents of Chapter 3 and 4}

We will only discuss the stability of linear time invariant systems with pointwise delay in the form of

$$\sum_{k=0}^{K} [A_k \dot{x}(t - r_k) + B_k x(t - r_k)] = 0, \tag{1.42}$$

where A_0 is nonsingular, and we can assume $A_0 = I$ without loss of generality. Let the characteristic quasipolynomial $\Delta(s)$ be defined as

$$\Delta(s) = \det\left(\sum_{k=0}^{K} e^{-r_k s}[sA_k + B_k]\right).$$

Let the all the solutions of the characteristic equation

$$\Delta(s) = 0$$

be denoted by s_k, $k = 1, 2, ...$ which are referred as the poles of the system. Then, we can state the following Theorem.

Theorem 1.6 *The system described by (1.42) is asymptotically stable if and only if there exists an $\alpha > 0$ such that all the poles of the system satisfy*

$$\mathrm{Re}(s_k) \leq -\alpha. \tag{1.43}$$

We will not give proof here. Interested readers are referred to the literature discussed in the *Notes and Reference* section. The above Theorem states that, in order for a neutral time delay system to be stable, all the poles must lie on the left of a verticle line $\mathrm{Re}(s) = -\alpha$. It is insufficient for all the poles to be on the left hand side of the imaginary axis. Recall that a retarded functional differential equation may only have a finite number of poles on the right of any vertical line, in which case, $\mathrm{Re}(s_k) < 0$, $k = 1, 2, ...$ implies (**??**). This is no longer the case for a system with neutral delay. It is indeed possible that a neutral functional differential equation may have an series of poles asymptotically approaching the imaginary axis, in which case, the system is not asymptotically stable.

1.8 Outline of the text

The focus of this book is the study of stability and robust stability of time-delay systems. In particular, we concentrate largely on linear delay systems, or systems that can be analyzed essentially using tools developed for linear systems. Within this scope, the rest of the book proceeds to develop time and frequency domain stability and robust stability conditions. The results can be grossly divided into three **two** categories: Chapter 2-Chapter 4 contain frequency domain stability conditions, Chapter 5-Chapter 7 present time domain results, and Chapter 8 explores relationships between time and frequency domain approaches **discusses robust stability under dynamic uncertainty which can be formulated both as time domain or frequency domain form**. Our aim is to report the main advances of last decade or so **fifteen to twenty years** on the stability and robust stability studies of time-delay systems, and to develop these results in a systematic as well as coherent manner. We intend to present a focused view rather than an exhaustive coverage.

Chapter 2 studies LTI systems with commensurate delays, and focuses on delay-dependent and delay-independent stability conditions. Here delay-independent stability, or stability independent of delay, is referred to as the property that a system is stable for all nonnegative delay values, while delay-dependent stability is characterized by the so-called *delay margin*. The chapter opens with a close examination of several prevalent classical results known collectively as two-variable criteria, which serves two useful purposes. First, it emphasizes the critical role that the continuity of the stability exponent plays in stability analysis. Secondly, it helps reveal the weakness of classical stability results. These two aspects thus provide the guide for the ensuing development, leading to the frequency-sweeping criteria of Section 2.3 and the constant matrix tests of Section 2.4, both of which are necessary and sufficient conditions for delay-dependent and delay-independent stability alike, and both require only the computations of matrix pencils. The frequency-sweeping tests are readily obtained based upon simple algebraic manipulations, and are reminiscent of small gain conditions. On the other hand, the derivation of the constant matrix tests draws upon heavily the theory of polynomials, notably the results concerning the Schur-Cohn stability test. While frequency-sweeping criteria are to be performed on the basis of frequency gridding, the constant matrix tests require only the computation of constant matrices.

Chapter 3 studies systems with incommensurate delays, which is both motivated by and built upon a small gain approach. This chapter begins with a concise introduction of an operator theoretic system description as well as a statement of the small gain theorem, the very rudiments needed in a small gain approach. It continues with a brief narrative of the so-called **{delete}** *structured singular value*, which is a key measure in robust control theory and in the characterization of stability for LTI systems with incommensurate delays. Indeed, it is shown that the necessary and sufficient condition for stability independent of delay requires computing a structured singular value; similarly, a sufficient condition is also obtained in the same manner for delay-dependent stability. While these provide useful conceptual results, the structured singular value is known to be difficult to compute, and in fact poses an intractable computational problem in general. Consequently, one is led to the contemplation on the inherent computational complexity of the stability problem, for which we give a formal analysis drawing upon concepts and techniques typically found in computing theory and operation research, with such notions as *NP-hard* and *NP-complete* problems. The analysis is preceded by a brief summary of necessary materials drawn from complexity theory, and it led to the classification of the stability problem as one of NP-hard. In light of this difficulty, we then develop a number of readily computable sufficient conditions. The chapter ends with an extension to neutral LTI delay systems, with both commensurate and incommensurate delays.

Chapter 4 examines the robust stability of LTI delay systems described by uncertain quasipolynomials with incommensurate delays. Here by robust stability, we mean that the stability is maintained for the entire family of quasipolynomials whose **as the** coefficients vary **with**in a prescribed set. Fundamental robust stability concepts and results such as the *zero exclusion principle*, the *edge theorem*, and the *phase growth* condition are extended to and further developed for uncertain quasipolynomials. More specifically, the chapter first presents the zero exclusion principle, which provides a general, geometrical characterization on the zeros of uncertain quasipolynomials. This leads to the first main result of the chapter, namely, the *edge theorem*. The result is analogous to its counterpart in robust stability theory, and it states that for a polytopic family of quasipolynomials, robust stability is achieved if the edge members of the family are all stable. It is worth noting that in the robust stability literature, the polytopic family remains to be the most general uncertain polynomial for which a tractable necessary and sufficient stability condition is available. As such, the edge theorem for the polytopic quasipolynomial presents an equally strong result. While it may be conservative for more special classes of quasipolynomials, the chapter goes on to develop more specialized conditions for interval quasipolynomials. **The chapter continues to identify some classes of polytopic quasipolynomial families such that it is sufficient to check the stability of a finite number of quasipolynomials, which can again be considered as an extension to the polynomial case.** Furthermore, it presents a multivariate polynomial approach for the robust stability of the interval and diamond multivariate polynomials, which in turn provide stability conditions for the interval and diamond quasipolynomial families. As a whole, the development in this chapter shares much in common with robust stability analysis of uncertain polynomials, and it too emphasizes robust stability tests given by finitely many conditions, in terms of edge and vertex quasipolynomials. **{delete}**

Beginning from Chapter 5, we present time domain stability and robust stability conditions. This is where Lyapunov-Krasovskii theorem and Razumikhin theorem come to play their roles. Chapter 5 treats systems with a single delay, but without uncertainty. The chapter discusses various Lyapunov-Krasovskii functionals as well as Razumikhin type stability criteria, resulting in various delay-independent, **and** delay-dependent, and mixed delay-independent/delay-dependent stability conditions, all **mostly** posed as solutions to LMIs. In deriving some of these results, a technique known as model transformation is used. **The relationships between various results are also explored.** It is shown that **the** use of **model transformation in deriving some of the delay-dependent stability results** this technique will inherently lead to conservatism, due to the resultant additional dynamics. **It is shown that for necessary and sufficient stability condition, a complete quadratic Lyapunov-Krasovskii functional needs to be used. Discretized Lyapunov functional method is used to discretize such Lyapunov-Krasovskii functional to a finite dimensional LMI. Numerical results shows that conservatism are usually very small even for rather coarse discretization.** To counter this conservatism, a discretized Lyapunov functional (DLF) method is developed, which, when applied to systems without uncertainty, is capable of reducing the conservatism and in fact, producing conditions that may reach very closely the analytical limit albeit at the expense of increased computation.

Chapter 6 presents extensions to uncertain delay systems. Unlike in Chapter 4, the system uncertainty description herein is given on the system's state matrices, which is either polytopic or norm-bounded. Both these uncertainty descriptions are widely found in state-space models. Robust delay-dependent and delay-independent stability conditions are developed in parallel to Chapter 5, which also admit the form of LMIs; in particular, for polytopic uncertainty, these LMI's are defined at the vertices of the uncertain family.

Chapter 7 is a culmination of the developments in Chapter 5 and Chapter 6, and it seeks to extend the results of these two chapters **Chapter 5 and 6** to systems with multiple delays **and distributed delays with piecewise constant coefficient**, which may or may not have uncertainty. The extensions follow the same conceptual line, and hence are mainly a technical outgrowth. Likewise, delay-dependent and delay-independent stability conditions are given in the form of LMIs. The chapter also contains a rather detailed study of systems with distributed delays. **The ideas are very similar to the previous two chapters but is technically more delicate.**

Chapter 8 seeks to link together time and frequency domain results. It opens with a more detailed exposure input-output stability description as well as the small gain theorem, in a way similar to Chapter 3 . It then discusses means of modeling delay dynamics via uncertainties, and presents the famed bounded real lemma and its scaled versions. The latter are the key link that connects small gain type conditions with Lyapunov-Krasovskii and Razumikhin approaches, thus connecting frequency-sweeping criteria to LMI tests. The general idea here is to treat delay dynamics as sources of uncertainty and reformulate the stability problem as one of robust stability. With the bounded real lemma in place, it is then possible to show that both the small gain type conditions and those stated in terms of LMIs are equivalent, both of which constitute sufficient robust stability conditions, and hence are sufficent stability conditions for the underlying time-delay systems. **{replace with the following paragraph}**

Chapter 8 tackle the problem of time delay systems subject to dynamical uncertainty. It starts with a brief discussion of input-output stability and small gain theorem similar to Chapter 3 but with more emphasis on state space formulation. Then it consider the special case with system with delay as the nominal system and time delay modeled as uncertainty. The results shows that a number of stability and robust stability conditions derived in Chapter 5 and 6 derived using Lyapunov-Krasovskii functional method can be derived and extended using small gain formulation. It further discusses the general setting with time delay systems as the nominal system with dynamical block-diagonal feedback uncertainty. As an application, it is shown how system with time-varying delay and general distributed delays can be approximated as systems with constant pointwise delays or distributed delays with piecewise constant coefficient, with the errors modeled as dynamic feedback uncertainty.

To make the book adequately self-contained, we have also included two appendices. Appendix A summarizes some key matrix facts and identities used throughout the book. Appendix B is intended as a quick reference to basic concepts and techniques of LMI and quadratic integral inequalities.

At the end of each chapter a section of Notes is provided. In addition to pointing to the most pertinent references, this section also summarizes and comments on the results of that chapter. In this vein, we will mainly limit our references to those most closely related, than to provide a broad survey. Nevertheless, the bibliography should contain most of the references needed.

1.9 Notes

1.9.1 A brief historic note

Time-delay systems are also known as hereditary systems, systems with aftereffects, or systems with time lags. The first functional differential equations (FDEs) were considered by, among others, such great mathematicians as Euler, Bernoulli, Lagrange, Laplace, and Poisson in the eighteenth century, which arose from various geometry problems. In the early twentieth century, a number of practical problems were also modelled using FDEs. These include viscoelasticity problems studied in 1909, the predator-prey model used in population dynamics in 1928-1931, both due to Volterra, mathematical biology problems studied in 1934 by Kostyzin, and ship stabilization problems in 1942 by Minorsky.

On the stability of time-delay systems, Pontryagin obtained some fundamental results concerning the zeros of quasipolynomials in 1942, and Chebotarev published a number of papers devoted to the Routh-Hurwitz criterion for quasipolynomials in early 1940's. A paper of Myshkis in 1949 formulated for the first time the initial-value problem. The attempts to extend Lyapunov's theory to time-delay systems initially encountered some difficulty, which was not resolved until the work of Krasovskii. Krasovskii was the first to emphasize the importance of adopting x_t rather than $x(t)$ as the state, and to develop further the idea to fruitation. This took place in 1956. The idea of bypassing functionals was due to Razumikhin, which led directly to the Razumikhin theorem. Further extensions along these lines were made by Kolmanovskii and Nosov to neutral functional differential equations. For a more comprehensive account of these early developments, we refer to the book by Kolmanovskii and Nosov [157]. Other treatises of time-delay systems include the books by Bellman and Cooke [12], Górecki, *et. al.* [83], and Hale and Verduyn Lunel [111], which, like Kolmanovskii and Nosov [157], detail the developments at their times and additionally cover other aspects of time-delay systems. The recent monograph by Niculescu [215] also documented briefly the historical developments.

Since 1990's there has been a substantial increase of research activities on time-delay systems in the systems and control society. The main achievements in this period are characterized by computational improvement and in the study of robust stability. These advances form the core of the later chapters, where brief reviews and references to the latest progress will be provided accordingly.

1.9.2 Application examples

The regenerative chatter system in Example 1.1 is discussed in Moon and Johnson [202], where a nice overview of the topic is given. For some linear analysis of such a system, see Stépán [262]. Other manufacturing processes also provide rich examples of time-delay systems. See Moon [201] for an overview, and Dorf and Kusiak [62] for additional examples.

The modeling of internal combustion engines similar to that in Example 1.2 is discussed by Kao and Moskwa [143]. Other parts of internal combustion engines, e.g., exhaust manifold, also involve time-delays. A comprehensive survey of internal combustion engine modeling is provided by Cook and Powell [47]. Steel rolling mill control is an example of measurement delay, which is found in Sbarbaro-Hofer [249].

The classical vibration absorber is found in most of undergraduate text-books on vibration. A comprehensive coverage on this subject can be found in Soong [260] and Tongue [275]. The delayed resonator was proposed by Olgac and Holm-Hansen in [227]. Analysis of the delayed resonator as well as its applications are reported in [225, 226, 70, 134]. For other examples of delays introduced intentionally in a system, we refer to, e.g., finite difference approximation to differentiation, and works on stabilization of chaotic attractors. The former is a very common industrial practice; for example, it is used in the adaptive control of robot manipulators where the angular acceleration is approximated by the finite difference of measured angular velocity [48]. The latter can be found in, e.g., the experimental work by Pyragas [239], which seeks to use delayed feedback to stabilize the unstable periodic trajectory embedded in a chaotic attractor.

Time-delay systems may also arise from simplification of partial differential equations. Such time-delay systems are usually of *neutral type*. For an example of this kind, see Brayton [27].

Time-delay systems are encountered in many different disciplines including engineering, biology, economy, and ecology. Examples in these disciplines are documented in such books as Hale and Verduyn Lunel [111], Kolmanovskii and Myshkis [158], Kolmanovskii and Nosov [157], Niculescu [215], Bellman and Cooke [12], and Górecki, *et. al.* [83].

1.9.3 Analysis of time-delay systems

We have chosen to bypass the existence and uniqueness issues concerning the solutions of functional differential equations for two reasons. First, these issues by themselves in general merit a comprehensive study and are evidently beyond the present scope. Secondly, for systems under consideration in this book, which are mainly linear systems, the existence and uniqueness is essentially insured and thus rendered moot. Nevertheless, the interested reader is referred to Hale and Verduyn Lunel [111] for an extensive treatment of these issues.

We are mainly concerned with asymptotic stability. For this reason other stability concepts and definitions have not been introduced. Among them, exponential stability provides a useful measure for decaying rate, or convergence speed. For this we refer to Kharitonov [144] and the references therein. We note that more elaborate notions, such as uniform boundedness and attraction to a set rather than a point can also be useful; see, e.g., [111]. For an LTI system, however, the notions of asymptotic and exponential stability are equivalent.

Lyapunov-Krasovskii and Razumikhin theorems are the cornerstones in time domain stability anaysis of delay systems. The use of Lyapunov functionals was first discussed by Krasovskii in his pioneering work [166]. Razumikhin theory originated in [242, 243]. A systematic presentation of these techniques is given by Hale and Verduyn Lunel [111], together with further extensions.

Frequency domain methods are a routine theme in stability analysis, which enjoy a vast literature. An extensive coverage of quasipolynomials can be found in Stépán [261], Hale and Verduyn Lunel [111], and Kogan [**?**]. The continuity of the stability exponent was discovered by Datko [56], who also pointed out that the property does not hold necessarily for neutral functional deferential equations. Frequency domain analysis and quasipolynomials will constitute a primary topic in the subsequent chapters and pertinent references will be provided therein.

From a broader perspective, RFDE is a subset of infinite-dimensional systems whose growth is "exponentially determined", *i.e.*, the grow rate is determined by the maximum real part of the poles, with an exponential bound (1.41). Infinite-dimensional systems are a well-studied branch of mathematics and systems science; for a glimpse of this subject, see, e.g., Luo, *et. al.* [190] Curtain and Pritchard [**?**], and Curtain and Zwart [**?**]. Other areas developed in this context, to name just a few, include control synthesis and stochastic delay systems; for the former we refer to Foias *et al.* [**?**], and for the latter we refer to Boukas and Liu [21].

The book is written under the presumption that the reader is familiar with the theory of ordinary (delay-free) dynamical systems, to which many excellent books are available; we recommend Bryson and Ho [30], Kailath [137], Vidyasagar [301], and Zhou, Doyle and Glover [318].

Part I: Frequency Domain Stability Criteria

New format for "Part" consistent with the style file is used instead in the final form.

2

Systems with Commensurate Delays

2.1 Introduction

Stability criteria based on frequency domain representations are time honored tools in the study of dynamical systems. Classical examples of frequency domain stability criteria include such results as Nyquist test and root-locus method. With the aid of the small gain theorem, frequency domain tests have become increasingly more prevalent in stability analysis, and have played especially a central role in the theory of robust control. More generally, while frequency domain methods are used predominantly in the analysis of linear systems, they have also found utilities in the studies of nonlinear systems, with such tools as describing functions, Popov and circle criterion, and as well the small gain theorem. Various frequency-sweeping tests are now commonplace. Generally, frequency-domain tests are often favored for their conceptual simplicity and computational ease, which typically can be checked in an efficient manner by plotting graphically a certain frequency-dependent measure.

This chapter develops frequency domain stability tests for LTI systems with pointwise commensurate delays. From Chapter 1, we have known that the stability of an LTI delay system can be completely characterized by its characteristic roots, i.e., the solutions to its characteristic equation. Two stability notions, known as *delay-independent stability* (or stability independent of delay) and *delay-dependent stability* (or stability dependent of delay), respectively, will be studied. The chapter opens with a statement of these two notions and proceed**s {grammar}** to examine a number of classical stability tests. Frequency domain delay-independent and delay-dependent, necessary and sufficient stability conditions will then be developed. Specifically, for systems with commensurate delays, a frequency-sweeping test will be provided which requires computing the spectral radius of a frequency-dependent matrix. This result is further developed to yield a test requiring only the computation of constant matrices. Systems with incommensurate delays will be discussed in Chapter 3.

Consider the LTI delay systems described by the state-space equation

$$\dot{x}(t) = A_0\ x(t) + \sum_{k=1}^{m} A_k\ x(t - r_k), \qquad r_k \geq 0, \tag{2.1}$$

where $A_0,\ A_k \in \mathsf{R}^{n\times n}$ are given system matrices, and r_k are delay times. As discussed in Chapter 1, the stability of this system is fully characterized by its characteristic function

$$p\left(s;\ e^{-r_1 s},\ \cdots,\ e^{-r_m s}\right) = \det\left(sI - A_0 - \sum_{k=1}^{m} A_k e^{-r_k s}\right). \tag{2.2}$$

Specifically, the system is stable if and only if $p\left(s;\ e^{-r_1 s},\ \cdots,\ e^{-r_m s}\right)$ has no zero, or root, in the closed right half plane $\overline{\mathsf{C}}_+$. We state this formally below, as a definition.

Definition 2.1 *The characteristic function (2.2) is said to be stable if*

$$p\left(s;\ e^{-r_1 s},\ \cdots,\ e^{-r_m s}\right) \neq 0, \qquad \forall s \in \overline{\mathsf{C}}_+. \tag{2.3}$$

It is said to be stable independent of delay if (2.3) holds for all $r_k \geq 0$, $k = 1,\ 2,\ \cdots,\ m$. The system (2.1) is said to be stable if its characteristic function (2.2) is stable, and is stable independent of delay if its characteristic function is stable independent of delay.

{all K are replaced by m for consistency in the chapter}Hence, the delay system (2.1) is stable independent of delaydelay-independent **{should be index, different convention in software}** if the stability persists with respect to all possible nonnegative delays. On the other hand, if it is stable only for a subset of nonnegative delays, then we say that the stability is delay-dependent

It is important to recognize that the characteristic function of a time-delay system defines a *real* quasipolynomial of s. This feature makes it possible to extend a number of key properties of polynomials to the study of time-delay systems. Indeed, a critical ingredient in our subsequent development dwells on the continuity of the stability exponent with respect to delays, a fact we have stated in Chapter 1. For a glimpse of its importance at the outset, let the system (2.1) be stable at the delay values r_k^*, $k = 1,\ 2,\ \cdots,\ K$. Then by the continuity property of the stability exponent, the system will remain stable within a neighborhood of r_k^*. Much of our effort in determining delay-independent and delay-dependent stability then amounts to expanding that neighborhood, so that the stability exponent remains negative. In turn, this amounts to finding the critical delay values at which the characteristic roots intersect the stability boundary, i.e., the imaginary axis, thus rendering the system unstable. In other words, when the delay values deviate from r_k^*, we want to determine the smallest deviation of r_k from r_k^*, such that

$$p\left(j\omega;\ e^{-jr_1\omega},\ \cdots,\ e^{-jr_K\omega}\right) = 0. \tag{2.4}$$

Evidently, if (2.4) admits no solution, i.e., $p\left(j\omega;\ e^{-jr_1\omega},\ \cdots,\ e^{-jr_K\omega}\right) \neq 0$ for all $\omega \in \mathsf{R}$, then the system is stable independent of delay. This observation, albeit a simple continuity argument, underlies the central idea in our frequency domain stability analysis of LTI delay systems, in both this chapter and Chapter 3. We shall elaborate further this point in the next section, using systems with commensurate delays.

For the rest of this chapter, we shall consider LTI delay systems with pointwise commensurate delays. Systems in this class can be represented by

$$\dot{x}(t) = A_0\, x(t) + \sum_{k=1}^{m} A_k\, x(t - k\tau), \qquad \tau \geq 0. \tag{2.5}$$

Its characteristic quasipolynomial possesses the form of

$$a\left(s,\ e^{-\tau s}\right) = \sum_{k=0}^{q} a_k(s) e^{-k\tau s}, \tag{2.6}$$

where

$$a_0(s) = s^n + \sum_{i=0}^{n-1} a_{0i} s^i, \qquad a_k(s) = \sum_{i=0}^{n-1} a_{ki} s^i, \quad k = 1,\ \cdots,\ q. \tag{2.7}$$

We note that $a\left(s,\ e^{-\tau s}\right)$ may also be the characteristic quasipolynomial of the system

$$y^{(n)}(t) + \sum_{i=1}^{n-1} a_{ki} y^{(i)}(t - k\tau) = 0. \tag{2.8}$$

The distinction between the two representations is thus inessential. It is convenient to introduce the variable $z = e^{-\tau s}$, and write (2.6) as a bivariate polynomial

$$a\left(s,\ z\right) = \sum_{k=0}^{q} a_k(s) z^{-k}, \qquad z = e^{-\tau s}. \tag{2.9}$$

Notice that the order of $a_0(s)$, often known as the "principal term", is higher than the order of any $a_k(s)$, $k = 1,\ 2,\ \cdots,\ q$. Throughout this chapter, we assume that the system (2.1) is stable for $\tau = 0$, or equivalently, $a(s,\ 1)$ is stable. Following the above continuity argument, the smallest deviation of τ from $\tau = 0$ such that the system becomes unstable can be determined as

$$\overline{\tau} := \min\left\{\tau \geq 0 \mid\ a\left(j\omega,\ e^{-j\tau\omega}\right) = 0 \text{ for some } \omega \in \mathsf{R}\right\}. \tag{2.10}$$

We call $\overline{\tau}$ the *delay margin* of the system. For any $\tau \in [0,\ \overline{\tau})$, the system is stable, and whenever $\overline{\tau} = \infty$, the system is stable independent of delay. Note that for any finite $\overline{\tau}$, the frequency at which $a\left(j\omega,\ e^{-j\overline{\tau}\omega}\right) = 0$ represents the first contact or crossing of the characteristic roots from the stable region to the unstable one. Note also that multiple crossings may exist. From Chapter 1, however, since only finitely many unstable roots may be in the right half plane, there are only a finite number of zero crossings.

Moreover, since $a\left(s,\ e^{-\tau s}\right)$ is a real quasipolynomial, all its complex roots appear in complex conjugate pairs; that is, it satisfies the *conjugate symmetry* property. Consequently, it suffices to consider only the zero crossings at positive frequencies. Let

$$a\left(j\omega_i,\ e^{-j\theta_i}\right) = 0, \qquad \omega_i > 0, \qquad \theta_i \in [0,\ 2\pi], \qquad i = 1,\ 2,\ \cdots,\ N.$$

Furthermore, define $\eta_i = \theta_i/\omega_i$. It is clear that

$$\overline{\tau} = \min_{1\leq i\leq N} \eta_i = \min\left\{ \frac{\theta_i}{\omega_i} \,\middle|\, \omega_i > 0 \right\}. \tag{2.11}$$

This gives a general formula for computing the delay margin. More generally, we may assume, with no loss of generality, that

$$\eta_1 < \eta_2 < \cdots < \eta_N.$$

It follows that the system is stable for all $\tau \in (\eta_i,\ \eta_{i+1})$ whenever it is stable for some $\tau^* \in (\eta_i,\ \eta_{i+1})$. This then allows us to ascertain a system's stability in the full range of delay values, beyond the interval determined by $\overline{\tau}$. It also indicates that the stability at $\tau = 0$ can be made without loss of generality.

With the bivariate polynomial representation (2.9), it is rather evident that $\overline{\tau}$ can be found by solving the imaginary roots $s \in \partial\mathsf{C}_+$ and the unitary roots $z \in \partial\mathsf{D}$ of $a(s,\ z)$, giving rise to a stability criterion commonly referred to as *two-variable criterion*. The two-variable criterion appears to be the origin of many classical stability tests for systems with commensurate delays, which attempt to solve the bivariate polynomial (2.9) in one way or another. In particular, most of the classical tests attempt to accomplish this by means of eliminating one variable, thus converting the stability problem to one free of delay, and seeking the solution of polynomials of one single variable. In the next section we discuss a number of tests in this spirit. These sample tests are rather representative of classical results and should give the reader the essential flavor of the two-variable criterion.

2.2 Some classical stability tests

2.2.1 2-D stability tests

The representation of the characteristic quasipolynomial via a bivariate polynomial lends the handy recognition that it may be treated as the characteristic polynomial of a 2-D system, and that the stability of a time-delay system may then be analyzed as in the case of a 2-D polynomial. Indeed, consider the bilinear transformation

$$s = \frac{1+\lambda}{1-\lambda}, \tag{2.12}$$

which maps s from the open right half complex plane C_+ to λ in the open unit disk D. Construct the 2-D polynomial

$$b(\lambda,\ z) := (1-\lambda)^n a\left(\frac{1+\lambda}{1-\lambda},\ z\right).$$

It is evident that $a(s,\ z) = 0$ for some $(s,\ z) \in \partial\mathsf{C}_+ \times \partial\mathsf{D}$ if and only if $b(\lambda,\ z) = 0$ for some $(\lambda,\ z) \in \partial\mathsf{D} \times \partial\mathsf{D}$. In addition, the quasipolynomial $a\left(s,\ e^{-\tau s}\right)$ has no root in $\overline{\mathsf{C}}_+$ if and only if $b(\lambda,\ z)$ is stable; here by stability of a 2-D polynomial, we mean that all its roots lie outside the closed region $\overline{\mathsf{D}} \times \overline{\mathsf{D}}$. Hence, to verify whether the system is stable independent of delay, it suffices to check whether the 2-D polynomial $b(\lambda,\ z)$ is stable, and to determine whether the stability is delay-dependent, it is necessary to calculate the roots of $b(\lambda,\ z)$. Stability of 2-D polynomials and 2-D systems is a well-studied subject in the theory of signal processing. The equivalence noted herein between the stability of a time-delay system and that of a 2-D polynomial enables us to draw upon the existing analysis techniques developed extensively for the latter, although 2-D stability tests themselves generally pose a rather formidable computational task.

In broader terms, one may view the bivariate polynomial $a(s,\ z)$ as a 2-D polynomial and tackle the stability problem directly. Define the conjugate polynomial

$$\bar{a}(s,\ z) := z^q a(-s,\ z^{-1}).$$

By the conjugate symmetry of $a(s,\ z)$, it follows that $(s,\ z) \in \partial\mathsf{C}_+ \times \partial\mathsf{D}$ is a root of $a(s,\ z)$ if and only if it is also a root of $\overline{a}(s,\ z)$. Thus, in order to find the roots of $a(s,\ z)$ on $\partial\mathsf{C}_+ \times \partial\mathsf{D}$, it suffices to solve the simultaneous polynomial equations

$$a(s,\ z) = 0, \tag{2.13}$$

$$\bar{a}(s,\ z) = 0. \tag{2.14}$$

When no solution exists, and when the system is stable in the delay-free case, it must also be stable independent of delay. Otherwise, when the two equations do admit a common solution, it is possible to eliminate one variable, resulting in a polynomial of one single variable. For example, we may eliminate s and obtain a polynomial in z, say $b(z)$. If $b(z)$ has no unitary root, we may again conclude that the system is stable independent of delay. Otherwise, we may proceed to find all the unitary roots z_i of $b(z)$. There are only a finite number of such roots since $b(z)$ is a polynomial. For each z_i, $a(s, z_i)$ is a polynomial of the variable s, which too admits only a finite number of possible roots $s_i \in \partial\mathsf{C}_+$. Thus, we find all the roots $(s_i,\ z_i) \in \partial\mathsf{C}_+ \times \partial\mathsf{D}$ such that $a(s_i,\ z_i) = 0$. Since the bivariate polynomial satisfies the conjugate symmetry property, only those s_i on the positive imaginary axis need to be considered. Thus, consider $s_i = j\omega_i$, $z_i = e^{-\theta_i}$, where $\omega_i > 0$, and $\theta_i \in [0,\ 2\pi]$. The delay margin can then be determined using (2.11).

Example 2.1 *Consider the scalar delay system*

$$\dot{x}(t) = -x(t-\tau) - x(t-2\tau), \qquad \tau \geq 0. \tag{2.15}$$

Evidently, the system is stable at $\tau = 0$. Its characteristic quasipolynomial is given by

$$a(s,\ e^{-\tau s}) = s + e^{-\tau s} + e^{-2\tau s}.$$

The corresponding bivariate polynomials are

$$\begin{aligned} a(s,\ z) &= s + z + z^2, \\ \bar{a}(s,\ z) &= 1 + z - sz^2. \end{aligned}$$

Eliminating s yields the polynomial equation

$$z^4 + z^3 + z + 1 = (z+1)^2(z^2 - z + 1) = 0,$$

which has the solutions $z_1 = -1$, $z_2 = (1 + j\sqrt{3})/2$, and $z_3 = (1 - j\sqrt{3})/2$. Correspondingly, $s_1 = 0$, $s_2 = -j\sqrt{3}$, and $s_3 = j\sqrt{3}$. As a result, $\overline{\tau}$ is found from s_3 and $z_3 = e^{-j\pi/3}$, as $\overline{\tau} = \pi/(3\sqrt{3})$.

2.2.2 Pseudo-delay methods

One of the standard techniques in the stability analysis of discrete-time systems is to transform the stability problem into one of continuous-time systems, and this is accomplished by mapping conformally the unit disc to the left half of the complex plane. The fact that the stability of a time-delay system is characterized by the roots of a bivariate polynomial raises the possibility that the same conformal map may be utilized on the variable z, which may facilitate the stability analysis. This idea leads us to consider the bilinear transformation

$$z = \frac{1 - sT}{1 + sT}, \qquad T > 0, \tag{2.16}$$

and the polynomial

$$c(T,\ s) := (1 + sT)^q a\left(s,\ \frac{1 - sT}{1 + sT}\right).$$

Since $c(0,\ s) = a(s,\ 1)$, the polynomial $c(0,\ s)$ is stable by assumption. Furthermore, for any $0 \leq T < \infty$, the factor $(1 + sT)^q$ has no effect on the imaginary roots of $a\left(s,\ \frac{1-sT}{1+sT}\right)$. Thus, for all $0 \leq T \leq \infty$, the imaginary roots of $a\left(s,\ \frac{1-sT}{1+sT}\right)$ consist of those of $c(T,\ s)$, $0 < T < \infty$, and those of $a(s,\ -1)$; the latter polynomial accounts for the case $T = \infty$. It then suffices to find the imaginary roots of $a(s,\ -1)$, and for finite values of T to calculate the roots of $c(T,\ s)$, which is a polynomial in s whose coefficients are parameterized by T. Clearly, if neither of the two polynomials has any imaginary root, then $a\left(s,\ \frac{1-sT}{1+sT}\right)$ has no imaginary zero for all $0 \le T \le \infty$. As a result, $a(s,\ z)$ has no root in $\partial \mathsf{C}_+ \times \partial \mathsf{D}$, and hence the delay system is stable independent of delay. However, if for some $T = T_i > 0$, the polynomial $c(T_i,\ s)$ has an imaginary root $s_i = j\omega_i$, $\omega_i > 0$, then there exists a unitary $z_i = e^{-j\theta_i} \in \partial \mathsf{D}$, $\theta_i \in [0,\ 2\pi]$, where

$$z_i = \frac{1 - j\omega_i T_i}{1 + j\omega_i T_i} \quad \text{or} \quad \theta_i = 2\tan^{-1}(\omega_i T_i),$$

such that $a(s_i,\ z_i) = 0$. These roots furnish an estimate on the delay margin, given by

$$\overline{\tau}_1 := \min\left\{ \frac{2}{\omega_i}\tan^{-1}(\omega_i T_i) \,\middle|\, \omega_i > 0 \right\}.$$

Additionally, $a(s,\ -1)$ may contain imaginary roots as well. At these roots,

$$a(j\omega_i,\ -1) = a\left(j\omega_i,\ e^{-j\tau\omega_i}\right), \qquad \tau = \frac{2K+1}{\omega_i}\pi, \quad K = 0,\ 1,\ \cdots.$$

They too yield an estimate on the delay margin, determined as

$$\overline{\tau}_2 := \min\left\{ \frac{\pi}{\omega_i} \,\middle|\, \omega_i > 0 \right\}.$$

Consequently, the delay margin is found as

$$\overline{\tau} = \min\left\{\overline{\tau}_1,\ \overline{\tau}_2\right\}.$$

The significance of the above technique, known broadly as *pseudo-delay method*, is quite self-evident. In essence, it reduces the stability problem effectively to one free of delay, which in turn requires calculating only roots of a single-variable polynomial. While this is accomplished at the expense of a parameter-dependent polynomial, it nonetheless allows the use of various classical stability tests such as the Routh-Hurwitz test and the root-locus method.

Example 2.2 *A second order delay system is described by*

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0.5 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix} x(t-\tau), \qquad \tau \ge 0. \qquad (2.17)$$

The corresponding characteristic quasipolynomial is

$$a\left(s,\ e^{-\tau s}\right) = s^2 + 4s + 4 - 0.25e^{-\tau s}.$$

The system is stable at $\tau = 0$. Form the polynomial $c(T,\ s)$ as

$$c(T,\ s) = Ts^3 + (1+4T)s^2 + (4+4.25T)s + 3.75.$$

To use the Routh-Hurwitz criterion for $c(T,\ s)$, we construct the Routh array

$$\begin{array}{lll} s^3: & T & 4+4.25T \\ s^2: & 1+4T & 3.75 \\ s^1: & \frac{4+16.5T+17T^2}{1+4T} & \\ s^0: & 3.75 & \end{array}$$

It follows that $c(T,\ s)$ is stable for any $0 < T < \infty$. Moreover, since $a(s,\ -1) = s^2 + 4s + 4.25$ is also stable, we conclude that $a\left(s,\ e^{-\tau s}\right)$ has no imaginary root for any $\tau \geq 0$. That is, the system is stable independent of delay.

Example 2.3 *The following system can be easily verified to be stable at $\tau = 0$:*

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x(t-\tau), \qquad \tau \geq 0. \tag{2.18}$$

The characteristic quasipolynomial is obtained as

$$a\left(s,\ e^{-\tau s}\right) = s^2 + s + 1 + se^{-\tau s},$$

and the polynomial $c(T,\ s)$ is found as

$$c(T,\ s) = Ts^3 + s^2 + (T+2)s + 1.$$

The Routh array is generated as

$$\begin{array}{lll} s^3: & T & T+2 \\ s^2: & 1 & 1 \\ s^1: & 2 & \\ s^0: & 1 & \end{array}$$

Hence for any $0 < T < \infty$, $c(T,\ s)$ is stable. Consider next $a(s,\ -1) = s^2 + 1$, which has a pair of imaginary roots $s = \pm j$. This leads to $\overline{\tau} = \pi$.

2.2.3 Direct method

It is yet further possible to find the zero-crossing frequencies directly based on the conjugate symmetry property of the quasipolynomial, without using transformation of any kind. To illustrate, it is instructive to begin with systems with a single delay. In this situation, the quasipolynomial (2.6) assumes the simpler form

$$a\left(s,\ e^{-\tau s}\right) = a_0(s) + a_1(s)e^{-\tau s}. \tag{2.19}$$

By conjugate symmetry, if $a(s,\ e^{-\tau s}) = 0$ for some $s = j\omega$, it is necessary that

$$\begin{aligned} a_0(j\omega) + a_1(j\omega)e^{-j\tau\omega} &= 0, \\ a_0(-j\omega) + a_1(-j\omega)e^{j\tau\omega} &= 0. \end{aligned}$$

Eliminating $e^{j\tau\omega}$ yields

$$|a_0(j\omega)|^2 - |a_1(j\omega)|^2 = 0. \tag{2.20}$$

This defines a polynomial equation in ω^2. As a consequence, a finite number of zero-crossing frequencies ω_i can be determined by solving the equation. We may then find the phase angle of $a_1(j\omega_i)/a_0(j\omega_i)$. By equating θ_i to

$$\theta_i = \begin{cases} \angle\frac{a_1(j\omega_i)}{a_0(j\omega_i)} + \pi, & 0 \le \angle\frac{a_1(j\omega_i)}{a_0(j\omega_i)} \le \pi \\ \angle\frac{a_1(j\omega_i)}{a_0(j\omega_i)} - \pi, & \pi < \angle\frac{a_1(j\omega_i)}{a_0(j\omega_i)} \le 2\pi \end{cases}$$

the delay margin can be duly computed according to (2.11). When the quasipolynomial is stable for $\tau = 0$ and (2.20) admits no solution, the system is stable independent of delay, corresponding to the conditions

$$\left|\frac{a_1(j\omega)}{a_0(j\omega)}\right| < 1, \qquad \forall\omega > 0, \tag{2.21}$$

or

$$\left|\frac{a_1(j\omega)}{a_0(j\omega)}\right| > 1, \qquad \forall\omega > 0.$$

The latter, however, is not possible since $a_0(s)$ is a higher order polynomial than $a_1(s)$. Thus, we conclude that the necessary and sufficient condition for stability independent of delay is that (2.21) holds, and that $a(s,\ 1) = a_0(s) + a_1(s)$ is stable. In this case, $a_0(s)$ is also stable, corresponding to $\tau = \infty$.

A closely related sufficient condition for the quasipolynomial (2.19) to be stable is that $a_0(s)$ is stable and (2.21) holds for all $\omega \ge 0$. This condition is of special interest for a number of reasons. First, it can be held as a special case of the well-known *Roche's Theorem*, a result routinely found in the complex variables theory. Indeed, Roche's Theorem states that for any function $f(s)$ analytic in C_+, the functions $a_0(s)$ and $a_0(s) + f(s)$ will have the same number of zeros in C_+ whenever $|f(j\omega)/a_0(j\omega)| < 1$ for all $\omega \in \mathsf{R}$. The sufficient condition alluded to above corresponds to the special case $f(s) = a_1(s)e^{-\tau s}$.

Secondly, this sufficient condition also has a linkage to a classical result known as Tsypkin's test. Consider a unity feedback system with an open-loop transfer function

$$G(s) = \frac{a_1(s)}{a_0(s)}e^{-\tau s}.$$

Tsypkin's theorem states that under the condition that $G(s)$, or equivalently $a_0(s)$, is stable, then the closed-loop system will be stable for all $\tau \geq 0$ whenever $|G(j\omega)| < 1$ for all $\omega \in \mathsf{R}$, which coincides with the above sufficient condition. Finally, we note that the condition provides a frequency-sweeping test in the form of a small gain theorem, which can be checked either numerically by inspecting the equation (2.20), or graphically by plotting the frequency-dependent measure $|a_1(j\omega)/a_0(j\omega)|$. The small gain theorem will play a predominant role in our development of frequency-sweeping tests and will be discussed in Chapter 3.

Example 2.4 *A well-studied example in classical stability analysis of time-delay systems is the first-order delay system*

$$\dot{x}(t) = -ax(t) - bx(t-\tau), \tag{2.22}$$

where a and b are real constants. The quasipolynomial for this system is

$$a\left(s,\ e^{-\tau s}\right) = s + a + be^{-\tau s}.$$

Under the assumption that $a+b > 0$, the system is stable at $\tau = 0$. Suppose that this is the case. Then, the equation (2.20) is given by

$$\omega^2 + a^2 - b^2 = 0,$$

which may have a nontrivial solution only when $|a| < |b|$, yielding the zero-crossing frequency $\omega_1 = \sqrt{b^2 - a^2}$. Clearly, this is possible only when either $a > 0$, $b > 0$, or $a < 0$, $b > 0$. In the former case,

$$\theta_1 = \angle \frac{b}{j\omega_1 + a} - \pi = \pi - \tan^{-1}\left(\frac{\omega_1}{a}\right).$$

Correspondingly,

$$\overline{\tau} = \frac{\pi - \cos^{-1}\left(\frac{a}{b}\right)}{\sqrt{b^2 - a^2}}.$$

In the latter case,

$$\theta_1 = \angle \frac{b}{j\omega_1 + a} - \pi = \tan^{-1}\left(\frac{\omega_1}{|a|}\right),$$

and thus

$$\overline{\tau} = \frac{\cos^{-1}\left(\frac{|a|}{b}\right)}{\sqrt{b^2 - a^2}}.$$

On the other hand, the system is stable independent of delay if and only if $a + b > 0$, and $a \geq |b|$. The latter guarantees that

$$\left|\frac{a_1(j\omega)}{a_0(j\omega)}\right| = \left|\frac{b}{j\omega + a}\right| < 1, \qquad \forall \omega > 0.$$

Note that when $a = b > 0$, $|a_1(j\omega)/a_0(j\omega)| < 1$ *for all* $\omega > 0$, *but* $|a_1(j\omega)/a_0(j\omega)| = 1$ *at* $\omega = 0$. *In this case, the system remains to be stable independent of delay.*

To further illustrate, consider the system given in Example 2.3, whose quasipolynomial is in the form of (2.19), with

$$a_0(s) = s^2 + s + 1, \qquad a_1(s) = s.$$

The equation (2.20) is simplified to

$$\omega^4 - 2\omega^2 + 1 = 0,$$

which yields the zero-crossing frequency $\omega_1 = 1$. *In addition,*

$$\angle \frac{a_1(j)}{a_0(j)} = 0.$$

Hence, $\theta_1 = \pi$. *This leads to the same* $\overline{\tau} = \pi$, *as obtained in Example 2.3.*

In the case of multiple commensurate delays, an iterative calculation can be employed to extend the above test. This proceeds as follows. With the quasipolynomial (2.6), define the new quasipolynomial

$$a^{(1)}\left(s,\ e^{-\tau s}\right) := a_0(s)a\left(s,\ e^{-\tau s}\right) - a_q(s)e^{-q\tau s}a\left(-s,\ e^{\tau s}\right),$$

A simple calculation yields

$$\begin{aligned} a^{(1)}\left(s,\ e^{-\tau s}\right) &= \sum_{k=0}^{q-1} a_k^{(1)}(s)e^{-k\tau s} \\ &= \sum_{k=0}^{q-1} \left[a_0(-s)a_k(s) - a_q(s)a_{q-k}(-s)\right] e^{-k\tau s}. \end{aligned}$$

Furthermore,

$$a^{(1)}\left(-s,\ e^{\tau s}\right) = \sum_{k=0}^{q-1} \left[a_0(s)a_k(-s) - a_q(-s)a_{q-k}(s)\right] e^{k\tau s}.$$

Note that $a^{(1)}\left(s,\ e^{-\tau s}\right)$ and $a^{(1)}\left(-s,\ e^{\tau s}\right)$ may be considered as a frequency-dependent linear transform of $a\left(s,\ e^{-\tau s}\right)$ and $a\left(-s,\ e^{\tau s}\right)$,

$$\begin{pmatrix} a^{(1)}\left(s,\ e^{-\tau s}\right) \\ a^{(1)}\left(-s,\ e^{\tau s}\right) \end{pmatrix} = \begin{pmatrix} a_0(s) & -a_q(s)e^{-q\tau s} \\ -a_q(-s)e^{q\tau s} & a_0(-s) \end{pmatrix} \begin{pmatrix} a\left(s,\ e^{-\tau s}\right) \\ a\left(-s,\ e^{\tau s}\right) \end{pmatrix}.$$

Thus, whenever

$$\begin{aligned} a\left(j\omega,\ e^{-j\tau\omega}\right) &= 0, \\ a\left(-j\omega,\ e^{j\tau\omega}\right) &= 0, \end{aligned}$$

it must be true that

$$
\begin{aligned}
a^{(1)}\left(j\omega,\ e^{-j\tau\omega}\right) &= 0, \\
a^{(1)}\left(-j\omega,\ e^{j\tau\omega}\right) &= 0.
\end{aligned}
$$

In other words, the imaginary roots of $a\left(s,\ e^{-\tau s}\right)$ constitute a subset of the imaginary roots of $a^{(1)}\left(s,\ e^{-\tau s}\right)$. Hence, the zero-crossing frequencies for the former can be obtained by finding the imaginary roots of the latter. In particular, under the condition that

$$
\left|\frac{a_q(j\omega)}{a_0(j\omega)}\right| < 1, \qquad \forall\omega \in \mathsf{R},
$$

the two sets of roots coincide. Nevertheless, the exponent order in $a^{(1)}\left(s,\ e^{-\tau s}\right)$ has been reduced to $q-1$. We may continue this procedure, by defining

$$
a^{(i+1)}\left(s,\ e^{-\tau s}\right) := a_0^{(i)}(s)a^{(i)}\left(s,\ e^{-\tau s}\right) - a_{q-i}^{(i)}(s)e^{-(q-i)\tau s}a^{(i)}\left(-s,\ e^{\tau s}\right).
$$

That is,

$$
a^{(i+1)}\left(s,\ e^{-\tau s}\right) = \sum_{k=0}^{q-i} a_k^{(i)}(s)e^{-k\tau s},
$$

with

$$
\begin{aligned}
a_k^{(0)}(s) &= a_k(s), \\
a_k^{(i+1)}(s) &= a_0^{(i)}(-s)a_k^{(i)}(s) - a_{q-i}^{(i)}(s)a_{q-i-k}^{(i)}(s).
\end{aligned}
$$

For $i = q-1$, the procedure generates the function

$$
a^{(q)}\left(s,\ e^{-\tau s}\right) = a_0^{(q-1)}(s) + a_1^{(q-1)}(s)e^{-\tau s}.
$$

This effectively reduces the problem to that with one single delay, from which all zero-crossing frequencies of $a^{(q)}\left(s,\ e^{-\tau s}\right)$, and hence those of $a\left(s,\ e^{-\tau s}\right)$, can be computed.

Example 2.5 *Let us invoke the iterative procedure to test the stability of the system (2.15). Note that* $a_0(s) = s$, $a_1(s) = a_2(s) = 1$. *Construct*

$$
\begin{aligned}
a_0^{(1)}(s) &= a_0(-s)a_0(s) - a_2(s)a_2(-s) = -s^2 - 1, \\
a_1^{(1)}(s) &= a_0(-s)a_1(s) - a_2(s)a_1(-s) = -s - 1.
\end{aligned}
$$

Next, solve the equation

$$
\left|a_0^{(1)}(j\omega)\right|^2 - \left|a_1^{(1)}(j\omega)\right|^2 = (\omega^2-1)^2 - (\omega^2+1) = \omega^4 - 3\omega^2 = 0.
$$

The positive zero-crossing frequencies are found to be $\omega_1 = \sqrt{3}$. *We then calculate*

$$\theta_1 = \angle \frac{a_1^{(1)}(j\sqrt{3})}{a_0^{(1)}(j\sqrt{3})} - \pi = \angle \frac{-j\sqrt{3}-1}{2} - \pi = \frac{\pi}{3}.$$

This gives the same value $\overline{\tau} = \pi/(3\sqrt{3})$.

In conclusion, it is critical to find the zero-crossing frequencies of the characteristic quasipolynomial. This indeed is a defining feature of frequency domain stability tests in general and will be a key step in our study of systems with commensurate delays in particular. The zero-crossing frequencies enable us to assert both delay-dependent and delay-independent stability; generally, if a system is stable in the delay-free case, then it will be stable independent of delay whenever its characteristic quasipolynomial exhibits no zero crossing, while if such crossing does occur, one may compute a delay margin below which the system remains stable. This thread is common to and has been well exhibited in the three classical stability tests alluded to above, which all seek to find the zero-crossing frequencies via an elimination procedure, reducing in essence a bivariate polynomial to that of a single variable. Two properties of quasipolynomials have been essential in these tests. The first is the continuity of the stability exponent with respect to delays, which is the fundamental cause motivating the computation of zero-crossing frequencies. This nature of frequency domain tests will continue to dominate our subsequent development. Secondly, the conjugate symmetry property of a real quasipolynomial facilitates the computation of zero-crossing frequencies. Nevertheless, the a careful reader may notice that the latter property is not necessary, and that a similar reduction procedure can still be executed for complex quasipolynomials that may not satisfy the conjugate symmetric property.

Useful notwithstanding, it appears that the classical tests are largely effective for simple systems of a low order and with a few delays; in fact, the past success appears to be mainly limited to examples in this category, as we so demonstrate in the present section. This does not come as a surprise, however. A little more thought reveals that the limitation is rooted in the elimination procedure. Indeed, for the three tests examined herein, we come to note the following aspects.

1. The 2-D tests require that one variable be eliminated and a resultant polynomial be generated from two bivariate polynomials. This can be difficult for high order systems with many delays, and there exists no systematic procedure for performing this elimination.

2. The pseudo-delay method requires first a bilinear transformation and next generating and solving a sequence of polynomials in the parameter T, that is, the polynomials forming the first column of the Routh

array. With the free parameter T, to generate the Routh array can be rather cumbersome a task.

3. Likewise, the direct method requires a rather complicated iterative procedure which involves algebraic manipulations of polynomials.

Most notably, the elimination step in each of the three tests requires symbolic computation, or calculation "by hand", which, from a computational perspective, is highly undesirable. Thus, for high order systems with many delays, their implementation may encounter a varying degree of difficulty, rendering the tests inefficient and less applicable.

2.3 Frequency sweeping tests

Our examination of the classical tests should have demonstrated in sufficient clarity the need for more efficient, computationally oriented stability criteria. This section develops frequency sweeping tests for systems of commensurate delays, including delay-dependent and delay-independent criteria. The next section develops tests that are based on the computation of constant matrices. Both types of tests emphasize implementational ease and computational efficiency.

To highlight the main ideas in our derivation, it will be instructive to begin with systems of a single delay. Thus, consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \qquad \tau \geq 0. \tag{2.23}$$

The quasipolynomial for the system is given by

$$a\left(s,\ e^{-\tau s}\right) = \det\left(sI - A_0 - e^{-\tau s} A_1\right). \tag{2.24}$$

We first provide a necessary and sufficient condition for stability independent of delay.

Theorem 2.1 *The system (2.23) is stable independent of delay if and only if*

(i) A_0 *is stable,*
(ii) $A_0 + A_1$ *is stable, and*
(iii)

$$\rho\left((j\omega I - A_0)^{-1} A_1\right) < 1, \qquad \forall \omega > 0, \tag{2.25}$$

where $\rho(\cdot)$ *denotes the spectral radius of a matrix.*

Proof. For the system to be stable independent of delay, it is necessary that it be stable for $\tau = \infty$ and $\tau = 0$, which mandates the conditions (i) and (ii), respectively. Assume then that A_0 is stable, so that $(j\omega I - A_0)^{-1} A_1$ is well-defined for all $\omega \in \mathsf{R}$. Suppose that (2.25) holds. This means that for any eigenvalue $\lambda_i\left((j\omega I - A_0)^{-1}A_1\right)$ of the matrix $(j\omega I - A_0)^{-1}A_1$, $i = 1, \cdots, n$,

$$\left|\lambda_i\left((j\omega I - A_0)^{-1}A_1\right)\right| < 1, \qquad \forall \omega > 0,$$

which in turn means that

$$\left|\lambda_i\left((j\omega I - A_0)^{-1}A_1 e^{-j\tau\omega}\right)\right| < 1, \qquad \forall \omega > 0$$

for any $\tau \in \mathsf{R}_+$. Since

$$\det\left(I - (j\omega I - A_0)^{-1}A_1 e^{-j\tau\omega}\right) = \prod_{i=1}^{n}\left[1 - \lambda_i\left((j\omega I - A_0)^{-1}A_1 e^{-j\tau\omega}\right)\right],$$

it follows that

$$\det\left(I - (j\omega I - A_0)^{-1}A_1 e^{-j\tau\omega}\right) \neq 0, \qquad \forall \omega > 0,$$

or equivalently,

$$\det\left(j\omega I - A_0 - A_1 e^{-j\tau\omega}\right) \neq 0, \qquad \forall \omega > 0.$$

Moreover, note that condition (ii) precludes the possibility of $\det(A_0 + A_1) = 0$. Therefore, we have shown that for all $\omega \in \mathsf{R}$,

$$\det\left(j\omega I - A_0 - A_1 e^{-j\tau\omega}\right) \neq 0,$$

that is, the quasipolynomial of the system does not intersect the imaginary axis. Hence, the system is stable independent of delay. This proves the sufficiency.

To establish the necessity, it suffices to show that (iii) is necessary. Toward this end, assume first that $\rho\left((j\omega_0 I - A_0)^{-1}A_1\right) = 1$ for some $\omega_0 > 0$. This implies that the matrix $(j\omega_0 I - A_0)^{-1}A_1$ has an eigenvalue $e^{j\theta_0}$, for some $\theta_0 \in [0,\ 2\pi]$. Let $\tau_0 = \theta_0/\omega_0$. It is clear that

$$\det\left(I - (j\omega_0 I - A_0)^{-1}A_1 e^{-j\tau_0\omega_0}\right) = 0,$$

or

$$\det\left(j\omega_0 I - A_0 - A_1 e^{-j\tau_0\omega_0}\right) = 0.$$

As such, the system becomes unstable at $\tau = \tau_0$, and hence cannot be stable independent of delay. Next, suppose that $\rho\left((j\omega I - A_0)^{-1}A_1\right) > 1$ for some $\omega > 0$. Since $\rho\left((j\omega I - A_0)^{-1}A_1\right)$ is a continuous function of ω, and since

$$\lim_{\omega\to\infty} \rho\left((j\omega I - A_0)^{-1}A_1\right) = 0,$$

there exists some $\omega_0 \in (\omega,\ \infty)$, such that $\rho\left((j\omega_0 I - A_0)^{-1}A_1\right) = 1$. This, as we have shown above, implies that the system cannot be stable independent of delay. Consequently, we conclude that the conditions (i-iii) are necessary for the system to be stable independent of delay, thus completing the proof. ■

The frequency-sweeping test given by Theorem 2.1 may be considered as an extended small gain condition, to be discussed later in Chapter 3. Stability conditions of this type are routinely found in robust control theory, and are generally held as efficient measures in stability analysis; a subsequent example will attest to this point. Evidently, our derivation of this test still draws fundamentally upon the continuity property of the stability exponent of quasipolynomials. Yet unlike the classical results, it rids of any variable elimination procedure and lends a readily implementable criterion. The test can be easily checked by computing essentially the frequency-dependent measure $\rho\left((j\omega I - A_0)^{-1}A_1\right)$, which is rather amenable to computation due to the ease in computing the spectral radius.

While this frequency-sweeping test is more numerically inclined and indeed has a justifiable advantage in computation, as a small gain type condition it may also be used for analysis purposes. This is to be seen in forthcoming examples. Moreover, it is of interest to point out that since $(sI - A_0)^{-1}A_1$ defines an analytic function in $\overline{\mathsf{C}}_+$, the spectral radius $\rho\left((sI - A_0)^{-1}A_1\right)$ enjoys stronger properties than merely the continuity required in establishing stability. In fact, it will be seen in Chapter 3 that $\rho\left((sI - A_0)^{-1}A_1\right)$ is a *subharmonic* function in $\overline{\mathsf{C}}_+$, and so is the function $\rho\left((sI - A_0)^{-1}A_1e^{-\tau s}\right)$. One direct consequence of this fact is that not only is $\rho\left((sI - A_0)^{-1}A_1e^{-\tau s}\right)$ continuous in $\overline{\mathsf{C}}_+$, for any $\tau \geq 0$, but also it satisfies the well-known *maximum modulus principle*; that is, it achieves its maximum on the boundary $\partial\mathsf{C}_+$ of C_+. In other words,

$$\begin{aligned}\sup_{s\in\overline{\mathsf{C}}_+} \rho\left((sI - A_0)^{-1}A_1e^{-\tau s}\right) &= \sup_{\omega\geq 0} \rho\left((j\omega I - A_0)^{-1}A_1e^{-j\tau\omega}\right) \\ &= \sup_{\omega\geq 0} \rho\left((j\omega I - A_0)^{-1}A_1\right).\end{aligned}$$

This observation thus provides an alternative explanation to the condition (2.25). More generally, it has more far-reaching implications and may be used to advantage for other purposes.

Example 2.6 *Let $A \in \mathsf{R}^{n\times n}$ be a stable matrix and consider the delay system*

$$\dot{x}(t) = A\,x(t) + \beta A\,x(t-\tau), \qquad \tau \geq 0, \quad \beta \in \mathsf{R}. \tag{2.26}$$

We want to determine the set of β and A such that the system is stable independent of delay. Note first that the condition (ii) dictates that $\beta > -1$. Let Λ be the Jordan form of A and let λ_i be its eigenvalues. Then,

$$\rho\left((j\omega I - A)^{-1}A\right) = \rho\left((j\omega I - \Lambda)^{-1}\Lambda\right) = \max_i \frac{|\lambda_i|}{\sqrt{(Re\lambda_i)^2 + (\omega - Im\lambda_i)^2}}\,.$$

Hence, if $\lambda_i \in \mathsf{R}$ *for all* i, *we have* $\rho\left(\beta(j\omega I - A)^{-1}A\right) < 1$ *for all* $\omega > 0$ *and all* $\beta \in (-1,\ 1]$. *In other words, the system is stable independent of delay for all* $\beta \in (-1,\ 1]$. *On the other hand, if* A *has complex eigenvalues, then*

$$\sup_{\omega>0} \rho\left(\beta(j\omega I - A)^{-1}A\right) = |\beta| \max_i \left|\frac{\lambda_i}{Re\lambda_i}\right|.$$

This suggests that the system will be stable independent of delay if

$$\beta \in \left(-\min_i \left|\frac{Re\lambda_i}{\lambda_i}\right|,\ \min_i \left|\frac{Re\lambda_i}{\lambda_i}\right|\right).$$

It is worth noting that this example has been studied elsewhere using classical stability tests, but the analysis was, understandably, restricted to 2×2 *matrices. In general, for a matrix* A *of an arbitrary dimension, the classical analysis becomes highly nontrivial.*

Example 2.7 *Consider Example 2.2, in which*

$$A_0 = \begin{pmatrix} -2 & 0 \\ 0.5 & -2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0.5 \end{pmatrix}.$$

It is trivial to verify that A_0 *and* $A_0 + A_1$ *are both stable. In addition,*

$$(sI - A_0)^{-1}A_1 = \frac{1}{(s+2)^2} \begin{pmatrix} s+2 \\ 0.5 \end{pmatrix} \begin{pmatrix} 0 & 0.5 \end{pmatrix}.$$

As a result,

$$\rho\left((sI - A_0)^{-1}A_1\right) = \left|\frac{1}{(s+2)^2} \begin{pmatrix} 0 & 0.5 \end{pmatrix} \begin{pmatrix} s+2 \\ 0.5 \end{pmatrix}\right| = \left|\frac{0.5^2}{(s+2)^2}\right|.$$

Evidently, $\rho\left((j\omega I - A_0)^{-1}A_1\right) < 1$ *for all* $\omega > 0$. *This leads to the same conclusion that the system (2.17) is stable independent of delay.*

When the system is not stable independent of delay, it remains possible to extend the above technique to compute the delay margin. In fact, the proof of Theorem 2.1 already shed some light on how this may be accomplished. Under the condition that A_0 has no eigenvalue on $\partial\mathsf{C}_+$, i.e., when $(j\omega I - A_0)$ is invertible, the zero crossings of the characteristic quasipolynomial occur at the frequencies ω where

$$\left|\lambda_i\left((j\omega I - A_0)^{-1}A_1\right)\right| = 1,$$

that is, when certain eigenvalues of $(j\omega I - A_0)^{-1}A_1$ have a unit modulus. In general, however, if only delay-dependent stability can be insured, A_0 may have eigenvalues on $\partial\mathsf{C}_+$. This leads us to the consideration of the frequency-dependent matrix pencil $(j\omega I - A_0) - \lambda A_1$. For a matrix pair

$(A,\ B)$, denote its ith generalized eigenvalue by $\lambda_i(A,\ B)$. Furthermore, define

$$\underline{\rho}(A,\ B) := \min\left\{|\lambda| \mid \det(A - \lambda B) = 0\right\}.$$

It is a well-known fact that the number of bounded generalized eigenvalues for $(A,\ B)$ is at most equal to the rank of B. Also, if the rank of B is constant, then $\lambda_i(A,\ B)$ is continuous with respect to the elements of A and B. The idea is then to use the frequency-dependent generalized eigenvalues $\lambda_i(j\omega I - A_0,\ A_1)$, to replace the eigenvalues $\lambda_i\left((j\omega I - A_0)^{-1}A_1\right)$, which may not exist at certain frequencies.

Theorem 2.2 *Suppose that the system (2.23) is stable at $\tau = 0$. Let*

$$\mathit{rank}\ (A_1) = q$$

Furthermore, define

$$\overline{\tau}_i := \begin{cases} \min\limits_{1\le k\le n} \frac{\theta_k^i}{\omega_k^i} & \text{if } \lambda_i\left(j\omega_k^i I - A_0,\ A_1\right) = e^{-j\theta_k^i} \\ & \quad\text{for some } \omega_k^i \in (0,\ \infty),\ \theta_k^i \in [0,\ 2\pi] \\ \infty & \text{if } \underline{\rho}\,(j\omega I - A_0,\ A_1) > 1,\ \forall\omega \in (0,\ \infty) \end{cases}$$

{added space in the above equation between two expressions}
Then,

$$\overline{\tau} := \min_{1\le i\le q} \overline{\tau}_i.$$

That is, system (2.23) is stable for all $\tau \in [0,\ \overline{\tau})$, but becomes unstable at $\tau = \overline{\tau}$.

Proof. Consider first the case $\overline{\tau} = \infty$. This corresponds to the condition

$$\underline{\rho}\,(j\omega I - A_0,\ A_1) = \underline{\rho}\left(j\omega I - A_0,\ A_1 e^{-j\tau\omega}\right) > 1,$$

for all $\tau \ge 0$ and all $\omega > 0$. Hence, it implies that for any $\tau \ge 0$,

$$\det\left(j\omega I - A_0 - A_1 e^{-j\tau\omega}\right) \neq 0, \qquad \forall\omega \in (0,\ \infty).$$

In addition, under the assumption that the system is stable at $\tau = 0$, $\det(A_0 + A_1) \neq 0$. This suggests that for any other τ,

$$\det\left(j\omega I - A_0 - A_1 e^{-j\tau\omega}\right) \neq 0$$

at $\omega = 0$ as well. It thus follows by the continuity of the stability exponent that

$$\det\left(sI - A_0 - A_1 e^{-\tau s}\right) \neq 0, \qquad \forall s \in \overline{\mathbb{C}}_+.$$

As a result, the system is stable for all $\tau \in [0,\ \infty)$; in other words, the system is stable independent of delay. Suppose now that $\overline{\tau} < \infty$. For any $\tau \in [0,\ \overline{\tau})$, we claim that

$$\det\left(j\omega I - A_0 - A_1 e^{-j\tau\omega}\right) \neq 0, \qquad \forall\omega \in [0,\ \infty).$$

$\{h$ replaced by τ in the above equation$\}$ This is clearly the case for $\omega \neq \omega_k^i$, for at these frequencies

$$|\lambda_i (j\omega I - A_0,\ A_1)| \neq 1$$

for all $i = 1, \ \cdots, \ n$. Additionally, at $\omega = \omega_k^i$, since for any $\tau \in [0, \ \overline{\tau})$, $\tau\omega_k^i \neq \theta_k$ by definition, we have $\det\left(j\omega_k^i I - A_0 - A_1 e^{-j\tau\omega_k^i}\right) \neq 0$. This proves the claim. Again, by an appeal to the continuity argument, we conclude that the system is stable for all $\tau \in [0, \ \overline{\tau})$. On the other hand, if $\tau = \overline{\tau}$, then there exists a pair $(\omega_k^i, \ \theta_k^i)$ such that $\overline{\tau} = \theta_k^i / \omega_k^i$, and

$$\det\left(j\omega_k^i I - A_0 - A_1 e^{-j\overline{\tau}\omega_k^i}\right) = \det\left(j\omega_k^i I - A_0 - A_1 e^{-j\theta_k^i}\right) = 0.$$

That is, the system is unstable. The proof is now complete. ■

It follows instantly that the delay-independent stability condition, Theorem 2.1, can be equivalently stated in terms of the measure $\underline{\rho}\,(j\omega I - A_0,\ A_1)$.

Corollary 2.3 *The system (2.23) is stable independent of delay if and only if*

(i) A_0 *is stable,*
(ii) $A_0 + A_1$ *is stable, and*
(iii)

$$\underline{\rho}\,(j\omega I - A_0,\ A_1) > 1, \qquad \forall \omega > 0, \tag{2.27}$$

In light of Theorem 2.2, an algorithm can be devised at once to compute the delay margin. First, we compute the generalized eigenvalues $\lambda_i\,(j\omega I - A_0,\ A_1)$. This is to be done by gridding the frequency axis, and at each gridding point, computing the generalized eigenvalues using standard numerical algorithms. If $\underline{\rho}\,(j\omega I - A_0,\ A_1) > 1$ for all $\omega \in (0, \ \infty)$, we conclude that the system is stable for all $\tau \in [0, \ \infty)$. Otherwise, the computation generates a pair $(\omega_k^i, \ \theta_k^i)$, which always yields a nonzero estimate of the delay margin. This can be seen by noting that whenever $\det\left(j\omega_k^i I - A_0 - A_1\right) \neq 0$, the inequality $0 < \theta_k^i < 2\pi$ holds, and hence $\overline{\tau}_i > 0$. One should note, unsurprisingly, that the computation of the delay margin is more demanding than to determine delay-independent stability. Note also that the computation needs to be performed only over a finite frequency interval, since the moduli of the generalized eigenvalues will inevitably exceed one beyond a certain frequency. This, together with the fact that the generalized eigenvalues are continuous functions of ω, is reassuring from a numerical standpoint.

We are now ready to tackle systems of multiple commensurate delays. Specifically, consider

$$\dot{x}(t) = A_0\ x(t) + \sum_{k=1}^{m} A_k\ x(t - k\tau), \qquad \tau \geq 0. \tag{2.28}$$

In light of Theorem 2.1 and Theorem 2.2, it is natural to seek to transform the stability problem to one with a single delay. This is readily accomplished, by the use of Schur complement. We call the reader's attention to the following matrix identity, which is a straightforward consequence of the Schur's complement for determinant, given in Appendix A.

Lemma 2.4 *For any* $z \in \mathbf{C}$ *and any matrices* $P_k \in \mathbf{C}^{n\times n}$, $k = 0,\ 1,\ \cdots,\ m$,

$$\begin{aligned} & \det\left(\sum_{k=0}^{m} P_k z^k\right) \\ = & \det\left\{ z \begin{pmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & P_m \end{pmatrix} - \begin{pmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -P_0 & -P_1 & \cdots & -P_{m-1} \end{pmatrix} \right\}. \end{aligned}$$

Theorem 2.5 *System (2.28) is stable independent of delay if and only if*

(i) A_0 *is stable,*

(ii) $A_0 + \sum_{k=1}^{m} A_k$ *is stable, and*

(iii)

$$\rho\left(M_m(j\omega)\right) < 1, \qquad \forall \omega > 0, \tag{2.29}$$

where

$$M_m(s) := \begin{pmatrix} (sI - A_0)^{-1}A_1 & \cdots & (sI - A_0)^{-1}A_{m-1} & (sI - A_0)^{-1}A_m \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix}.$$

Theorem 2.6 *Suppose that the system (2.28) is stable at* $\tau = 0$, *and let* $q = \mathrm{rank}(A_m)$. *Furthermore, define*

$$\overline{\tau}_i := \begin{cases} \min\limits_{1\le k\le n} \frac{\theta_k^i}{\omega_k^i} & \text{if } \lambda_i\left(G(j\omega_k^i),\ H(j\omega_k^i)\right) = e^{-j\theta_k^i} \\ & \quad \text{for some } \omega_k^i \in (0,\ \infty),\ \theta_k^i \in [0,\ 2\pi] \\ \infty & \text{if } \underline{\rho}\left(G(j\omega),\ H(j\omega)\right) > 1,\ \forall \omega \in (0,\ \infty) \end{cases}$$

{space added in the above equation} *where*

$$G(s) := \begin{pmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -(sI - A_0) & A_1 & \cdots & A_{m-1} \end{pmatrix},$$

$$H(s) := \mathit{diag}\begin{pmatrix} I & \cdots & I & -A_m \end{pmatrix}.$$

Then,

$$\overline{\tau} := \min_{1 \le i \le q+n(m-1)} \overline{\tau}_i.$$

The system (2.28) is stable for all $\tau \in [0,\ \overline{\tau})$, but becomes unstable at $\tau = \overline{\tau}$.

Proof. Let $z = e^{-j\tau\omega}$, $P_0 = j\omega I - A_0$, and $P_k = -A_k$. It follows from Lemma 2.4 that

$$\det\left(j\omega I - \sum_{k=0}^{m} A_k e^{-jk\tau\omega}\right) = (-1)^{nm} \det\left(G(j\omega) - e^{-j\tau\omega} H(j\omega)\right).$$

This effectively reduces the stability problem to one with a single delay. Thus, the proof for Theorem 2.6 follows as in that for Theorem 2.2. The proof for Theorem 2.5 may proceed by recognizing that the condition for stability independent of delay is given by Theorem 2.5 (i-ii), and additionally,

$$\underline{\rho}\left(G(j\omega),\ H(j\omega)\right) > 1, \qquad \forall \omega \in (0,\ \infty).$$

Under the condition that $G(j\omega)$ is invertible,

$$\underline{\rho}\left(G(j\omega),\ H(j\omega)\right) = \frac{1}{\rho\left(G^{-1}(j\omega)H(j\omega)\right)}.$$

In view of the matrix identity

$$\begin{pmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -P_0 & -P_1 & \cdots & -P_{m-1} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -P_0^{-1}P_1 & \cdots & -P_0^{-1}P_{m-1} & -P_0^{-1} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix}, \tag{2.30}$$

we conclude that $M_m(j\omega) = G^{-1}(j\omega)H(j\omega)$. The proof for Theorem 2.5 is thus completed. ■

The following examples serve to illustrate the computational advantage of these results. It demonstrates that the frequency-sweeping tests obtained herein can be fairly effective for systems with high dimensions and many delays. Such systems are not easily handled using classical stability tests.

Example 2.8 *Consider the system (2.23), with* 4 *states and* 3 *commensurate delays, given by*

$$
\begin{aligned}
A_0 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{pmatrix}, \\
A_1 &= \begin{pmatrix} -0.05 & 0.005 & 0.25 & 0 \\ 0.005 & 0.005 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -0.5 & 0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0.005 & 0.0025 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.0005 \\ -1 & -0.5 & -0.5 & 0 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 0.0375 & 0 & 0.075 & 0.125 \\ 0 & 0.05 & 0.05 & 0 \\ 0.05 & 0.05 & 0 & 0 \\ 0 & -2.5 & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

A simple computation shows that the eigenvalues of A_0 are located at

$$
\begin{array}{c} -0.6887 \pm 1.7636j \\ -0.3113 \pm 0.6790j. \end{array}
$$

Hence, A_0 is stable. Similarly, $A_0 + \sum_{k=1}^{3} A_k$ is also found stable. As such, we need to examine the condition (2.28) to check whether the system is stable independent of delay. The spectral radius $\rho(M_m(j\omega))$ is shown in 2.1. From this plot, it is clear that the condition (2.28) is not satisfied. Therefore, we conclude that the system is not stable independent of delay.

Example 2.9 *Consider the same system in Example 2.8, with A_k, $k = 1,\ 2,\ 3$ modified to*

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & -2 \end{pmatrix}.
\end{aligned}
$$

It can be easily verified that the system is stable at $\tau = 0$. Let us compute $\overline{\tau}$ based on Theorem 2.6. First, an initial computation of $\lambda_i(G(j\omega),\ H(j\omega))$ is performed on a crude frequency grid, to determine an approximate interval

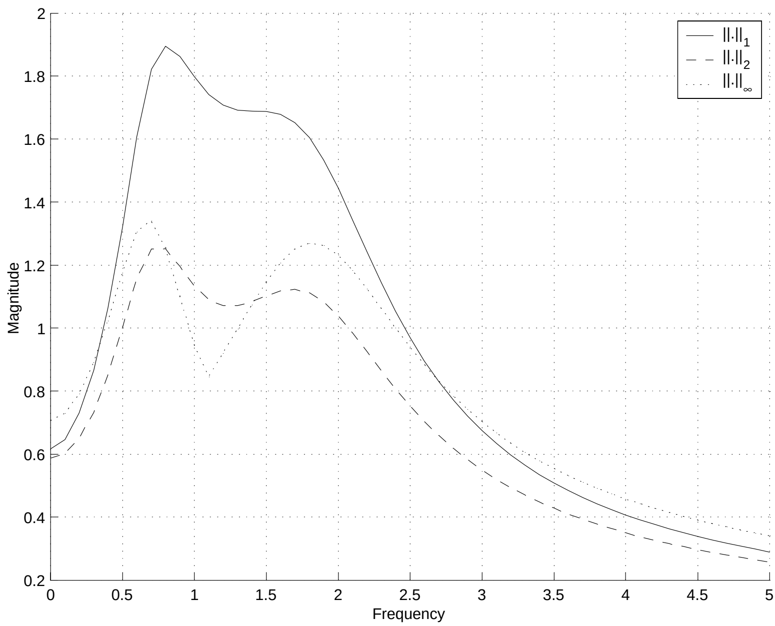

FIGURE 2.1. The spectral radius $\rho(M_{\mathsf{m}}(j\omega))$

in which $|\lambda_i(G(j\omega),\ H(j\omega))|$ *cross the value of* 1*. This interval was found confined to the interval* $[0,\ 2]$*. The computation is then performed using a much finer grid over* $[0,\ 2]$ *with* 1000 *grid points for the purpose of achieving a higher precision. This yields the critical frequency-angle pairs* $(\omega_k^i,\ \theta_k^i)$*:*

$$(0.2743,\ 3.5952),\quad (0.9149,\ 2.9014),\quad (0.9169,\ 2.9001),\quad (1.3033,\ 0.4931),$$

from which we obtain $\overline{\tau} = 0.4931/1.3033 = 0.3783$*.*

We end this section with a comment on systems described by the scalar differential-difference equation

$$y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{k=0}^{q} a_{ki} y^{(i)}(t - k\tau) = 0, \qquad \tau \geq 0. \tag{2.31}$$

The characteristic quasipolynomial for such a system is given by (2.6) and (2.7). Clearly, both Theorem 2.5 and Theorem 2.6 can be immediately applied to (2.31), by defining

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{00} & -a_{01} & \cdots & -a_{0,n-1} \end{pmatrix},$$

$$A_k = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ -a_{k0} & -a_{k1} & \cdots & -a_{k,n-1} \end{pmatrix}.$$

The tests, however, can be further simplified computationally. Indeed, we recognize with ease the following conditions for the stability of such a system.

Corollary 2.7 *The system (2.31) is stable independent of delay if and only if*

(i) $a_0(s)$ *is stable,*

(ii) $a_0(s) + \sum_{k=1}^{q} a_k(s)$ *is stable, and*

(iii)

$$\rho\left(M_a(j\omega)\right) < 1, \qquad \forall \omega > 0, \tag{2.32}$$

where

$$M_a(s) := \begin{pmatrix} -\frac{a_1(s)}{a_0(s)} & \cdots & -\frac{a_{q-1}(s)}{a_0(s)} & -\frac{a_q(s)}{a_0(s)} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Corollary 2.8 *Suppose that system (2.31) is stable at* $\tau = 0$*. Define*

$$\overline{\tau}_i := \begin{cases} \min\limits_{1\le k\le n} \frac{\theta_k^i}{\omega_k^i} & \text{if } \lambda_i\left(G_a(j\omega_k^i),\ H_a(j\omega_k^i)\right) = e^{-j\theta_k^i} \\ & \quad \text{for some } \omega_k^i \in (0,\ \infty),\ \theta_k^i \in [0,\ 2\pi] \\ \infty & \text{if } \underline{\rho}\left(G_a(j\omega),\ H_a(j\omega)\right) > 1,\ \forall \omega \in (0,\ \infty) \end{cases}$$

where

$$G_a(s) := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0(s) & -a_1(s) & \cdots & -a_{q-1}(s) \end{pmatrix},$$

$$H_a(s) := \mathit{diag}\begin{pmatrix} 1 & 1 & \cdots & a_q(s) \end{pmatrix}.$$

Then,

$$\overline{\tau} := \min_{1\le i\le q} \overline{\tau}_i.$$

The system (2.31) is stable for all $\tau \in [0,\ \overline{\tau})$*, but becomes unstable at* $\tau = \overline{\tau}$*.*

Note that the dimensions of these matrices are equal to q, which may be considerably lower than that of the matrices in Theorem 2.5 and Theorem 2.6. As such, it is possible to check both conditions more efficiently.

2.4 Constant matrix tests

In spite of their implementational simplicity, frequency-sweeping tests by nature may not be executed in finite computation, and the computational accuracy hinges on the fineness of the frequency grids. Thus, they are likely to be at disadvantage if high computational precision is sought after. This issue can be especially acute in the computation of the delay margin. The consideration thus leads us to search for alternative stability tests, tests that are both readily implementable and can be performed via finite step algorithms. In particular, for numerical precision it will be highly desirable to rid of any need of frequency sweep, while retaining the merits of computing eigenvalues and generalized eigenvalues, for the computational ease of the latter. The stability tests to be developed below seek to combine these advantageous features, which will require only the computation of *constant matrices*.

We begin with a re-examination of the quasipolynomial (2.6), while leaving aside its interpretation either as one for the state-space description (2.5), or that for the differential-difference equation model (2.31). We have seen that the stability of the quasipolynomial, either independent or dependent of delay, can be assessed by finding its zero-crossing frequencies. More specifically, it amounts to finding the roots $(s,\ z) \in \partial \mathsf{C}_+ \times \partial \mathsf{D}$, so that the bivariate polynomial $a(s,\ z) = 0$, or equivalently, all such $\omega > 0$ that $a(j\omega,\ z) = 0$ for some $z \in \partial \mathsf{D}$. A key, albeit straightforward, observation is that for each $\omega \in (0,\ \infty)$, $a(j\omega,\ z)$ defines a polynomial in z, whose root locations can be analyzed using established tools. Our development is built upon this recognition. With D being the stability region of concern, we come to note the well-known *Schur-Cohn criterion*, which gives a necessary and sufficient condition for the roots of a polynomial to lie in D. For ease of reference, we make a slight digression to the subject, but choose to leave the proofs to the relevant literature discussed in the Notes section at the end of this chapter.

Consider a complex polynomial

$$p(z) = \sum_{k=0}^{q} p_k z^k.$$

Define its conjugate polynomial $\overline{p}(z)$ by

$$\overline{p}(z) := z^q \sum_{k=0}^{q} \overline{p}_k z^{-k}.$$

Furthermore, define the *Schur-Cohn-Fujiwara matrix* as

$$K := \overline{p}^H(S)\overline{p}(S) - p^H(S)p(S),$$

where $p(S)$ is the matrix polynomial

$$p(S) := p_0 I + p_1 S + \cdots + p_q S^q,$$

and S is the shift matrix given by

$$S := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The Schur-Cohn-Fujiwara matrix $K = [k_{ij}]$ is Hermitian and consists of elements

$$k_{ij} = \sum_{r=0}^{i-1} \left(p_{n-i-r-1}\overline{p}_{n-j-r-1} - \overline{p}_{i+r+1}p_{j+r+1} \right), \qquad i \leq j.$$

It is known that the polynomial $p(z)$ has all its roots in D if and only if K is positive definite. In particular, the roots of $p(z)$, z_i, $i = 1, \ \cdots, \ q$, are related to K via the following *Orlando formula*.

Lemma 2.9 (Orlando Formula) *Let z_i, $i = 1, \ \cdots, \ q$, be the roots of the complex polynomial $p(z) = \sum_{k=0}^{q} p_k z^k$, whose corresponding Schur-Cohn-Fujiwara matrix is defined by K. Then,*

$$\det(K) = |p_q|^{2q} \prod_{i,j=1}^{q} (1 - z_i \overline{z}_j).$$

We next define the matrix

$$C := \begin{pmatrix} p^T(S) & \left(\overline{p}^T(S)\right)^H \\ \overline{p}^T(S) & \left(p^T(S)\right)^H \end{pmatrix}.$$

Note that

$$p^T(S) = \begin{pmatrix} p_0 & 0 & \cdots & 0 \\ p_1 & p_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{q-1} & p_{q-2} & \cdots & p_0 \end{pmatrix}, \quad \left(\overline{p}^T(S)\right)^H = \begin{pmatrix} p_q & p_{q-1} & \cdots & p_1 \\ 0 & p_q & \cdots & p_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_q \end{pmatrix}.$$

Note also that C is the test matrix in the Schur-Cohn criterion. The following fact establishes a relationship between C and K.

Lemma 2.10

$$\det(K) = (-1)^q \det(C).$$

Proof. Suppose first that $p_0 \neq 0$. Then by definition $p(S)$ is invertible. Since the polynomial matrices $p(S)$ and $\overline{p}(S)$ commute, we have $\overline{p}(S)p^{-1}(S) = p^{-1}(S)\overline{p}(S)$. Based on this fact and the Schur determinant formula, we find that

$$\begin{aligned}
\det(K) &= \det\left(\overline{p}^H(S)\overline{p}(S) - p^H(S)p(S)\right) \\
&= \det\left(\overline{p}^H(S)\overline{p}(S)p^{-1}(S) - p^H(S)\right)\det(p(S)) \\
&= (-1)^q \det\left(p^H(S) - \overline{p}^H(S)p^{-1}(S)\overline{p}(S)\right)\det(p(S)) \\
&= (-1)^q \det\begin{pmatrix} p(S) & \overline{p}(S) \\ \overline{p}^H(S) & p^H(S) \end{pmatrix} \\
&= (-1)^q \det\begin{pmatrix} p(S) & \overline{p}(S) \\ \overline{p}^H(S) & p^H(S) \end{pmatrix}^T \\
&= (-1)^q \det\begin{pmatrix} p^T(S) & \left(\overline{p}^T(S)\right)^H \\ \overline{p}^T(S) & \left(p^T(S)\right)^H \end{pmatrix}.
\end{aligned}$$

Hence, $\det(K) = (-1)^q \det(C)$. Additionally, in case $p_0 = 0$, the equality remains valid; this is seen by an appeal to the limit $p_0 \to 0$. ∎

It follows from Lemma 2.9 that $p(z)$ has a root on the unit circle $\partial\mathsf{D}$ if and only if $\det(K) = 0$, and from Lemma 2.10 that $p(z)$ has a unitary root if and only if $\det(C) = 0$. The latter sheds light on how zero-crossing frequencies may be found. Essentially, for each $\omega \in \mathsf{R}_+$, one may view $a(j\omega,\ z)$ as a complex polynomial in z, and construct accordingly its Schur-Cohn-Fujiwara matrix or Schur-Cohn test matrix, both of which are frequency dependent. It then follows that at that ω, $a(j\omega,\ z)$ has a root $z \in \partial\mathsf{D}$ whenever the determinant of these matrices is equal to zero. This consequently insures the existence of a pair $(j\omega,\ z) \in \partial\mathsf{C}_+ \times \partial\mathsf{D}$ such that $a(j\omega,\ z) = 0$. In other words, if ω is a zero-crossing frequency, then it may be found from the condition that the determinant of the frequency-dependent Schur-Cohn-Fujiwara matrix or Schur-Cohn test matrix is equal to zero.

More specifically, for any fixed s consider the polynomial

$$a(z) := a(s,\ z) = \sum_{k=0}^{q} a_k(s)z^k.$$

Define its conjugate polynomial as

$$\overline{a}(z) := z^q \sum_{k=0}^{q} \overline{a}_k(s)z^{-k}.$$

Furthermore, define the matrices

$$\Sigma_1(s) \;:=\; \begin{pmatrix} a_0(s) & 0 & \cdots & 0 \\ a_1(s) & a_0(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{q-1}(s) & a_{q-2}(s) & \cdots & a_0(s) \end{pmatrix},$$

$$\Sigma_2(s) \;:=\; \begin{pmatrix} a_q(s) & a_{q-1}(s) & \cdots & a_1(s) \\ 0 & a_q(s) & \cdots & a_2(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_q(s) \end{pmatrix}, \tag{2.33}$$

and

$$\Sigma(s) := \begin{pmatrix} \Sigma_1(s) & \Sigma_2(s) \\ \Sigma_2^H(s) & \Sigma_1^H(s) \end{pmatrix}.$$

In view of Lemma 2.9 and Lemma 2.10, we assert that that for any s,

$$\det\left(\Sigma(s)\right) = (-1)^q \left|a_q(s)\right|^{2q} \prod_{i,j=1}^{n} \left(1 - z_i \overline{z}_j\right), \tag{2.34}$$

where z_i, $i = 1, \; \cdots, \; q$, are the roots of $a(z)$ for fixed s. Therefore, for any $\omega > 0$, there is a $z \in \partial\mathsf{D}$ such that $a(j\omega, \; z) = a(z) = 0$ whenever $\det\left(\Sigma(j\omega)\right) = 0$. The implication is then rather clear: By solving the equation $\det(\Sigma(j\omega)) = 0$, we find all such $\omega > 0$ that $a(z) = a(j\omega, \; z)$ has a root on $\partial\mathsf{D}$. Since $\det(\Sigma(j\omega))$ defines a polynomial in ω, the solutions can be found by solving the eigenvalues of a constant matrix. Clearly, there are only a finite number of solutions, and among them the positive solutions constitute the zero-crossing frequencies. Once the zero-crossing frequencies are found, the unitary roots of $a(j\omega, \; z)$ can also be determined at each zero-crossing frequency by solving the complex polynomial $a(z) = a(j\omega, \; z)$. The delay margin $\overline{\tau}$ can then be computed immediately from the solutions of these two problems. The following result implements this solution procedure.

Theorem 2.11 *Suppose that the quasipolynomial (2.6) is stable for $\tau = 0$. Let $H_n := 0$, $T_n := I$, and*

$$H_i := \begin{pmatrix} a_{qi} & a_{q-1,i} & \cdots & a_{1i} \\ 0 & a_{qi} & \cdots & a_{2i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{qi} \end{pmatrix}, \qquad i = 0, \; ,1, \; \cdots, \; n-1,$$

$$T_i := \begin{pmatrix} a_{0i} & 0 & \cdots & 0 \\ a_{1i} & a_{0i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{q-1,i} & a_{q-2,i} & \cdots & a_{0i} \end{pmatrix}, \qquad i = 0, \; ,1, \; \cdots, \; n-1,$$

$$P_i := \begin{pmatrix} (j)^i T_i & (j)^i H_i \\ (-j)^i H_i^T & (-j)^i T_i^T \end{pmatrix}, \quad i = 0, \ ,1, \ \cdots, \ n.$$

Furthermore, define

$$P := \begin{pmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -P_n^{-1}P_0 & -P_n^{-1}P_1 & \cdots & -P_n^{-1}P_{n-1} \end{pmatrix}.$$

Then, $\overline{\tau} = \infty$ *if* $\sigma(P) \cap \mathsf{R}_+ = \emptyset$ *or* $\sigma(P) \cap \mathsf{R}_+ = \{0\}$*. Additionally, let*

$$F(s) := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0(s) & -a_1(s) & \cdots & -a_{q-1}(s) \end{pmatrix},$$

$$G(s) := diag\begin{pmatrix} 1 & 1 & \cdots & a_q(s) \end{pmatrix}.$$

Then $\overline{\tau} = \infty$ *if* $\sigma(F(j\omega_k), \ G(j\omega_k)) \cap \partial\mathsf{D} = \emptyset$ *for all* $0 \neq \omega_k \in \sigma(P) \cap \mathsf{R}_+$*. In these cases the quasipolynomial (2.6) is stable independent of delay. Otherwise,*

$$\overline{\tau} = \min_{1 \le k \le 2nq} \frac{\theta_k}{\omega_k},$$

where $0 \neq \omega_k \in \sigma(P) \cap \mathsf{R}_+$ *and* $\theta_k \in [0, \ 2\pi]$ *satisfy the relation* $e^{-j\theta_k} \in \sigma(F(j\omega_k), \ G(j\omega_k))$*. The quasipolynomial (2.6) is stable for all* $\tau \in [0, \ \overline{\tau})$*, but is unstable at* $\tau = \overline{\tau}$*.*

Proof. Suppose first that $\sigma(P) \cap \mathsf{R}_+ = \emptyset$. This implies that for any $\omega \in \mathsf{R}_+$, $\det(\omega I - P) \neq 0$. In view of Lemma 2.4,

$$\begin{aligned} \det(\omega I - P) &= \det\left(P_n^{-1}\right) \det\left(\sum_{i=0}^{n} \omega^i P_i\right) \\ &= \det\left(P_n^{-1}\right) \det\begin{pmatrix} \Sigma_1(j\omega) & \Sigma_2(j\omega) \\ \Sigma_2^H(j\omega) & \Sigma_1^H(j\omega) \end{pmatrix}. \end{aligned}$$

Hence, the condition that $\sigma(P) \cap \mathsf{R}_+ = \emptyset$ is equivalent to $\det(\Sigma(j\omega)) \neq 0$ for all $\omega \in \mathsf{R}_+$. This implies that $a(j\omega, \ z)$ has no root on $\partial\mathsf{D}$, that is, $a\left(j\omega, \ e^{-j\tau\omega}\right) \neq 0$ for all $\omega \in \mathsf{R}_+$ and $\tau \in [0, \ \infty)$. Since $a\left(s, \ e^{-\tau s}\right)$ is stable at $\tau = 0$, it is stable for all $\tau \in [0, \ \infty)$. Note also that if $\sigma(P) \cap \mathsf{R}_+ = \{0\}$, then $a\left(j\omega, \ e^{-j\tau\omega}\right) \neq 0$ for all $\omega \in \mathsf{R}_+$ and $\tau \in [0, \ \infty)$, despite that $a(0, \ z) = 0$ may hold for some z on $\partial\mathsf{D}$. This follows from the fact that $a(j\omega, \ z)$ has no root on $\partial\mathsf{D}$ for any $\omega \neq 0$, and that $a\left(j\omega, \ e^{-j\tau\omega}\right) \neq 0$ at $\omega = 0$, by the assumption that $a\left(s, \ e^{-\tau s}\right)$ is stable at $\tau = 0$; the latter implies that $a(j\omega, \ 1) \neq 0$ for all $\omega \in \mathsf{R}_+$. Hence, again, we conclude that $a\left(s, \ e^{-\tau s}\right)$ is stable for all $\tau \in [0, \ \infty)$ if $\sigma(P) \cap \mathsf{R}_+ = \{0\}$.

Suppose now that $\sigma(P) \cap \mathsf{R}_+ \neq \emptyset$ and $\sigma(P) \cap \mathsf{R}_+ \neq \{0\}$. In other words, P does have positive real roots. Let $0 \neq \omega_k \in \sigma(P) \cap \mathsf{R}_+$. Since according to Lemma 2.4,

$$\det\left(zG(s) - F(s)\right) = \sum_{i=0}^{q} a_i(s) z^i = a(s,\ z),$$

it follows that $a(j\omega_k,\ z)$ has no root on $\partial\mathsf{D}$ if $\sigma(F(j\omega_k),\ G(j\omega_k)) \cap \partial\mathsf{D} = \emptyset$. As a result, we may conclude that $\overline{\tau} = \infty$ if $\sigma(F(j\omega_k),\ G(j\omega_k)) \cap \partial\mathsf{D} = \emptyset$ for all $0 \neq \omega_k \in \sigma(P) \cap \mathsf{R}_+$. If, however, $\sigma(F(j\omega_k),\ G(j\omega_k)) \cap \partial\mathsf{D} \neq \emptyset$ for some $0 \neq \omega_k \in \sigma(P) \cap \mathsf{R}_+$, then there exist some $z_k \in \partial\mathsf{D}$ such that $z_k \in \sigma(F(j\omega_k),\ G(j\omega_k))$. Consequently, there exist some $\theta_k \in [0,\ 2\pi]$ such that $z_k = e^{-j\theta_k}$ and $a\left(j\omega_k,\ e^{-j\theta_k}\right) = 0$, or equivalently, $e^{-j\theta_k} \in \sigma(F(j\omega_k),\ G(j\omega_k))$. It follows that $a\left(j\omega_k,\ e^{-j\tau\omega_k}\right) = 0$ for $\tau = \theta_k/\omega_k$, and hence the system is unstable at $\tau = \overline{\tau}$. However, for any $\tau \in [0,\ \overline{\tau})$, since $\tau\omega_k \neq \theta_k$ for all $(\omega_k,\ \theta_k)$ such that $e^{-j\theta_k} \in \sigma(F(j\omega_k),\ G(j\omega_k))$, we have $a\left(j\omega_k,\ e^{-j\tau\omega_k}\right) \neq 0$. This implies that $a\left(j\omega,\ e^{-j\tau\omega}\right) \neq 0$ for all $\omega \in \mathsf{R}_+$. Therefore, $a\left(s,\ e^{-\tau s}\right)$ is stable for all $\tau \in [0,\ \overline{\tau})$. The proof is now completed. ∎

Theorem 2.11 suggests that a two-step procedure may be employed to test the stability of the quasipolynomial (2.6). First compute the eigenvalues of the $2nq \times 2nq$ matrix P. If P has no real eigenvalue or only one real eigenvalue at zero, we conclude that the quasipolynomial is stable independent of delay. If this is not the case, compute next the generalized eigenvalues of the $q \times q$ matrix pair $(F(j\omega_k), G(j\omega_k))$, with respect to each positive real eigenvalue ω_k of P. If for all such eigenvalues the pair $(F(j\omega_k), G(j\omega_k))$ has no generalized eigenvalue on the unit circle, we again conclude that the quasipolynomial is stable independent of delay. Otherwise, we obtain the delay margin $\overline{\tau}$. Note that if $a_q(j\omega) \neq 0$ for all $\omega \in \mathsf{R}_+$, the pair $(F(j\omega), G(j\omega))$ will always have generalized eigenvalues on $\partial\mathsf{D}$ whenever $\sigma(P) \cap \mathsf{R}_+ \neq \emptyset$. Indeed, this implies that $\det(\Sigma(j\omega_k)) = 0$ for some $\omega_k \in \sigma(P) \cap \mathsf{R}_+$. According to (2.34), we may conclude that $a(j\omega_k,\ z)$ will always have roots on or symmetric about $\partial\mathsf{D}$. Since the roots of $a(j\omega,\ z)$ are continuous functions of ω, there must exist also some ω_l such that $a(j\omega_l,\ z)$ has roots on $\partial\mathsf{D}$. Thus, under this circumstance, we may assert, without further computing the generalized eigenvalues of $F(j\omega)$ and $G(j\omega)$, that the quasipolynomial cannot be stable independent of delay.

The following result gives an alternative test in the same spirit. It may be executed by computing sequentially the eigenvalues of two matrices. A minor restriction, however, is that the assumption $a_0(j\omega) \neq 0$ needs to be imposed for all $\omega \in \mathsf{R}_+$.

Theorem 2.12 *Suppose that $a_0(j\omega) \neq 0$ for all $\omega \in \mathsf{R}_+$. Suppose also that the quasipolynomial (2.6) is stable for $\tau = 0$. Let $(A,\ B,\ C)$ be a minimal* *realization of* $\Sigma_1^{-1}(s)\Sigma_2^T(-s)$*, where* $\Sigma_1(s)$ *and* $\Sigma_2(s)$ *are given by (2.33). Furthermore, define the Hamiltonian matrix*

$$H := \begin{pmatrix} A & BB^T \\ -C^T C & -A^T \end{pmatrix} \in \mathsf{R}^{2nq\times 2nq}.$$

Then, $\overline{\tau} = \infty$ *if* $\sigma(H) \cap \partial\mathsf{C}_+ = \emptyset$ *or* $\sigma(H) \cap \partial\mathsf{C}_+ = \{0\}$*. Additionally, define*

$$M_a(s) := \begin{pmatrix} -\frac{a_1(s)}{a_0(s)} & \cdots & -\frac{a_{q-1}(s)}{a_0(s)} & -\frac{a_q(s)}{a_0(s)} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then, if $\sigma\left(M_a(j\omega_k)\right) \cap \partial\mathsf{D} = \emptyset$ *for all* $0 \neq \omega_k \in \mathsf{R}_+$ *such that* $j\omega_k \in \sigma(H)$*,* $\overline{\tau} = \infty$*. In these cases the quasipolynomial (2.6) is stable independent of delay. Otherwise,*

$$\overline{\tau} = \min_{1\le k\le 2nq} \frac{\theta_k}{\omega_k},$$

where $j\omega_k \in \sigma(H)$*,* $\omega_k \in \mathsf{R}_+$*,* $\omega_k \neq 0$*, and* $\theta_k \in [0,\ 2\pi]$ *satisfy the relation* $e^{-j\theta_k} \in \sigma(M_a(j\omega_k))$*.*

Proof. First, since $a_0(j\omega) \neq 0$ for all $\omega \in \mathsf{R}_+$, it follows that

$$\begin{aligned} \det(j\omega I - A) &= a_0(j\omega) \neq 0, \\ \det(j\omega I + A^T) &= (-1)^n a_0(-j\omega) \neq 0, \end{aligned}$$

and $\Delta^{-1}(j\omega)$ is well-defined, for all $\omega \in \mathsf{R}$. Consider the determinant $\det(j\omega I - H)$. A straightforward calculation shows that

$$\begin{aligned} & \det(j\omega I - H) \\ = \;& \det\begin{pmatrix} j\omega I - A & -BB^T \\ C^T C & j\omega + A^T \end{pmatrix} \\ = \;& \det(j\omega I - A)\det\left(I - B^T(-j\omega I - A)^{-T}C^T C(j\omega I - A)^{-1}B\right) \\ & \det(j\omega I + A^T) \\ = \;& \det(j\omega I - A)\det\left(I - \Sigma_2(j\omega)\Sigma_1^{-H}(j\omega)\Sigma_1^{-1}(j\omega)\Sigma_2^H(j\omega)\right) \\ & \det(j\omega I + A^T) \\ = \;& (-1)^n\left|a_0(j\omega)\right|^2 \det\left(I - \Sigma_2(j\omega)\Sigma_1^{-H}(j\omega)\Sigma_1^{-1}(j\omega)\Sigma_2^H(j\omega)\right) \\ = \;& (-1)^n\left|a_0(j\omega)\right|^2 \det\left(\Sigma_1^{-H}(j\omega)\right)\det\left(\Sigma_1^H - \Sigma_2^H(j\omega)\Sigma_1^{-1}(j\omega)\Sigma_2(j\omega)\right) \\ = \;& (-1)^n\left|a_0(j\omega)\right|^2\left|a_0^{-1}(j\omega)\right|^2 \det\begin{pmatrix} \Sigma_1(j\omega) & \Sigma_2(j\omega) \\ \Sigma_2^H(j\omega) & \Sigma_1^H(j\omega) \end{pmatrix} \\ = \;& (-1)^n\det(\Sigma(j\omega)). \end{aligned}$$

In this derivation, we have used repeatedly the Schur determinant formula, and relied on the facts that $\Sigma_1(s)$ and $\Sigma_2(s)$ commute, and that

$\det(\Sigma_1(s)) = a_0(s)$. Hence, $\det(j\omega I - H) = 0$ if and only if $\det(\Sigma(j\omega)) = 0$. Let $F(s)$ and $G(s)$ be defined as in Theorem 2.11. Since $a_0(j\omega) \neq 0$, $F(j\omega)$ is invertible. In light of the fact that $M_a(s) = F^{-1}(s)G(s)$, the result follows at once. ∎

Similarly, Theorem 2.12 indicates that the stability of the quasipolynomial (2.6) can be checked by first computing the eigenvalues of the $2qn \times 2qn$ Hamiltonian matrix H, and next the eigenvalues of the constant $q \times q$ matrix $M_a(j\omega_k)$, for each $\omega_k \in \mathsf{R}_+$ such that $j\omega_k$ is a pure imaginary eigenvalue of H. It is clear that the imaginary eigenvalues of H correspond to exactly the real eigenvalues of P. It is also useful to note that the computation of the imaginary eigenvalues of a Hamiltonian matrix is closely related to that of computing the $\mathcal{H}_\infty$ norm of a certain transfer function matrix; in the present setting, the transfer function matrix in question is $\Delta_1^{-1}(s)\Delta_2^T(-s)$. This connection has additional implications, which will be revisited in Chapter 3.

Thus, a notable feature with the stability tests in both Theorem 2.11 and Theorem 2.12 is that they require only the computation of constant matrices. As a result, while retaining the computational ease of the frequency-sweeping tests developed in the preceding section, both tests enjoy superior numerical precision, though the benefit is gained at the expense of increased matrix dimensions. The following example serves to demonstrate the numerical advantage.

Example 2.10 *The following quasipolynomial, with* $n = 4$ *and* $q = 3$*, corresponds to the system in Example 2.9, which is known to be stable at* $\tau = 0$*, but not stable independent of delay:*

$$\begin{aligned} a\left(s,\ e^{-hs}\right) &= \left(s^4 + 2s^3 + 5s^2 + 3s + 2\right) + \left(s^2 + 2\right) e^{-\tau s} \\ &\quad + \left(s^2 + s + 2\right) e^{-2\tau s} + \left(2s^3 + 5s\right) e^{-3\tau s}. \end{aligned}$$

We may compute $\overline{\tau}$ *based on Theorem 2.11 and Theorem 2.12. Construct first the matrices* P *and* H*, both of which are* 24×24 *matrices. Using Theorem 2.11, we found the positive real eigenvalues of* P *to be*

$$\begin{gathered} 0.27440341977960 - 0.00000000000000j, \\ 0.91548178048665 - 0.00000000000000j, \\ 1.30259246725682 + 0.00000000000000j, \end{gathered}$$

and the corresponding unitary roots are

$$\begin{gathered} -0.89898090347872 + 0.43798782537941j, \\ -0.97120708373425 - 0.23823685798887j, \\ 0.88081464844625 - 0.47346124982147j, \end{gathered}$$

respectively. The reader may want to verify that the modulus of all the roots is 1.00000000000000*. From these results, we found* $\overline{\tau} = 0.37864202327266$*.*

The computation results based on Theorem 2.12 turn out to be identical up to the indicated precision (16 digits), and thus the two sets of results demonstrate for one another the numerical accuracy of the tests.

The final part of this section is devoted to systems described by the state-space representation (2.5). At the outset, we note that the stability of such a system can always be analyzed by studying its characteristic quasipolynomial, whereas the latter can be computed routinely using, e.g., *Leverrier-Faddeeva algorithm.* The constant matrix tests developed above are then applicable. Often, however, it may be numerically cumbersome to compute the characteristic quasipolynomial, and hence it may be more advantageous to tackle the stability problem based on the state-space description directly. In what follows we develop one such test, in which the computation of the characteristic quasipolynomial is rendered unnecessary.

We recall that for the delay system of commensurate delays given by the state-space form (2.5), its stability is fully determined by the root locations of the bivariate polynomial

$$a(s,\ z) = \det\left(sI - \sum_{k=0}^{m} A_k z^k\right).$$

Assume that the system is stable at $\tau = 0$. Then by fixing $z \in \partial\mathsf{D}$, the stability problem amounts to determining whether the matrix $\left(\sum_{k=0}^{m} A_k z^k\right)$ has eigenvalues on $\partial\mathsf{C}_+$. Indeed, if it has no imaginary eigenvalue for all $z \in \partial\mathsf{D}$, the system is stable independent of delay. Suppose otherwise that it does have imaginary eigenvalues for some $z_i \in \partial\mathsf{D}$, $z_i = e^{-j\theta_i}$. That is, there exist some $j\omega_i \in \partial\mathsf{C}_+$ such that

$$j\omega_i \in \sigma\left(\sum_{k=0}^{m} A_k z_i^k\right).$$

In this case we may compute the delay margin once finding the pairs $(\omega_i,\ \theta_i)$. Consequently, we are led to the problem of finding all such critical $z \in \partial\mathsf{D}$ that $\sigma\left(\sum_{k=0}^{m} A_k z^k\right)$ has an imaginary eigenvalue, and that of solving the imaginary eigenvalues for all such $z \in \partial\mathsf{D}$. The *Kronecker sum* of matrices discussed in Appendix A comes to our aid in this task.

Recall that for square matrices A and B,

$$\sigma(A \oplus B) = \{\lambda + \mu :\ \lambda \in \sigma(A),\ \mu \in \sigma(B)\}.$$

Hence, whenever $\sigma\left(\sum_{k=0}^{m} A_k z^k\right) \cap \partial\mathsf{C}_+ \neq \emptyset$, it is necessary that

$$\det\left[\left(\sum_{k=0}^{m} A_k z^k\right) \oplus \left(\sum_{k=0}^{m} A_k z^k\right)^H\right] = 0. \tag{2.35}$$

This underlies the main idea in our derivation.

Theorem 2.13 *Suppose that the system (2.5) is stable at* $\tau = 0$*. Define the matrices* $B_k \in \mathsf{R}^{n^2}$, $k = 0,\ 1,\ \cdots,\ 2m$ *by* $B_m := A_0 \oplus A_0^T$*, and*

$$B_{m-k} := I \otimes A_k^T, \qquad B_{m+k} = A_k \otimes I, \qquad k = 1,\ \cdots,\ m.$$

Furthermore, define

$$U := \begin{pmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & B_{2m} \end{pmatrix}, \qquad V := \begin{pmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -B_0 & -B_1 & \cdots & -B_{2m-1} \end{pmatrix}.$$

Then, $\overline{\tau} = \infty$ *if* $\sigma(V,\ U) \cap \partial\mathsf{D} = \emptyset$*. If, however,* $\sigma(V,\ U) \cap \partial\mathsf{D} \neq \emptyset$ *and* $\sigma\left(\sum_{k=0}^{m} A_k z_i^k\right) = \{0\}$ *for all* $z_i \in \sigma(V,\ U) \cap \partial\mathsf{D}$*, then* $\overline{\tau} = \infty$ *as well. In these cases the system (2.5) is stable independent of delay. Otherwise,*

$$\overline{\tau} := \min_{1 \le i \le 2n^2 m} \frac{\theta_i}{\omega_i},$$

where $\theta_i \in [0,\ 2\pi]$*,* $\omega_i \in \mathsf{R}_+$*,* $\omega_i \neq 0$*, and* $e^{-j\theta_i} \in \sigma(V,\ U)$ *satisfy the relation* $j\omega_i \in \sigma\left(\sum_{k=0}^{m} A_k e^{-jk\theta_i}\right)$*. The system (2.5) is stable for all* $\tau \in [0,\ \overline{\tau})$*, but is unstable at* $\tau = \overline{\tau}$*.*

Proof. In view of Lemma 2.4, it is straightfoward to verify that

$$\begin{aligned} \det(zU - V) &= \det\left(\sum_{k=0}^{2m} B_k z^k\right) \\ &= z^{n^2 m} \det\left(\sum_{k=1}^{m} B_{m+k} z^k + B_m + \sum_{k=1}^{m} B_{m-k} z^{-k}\right). \end{aligned}$$

Thus, for any $z \in \partial\mathsf{D}$,

$$\begin{aligned} &\det(zU - V) \\ = \quad & z^{n^2 m} \det\left(\sum_{k=1}^{m} (A_k \otimes I) z^k + A_0 \oplus A_0^T + \sum_{k=1}^{m} (I \otimes A_k^T) z^{-k}\right) \\ = \quad & z^{n^2 m} \det\left(\sum_{k=0}^{m} (A_k z^k \otimes I) + \sum_{k=0}^{m} (I \otimes A_k^T z^{-k})\right) \\ = \quad & z^{n^2 m} \det\left[\left(\sum_{k=0}^{m} A_k z^k\right) \oplus \left(\sum_{k=0}^{m} A_k z^k\right)^H\right]. \end{aligned}$$

Suppose that $\sigma(V,\ U) \cap \partial\mathsf{D} = \emptyset$. Then, for any $z \in \partial\mathsf{D}$,

$$\det\left[\left(\sum_{k=0}^{m} A_k z^k\right) \oplus \left(\sum_{k=0}^{m} A_k z^k\right)^H\right] \neq 0.$$

This implies that for any $z \in \partial\mathsf{D}$ and any $\omega \in \mathsf{R}_+$, $\det\left(j\omega I - \sum\limits_{k=0}^{m} A_k z^k\right) \neq 0$. Thus, for any $\tau \in [0,\ \infty)$ and any $\omega \in \mathsf{R}_+$,

$$\det\left(j\omega I - \sum_{k=0}^{m} A_k e^{-jk\tau\omega}\right) \neq 0. \tag{2.36}$$

Consequently, the system is stable independent of delay, i.e., $\overline{\tau} = \infty$. Suppose next that $\sigma(V,\ U) \cap \partial\mathsf{D} \neq \emptyset$, but $\sigma\left(\sum\limits_{k=0}^{m} A_k z_i^k\right) = \{0\}$ for all $z_i \in \sigma(V,\ U) \cap \partial\mathsf{D}$. This means that (2.36) holds for all $\omega > 0$. By the assumption that the system is stable at $\tau = 0$, the condition must also hold at $\omega = 0$; namely, it holds for all $\omega \in \mathsf{R}_+$. Thus, the system remains stable independent of delay. On the other hand, let $\theta_i \in [0,\ 2\pi]$ such that $z_i = e^{-j\theta_i} \in \sigma(V,\ U)$, and let $\omega_i > 0$ such that $j\omega_i \in \sigma\left(\sum\limits_{k=0}^{m} A_k z_i^k\right)$. It follows that

$$\det\left(j\omega_i I - \sum_{k=0}^{m} A_k e^{-jk\tau\omega_i}\right) = 0$$

at $\tau = \theta_i/\omega_i$. Thus, the system is unstable at $\tau = \overline{\tau}$. In contrast, for any $\tau \in [0,\ \overline{\tau})$, one can show similarly that the system is stable, as in the proof for Theorem 2.11. ∎

As a consequence of Theorem 2.13, we see that the stability of a system given in a state-space description, such as that in (2.5), can also be determined by computing first the generalized eigenvalues z_i of the $2n^2m \times 2n^2m$ matrices V and U, and next the eigenvalues of the $n \times n$ matrix $\sum\limits_{k=0}^{m} A_k z_i^k$, without resorting to the system's quasipolynomial. The computation is to be performed on constant matrices as well, and thus a high computational precision can be insured. The downside of this result lies in the significant increase in the size of the matrices involved in the computation.

Example 2.11 *While mainly of computational significance, the constant matrix tests developed in this section may also be used as analytical tools for simple systems. We demonstrate this point by revisiting Example 2.1:*

$$\dot{x}(t) = -x(t-\tau) - x(t-2\tau), \qquad \tau \geq 0.$$

In order to apply Theorem 2.13, we first obtain

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

which in turn yields $\sigma(V,\ U) = \{-1,\ 0.5 \pm 0.866j\}$. *Clearly,* $\sigma(V,\ U) \subset \partial\mathsf{D}$. *For* $z = -1$, $\sigma\left(\sum_{k=0}^{2} A_k z^k\right) = \{0\}$. *However, for* $z = 0.5 \pm 0.866j = e^{\pm j\pi/3}$, $\sigma\left(\sum_{k=0}^{2} A_k z^k\right) = \{\mp\sqrt{3}\ j\}$. *Hence,* $\overline{\tau} = (\pi/3)/\sqrt{3} = \pi/(3\sqrt{3})$. *This value is identical to that obtained in Example 2.1. Similarly, if we are to invoke Theorem 2.11, we need to construct*

$$P = j\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \end{pmatrix}.$$

This results in $\sigma(P) = \{0,\ \pm 1.732j\}$. *Furthermore, we have*

$$F(s) = \begin{pmatrix} 0 & 1 \\ -s & -1 \end{pmatrix}, \qquad G(s) = I.$$

It follows that $\sigma(F(1.732j),\ I) = \{0.5 - 0.866j,\ \ -1.5 + 0.866j\}$. *This, again, yields* $\overline{\tau} = \pi/(3\sqrt{3})$.

In summary, the development in this section should have demonstrated sufficiently well how classical stability criteria for polynomials of one variable can be applied to lead to efficient stability tests in the study of time-delay systems. Theorem 2.11 and Theorem 2.12 are just two of such applications. Likewise, matrix analysis techniques, which also play an important role in classical stability analysis, can be used to advantage; Theorem 2.13 provides one such example. In light of the wide variety of classical results, one anticipates that more stability tests for time-delay systems can be made available in this manner. Indeed, as an example, one may consider treating the bivariate polynomial $a(s,\ z)$ as a complex polynomial in s with coefficients as functions of z. This makes it possible to analyze the stability of $a(s,\ z)$ as a Hurwitz stability problem, one that can be tackled by using such classical stability results as Hurwitz and Hermite criteria. It is then possible to develop, in parallel to Theorem 2.11, stability tests of similar nature. Nevertheless, as a cautionary note, one should exercise care to ascertain that the criteria in use be applicable to complex polynomials; for example, in checking the Hurwitz stability of a complex polynomial, the Hurwitz test must be replaced by the more general Bilherz criterion. We leave further deliberations of these aspects to the reader.

In closing this chapter, we emphasize that the classical methods and the more recent stability tests are rooted in the same fundamental cause and both seek to compute the zero-crossing frequencies of characteristic functions. Indeed, they only differ in ways of computation. While in the classical methods the key lies in a variable elimination procedure which by nature requires symbolic computation and hence can be difficult, the results developed in the present chapter bypass any such procedure and seek to compute matrix measures whose computation can be systematically implemented. For systems with commensurate delays, this comes down to the computation of eigenvalues and generalized eigenvalues of frequency-dependent or constant matrices, resulting in numerically oriented, more computable delay-independent and delay-dependent stability tests. The main asset of these results lies in their computational efficiency. This can be rather significant for high order systems with many delays, for which the classical methods appear to be ill-equipped.

2.5 Notes

2.5.1 Classical results

Frequency domain stability criteria have been long in existence for time-delay systems. Notable classical examples include such results as the Tsypkin's test and the Pontryagin criterion; a concise elaboration of these and other classical results can be found in the monographs by Górecki *et. al.* [83], Stépán [261], Niculescu [215], and the survey article by Niculescu *et. al.* [224]. The term "stability independent of delay" appears to be formally introduced by Kamen [140], which has since become a term routinely used in the literature, though delay-independent and delay-dependent stability problems had been heavily studied much earlier. Kamen [139, 140, 141] also advocated the use of two-variable criterion. Further development of the two-variable criterion led to a variety of stability tests for systems with commensurate delays, among which we note the polynomial elimination methods by Brierley *et. al.* [28], Chiasson [43], Chiasson *et. al.* [41, 42], Hertz and co-workers [118], the pseudo-delay methods advanced by Rekasius [?], Thowsen [270, 271, 272], Hertz *et. al.* [119], and the iterative method developed in Walton and Marshall [306], Marshall *et. al.* [198]. Other closely related techniques include an optimization algorithm suggested by Lewis and Anderson [179], and graphical criteria by Mori and Kokame [206], Mori *et. al.* [207]. For an examination of Roche's theorem, we recommend Levinson and Redheffer [178]. Two-dimensional systems and stability of 2-D systems are extensively documented in Bose [20].

2.5.2 Frequency-sweeping and constant matrix tests

The idea of using spectral radius and eigenvalues as stability tests were advanced in Chen [35], Chen and Latchman [38], and Chen *et. al.* [37]. Section 2.3 draws essentially upon these references. More specifically, frequency-sweeping conditions for stability independent of delay and those for computing the delay margin are taken from [35, 38], while the constant matrix tests from [37]. Similar results have been subsequently pursued by Su [263], and Niculescu and Ionescu [211], which share the essential spirit of [37]. These efforts together bring the stability problem in the case of commensurate delays to a satisfactory closure. The matrix background for this section is readily found in many texts; we refer to Golub and Van Loan [81], Horn and Johnson [122]. For a comprehensive coverage of classical stability results such as Schur-Cohn criterion and Orlando formula, we recommend Barnett [9] and Marden [196].

3

Systems with Incommensurate Delays

General comment:

1. I have some trouble displaying smgfig01.eps and smgfig02.eps. Please check them and send me another copy.

2. I have some doubts regarding the descriptions in the neutral delay part. The bode face is my suggestion.

3.1 Introduction

Unlike for those with commensurate delays, stability analysis for systems with incommensurate delays proves far more difficult. In particular, most of the results known to date are sufficient conditions, and seldom are they non-conservative. Necessary and sufficient conditions are scarce, and are hardly computable; indeed, many such results merely fall into a restatement–in one form or another–of the definition of stability, thus rendering the conditions of little use. The reader will quickly find out that this lack of strong results is by no means coincidental.

In this chapter we study delay systems with multiple pointwise incommensurate delays. In particular, we focus on the LTI systems described by the state-space equation

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - r_k), \qquad r_k \geq 0, \tag{3.1}$$

where the delays r_k are assumed to be independent of each other. A special case in this class is the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau), \qquad \tau \geq 0, \tag{3.2}$$

that is, the system contains only one delay. We shall first develop small gain type, frequency-sweeping stability conditions for general LTI systems in the form of (3.1). The results employ a matrix measure known as the *structured singular value*, a measure used widely in robust control analyis and design. The structured singular value approach constitutes a step forward in the study of systems with incommensurate delays, in that it provides a necessary and sufficient condition by means of a well-developed theoretical notion as well as a computational tool, for which standard numerical algorithms and commercial software programs are available. It thus enables a systematic analysis of the stability problem from a computational perspective, which can be accomplished at least approximately. On the other hand, the exact computation of the structured singular value is also known to be difficult, and in fact is known to be intractable in general. This raises the speculation that the stability analysis for systems of incommensurate delays may be intrinsically difficult. The conjecture consequently leads to a formal analysis of the computational complexity of the stability problem, drawing upon concepts and techniques found in decision theory. We shall attempt to provide a formal proof that the exact stability check would require solving a so-called NP-hard decision problem. This, in the language of computing theory and operation **operations** research, is essentially equated to problems whose computational complexity increases exponentially with problem size. Problems in this category are commonly believed to be, and hence regarded as computationally intractable, especially when the problem size is large. For this intrinsic difficulty, we then develop a number of sufficient stability conditions. The chapter ends with a brief discussion on neutral delay systems, together with a summary note on frequency domain methods.

3.2 Small gain/μ theorem

We first provide a brief narrative on the classical small gain theorem and its extensions; the proof of this result together with other relevant details is delegated to references given at the end of the chapter. The exposure commences with an input-output description of systems.

3.2.1 Small gain theorem

In studying signals and systems, it has been found useful to identify signals with function spaces. Of particular pertinence here is the class of energy-bounded signals, which, in mathematical terms, are defined as square integrable functions. More specifically, n-dimensional energy-bounded signals form the function space

$$L_2 := \{ f : \ \mathsf{R} \to \mathsf{R}^n \mid f(t) \text{ is measurable}, \ \|f\|_{L_2} < \infty \},$$

where

$$\|f\|_{L_2} := \left(\int_{-\infty}^{\infty} \|f(t)\|_2^2 dt \right)^{1/2},$$

and $\|\cdot\|_2$ is the Hölder ℓ_2 norm,

$$\|f(t)\|_2 := \left(\sum_{i=1}^{n} |f_i(t)|^2\right)^{1/2}.$$

Define the Fourier transform of $f(t)$ by

$$\hat{f}(j\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt,$$

and the space of Lebesgue integrable functions

$$\mathcal{L}_2 := \left\{\hat{f} : \mathsf{R} \to \mathsf{C}^n \mid \hat{f}(j\omega) \text{ is measurable, } \|\hat{f}\|_{\mathcal{L}_2} < \infty\right\},$$

where

$$\|\hat{f}\|_{\mathcal{L}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{f}(j\omega)\|_2^2 d\omega\right)^{1/2}.$$

The well-known Parseval identity states that

$$\|\hat{f}\|_{\mathcal{L}_2} = \|f\|_{L_2}. \tag{3.3}$$

In turn, the function spaces L_2 and $\mathcal{L}_2$ form an isometrical isomorphism. The Fourier transform thus provides an equivalent characterization of a signal via its frequency response **domain representation.**

By so identifying input-output **the input and output** signals with **elements of** function spaces, a system can be viewed as an operator mapping the input space to the output space, as shown in Figure 3.1. In particular, a linear system can be viewed as a linear operator. Let $H : L_2 \to L_2$

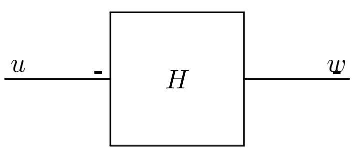

FIGURE 3.1. Input-output system description

be a linear system. Then its induced norm can be defined as

$$\|H\|_{2,2} := \sup_{u \neq 0} \frac{\|w\|_{L_2}}{\|u\|_{L_2}},$$

where $u \in L_2$ denotes the system's input and $w \in L_2$ the output. Such a **When this induced norm is well defined and finite, the** system is said to be BIBO (*bounded-input bounded-output*) L_2-stable, whose induced norm $\|H\|_{2,2}$ is finite. For an LTI system, the induced norm can be quantified by computing the system's frequency response. Let $\hat{H}(s)$ be the transfer function matrix of an LTI BIBO L_2-stable system H. In light of the Parseval identity (3.3), the induced norm $\|H\|_{2,2}$ can be alternativly **alternatively** characterized as

$$\|H\|_{2,2} = \sup_{\hat{u}\neq 0} \frac{\|\hat{H}\hat{u}\|_{\mathcal{L}_2}}{\|\hat{u}\|_{\mathcal{L}_2}} = \sup_{\omega \in \mathsf{R}} \overline{\sigma}(\hat{H}(j\omega)).$$

In other words, the induced norm coincides with the peak magnitude of the system's frequency response, whereas the magnitude is measured by the largest singular value and has the interpretation as the system's *gain*. It follows that an LTI system is BIBO L_2-stable if and only if its frequency response is essentially bounded on $\partial\mathsf{C}_+$. If in addition an LTI BIBO L_2-stable system is also causal, then its transfer function matrix $\hat{H}(s)$ will be analytic in C_+ and essentially bounded on $\partial\mathsf{C}_+$. It is known that the collection of all such functions forms another function space, known as the Hardy space of $\mathcal{H}_\infty$ functions, endowed with the $\mathcal{H}_\infty$ norm and defined as

$$\mathcal{H}_\infty := \left\{ \hat{H} \middle|\ \hat{H}(s) \text{ analytic in } \mathsf{C}_+,\ \|\hat{H}\|_\infty := \sup_{s\in\mathsf{C}_+} \overline{\sigma}(\hat{H}(s)) < \infty \right\}$$

Thus, LTI causal BIBO L_2-stable systems correspond to functions in $\mathcal{H}_\infty$, and the maximum gain of an LTI causal BIBO L_2-stable system is equal to the $\mathcal{H}_\infty$ norm of its transfer function matrix. This consequently gives an operator theoretic interpretation of the familiar notion of gain defined in the frequency domain. One should note that in the present book only causal systems are considered, and thus for an LTI causal system, the notion of stability is identified with that of BIBO L_2-stability; that is, a causal LTI system is BIBO L_2-stable if and only if its transfer function has no pole in C_+.

Functions in the space of $\mathcal{H}_\infty$ **space** constitute a well-studied mathematical subject of deep and rich results. The linkage between system analysis and $\mathcal{H}_\infty$ theory has an immediate yet profound impact: it furnishes a well-developed mathematical theory, a powerful tool to the study of causal, BIBO L_2-stable systems. The stage is thus set to usher in the small gain theorem. Consider the linear feedback system given in Figure 3.2. The configuration, often referred to as M-Δ loop, is the standard feedback interconnection of two systems M and Δ.

Lemma 3.1 (Small Gain Theorem) *Suppose that M and Δ are both causal, linear, and BIBO L_2-stable. Then the M-Δ loop is BIBO L_2-stable if*

$$\|M\Delta\|_{2,2} < 1. \tag{3.4}$$

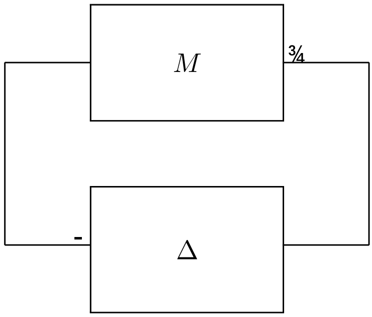

FIGURE 3.2. Feedback interconnection

Hence, the small gain theorem provides a sufficient condition that governs the stability of feedback interconnections. One should note that in its full generality, and modulo to aprropriate **modulo appropriate** modifications, the small gain theorem in fact is applicable in a much broader sense. Nevertheless, the statement given in Lemma 3.1 is sufficiently general for our present purpose. In particular, with a more far-reaching implication, it allows the study of time-varying systems and can therefore be applied to systems containing time-varying delays; we will illustrate this point shortly via examples.

The small gain theorem has played an especially prominent role in the robust control of uncertain LTI systems. In robust stability analysis, one typically models a system's uncertainty by deterministic, unknown but bounded perturbations to its parameters or transfer function matrix. A specific uncertainty model, known as *unstructured uncertainty*, describes unknown perturbations as stable transfer function matrices with a prescribed $\mathcal{H}_\infty$ norm bound, by means of the set description

$$\overline{\mathcal{B}}\mathcal{H}_\infty := \left\{ \hat{\Delta} \in \mathcal{H}_\infty : \ \|\hat{\Delta}\|_\infty \leq 1 \right\}.$$

In other words, for an unstructured uncertainty, an element of $\overline{\mathcal{B}}\mathcal{H}_\infty$, no other information is known than **the fact** that its transfer function matrix is bounded in the $\mathcal{H}_\infty$ norm; without loss of generality, the $\mathcal{H}_\infty$ norm bound can be taken as one. While simplistic, this uncertainty description nevertheless enables us to strengthen the small gain theorem to a necessary and sufficient condition for the robust stability of the M-Δ loop, when Δ is assumed to be an unstructured uncertainty.

Lemma 3.2 (Small Gain Theorem-Robust Stability) *Let $\hat{M} \in \mathcal{H}_\infty$. Then the M-Δ loop is BIBO L_2-stable for all $\hat{\Delta} \in \overline{\mathcal{B}}\mathcal{H}_\infty$ if and only if*

$$\|\hat{M}\|_\infty < 1. \tag{3.5}$$

In particular, if $\hat{M}(j\omega) \geq 1$ for some ω, then it is possible to construct a "destabilizing" uncertainty $\hat{\Delta}$ **{Δ was replaced by $\hat{\Delta}$}** such that $\hat{\Delta} \in$

FIGURE 3.3. Realization of systems with one delay

$\overline{\mathcal{B}}\mathcal{H}_\infty$, and

$$\det\left(I - \hat{M}(j\omega)\hat{\Delta}(j\omega)\right) = 0,$$

so that the $M - \Delta$ loop becomes unstable.

Example 3.1 *Consider the linear delay system*

$$\dot{x}(t) = A_0\ x(t) + A_1\ x(t - \tau(t)),$$

where the delay $\tau :\ \mathsf{R} \to \mathsf{R}_+$ *is time-varying. This system can be realized as in Figure 3.3, in which* $\Delta :\ L_2 \to L_2$ *is a linear operator such that* $\Delta x(t) = x(t - \tau(t))$.

More generally, the idea of representing a delay by a fictitious uncertainty extends to systems with multiple delays. Consdier the system

$$\dot{x}(t) = A_0\ x(t) + \sum_{k=1}^{m} A_k\ x\left(t - r_k(t)\right).$$

Likewise, the delays $r_k :\ \mathsf{R} \to \mathsf{R}_+$ *may be time-varying. The system can be realized by Figure 3.4, and in turn by the* M-Δ *loop.*

We will mainly discuss the case $\tau(t)$ =constant in this Chapter. In this case, it can be easily seen that $||\Delta||_{2,2} = 1$. The time varying case with $\dot{\tau}(t) \leq \rho < 1$ will be discussed in Chapter 8.

Example 3.2 *Tsypkin's test is concerned with closed loop stability of the system depicted in Figure 3.5, where* $P(s)$ *is a stable, scalar transfer function, and* $\tau \geq 0$ *is a delay constant. Suppose that* $|P(0)| < 1$. *Then Tsypkin's test states that the system will be stable independent of delay if and only if*

FIGURE 3.4. Realization of systems with multiple delays

$|P(j\omega)| < 1$, for all $\omega > 0$. This condition, however, is equivalent to that $\|P\|_\infty < 1$, which, in light of Lemma 3.2, is necessary and sufficient for the M-Δ loop to be stable, with $\hat{M}(s) = P(s)$ and $\hat{\Delta} \in \overline{\mathcal{B}}\mathcal{H}_\infty$. The implication then is that under the assumption $|P(0)| < 1$, the system can be represented equivalently by the M-Δ loop and its stability be given by the small gain condition (3.5), despite that the delay may not be fully equated to an uncertainty. Indeed, though we may represent the delay by an LTI uncertainty $\hat{\Delta}(s) = e^{-\tau s} \in \overline{\mathcal{B}}\mathcal{H}_\infty$, $\hat{\Delta}$ cannot be completely uncertain as required, for $\hat{\Delta}(0) = 1$. Otherwise, at any other ω, $\hat{\Delta}(j\omega)$ can fully replicate an unstructured uncertainty, since τ can be chosen arbitrarily. In summary, the small

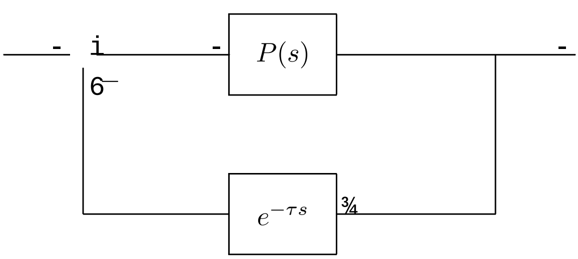

FIGURE 3.5. Closed loop delay system

gain condition (3.5) will remain necessary and sufficient when the additional assumption $|P(0)| < 1$ *is imposed at* $\omega = 0$*, but the necessity of the condition will in general vanish otherwise. This suggests that in modeling delay operators by uncertainties, caution must be paid to the zero frequency* $\omega = 0$.

The examples given above underscore the role that the small gain type results may play in stability analysis of time-delay systems. They demonstrate that the small gain theorem can be applied effectively to yield a sufficient condition for delay-independent stability; this is true even for systems with time-varying delays. In particular, Example 3.2 shows that in certain situations it is possible to obtain a necessary and sufficient condition, specifically when the delay can be modeled as an unstructured uncertainty modulo to **modulo** a side constraint at $s = 0$, as in the case of the Tsypkin criterion. This, however, ceases to be true when the system contains multiple delays. **Indeed, in the multiple delay case of example** Example 3.1 clearly exhibits that in the latter case one unstructured uncertainty can no longer represent fully several delays **without undue conservatism{I think it was not "clearly shown", rather, here is we only "point out"}**. It thus calls for a more refined description that can incorporate the structural information of the uncertainty which results when several such delays are lumped together. For this purpose, the so-called *structured uncertainty* provides a remedy.

3.2.2 Structured singular value

In general, structured uncertainty is a broad description and may refer to any uncertainty which is not unstructured, and which is meant to provide such additional, more specific information that a crude description like unstructured uncertainty fails to capture. Thus, let $\mathbf{\Delta}$ be a strict subset in the vector space $\mathbf{C}^{l\times l}$. Any $\Delta \in \mathbf{\Delta}$ may then be considered a structured uncertainty of a **{delete "a"}** dimension l. The *structured singular value*, or μ, provides a matrix measure contingent on the structure defined by $\mathbf{\Delta}$.

Definition 3.1 *The structured singular value* $\mu_{\mathbf{\Delta}}(M)$ *of a matrix* $M \in \mathbf{C}^{l\times l}$ *with respect to the block structure* $\mathbf{\Delta}$ *is defined to be zero if there is no* $\Delta \in \mathbf{\Delta}$ *such that* $\det(I - M\Delta) = 0$*, and otherwise*

$$\mu_{\mathbf{\Delta}}(M) = \left(\min_{\Delta \in \mathbf{\Delta}} \{ \overline{\sigma}(\Delta) | \ \det(I - M\Delta) = 0 \} \right)^{-1}.$$

When applied to the $M - \Delta$ loop of LTI systems, the structured singular value defines a precise frequency-dependent robust stability measure with respect to uncertainties of the structure $\mathbf{\Delta}$, for any structure $\mathbf{\Delta}$. The exact robust stability condition is furnished by an extended form of the small gain theorem, sometimes dubbed as "small μ theorem".

Lemma 3.3 (Small μ Theorem) *Let $\hat{M} \in \mathcal{H}_\infty$ and define*

$$\mathbf{\Delta}_\infty := \left\{ \hat{\Delta} \in \mathcal{H}_\infty | \ \|\hat{\Delta}\|_\infty \leq 1, \ \hat{\Delta}(j\omega) \in \mathbf{\Delta}, \ \forall \omega \in \mathsf{R} \right\}.$$

Then the M-Δ loop is BIBO L_2-stable for all $\hat{\Delta} \in \mathbf{\Delta}_\infty$ if and only if

$$\|\hat{M}\|_\mu < 1, \tag{3.6}$$

where

$$\|\hat{M}\|_\mu := \sup_{\omega \in \mathsf{R}} \mu_{\mathbf{\Delta}} \left(\hat{M}(j\omega) \right).$$

A canonical uncertainty structure is that of block diagonal matrices; it is known that an uncertainty of any arbitrary structure can be transformed readily into a block diagonal one by simple, algebraic manipulation. For a block diagonal uncertainty, every $\Delta \in \mathbf{\Delta}$ may contain nonzero entries only on its main diagonal, while all off-diagonal blocks are zero. Example 3.1 shows that an even more specified diagonal structure is particularly pertinent in dealing with systems of multiple delays. More specifically, the block diagonal uncertainties possess the form of

$$\mathsf{X}_m := \left\{ \mathrm{diag}\left(\delta_1 I_{k_1}, \ \cdots, \ \delta_m I_{k_m} \right) | \ \delta_k \in \mathsf{C} \right\}, \qquad \sum_{i=1}^{m} k_i = l. \tag{3.7}$$

Uncertainties of this structure, known as "repeated complex scalars" in the robust control literature, will be considered in the remainder of this chapter. One should note, nevertheless, that it may be made more general to include parts or all of the so-called ''full complex'', ''repeated real scalar'', ''full real'' blocks of uncertainty. **One should note that other than the uncertainty type used to model time delays mentioned here, there are other types of uncertainty structures such as full complex", "repeated real scalar", "full real" blocks of uncertainty. {other types of uncertainty are not relevant to model time delay, and should be made clear}.** Such uncertainty components can be augmented in X_m whenever needed.

It follows from Lemma 3.3 that in the presence of structured uncertainties, the stability of the M-Δ loop can be ascertained by computing pointwise **in frequency** $\omega \in (0, \infty)$ the structured singular value $\mu_{\mathbf{\Delta}} \left(\hat{M}(j\omega) \right)$. Unfortunately, this computation is known to be difficult; in general, the computational problem has been found to be as difficult as one of NP-hard decision problems, and hence is unlikely to be tractable computationally. The standard practice in robust control then is to resort to approximate **approximating {here "to" is a preposition}** computation via upper and lower bounds. Several such bounds are available, which are summarized below together with other relevant properties. We note that most of these properties are rather handy, and are thus provided without proof.

Lemma 3.4 *The following properties hold for* $\mu_{\mathbf{\Delta}}(M)$.

(i) *If* $\mathbf{\Delta} = \{\delta I |\ \delta \in \mathsf{C}\}$, *then* $\mu_{\mathbf{\Delta}}(M) = \rho(M)$.

(ii) *If* $\mathbf{\Delta} = \mathsf{C}^{l\times l}$, *then* $\mu_{\mathbf{\Delta}}(M) = \overline{\sigma}(M)$.

(iii) *For any block diagonal* $\mathbf{\Delta}$, $\rho(M) \leq \mu_{\mathbf{\Delta}}(M) \leq \overline{\sigma}(M)$.

(iv) *Let* $\hat{M} \in \mathcal{H}_\infty$. *Then for any block diagonal* $\mathbf{\Delta}$, $\mu_{\mathbf{\Delta}}\left(\hat{M}(s)\right)$ *is a subharmonic function in* C_+ *and its maximum is achieved on* $\partial\mathsf{C}_+$.

(v) *For any block diagonal* $\mathbf{\Delta}$, *define*

$$\begin{aligned} \overline{\mathcal{B}}\mathbf{\Delta} : &= \{\Delta \in \mathbf{\Delta} |\ \overline{\sigma}(\Delta) \leq 1\}, \\ \mathcal{U} : &= \{U \in \mathbf{\Delta} |\ \ UU^* = I\}. \end{aligned}$$

In particular, denote $\mathcal{U}$ *by* $\mathcal{U}_m$ *if* $\mathbf{\Delta} = \mathsf{X}_m$, *i.e.*,

$$\mathcal{U}_m : = \{U \in \mathsf{X}_m |\ \ UU^* = I\}.$$

Then, $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(UM) = \mu_{\mathbf{\Delta}}(MU)$. *Furthermore,*

$$\mu_{\mathbf{\Delta}}(M) = \max_{\Delta \in \overline{\mathcal{B}}\mathbf{\Delta}} \rho(M\Delta) = \max_{U \in \mathcal{U}} \rho(MU). \tag{3.8}$$

(vi) *Let* $\mathbf{\Delta} = \mathsf{X}_m$. *Define*

$$\mathcal{D}_m : = \left\{ diag\,(D_1,\ \cdots,\ D_m) |\ D_i \in \mathsf{C}^{k_i \times k_i},\ D_i = D_i^H\ \right\}.$$

Then, $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1})$ *for any* $D \in \mathcal{D}_m$. *In addition,*

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathcal{D}_m} \overline{\sigma}(DMD^{-1}). \tag{3.9}$$

(vii) *Let* $\mathbf{\Delta} = \mathsf{X}_m$. *Then* $\inf_{D\in\mathcal{D}_m} \overline{\sigma}(DMD^{-1}) \leq 1$ *if and only if there exists some* $D \in \mathcal{D}_m$ *such that*

$$D - M^H DM \geq 0. \tag{3.10}$$

(viii) *Let* $\mathbf{\Delta} = \mathsf{X}_m$. *Suppose that* $M = ba^H$ *with* $a, b \in \mathsf{C}^l$, *and partition* a, b *compatibly with* X_m. *Then,*

$$\mu_{\mathbf{\Delta}}(M) = \sum_{i=1}^{m} \left| a_i^H b_i \right|. \tag{3.11}$$

Hence, $\mu_{\mathbf{\Delta}}(M)$ can be approximately computed by solving the minimization problem in (3.9), or the maximization problem in (3.8). The property (vii) indicates that the former is equivalent to an LMI problem, which has a unique minimum and can be solved systematically using available software programs. The latter, while having multiple local maxima, can be approached by means of a power algorithm. The property (iv) reveals that $\mu_{\mathbf{\Delta}}\left(\hat{M}(s)\right)$ still possesses some desirable analytic properties which aid in checking robust stability. For example, it is subharmonic and hence a continuous function, wherever $\hat{M}(s)$ is analytic. In particular, since every subharmonic function satisfies the maximum modulus principle, $\mu_{\mathbf{\Delta}}\left(\hat{M}(s)\right)$ achieves its maximum on $\partial\mathsf{C}_+$, i.e., the boundary of C_+, whenever $\hat{M}(s)$ is stable. Finally, the properties (i), (ii), and (viii) show that in some special cases closed-form expressions can be found for $\mu_{\mathbf{\Delta}}(M)$.

3.3 Frequency-sweeping conditions

Several leads now point to the possibility that the structured singular value may be used as a measure for delay-independent stability of systems with multiple incommensurate delays. Indeed, Theorem 2.1 indicates that in the more specialized case of a single delay, the structured singular value offers a necessary and sufficient condition, for as demonstrated in Example 3.2, a single delay can be treated as a repeated complex scalar uncertainty, for which μ reduces to the spectral radius. Example 3.1 also shows that it is possible to model multiple delays by repeated complex scalar uncertainties, collected in a structured uncertainty of the block diagonal form described by X_m. That the small μ theorem may serve as a necessary and sufficient condition in general then becomes intuitively clear, though the condition must be addressed separately at $\omega = 0$, as we have been so alerted by Example 3.2. In fact, based on Example 3.1 and Lemma 3.3, the reader may already come to recognize the following result.

Theorem 3.5 *Let $r_k \geq 0$, $k = 1,\ 2,\ \cdots,\ m$ be independent, incommensurate delays. Then the system (3.1) is stable independent of delay if and only if*

(i) A_0 is stable,

(ii) $\sum\limits_{k=0}^{m} A_k$ is stable, and

(iii)

$$\mu_{\mathsf{X}_m}\left(M(j\omega)\right) < 1, \qquad \forall\omega > 0, \tag{3.12}$$

where

$$M(s) := \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} (sI - A_0)^{-1} \begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}. \tag{3.13}$$

Proof. The conditions (i) and (ii) are required to insure that the system be stable for $r_k = \infty$ and $r_k = 0$, for all $k = 1, \cdots, m$, and hence are necessary. Under the condition (i), the system is stable independent of delay if and only if for all $s \in \overline{\mathsf{C}}_+$

$$\det \left(I - (sI - A_0)^{-1} \sum_{k=1}^{m} A_k e^{-r_k s} \right) \neq 0, \quad \forall r_k \geq 0. \tag{3.14}$$

Let

$$\Delta(s) := \operatorname{diag}\left(e^{-r_1 s} I, \ \cdots, \ e^{-r_m s} I\right).$$

It follows that (3.14) is satisfied for all $s \in \overline{\mathsf{C}}_+$ if and only if

$$\det\left(I - M(s)\Delta(s)\right) \neq 0, \qquad \forall s \in \overline{\mathsf{C}}_+, \ \ \forall r_k \geq 0.$$

Since $M(s)$ and $\Delta(s)$ are analytic in $\overline{\mathsf{C}}_+$, the spectral radius $\rho\left(M(s)\Delta(s)\right)$, either in its own right or as a special case of $\mu_{\mathsf{X}_m}\left(M(s)\Delta(s)\right)$, defines a subharmonic function in $\overline{\mathsf{C}}_+$. As a result, it achieves its maximum on $\partial \mathsf{C}_+$. That is,

$$\sup_{s \in \overline{\mathsf{C}}_+} \rho\left(M(s)\Delta(s)\right) = \sup_{\omega \in \mathsf{R}} \rho\left(M(j\omega)\Delta(j\omega)\right).$$

Since r_k are arbitrary, independent real constants, we may replace $\Delta(j\omega)$ by an arbitrary constant matrix $U \in \mathcal{U}_m$, for all $\omega > 0$. Consequently,

$$\begin{aligned} \sup_{r_k \geq 0} \sup_{\omega > 0} \rho\left(M(j\omega)\Delta(j\omega)\right) &= \sup_{\omega > 0} \sup_{U \in \mathcal{U}_m} \rho\left(M(j\omega)U\right) \\ &= \sup_{\omega > 0} \mu_{\mathsf{X}_m}\left(M(j\omega)\right). \end{aligned}$$

Therefore, whenever (3.12) is true, the condition (3.14) holds for all $s \in \overline{\mathsf{C}}_+$ except at $s = 0$. However, under the condition (ii), it is necessary that

$$\det\left(\sum_{k=0}^{m} A_k\right) \neq 0.$$

This implies that (3.14) also holds at $s = 0$, and hence it does for all $s \in \overline{\mathsf{C}}_+$. In other words, the system is stable independent of delay. The proof for the sufficiency part is completed. To show the necessity, assume first that $\mu_{\mathsf{X}_m}\left(M(j\omega_0)\right) = 1$ for some $\omega_0 > 0$. This means that a matrix $U \in \mathcal{U}_m$ exists such that

$$U = \operatorname{diag}\left(e^{-j\theta_1} I, \ \cdots, \ e^{-j\theta_m} I\right), \quad \theta_k \in [0, \ 2\pi], \ \ k = 1, \ \cdots, \ m,$$

and that $\det(I - M(j\omega_0)U) = 0$. A set of delay constants can be constructed as $r_k = \theta_k/\omega_0$, so that the condition (3.14) is violated at $s = j\omega_0$. Thus, the system cannot be stable independent of delay. Additionally, if $\mu_{\mathsf{X}_m}(M(j\omega)) > 1$ for some $\omega > 0$, then by the continuity of $\mu_{\mathsf{X}_m}(M(j\omega))$, and in addition the fact that

$$\lim_{\omega\to\infty} \mu_{\mathsf{X}_m}(M(j\omega)) = 0,$$

there must exist some $\omega_0 \in (\omega, \infty)$ such that $\mu_{\mathsf{X}_m}(M(j\omega_0)) = 1$. As such, we may also conclude that the system is not stable independent of delay. The proof is now completed. ■

Just as the structured singular value serves as a nonstandard matrix measure generalizing the common notion of singular values, when applied to a transfer function matrix, it extends the concept of gain defined on the basis of singular values. Theorem 3.5 then provides a small gain type condition for delay-independent stability. There is, however, a subtle difference between the condition (3.12) and the small μ condition, which concerns the frequency $\omega = 0$. We have repeatedly warned against this difference. For a further clarification, we state the following alternative to Theorem 3.5. The proof is similar and hence omitted.

Theorem 3.6 *Let $r_k \geq 0$, $k = 1, 2, \cdots, m$ be independent, incommensurate delays. Then the system (3.1) is stable independent of delay if and only if*

(i) A_0 is stable;
(ii) $\mu_{\mathsf{X}_m}(M(0)) < 1$, or $\mu_{\mathsf{X}_m}(M(0)) = 1$ but $\det(I - M(0)) \neq 0$;
(iii) $\mu_{\mathsf{X}_m}(M(j\omega)) < 1$, $\forall \omega > 0$.

Consequently, while the small μ theorem requires that $\mu_{\mathsf{X}_m}(M(0)) < 1$, it is possible that $\mu_{\mathsf{X}_m}(M(0)) \leq 1$ while the system may still be stable independent of delay.

It is then clear that in general the condition for delay-independent stability does not match exactly that for robust stability; the latter is stronger and in fact provides only a sufficient condition for stability independent of delay. The reason, evidently, points to the fact that the delays cannot be taken exactly as uncertainties, a phenomenon we have already observed in Example 3.2. This difference helps explain a discrepancy between any necessary and sufficient condition and the classical two-variable criterion, which was perceived for some time as a necessary and sufficient condition for stability independent of delay. To further illustrate, consider the case of one delay. The following corollary is a counterpart to Theorem 2.1 and follows immediately from Theorem 3.6.

Corollary 3.7 *The system (3.2) is stable independent of delay if and only if*

(i) A_0 *is stable;*
(ii) $\rho\left(A_0^{-1}A_1\right) < 1$, *or* $\rho\left(A_0^{-1}A_1\right) = 1$ *but* $\det\left(A_0 + A_1\right) \neq 0$*;*
(iii) $\rho\left((j\omega I - A_0)^{-1}A_1\right) < 1$, $\forall\omega > 0$

Suppose that A_0 is stable. A sufficient condition for stability independent of delay is then given by

$$\rho\left((j\omega I - A_0)^{-1}A_1\right) < 1, \qquad \forall\omega \geq 0. \tag{3.15}$$

This, however, coincides with the condition

$$\mu_{\mathbf{\Delta}}\left((j\omega I - A_0)^{-1}A_1\right) < 1, \qquad \forall\omega \geq 0,$$

for $\mathbf{\Delta} = \{\delta I|\ \delta \in \mathsf{C}\}$. It thus follows that the condition (3.15) is equivalent to that

$$\det\left(I - (sI - A_0)^{-1}A_1 z\right) \neq 0, \qquad \forall s \in \overline{\mathsf{C}}_+, \quad \forall z \in \overline{\mathsf{D}},$$

which is equivalent to the standard two-variable criterion

$$\det\left(sI - A_0 - A_1 z\right) \neq 0, \qquad \forall s \in \overline{\mathsf{C}}_+, \quad \forall z \in \overline{\mathsf{D}}, \tag{3.16}$$

In other words, the two-variable criterion, which coincides with a small μ condition, is only a sufficient condition. One notes that for such a condition to be also necessary, there must exist an "uncertainty" $\Delta(s) \in \mathbf{\Delta}_\infty$ with which the matrix $I - M(s)\Delta(s)$ loses rank. Since $\tau \geq 0$ can be selected arbitrarily, for $s \neq 0$, such an uncertainty can be constructed as $\Delta(s) = e^{-\tau s}I$, so that the matrix $I - (sI - A_0)^{-1}A_1\Delta(s)$ loses rank. However, the construction fails at $s = 0$, for $e^{-\tau s}I = I$ for any τ. This is precisely where the two-variable criterion (3.16), or more generally the small μ condition, loses the necessity; the *fictitious* uncertainty we use to model the delay is fixed, rather than uncertain, at $s = 0$.

We emphasize, in light of the difficulty in computing the structured singular value, that Theorem 3.5 and Theorem 3.6 only provide *in effect* sufficient or necessary conditions for stability independent of delay, for in general only upper and lower bounds of the structured singular value may be efficiently computed. Nevertheless, while computationally less encouraging, the results do provide an exact characterization that can be analyzed systematically and computed approximately. As a direct benefit, it will enable the derivation of many sufficient or necessary conditions; a series of such sufficient conditions will be presented later in this chapter. Moreover, for certain special cases where the structured singular value admits closed-form expressions, the characterization leads to exact, closed-form stability criteria. The case of a single delay is one such example. Another case of interest arises when the transfer function matrix $M(s)$ is rank-one. The scalar delay system described by the differential-difference equation

$$y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{k=0}^{m} a_{ki}y^{(i)}(t - r_k) = 0, \qquad r_k \geq 0 \tag{3.17}$$

belongs to this category. Define the polynomials

$$a_0(s) = s^n + \sum_{i=0}^{n-1} a_{0i} s^i, \tag{3.18}$$

$$a_k(s) = \sum_{i=0}^{n-1} a_{ki} s^i, \qquad k = 1,\ 2,\ \cdots,\ m. \tag{3.19}$$

Corollary 3.8 *Let $r_k \geq 0$, $k = 1,\ 2,\ \cdots,\ m$ be independent, incommensurate delays. Then the following statements are true.*
(1) Suppose that for some $u \in \mathsf{R}^n$ and $v = \begin{bmatrix} v_1^T & \cdots & v_m^T \end{bmatrix}^T \in \mathsf{R}^l$,

$$\begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix} = uv^T.$$

Then the system (3.1) is stable independent of delay if and only if

(i) A_0 is stable;

(ii) $\sum_{k=0}^{m} A_k$ is stable;

(iii) $\sum_{k=1}^{m} \left| v_k^T (j\omega I - A_0)^{-1} u \right| < 1$, $\forall \omega > 0$.

(2) Let $a_k(s)$, $k = 0,\ 1,\ \cdots,\ m$, be defined in (3.18) and (3.19). Then the system (3.17) is stable independent of delay if and only if

(i) $a_0(s)$ is stable;

(ii) $\sum_{k=0}^{m} a_k(s)$ is stable;

(iii)

$$\frac{\sum_{k=1}^{m} |a_k(j\omega)|}{|a_0(j\omega)|} < 1, \qquad \forall \omega > 0.$$

(3) The first-order scalar delay system

$$\dot{y}(t) + a\ y(t) + \sum_{k=1}^{q} a_k\ y(t - r_k) = 0, \qquad r_k \geq 0.$$

is stable independent of delay if and only if

(i) $a > 0$;

(ii) $a - \sum_{k=1}^{q} |a_k| > 0$, or $a - \sum_{k=1}^{q} |a_k| = 0$ but $a + \sum_{k=1}^{q} a_k \neq 0$.

Proof. The statement (1) follows from Lemma 3.4 (viii), which leads to

$$\mu_{\mathsf{X}_m}(M(j\omega)) = \sum_{k=1}^{m} \left| v_k^T (j\omega I - A_0)^{-1} u \right|.$$

To prove the statement (2), note that it constitutes a special case of (1), with

$$A_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{00} & -a_{01} & \cdots & -a_{0,n-1} \end{bmatrix}, \quad v_k = \begin{bmatrix} -a_{k0} \\ -a_{k1} \\ \vdots \\ -a_{k,n-1} \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

This gives rise to

$$\mu_{\mathsf{X}_q}(M(j\omega)) = \frac{\sum_{k=1}^{q} |a_k(j\omega)|}{|a_0(j\omega)|}.$$

Finally, the statement (3) is easily recognized as a special case of (2), with

$$\mu_{\mathsf{X}_q}(M(j\omega)) = \frac{\sum_{k=1}^{q} |a_k|}{\sqrt{\omega^2 + a^2}}.$$

The proof is then completed. ■

We conclude by presenting a small gain type sufficient condition for delay-dependent stability, using also the structured singular value. Before proceeding, we note that in principle a delay-dependent stability condition can be obtained in the process of computing $\mu_{\mathsf{X}_m}(M(j\omega))$, in verifying (3.12). Alternatively, it may also be obtained by solving a maximization problem of the form described by (3.8). In either case, the solution, were it attainable, would yield certain critical, "destabilizing" $\Delta \in \mathcal{U}_m$, at the frequencies ω where $\mu_{\mathsf{X}_m}(M(j\omega)) = 1$. The critical delay values may then be estimated from Δ and ω. Evidently, while the estimates would provide an exact range of delay values for which the system is stable, the computation of such critical Δ is, unfortunately, as difficult as that of $\mu_{\mathsf{X}_m}(M(j\omega))$, if not more, and hence from a computational standpoint is unlikely to be tractable in general. The condition to be developed below inherits the same complexity in computing μ, but is more amenable to analysis.

Consider the system (3.1) with incommensurate delays r_k. Noting that

$$x(t) - x(t-r_i) = \int_{t-r_i}^{t} \dot{x}(u)du = \int_{t-r_i}^{t} \left(\sum_{k=0}^{m} A_k x(u-r_k) \right) du,$$

we may rewrite the state-space equation as

$$\begin{aligned} \dot{x}(t) &= \left(\sum_{i=0}^{m} A_i \right) x(t) + \sum_{i=1}^{m} A_i \left(x(t-r_i) - x(t) \right) \\ &= \left(\sum_{i=0}^{m} A_i \right) x(t) - \sum_{i=1}^{m} A_i \int_{t-r_i}^{t} \left(\sum_{k=0}^{m} A_k x(u-r_k) \right) du. \end{aligned}$$

This gives an alternative representation of the system (3.1), known as a model tranform. **{the above are modified as follows} Consider the system (3.1) with incommensurate delays $\mathbf{r}_k$. Using**

$$x(t-r_i) = x(t) - \int_{t-r_i}^{t} \dot{x}(u)du = x(t) - \int_{t-r_i}^{t} \left(\sum_{k=0}^{m} A_k x(u-r_k) \right) du,$$

we may rewrite the state-space equation (3.1) as

$$\dot{x}(t) = \left(\sum_{i=0}^{m} A_i \right) x(t) - \sum_{i=1}^{m} A_i \int_{t-r_i}^{t} \left(\sum_{k=0}^{m} A_k x(u-r_k) \right) du. \tag{3.20}$$

The process of transforming the original equation (3.1) to the new representation (??) is known as model transformation**, and is widely used in both frequency domain and time domain stability analysis of time delay systems. It should be pointed out that although the stability of (??) implies that of (3.1), the reverse is not necessarily true due to presence of** additional dynamics. **This subtle point will be illustrated in Chapter 5.** The Laplace transform of the integral term is given by

$$\mathcal{L}\left\{ \int_{t-r_i}^{t} \left(\sum_{k=0}^{m} A_k x(u-r_k) \right) du \right\} = \frac{1-e^{-r_i s}}{s} \sum_{k=0}^{m} A_k e^{-r_k s}.$$

Hence, for any r_k, $k = 1, \cdots, m$, the system is stable if and only if **{delete}**

$$\det \left[sI - \left(\sum_{i=0}^{m} A_i \right) - \sum_{i=1}^{m} A_i \frac{e^{-r_i s} - 1}{s} \left(\sum_{k=0}^{m} A_k e^{-r_k s} \right) \right] \neq 0, \quad \forall s \in \overline{\mathbb{C}}_+.$$

Under the assumption that $A := \sum_{i=0}^{m} A_i$ is stable, this condition is equivalent to

$$\det \left[I - (sI-A)^{-1} \sum_{i=1}^{m} A_i \frac{e^{-r_i s} - 1}{s} \left(\sum_{k=0}^{m} A_k e^{-r_k s} \right) \right] \neq 0, \quad \forall s \in \overline{\mathbb{C}}_+. \tag{3.21}$$

This consequently leads to a delay-dependent stability condition characterized by the structured singular value.

Theorem 3.9 *Let $r_k \geq 0$, $k = 1, 2, \cdots, m$ be independent, incommensurate delays. Then the system (3.1) is stable for all $r_k \in [0, \overline{r}_k)$, $k = 1, \cdots, m$, if*

(i) $\sum_{k=0}^{m} A_k$ *is stable,*

(ii)

$$\mu_{\mathbf{\Delta}}\left(M(j\omega;\ \overline{r})\right) < 1, \qquad \forall \omega \geq 0, \tag{3.22}$$

where $\mathbf{\Delta} = \ \mathsf{X}_{m^2+m}$,

$$\begin{aligned} M(s;\ \overline{r}):\ &= \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \left[sI - \left(\sum_{i=0}^{m} A_i \right) \right]^{-1} \begin{bmatrix} \overline{r}_1 A_1 C & \cdots & \overline{r}_m A_m C \end{bmatrix}, \\ C:\ &= \begin{bmatrix} A_0 & A_1 & \cdots & A_m \end{bmatrix}. \end{aligned}$$

Proof. For any $\overline{r}_k > 0$, $k = 1, \ \cdots, \ m$, we may first rewrite the condition (3.21) as

$$\det\left(I - \left(sI - \left(\sum_{i=0}^{m} A_i \right)\right)^{-1} \sum_{i=1}^{m} \overline{r}_i A_i C \Delta_i(s) \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \right) \neq 0, \quad \forall s \in \overline{\mathbb{C}}_+,$$

or equivalently

$$\det\left(I - M(s;\ \overline{r})\Delta(s)\right) \neq 0, \qquad \forall s \in \overline{\mathbb{C}}_+,$$

where

$$\begin{aligned} \Delta(s):\ &= \operatorname{diag}\left(\Delta_1(s),\ \Delta_2(s),\ \cdots,\ \Delta_m(s)\right), \\ \Delta_i(s):\ &= \delta_i(s)\operatorname{diag}\left(I,\ e^{-r_1 s}I,\ \cdots,\ e^{-r_m s}I\right), \\ \delta_i(s):\ &= \frac{e^{-r_i s} - 1}{\overline{r}_i s}. \end{aligned}$$

It thus suffices to show that $\delta_i(s) \in \overline{\mathcal{B}}\mathcal{H}_\infty$, for all $i = 1, \ \cdots, \ m$, and to involke the small μ theorem. Evidently, $\delta_i(s) \in \mathcal{H}_\infty$. Furthermore, since for any $r_k \in [0, \ \overline{r}_k)$, $k = 1, \ \cdots, \ m$,

$$|\delta_i(j\omega)| = \left| \frac{\sin(r_i\omega/2)}{(\overline{r}_i\omega/2)} \right| \leq \left| \frac{\sin(r_i\omega/2)}{(r_i\omega/2)} \right| \leq 1,$$

we have $\|\delta_i\|_\infty \leq 1$. This completes the proof. ∎

The idea behind the above derivation is rather clear, and remains familiar; it also seeks to approximate delay effects by a structured uncertainty, though in a different way. It should be rather clear that the model transformation technique can be applied to further depth to yield more sophisticated representations of the delay system, hence leading to other forms of uncertainty to model the delay effects and consequently to additional delay-dependent stability conditions of a similar nature. More generally, such approximations need not be restricted to model transformation alone; in fact, the delay effects may be modeled by uncertainties in a variety of ways. Nevertheless, fundamental computational difficulty remains. Clearly, the barrier lies at the computation of the structured singular value, thus necessitating a tradeoff between the conservatism of a test and its computational complexity. Moreover, while Theorem 3.9 does provide a delay-dependent stability condition, it may or may not be less conservative than the delay-independent condition given in Theorem 3.5.

3.4 Computational complexity analysis

The computation of structured singular values is known to be difficult. That a stability test is equated to a structured singular value problem casts doubts on the test's practical feasibility. For systems with incommensurate delays, we have seen, in light of Theorem 3.5, that the exact verification of delay-independent stability must bear a same degree of difficulty as that of computing a specific μ, corresponding to a specific $M(s)$ and the specific uncertainty structure X_m. Delay-dependent stability will only prove more difficult. Can then the computation of that particular μ, as a special case of the general structured singular value problem, be tractable? Or is the stability problem inherently difficult? While it would be clearly invalid to conclude anything based on the known computational difficulty of the general μ problem, we provide a formal, conclusive proof to show that the delay-independent stability problem is in general equivalent to one of NP-hard decision problems, whereas the notion of NP-hardness will be defined in a precise manner. The proof, which requires concepts and techniques found in computational complexity theory, thus translates into the statement that the stability problem is indeed fundamentally difficult, and is unlikely to be tractable computationally.

3.4.1 Basic complexity concepts

The computational complexity theory we refer to herein is a paradigm concerning the study of the intrinsic computational difficulty, and thereupon the classification of problems into classes that are *tractable* or *intractable*. Here whether a problem is tractable or not is measured by the computational requirement in computing a solution, that is, the demand in computer time and space for solving the problem; an intractable problem may be solvable in theory provided that unlimited resources of computation are available, but is unlikely to be solvable in practice in any realistic sense. The theory has for long become an extensively developed branch of computing science and operation research, built on languages and concepts that seem rather unfamiliar, though not uncommon, to the field of systems and control, and as such are rather removed from the general subject of stability analysis. We will therefore attempt to provide only a minimal glimpse into the theory required in the present context, and focus exclusively on its pertinence to our stability problem. The exposure, for this reason, will be mainly informative though deliberately informal, in order not to labor the reader with additional, unnecessary complications.

The central issue in the complexity theory dwells on the notions of efficient algorithms and tractable problems solvable by efficient algorithms. An efficient algorithm, also known as a polynomial-time algorithm, is one that can perform the computation in polynomial time; in other words, the required computation time, or the number of *basic* operations, can be bounded by a polynomial function of the problem size, or the length of input data. A problem is deemed tractable if it can be solved by a polynomial-time algorithm in the worst case. More specifically, let $\phi(x)$ denote the number of *basic* operations required for computing a solution from the available data x. Let also n be the size of the data length, or more generally, the problem size, which can be thought as the dimension of a problem or the number of unknowns. Under some common, standard measure of computation efficiency (known as the *Turing machine* in computer science), the problem is said to be polynomial-time solvable if there exists a positive integer q such that for all possible data x,

$$\phi(x) = \mathcal{O}(n^q),$$

that is, asymptotically, the computation cost will not grow faster than at a polynomial rate, in the worst case and hence in all instances of the problem. It is customary to refer to this class of problems the class $\mathbf{P}$. For a problem not in the class $\mathbf{P}$, there exists at least one instance for which the required computation time cannot be bounded by a polynomial function. Problems in the latter category are generally held intractable. One should be cautious in interpreting the *practical* difficulty in computing problems in $\mathbf{P}$. The formalism gives no indication and indeed de-emphasizes the importance of n and q. It could be well possible that for all practical purposes, a problem of a complexity $\mathcal{O}(n^{100})$ cannot be realistically solved even for a moderate n, though it belongs to $\mathbf{P}$ and therefore is considered tractable.

There is yet another class of decision problems whose membership in $\mathbf{P}$ cannot be–in fact, has never been–ascertained, but whose solution can be verified via a polynomial-time algorithm, if such a solution has been postulated *a priori* in some manner. This class of problems are referred to as the class $\mathbf{NP}$. Intuitively, it is easier to verify than to find a solution. Indeed, it is an immediate fact that

$$\mathbf{P} \subset \mathbf{NP}.$$

An analogy can be made here between finding a difficult proof for a mathematical theorem versus checking an existing proof. A longstanding problem in the complexity theory, one that remains unresolved today and seemingly will be open in the foreseeable future, questions whether

$$\mathbf{P} \neq \mathbf{NP}.$$

While no conclusion has been reached in either way, it has been generally believed that this is likely to be true, a *de facto* proposition supported by ample empirical evidences. The fact, under the premise that it does hold, commands fundamental importance in complexity studies, due to a strong property of problems in **NP**. It turns out that **NP** contains a subclass in which every problem can be transformed via a polynomial-time algorithm to another problem in the same subclass. Such a problem is said to be *NP-complete*. More specifically, we say that a problem $\wp$ is NP-hard if every problem in **NP** is polynomially reducible to $\wp$, and NP-complete if it is NP-hard and $\wp \in \mathbf{NP}$. Here by the polynomial reducibility of a problem $\wp_1$ to another problem $\wp_2$, we mean that every instance p_1 of $\wp_1$ can be transformed by a polynomial-time algorithm to an instance p_2 of $\wp_2$, so that p_1 admits a solution if and only if p_2 does, and p_1 does not admit a solution if and only if p_2 does not. The implication then is that it is unlikely to find a polynomial-time algorithm for an NP-hard problem $\wp$, for otherwise every problem in **NP** can be polynomially reduced to $\wp$ and consequently solved via a polynomial-time algorithm, but this would necessarily contradict the fact that $\mathbf{P} \neq \mathbf{NP}$. Similarly, since an NP-complete problem is also NP-hard, NP-complete problems are unlikely to be tractable as well.

That every problem in **NP** can be polynomially reduced to an NP-hard problem suggests a proof for NP-hardness. It follows that if a problem $\wp_1$ is NP-hard and polynomially reducible to some problem $\wp_2$, then $\wp_2$ must also be NP-hard. In other words, if one is to prove that $\wp_2$ is NP-hard, one may attempt to reduce polynomially a known NP-hard problem $\wp_1$ to $\wp_2$. In particular, it suffices to polynomially reduce $\wp_1$ to just *one* instance of $\wp_2$, since that particular instance of $\wp_2$ is as difficult as any instance of $\wp_1$, and therefore in the worst case, $\wp_2$ must be as difficult as $\wp_1$. A vast number of NP-hard and NP-complete problems have been discovered in this manner, among them notably are such classical problems as the satisfiability problem and the traveling salesman problem. The same strategy is adopted in our proof of the NP-hardness of the stability problem.

3.4.2 Proof of NP-hardness

In proving that the stability problem is NP-hard, we attempt to reduce polynomially a well-established NP-hard problem, known as the Knapsack problem, to a sequence of NP-hard problems leading finally to the stability problem. In the reduction procedure, we construct one specific instance for each problem and show that the Knapsack problem reduces polynomially to that instance.

We begin with the statement of the Knapsack problem.

Knapsack Problem *Given a nonzero vector of integers $c \in \mathsf{Z}^n$, determine if there exists a vector $X \in \{-1,\ 1\}^n$ such that $c^T X = 0$? This problem is called the Knapsack problem.*

The Knapsack problem is a classical integer programming problem that is known to be NP-complete. We need to expand it to complex numbers of unit modulus, a problem called T-Knapsack problem.

T-Knapsack Problem *Given a sequence of nonzero vectors $a_i \in \mathsf{Z}^n$, $i = 1, \cdots, q$, determine if there exists a vector $Z \in \mathsf{T}^n$ such that $a_i^T Z = 0$, for all $i = 1, \cdots, q$? This problem is called the T-Knapsack problem.*

Lemma 3.10 *The T-Knapsack problem is NP-hard.*

Proof. Consider first the equation

$$1 + z_1 + z_2 = 0, \tag{3.23}$$

for some $z_1 = e^{j\theta_1}$, $z_2 = e^{j\theta_2}$. Evidently, it admits a solution if and only if

$$\begin{aligned} \cos\theta_1 + \cos\theta_2 &= -1 \\ \sin\theta_1 + \sin\theta_2 &= 0. \end{aligned}$$

By solving the latter equations, we obtain the solution for (3.23):

$$z_1 = e^{\pm j(2\pi/3)}, \qquad z_2 = \overline{z}_1.$$

Next, consider the equations

$$z_0 + z_{i,1} + z_{i,2} = 0, \qquad i = 1, \ \cdots, \ n, \tag{3.24}$$

and

$$\sum_{i=1}^{n} c_i \left(z_{i,1} - z_{i,2} \right) = 0, \tag{3.25}$$

where $c = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T \in \mathsf{Z}^n$ is a nonzero vector, and

$$Z := \begin{bmatrix} z_0 & z_{1,1} & z_{1,2} & \cdots & z_{n,1} & z_{n,2} \end{bmatrix}^T \in \mathsf{T}^{2n+1}.$$

Construct the vector sequence $a_1, \ \cdots, \ a_{n+1} \in \mathsf{Z}^{2n+1}$ as

$$\begin{aligned} a_i &= \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \end{bmatrix}^T, \quad i = 1, \ \cdots, \ n, \\ a_{n+1} &= \begin{bmatrix} 0 & c_1 & -c_1 & \cdots & c_n & -c_n \end{bmatrix}^T. \end{aligned}$$

It follows that (3.24-3.25) hold if and only if

$$a_i^T Z = 0, \qquad i = 1, \ \cdots, \ n+1. \tag{3.26}$$

Suppose that the Knapsack problem admits a solution $X \in \{-1,\ 1\}^n$ for some nonzero $c \in \mathsf{Z}^n$. Let $z_0 = 1$, and

$$(z_{i,1},\ z_{i,2}) = \begin{cases} \left(e^{j(2\pi/3)},\ e^{-j(2\pi/3)}\right) & \text{if } X_i = 1 \\ \left(e^{-j(2\pi/3)},\ e^{j(2\pi/3)}\right) & \text{if } X_i = -1 \end{cases} \qquad i = 1,\ \cdots,\ n.$$

In view of the solution to (3.23), it is clear that Z satisfies the equations (3.24-3.25) and hence is a solution to (3.26). Conversely, suppose that (3.26) admits a solution $Z \in \mathsf{T}^{2n+1}$. This implies that

$$1 + \frac{z_{i,1}}{z_0} + \frac{z_{i,2}}{z_0} = 0, \qquad i = 1,\ \cdots,\ n,$$

and in turn

$$\frac{z_{i,1}}{z_0} = e^{\pm j(2\pi/3)}, \qquad \frac{z_{i,2}}{z_0} = \overline{\left(\frac{z_{i,1}}{z_0}\right)}.$$

We may then find

$$X_i = \frac{z_{i,1} - z_{i,2}}{z_0\left(e^{j(2\pi/3)} - e^{-j(2\pi/3)}\right)} = \pm 1, \qquad i = 1,\ \cdots,\ n,$$

and hence an $X \in \{-1,\ 1\}^n$, so that $c^T X = 0$. In other words, we have shown that the Knapsack problem can be polynomially reduced to an instance of the T-Knapsack problem. Based on the NP-hardness of the former, we conclude that the T-Knapsack problem is NP-hard. ■

The next result shows that the T-Knapsack problem is polynomially reducible to a number of optimization problems, known respectively as sesquilinear and bilinear programming problems.

Lemma 3.11 *For a given positive definite matrix $A \in \mathsf{R}^{n\times n}$, the following problems are NP-hard.*

(i) Sesquilinear programming: Determine if $\sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H A Z\right| < 1$.

(ii) Bilinear programming: Determine if $\sup_{Z_1,Z_2\in\overline{\mathsf{D}}^n} \left|Z_1^T A Z_2\right| < 1$.

Proof. For a positive definite A, we may write $A = Q^H Q$. Furthermore, for any $Z_1,\ Z_2 \in \overline{\mathsf{D}}^n$,

$$\left|Z_1^T A Z_2\right| = \left|Z_1^T Q^H Q Z_2\right| \le \|QZ_1\|_2 \|QZ_2\|_2.$$

Hence,

$$\sup_{Z_1,Z_2\in\overline{\mathsf{D}}^n} \left|Z_1^T A Z_2\right| \le \sup_{Z\in\overline{\mathsf{D}}^n} \|QZ\|_2^2 = \sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H A Z\right|.$$

Additionally, it is evident that

$$\sup_{Z_1,Z_2\in\overline{\mathsf{D}}^n} \left|Z_1^T A Z_2\right| \ge \sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H A Z\right|.$$

This establishes the fact that

$$\sup_{Z_1, Z_2 \in \overline{\mathsf{D}}^n} \left| Z_1^T A Z_2 \right| = \sup_{Z \in \overline{\mathsf{D}}^n} \left| Z^H A Z \right|.$$

Similarly,

$$\sup_{Z_1, Z_2 \in \mathsf{T}^n} \left| Z_1^T A Z_2 \right| = \sup_{Z \in \mathsf{T}^n} \left| Z^H A Z \right|.$$

We claim that

$$\sup_{Z_1, Z_2 \in \overline{\mathsf{D}}^n} \left| Z_1^T A Z_2 \right| = \sup_{Z_1, Z_2 \in \mathsf{T}^n} \left| Z_1^T A Z_2 \right|.$$

This can be seen either as a consequence of the maximum modulus principle, whereas $Z_1^T A Z_2$ is viewed as a multivariate complex function, or more simply by invoking the properties of the structured singular value. Indeed, define

$$M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}.$$

Then in light of Lemma 3.4 (v), it is easy to observe that with $k_1 = \cdots = k_{2n} = 1$,

$$\begin{aligned}
\sup_{Z_1, Z_2 \in \overline{\mathsf{D}}^n} \left| Z_1^T A Z_2 \right| &= \sup_{\Delta_1, \Delta_2 \in \overline{\mathcal{B}}\ \mathsf{X}_n} \rho\left(A \Delta_1 B \Delta_2\right) \\
&= \sup_{\Delta \in \overline{\mathcal{B}}\ \mathsf{X}_{2n}} \rho^2\left(M \Delta\right) \\
&= \sup_{U \in \mathcal{U}_{2n}} \rho^2\left(M U\right) \\
&= \sup_{U_1, U_2 \in \mathcal{U}_n} \rho\left(A U_1 B U_2\right) \\
&= \sup_{Z_1, Z_2 \in \mathsf{T}^n} \left| Z_1^T A Z_2 \right|.
\end{aligned}$$

We have thus shown that

$$\begin{aligned}
\sup_{Z \in \overline{\mathsf{D}}^n} \left| Z^H A Z \right| &= \sup_{Z_1, Z_2 \in \overline{\mathsf{D}}^n} \left| Z_1^T A Z_2 \right| \\
&= \sup_{Z_1, Z_2 \in \mathsf{T}^n} \left| Z_1^T A Z_2 \right| \qquad (3.27) \\
&= \sup_{Z \in \mathsf{T}^n} \left| Z^H A Z \right|.
\end{aligned}$$

Next, we construct a specific positive definite matrix A. Let $c \in \mathsf{Z}^n$ be a nonzero vector, and let a_i be given as in the proof of Lemma 3.10. Define

$$A := \frac{1}{2n+1} \left(I - \alpha \sum_{i=1}^{n+1} a_i a_i^T \right), \qquad \alpha := \frac{1}{2 \sum_{i=1}^{n+1} \|a_i\|_2^2}. \qquad (3.28)$$

It follows from (3.27) that

$$\sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H AZ\right| = \sup_{Z\in\mathsf{T}^n} \left|Z^H AZ\right| = 1 - \frac{\alpha}{2n+1} \inf_{Z\in\mathsf{T}^{2n+1}} \sum_{i=1}^{n+1} \left|a_i^T Z\right|^2 .$$

Hence, if the T-Knapsack problem has a solution, then $\sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H AZ\right| = 1$. Otherwise, $\sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H AZ\right| < 1$. Consequently, the T-Knapsack problem has been polynomially reduced to a special instance of the sesquilinear programming problem, and similarly the bilinear programming problem. We then conclude that both problems are NP-hard. ■

The above development finally culminates at the main result of this section. We are now ready to construct one specific system with incommensurate delays, for which we show that it is stable independent of delay if and only if the T-Knapsack problem admits a solution. The implication then is that the T-Knapsack problem is polynomially reducible to a particular instance of the stability problem, thereby enabling us to conlcude the NP-hardness of the latter.

Theorem 3.12 *Let $r_k \geq 0$, $k = 1,\ 2,\ \cdots,\ m$ be independent, incommensurate delays. Then the following problems are NP-hard.*

(i) Is the system (3.1) stable independent of delay?

(ii) Is the system (3.1) stable for all $r_k \in [0,\ \overline{r}_k)$, where $\overline{r}_k > 0$, $k = 1,\ \cdots,\ m$?

Proof. Let $\omega_0 > 0$ and $A \in \mathsf{R}^{n\times n}$ be the positive definite matrix in (3.28). Define

$$A_0 := \begin{bmatrix} -A^{-1} & 0 & \omega_0 I & 0 \\ 0 & -1 & 0 & \omega_0 I \\ -\omega_0 I & 0 & -A^{-1} & 0 \\ 0 & -\omega_0 I & 0 & -1 \end{bmatrix}, \tag{3.29}$$

and

$$A_k := \begin{cases} e_k e_{n+1}^T & k = 1,\ \cdots,\ n \\ e_{n+1} e_{k-n}^T & k = n+1,\ \cdots,\ 2n \\ e_{k-n+1} e_{2n+2}^T & k = 2n+1,\ \cdots,\ 3n \\ e_{2n+2} e_{k-2n+1}^T & k = 3n+1,\ \cdots,\ 4n \end{cases} \tag{3.30}$$

where $e_k \in \mathsf{R}^{2n+2}$ is the k-th Euclidean coordinate. Denote $m = 2n+2$, and $\Phi_z := A_0 + \sum_{k=1}^{m} A_k z_k$. It is clear that for any $z_k \in \overline{\mathsf{D}}$, $k = 1,\ \cdots,\ m$,

$$\Phi_z = \begin{bmatrix} -A^{-1} & Z_1 & \omega_0 I & 0 \\ Z_2^T & -1 & 0 & \omega_0 I \\ -\omega_0 I & 0 & -A^{-1} & Z_3 \\ 0 & -\omega_0 I & Z_4^T & -1 \end{bmatrix},$$

for some $Z_i \in \overline{\mathsf{D}}^n$, $i = 1,\ 2,\ 3,\ 4$. Suppose that $\sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H A Z\right| < 1$. This implies that

$$\begin{bmatrix} -A^{-1} & Z \\ Z^H & -1 \end{bmatrix} < 0, \qquad \forall Z \in \overline{\mathsf{D}}^n.$$

It then follows that

$$\frac{\Phi_z + \Phi_z^H}{2} = \begin{bmatrix} -A^{-1} & Z_a & 0 & 0 \\ Z_a^H & -1 & 0 & 0 \\ 0 & 0 & -A^{-1} & Z_b \\ 0 & 0 & Z_b^H & -1 \end{bmatrix} < 0,$$

where

$$Z_a = \frac{Z_1 + Z_2}{2} \in \overline{\mathsf{D}}^n, \qquad Z_b = \frac{Z_3 + Z_4}{2} \in \overline{\mathsf{D}}^n.$$

Since for any $\lambda \in \sigma(\Phi_z)$,

$$2\mathrm{Re}\ (\lambda) \le \sup_{x\neq 0} \frac{x^H \left(\Phi_z + \Phi_z^H\right) x}{x^H x} < 0,$$

we conclude that $\sigma(\Phi_z) \cap \overline{\mathsf{C}}_+ = \emptyset$ for all $Z_i \in \overline{\mathsf{D}}^n$, $i = 1,\ 2,\ 3,\ 4$. In other words, Φ_z has no eigenvalue in the closed right half plane, or equivalently,

$$\sigma\left(A_0 + \sum_{k=1}^{m} A_k z_k\right) \cap \overline{\mathsf{C}}_+ = \emptyset, \qquad \forall z_k \in \overline{\mathsf{D}},\ k = 1,\ \cdots,\ m.$$

Evidently, the condition is equivalent to

$$\det\left(sI - A_0 - \sum_{k=1}^{m} A_k e^{-r_k s}\right) \neq 0, \qquad \forall s \in \overline{\mathsf{C}}_+,$$

for all $r_k \ge 0$. Consequently, with the matrices A_k, $k = 0,\ 1,\ \cdots,\ m$ constructed in (3.29) and (3.30), the system (3.1) is stable independent of delay. On the other hand, suppose that $\sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H A Z\right| = 1$. Define

$$\Psi_z := \begin{bmatrix} -A^{-1} & Z \\ Z^H & -1 \end{bmatrix}.$$

Since, in light of (3.27),

$$\sup_{Z\in\mathsf{T}^n} \left|Z^H A Z\right| = \sup_{Z\in\overline{\mathsf{D}}^n} \left|Z^H A Z\right| = 1,$$

there exists a $Z \in \mathsf{T}^n$ such that $\det(\Psi_z) = 0$. This is easily seen by an appeal to Schur determinant formula. Using the same formula, it is straightforward to verify that

$$\det\left(j\omega_0 I - \begin{bmatrix} \Psi_z & \omega_0 I \\ -\omega_0 I & \Psi_z \end{bmatrix}\right) = 0.$$

Let $Z_1 = Z_3 = Z$, and $Z_2 = Z_3 = \overline{Z}$. Then,

$$\Phi_z = \begin{bmatrix} \Psi_z & \omega_0 I \\ -\omega_0 I & \Psi_z \end{bmatrix}.$$

Consequently, there exist $\theta_k \in [0,\ 2\pi]$, $k = 1,\ \cdots,\ n$, such that

$$z_k := \begin{cases} e^{-j\theta_k} & k = 1,\ \cdots,\ n \\ e^{-j(2\pi-\theta_k)} & k = n+1,\ \cdots,\ 2n \\ e^{-j\theta_k} & k = 2n+1,\ \cdots,\ 3n \\ e^{-j(2\pi-\theta_k)} & k = 3n+1,\ \cdots,\ 4n \end{cases}$$

and $\det(j\omega_0 I - \Phi_z) = 0$, or equivalently,

$$\det\left(j\omega_0 I - A_0 - \sum_{k=1}^{m} A_k z_k\right) = 0.$$

By equating

$$r_k := \begin{cases} \theta_k/\omega_0 & k = 1,\ \cdots,\ n \\ (2\pi - \theta_k)/\omega_0 & k = n+1,\ \cdots,\ 2n \\ \theta_k/\omega_0 & k = 2n+1,\ \cdots,\ 3n \\ (2\pi - \theta_k)/\omega_0 & k = 3n+1,\ \cdots,\ 4n \end{cases}$$

we have

$$\det\left(j\omega_0 I - A_0 - \sum_{k=1}^{m} A_k e^{-jr_k\omega_0}\right) = 0.$$

That is, with the matrices A_k in (3.29) and (3.30), the system (3.1) cannot be stable independent of delay. To summarize, we have shown that if the T-Knapsack problem admits a solution, then we may construct via a polynomial-time algorithm the matrix A in (3.28), and in turn a delay system (3.1) with A_k given by (3.29) and (3.30), so that it is stable independent of delay. Conversely, if the T-Knapsack problem has no solution, then the same delay system cannot be stable independent of delay. In other words, we have reduced polynomially the NP-hard T-Knapsack problem to a specific instance of the problem of determining whether the system (3.1) is stable independent of delay. Consequently, we have proven Theorem 3.12 (i); that is, for systems with independent, incommensurate delays, the problem of checking delay-independent stability is NP-hard. Since the case (i) is a special case of (ii), it follows that Theorem 3.12 (ii) is also true: the problem of checking delay-dependent stability, for any delay region bounded by $\overline{r}_k$, is NP-hard as well. ■

3.5 Sufficient stability conditions

Due to the computational requirement of necessary and sufficient conditions, which can be demanding even for systems with commensurate delays and has been proven impractical for systems with incommensurate delays, efficient, less demanding sufficient conditions become an attractive alternative. Such conditions provide a complementary tool in general, and indeed appears to be the sole means for systems of incommensurate delays in particular. With these incentives, we now develop sufficient stability conditions. For the time being, we shall be mainly interested in simple, readily computable conditions that require only a moderate amount of computation. Many sufficient conditions the reader will come across in the later chapters will be more complex and will generally require solving an LMI optimization problem; these conditions will be developed based on time domain techniques. The reader will witness that small gain type stability criteria developed herein provide a potent vehicle in the development of sufficient conditions; with a variety of matrix norms and measures coming to aid, it leads readily to many sufficient conditions of interest.

3.5.1 Systems of one delay

We begin our development with systems of one delay. The following result provides a sequence of sufficient conditions for delay-independent stability.

Theorem 3.13 *Suppose that A_0 is stable. Then the system (3.2) is stable independent of delay if one of the following conditions holds.*

(i) $\rho\left((j\omega I - A_0)^{-1}A_1\right) < 1, \quad \forall\omega \geq 0.$

(ii) For any induced matrix norm $\|\cdot\|$,

$$\|(j\omega I - A_0)^{-1}A_1\| < 1, \qquad \forall\omega \geq 0.$$

(iii) $\overline{\sigma}\left((j\omega I - A_0)^{-1}A_1\right) < 1, \quad \forall\omega \geq 0,$ *or equivalently,*

$$\|(sI - A_0)^{-1}A_1\|_\infty < 1.$$

(iv) For any induced matrix norm $\|\cdot\|$ and the corresponding matrix measure $\nu(\cdot)$, $\|A_1\| < -\nu(A_0)$.

(v) For any induced matrix norm $\|\cdot\|$ such that $\|e^{A_0 t}\| \leq \zeta e^{-\eta t}$, $\forall t \geq 0$ for some $\eta \geq 1$ and some $\zeta \geq 1$ and $\eta > 0$, $\|A_1\| < \eta/\zeta$.

Proof. The condition (i) is a restatement of the sufficient condition (3.15), while (ii) and (iii) are a direct consequence of (i), in view of the fact that $\rho(\cdot) \leq \|\cdot\|$ for any induced matrix norm $\|\cdot\|$. To establish (iv), we first invoke the properties of matrix measures, which give rise to

$$-\nu(-j\omega I + A_0) \leq \frac{1}{||(j\omega I - A_0)^{-1}||}.$$

and

$$-\nu(j\omega I) + \nu(A_0) \leq \nu(-j\omega I + A_0) \leq \nu(-j\omega I) + \nu(A_0).$$

We claim that $\nu(j\omega I) = \nu(-j\omega I) = 0$, and hence $\nu(-j\omega I + A_0) = \nu(A_0)$. Indeed, by definition, we have

$$\nu(j\omega I) = \lim_{\sigma \to 0^+} \frac{\|I + \sigma j\omega I\| - 1}{\sigma} = \lim_{\sigma \to 0^+} \frac{|1 + j\sigma\omega| - 1}{\sigma} = 0.$$

Similarly, $\nu(-j\omega I) = 0$. We are thus led to the inequality

$$-\nu(A_0) \leq \frac{1}{||(j\omega I - A_0)^{-1}||}.$$

Since by (ii) the system is stable whenever

$$\|A_1\| < \frac{1}{||(j\omega I - A_0)^{-1}||},$$

the condition (iv) follows. Finally, to prove (v), we use the Laplace transform

$$(sI - A_0)^{-1} = \int_0^\infty e^{A_0 t} e^{-st} dt.$$

For any $s \in \overline{\mathbb{C}}_+$, it follows that

$$||(sI - A_0)^{-1}|| = \left|\left|\int_0^\infty e^{A_0 t} e^{-st} dt\right|\right| \leq \int_0^\infty ||e^{A_0 t}||\, dt \leq \int_0^\infty \zeta e^{-\eta t} dt = \frac{\zeta}{\eta}.$$

This completes the proof. ■

We note that of all the conditions in Theorem 3.13, the conservatism rises in ascending order, with (i) the least conservative and (iv-v) the most. A payoff, however, is that the conditions (iii), (iv), and (v) do not require frequency sweep. In particular, the $\mathcal{H}_\infty$ norm of $(sI - A_0)^{-1} A_1$ can be computed using a bisection method, which amounts to checking whether the Hamiltonian matrix

$$H := \begin{bmatrix} A_0 & A_1 A_1^T \\ -I & -A_0^T \end{bmatrix}$$

has an eigenvalue on the imaginary axis, including the origin. The relation between the $\mathcal{H}_\infty$ norm and the Hamiltonian matrix H is well-known, and the bisection method is well-documented in the $\mathcal{H}_\infty$ control literature. The connection between the $\mathcal{H}_\infty$ norm and other time-domain measures will be further explored in Chapter 8.

The next result also draws upon frequency-sweeping tests, which give conditions based on the solution of the Lyapunov equation

$$A_0^T P + P A_0 = -2I. \tag{3.31}$$

Theorem 3.14 *Suppose that A_0 is stable, and let P be the unique positive definite solution of (3.31). Then the system (3.2) is stable independent of delay if one of the following conditions holds.*

(i) $\overline{\sigma}(PA_1) < 1$.

(ii)

$$\overline{\lambda}\left(\frac{|PA_1|^T + |PA_1|}{2}\right) < 1. \tag{3.32}$$

Proof. The existence and uniqueness of the positive definite solution P of (3.31) is guaranteed by the fact that A_0 is stable. Let $\Phi(s) := (sI - A_0)^{-1}$. Then, by adding and subtracting $j\omega P$ on the left hand side of (3.31) followed by pre- and post-multiplication of the resulting equation by $\Phi^H(j\omega)$ and $\Phi(j\omega)$, it follows that

$$\Phi^H(j\omega)P + P\Phi(j\omega) = 2\Phi^H(j\omega)\Phi(j\omega).$$

This in turn leads to

$$(\Phi(j\omega)A_1)^H (PA_1) + (PA_1)^H (\Phi(j\omega)A_1) = 2\,(\Phi(j\omega)A_1)^H (\Phi(j\omega)A_1)\,. \tag{3.33}$$

Therefore, by applying the triangle inequality, we have

$$\overline{\sigma}\,(\Phi(j\omega)A_1) \leq \overline{\sigma}(PA_1).$$

The condition (i) then follows from Theorem 3.13 (iii). To establish (ii), let $x \in \mathsf{C}^n$ be the eigenvector of $\Phi(j\omega)A_1$ associated with the eigenvalue λ such that $|\lambda| = \rho\,(\Phi(j\omega)A_1)$. Then, pre- and post-multiplication of (3.33) by x^H and x leads to

$$x^H\left(\frac{\overline{\lambda}PA_1 + \left(\overline{\lambda}PA_1\right)^H}{2}\right)x = |\lambda|^2 x^H x.$$

Since

$$\begin{aligned} x^H\left(\frac{\overline{\lambda}PA_1 + \left(\overline{\lambda}PA_1\right)^H}{2}\right)x &\leq \overline{\lambda}\left(\frac{\overline{\lambda}PA_1 + \left(\overline{\lambda}PA_1\right)^H}{2}\right)x^H x \\ &\leq \rho\left(\frac{\overline{\lambda}PA_1 + \left(\overline{\lambda}PA_1\right)^H}{2}\right)x^H x, \end{aligned}$$

we have

$$|\lambda|^2 \leq \rho\left(\frac{\overline{\lambda}PA_1 + \left(\overline{\lambda}PA_1\right)^H}{2}\right).$$

However,

$$\begin{aligned}
\rho\left(\frac{\overline{\lambda}PA_1+\left(\overline{\lambda}PA_1\right)^H}{2}\right) &\leq \rho\left(\frac{\left|\overline{\lambda}PA_1+(\overline{\lambda}PA_1)^H\right|}{2}\right)\\
&\leq \rho\left(\frac{|\overline{\lambda}||PA_1|+|\lambda||PA_1|^T}{2}\right)\\
&= |\lambda|\rho\left(\frac{|PA_1|^T+|PA_1|}{2}\right).
\end{aligned}$$

This gives

$$\rho\left(\Phi(j\omega)A_1\right)=|\lambda|\leq\rho\left(\frac{|PA_1|^T+|PA_1|}{2}\right).$$

The condition (ii) then follows from Theorem 3.13 (i). ■

The conditions in Theorem 3.14 are a consequence of Theorem 3.13 (i), (ii), and (iii), but may improve the other conditions. For example, it is rather straightforward to show that

$$-\nu_2(A_0)\leq\frac{1}{\overline{\sigma}(P)}, \tag{3.34}$$

indicating that Theorem 3.14 (i) is less conservative than Theorem 3.13 (iv), when the norm in question is $\|\cdot\|_2$. Indeed, let $x\in\mathsf{R}^n$ be the eigenvector of P associated with the largest eigenvalue $\lambda_{\max}(P)=\overline{\sigma}(P)$. Pre- and post-multiplying (3.31) by x^T and x yields

$$-x^T\left(\frac{A_0^T+A_0}{2}\right)x=\frac{x^Tx}{\overline{\sigma}(P)}.$$

Since

$$x^T\left(\frac{A_0^T+A_0}{2}\right)x\leq\lambda_{\max}\left(\frac{A_0^T+A_0}{2}\right)x^Tx=\nu_2(A_0)x^Tx,$$

the inequality (3.34) follows. Thus, whenever Theorem 3.13 (iv) is true, so is Theorem 3.14 (i). The example below demonstrates the merits and weaknesses of these sufficient conditions.

Example 3.3 *The following delay system is modified after Example 2.8:*

$$\dot{x}(t)=A_0\,x(t)+\beta A_1\,x(t-\tau),$$

where

$$A_0=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-2&-3&-5&-2\end{bmatrix},\qquad A_1=\begin{bmatrix}-0.05&0.005&0.25&0\\0.005&0.005&0&0\\0&0&0&0\\-1&0&-0.5&0\end{bmatrix},$$

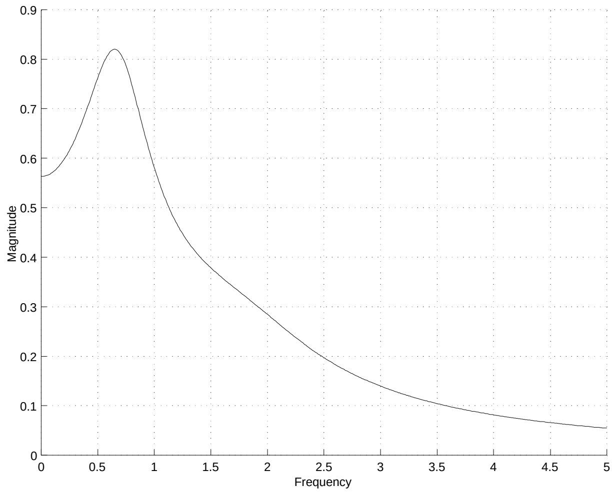

FIGURE 3.6. The spectral radius $\rho((j\omega I - A_0)^{-1}A_1)$

and $\beta \in \mathsf{R}$ *is a parameter whose range is to be determined so that the system is stable independent of delay. It is known from Example 2.8 that* A_0 *is stable. Figure 3.6 gives the plot of* $\rho\left((j\omega I - A_0)^{-1}A_1\right)$*, which allows the calculation of the maximal permissible range of* β*,*

$$\overline{\beta} := \sup_{\omega \geq 0} \left\{ |\beta| : \ |\beta| \rho\left((j\omega I - A_0)^{-1}A_1\right) < 1 \right\} = 1.2196.$$

Figure 3.7 plots the bound $\|(j\omega I - A_0)^{-1}A_1\|$*, for Hölder* ℓ_1*,* ℓ_2*, and* ℓ_∞ *induced matrix norms, respectively. The reader may compare these plots to assess the conservatism of Theorem 3.13 (ii-iii), with the three different norms in use. A simple calculation shows that in all three cases,* $\nu(A_0) > 0$*. Thus, Theorem 3.13 (iv) ceases to be useful. The estimates of* $\overline{\beta}$ *based upon other sufficient conditions are summarized in the following table.*

	$\rho(\cdot)$	$\|\cdot\|_1$	$\|\cdot\|_2$	$\|\cdot\|_\infty$	$\overline{\sigma}(PA_1)$	$\rho\left(\frac{\|PA_1\|^T + \|PA_1\|}{2}\right)$
$\overline{\beta}$	1.2196	0.5279	0.7983	0.7453	0.256	0.3357

Example 3.4 *Consider the scalar differential-difference equation*

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \sum_{i=0}^{n-1} b_i y^{(i)}(t-\tau) = 0,$$

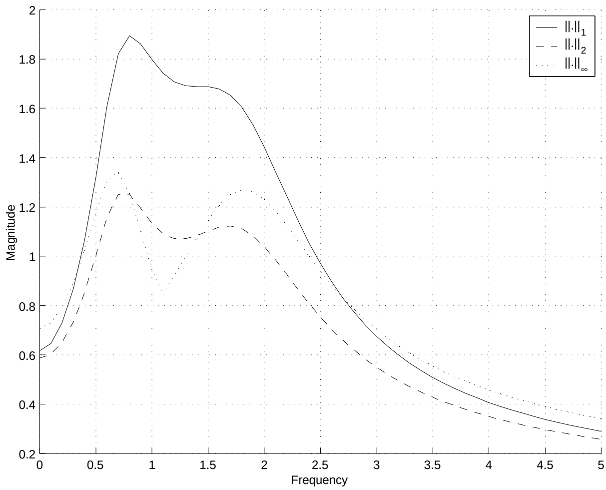

FIGURE 3.7. The norms $||(j\omega I - A_0)^{-1}A_1||$

which, when posed in the form of (3.2), corresponds to

$$A_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix},$$

$$A_1 = B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ -b_0 & -b_1 & \cdots & -b_{n-1} \end{bmatrix}.$$

The conditions in Theorem 3.13 and Theorem 3.14 can be further simplified for this class of systems. The following table lists the relevant quantities, where $\Phi(s) := (sI - A)^{-1}$, $P = [p_{ij}]$, $\alpha := \sum_{i=0}^{n-1} |b_i p_{in}|$, *and*

$$\beta := \sqrt{\left(\sum_{i=0}^{n-1} |b_i|^2\right)\left(\sum_{i=0}^{n-1} |p_{in}|^2\right)},$$

$$\gamma(s) := \sqrt{\left(\sum_{i=0}^{n-1} |s|^{2i}\right)\left(\sum_{i=0}^{n-1} |b_i|^2\right)}.$$

The second column in the table corresponds to a general rank-one matrix $A_1 = cb^T$, *and the third to the specific* $A_1 = B$; *correspondingly,* $B = cb^T$

with

$$\begin{aligned} c &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T, \\ b &= \begin{bmatrix} -b_0 & -b_1 & \cdots & -b_{n-1} \end{bmatrix}^T. \end{aligned}$$

A_1	cb^T	B
$\rho(\Phi(s)A_1)$	$\left\|b^T\Phi(s)c\right\|$	$\left\|\frac{b_{n-1}s^{n-1}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_0}\right\|$
$\overline{\sigma}(\Phi(s)A_1)$	$\\|\Phi(s)c\\|_2\\|b\\|_2$	$\frac{\gamma(s)}{\|s^n+a_{n-1}s^{n-1}+\cdots+a_0\|}$
$\\|A_1\\|_1$	$\max_i \|b_i\| \sum_i \|c_i\|$	$\max_i \|b_i\|$
$\overline{\sigma}(PA_1)$	$\\|Pc\\|_2\\|b\\|_2$	β
$\rho\left(\frac{\|PA_1\|^T+\|PA_1\|}{2}\right)$	$\frac{\|b\|^T\|Pc\|+\\|Pc\\|_2\\|b\\|_2}{2}$	$\frac{\alpha+\beta}{2}$

It is useful to observe that when A_1 is a rank-one matrix, the $\mathcal{H}_\infty$ norm of the scalar transfer function $b^T\Phi(s)c$ actually gives the exact delay-independent stability condition.

3.5.2 Systems of multiple delays

It is rather straightforward to extend the above sufficient conditions to systems with multiple incommensurate delays. The following theorem provides the counterparts to Theorem 3.13 and Theorem 3.14.

Theorem 3.15 *Let $r_k \geq 0$, $k = 1, 2, \cdots, m$ be independent, incommensurate delays. Suppose that A_0 is stable. Then the system (3.1) is stable independent of delay if one of the following conditions holds.*

(i) $\mu_{\mathsf{X}_m}(M(j\omega)) < 1, \quad \forall\omega \geq 0.$

(ii) For any absolute or unitarily invariant induced matrix norm $\|\cdot\|$,

$$\|M(j\omega)\| < 1, \qquad \forall\omega \geq 0.$$

(iii) $\left\|(sI - A_0)^{-1}\begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}\right\|_\infty < 1/\sqrt{m}.$

(iv) $\overline{\sigma}\left(\begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}\right) < -\nu_2(A_0)/\sqrt{m}.$ {subscript K replaced by m}

(v) If for some $\zeta \geq 1$ and $\eta > 0$, $\|e^{A_0 t}\|_2 \leq \zeta e^{-\eta t}$, $\forall t \geq 0$,

$$\overline{\sigma}\left(\begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}\right) < \eta/(\sqrt{m}\,\zeta).$$

(vi) $\overline{\sigma}\left(P\begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}\right) < 1/\sqrt{m}.$

(vii)

$$\overline{\lambda}\left(\frac{|R|^T + |R|}{2}\right) < 1.$$

Here $M(s)$ is given by (3.13), P is the solution to (3.31), and

$$R := \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} P \begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}.$$

Proof. The conditions (i) is a direct consequence of Theorem 3.5. To prove (ii), it suffices to invoke Lemma 3.4 (v), and to note that for any absolute or unitarily invariant norm,

$$\rho\left(M(j\omega)U\right) \leq \|M(j\omega)U\| = \|M(j\omega)\|$$

holds for any $U \in \mathcal{U}_m$. The proofs for (iii-vi) are similar to those found in the proofs for Theorem 3.13 and Theorem 3.14, and hence are omitted. To establish (vii), we first show that

$$M^H(j\omega)R + R^T M(j\omega) = 2M^H(j\omega)M(j\omega).$$

This can be done in the same manner as in the proof for Theorem 3.14 (ii). Accordingly, it follows that for any $U \in \mathcal{U}_m$,

$$(M(j\omega)U)^H(RU) + (RU)^H(M(j\omega)U) = 2(M(j\omega)U)^H(M(j\omega)U).$$

Let U be so chosen that $\mu_{\mathsf{X}_m}(M(j\omega)) = \rho(M(j\omega)U)$. Moreover, let x be the eigenvector of $M(j\omega)U$ corresponding to the eigenvalue λ such that $|\lambda| = \rho\left(M(j\omega)U\right)$. Then, as in the proof for Theorem 3.14 (ii), we have

$$x^H\left(\frac{\overline{\lambda}RU + \left(\overline{\lambda}RU\right)^H}{2}\right)x = |\lambda|^2 x^H x$$

and

$$|\lambda|^2 \leq \rho\left(\frac{\overline{\lambda}RU + \left(\overline{\lambda}RU\right)^H}{2}\right).$$

Since $|U| = [|u_{ij}|] = I$ and

$$\rho\left(\frac{\overline{\lambda}RU + \left(\overline{\lambda}RU\right)^H}{2}\right) \leq \rho\left(\frac{|\lambda||R||U| + |\lambda||R|^T|U|}{2}\right) = |\lambda|\rho\left(\frac{|R| + |R|^T}{2}\right),$$

the result follows. ■

Much of Theorem 3.15 is built upon the rather crude bound $\mu_{\mathbf{\Delta}}(\cdot) \leq \overline{\sigma}(\cdot)$, which is valid for any block structure $\mathbf{\Delta}$. Clearly, the D-scaled upper bound for $\mu_{\mathsf{X}_m}(\cdot)$ can be used to further tighten the conditions. For this purpose, it is of interest to consider positive diagonal scaling matrices in the set

$$\mathcal{D}_m^d := \left\{\mathrm{diag}\left(d_1 I_{k_1} \ \cdots \ d_m I_{k_m}\ \right) : \ d_k > 0\right\}.$$

Let $d := \sqrt{\sum\limits_{k=1}^{m} d_k^2}$. It is straightforward to verify that for any $D \in \mathcal{D}_m^d$,

$$\overline{\sigma}\left(DM(j\omega)D^{-1}\right) = \overline{\sigma}\left((j\omega I - A_0)^{-1}\left[\begin{array}{ccc} \frac{d}{d_1}A_1 & \cdots & \frac{d}{d_q}A_m \end{array}\right]\right).$$

The following conditions are then clear. Note that by setting $d_k = 1/\sqrt{m}$, $k = 1, \ \cdots, \ m$, these conditions replicate some of those given in Theorem 3.15.

Corollary 3.16 *Let $r_k \geq 0$, $k = 1, \ 2, \ \cdots, \ m$ be independent, incommensurate delays. Suppose that A_0 is stable. Then the system (3.1) is stable independent of delay provided that*

$$\inf_{D \in \mathcal{D}_m^d} \overline{\sigma}\left(DM(j\omega)D^{-1}\right) < 1, \qquad \forall \omega \geq 0.$$

Furthermore, it is stable independent of delay if there exist $d_k > 0$, $k = 1, \ \cdots, \ m$, $\sum\limits_{k=1}^{m} d_k^2 = 1$, such that one of the following conditions holds.

(i) $\left\|(sI - A_0)^{-1}\left[\begin{array}{ccc} (1/d_1)A_1 & \cdots & (1/d_m)A_m \end{array}\right]\right\|_\infty < 1.$

(ii) $\overline{\sigma}\left(\left[\begin{array}{ccc} (1/d_1)A_1 & \cdots & (1/d_m)A_m \end{array}\right]\right) < -\nu_2(A_0).$

(iii) If for some $\zeta \geq 1$ and $\eta > 0$, $\|e^{A_0 t}\|_2 \leq \zeta e^{-\eta t}$, $\forall t \geq 0$,

$$\overline{\sigma}\left(\left[\begin{array}{ccc} (1/d_1)A_1 & \cdots & (1/d_m)A_m \end{array}\right]\right) < \eta/\zeta.$$

(iv) $\overline{\sigma}\left(P\left[\begin{array}{ccc} (1/d_1)A_1 & \cdots & (1/d_m)A_m \end{array}\right]\right) < 1.$

Here P is the solution to (3.31).

In light of Theorem 3.9, simple sufficient conditions can also be obtained for delay-dependent stability. We note that similarity scaling can be employed as well to improve further the following conditions. This is rather straightforward and thus is not pursued herein.

Theorem 3.17 *Let $r_k \geq 0$, $k = 1, \ 2, \ \cdots, \ m$ be independent, incommensurate delays. Suppose that $\sum\limits_{k=0}^{m} A_k$ is stable. Then the system (3.1) is stable for all $r_k \in [0, \ \overline{r}_k)$, $k = 1, \ \cdots, \ m$, if one of the following conditions holds.*

(i) For any absolute or unitarily invariant induced matrix norm $\|\cdot\|$,

$$\|M(j\omega;\ \overline{r})\| < 1, \qquad \forall\omega \geq 0.$$

(ii) $$\left\| \left(sI - \sum_{k=0}^{m} A_k \right)^{-1} \begin{bmatrix} \overline{r}_1 A_1 C & \cdots & \overline{r}_m A_m C \end{bmatrix} \right\|_{\infty} < 1/\sqrt{m(m+1)}.$$

(iii) $\overline{\sigma}\left(\begin{bmatrix} \overline{r}_1 A_1 C & \cdots & \overline{r}_m A_m C \end{bmatrix} \right) < -\nu_2 \left(\sum_{k=0}^{m} A_k \right) / \sqrt{m(m+1)}$.

(iv) If for some $\zeta \geq 1$ *and* $\eta > 0$, $\overline{\sigma}\left[\exp\left(\sum_{k=0}^{m} A_k t \right) \right] \leq \zeta e^{-\eta t},\ \forall t \geq 0,$

$$\overline{\sigma}\left(\begin{bmatrix} \overline{r}_1 A_1 C & \cdots & \overline{r}_m A_m C \end{bmatrix} \right) < \frac{\eta}{\zeta\sqrt{m(m+1)}}.$$

(v) $\overline{\sigma}\left(P \begin{bmatrix} \overline{r}_1 A_1 C & \cdots & \overline{r}_m A_m C \end{bmatrix} \right) < 1/\sqrt{m(m+1)}$.

(vi)

$$\overline{\lambda}\left(\frac{|Q|^T + |Q|}{2} \right) < 1.$$

Here $M(s;\ \overline{r})$ *and* C *are given as in Theorem 3.9,* P *is the solution to the Lyapunov equation*

$$\left(\sum_{k=0}^{m} A_k \right)^T P + P \left(\sum_{k=0}^{m} A_k \right) = -2I,$$

and

$$Q := \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} P \begin{bmatrix} \overline{r}_1 A_1 C & \cdots & \overline{r}_m A_m C \end{bmatrix}.$$

The proof of this result resembles `to` **{delete}** that for Theorem 3.15, and hence is omitted.

3.6 Neutral delay systems

Neutral delay systems constitute more general a class than those of the retarded type. An LTI neutral delay system with pointwise delays can be generally described by the differential-difference equation

$$\dot{x}(t) - \sum_{k=1}^{m} B_k\ \dot{x}(t - r_k) = A_0\ x(t) + \sum_{k=1}^{m} A_k\ x(t - r_k), \qquad r_k \geq 0. \quad (3.35)$$

We note that the two sides of (3.35) need not contain an equal number of delays, but this can nevertheless be assumed with no loss of generality. Moreover, as in retarded systems, the delays can be commensurate or incommensurate. For neutral systems of commensurate delays, the differential-difference equation is specified to

$$\dot{x}(t) - \sum_{k=1}^{m} B_k \; \dot{x}(t - k\tau) = A_0 \; x(t) + \sum_{k=1}^{m} A_k \; x(t - k\tau), \qquad \tau \geq 0. \quad (3.36)$$

We show in this section how the preceding results may be extended to neutral delay systems, with both commensurate and incommensurate delays.

Stability of neutral delay systems proves to be a more complex issue, evidently, due to the fact that the system involves the derivative of the delayed state. In fact, the stability notion itself needs to be modified in the first place. Define the characteristic function by

$$p\left(s;\ e^{-r_1 s},\ \cdots,\ e^{-r_m s}\right) := \det\left(s\left(I - \sum_{k=1}^{m} B_k e^{-r_k s}\right) - \sum_{k=0}^{m} A_k e^{-r_k s}\right). \quad (3.37)$$

Based on Theorem 1.6 in Chapter 1, we give the following stability definition of this characteristic function:

Definition 3.2 *The characteristic function (3.37) is said to be stable if there exists an* $\alpha < 0$ *such that*

$$p\left(s;\ e^{-r_1 s},\ \cdots,\ e^{-r_m s}\right) \neq 0, \qquad \forall s \in \overline{\mathsf{C}}_{\alpha+}, \quad (3.38)$$

where $\mathsf{C}_{\alpha+} := \{s :\ \mathrm{Re}(s) > \alpha\}$. *It is said to be stable independent of delay if (3.38) holds for all* $r_k \geq 0$, $k = 1, \cdots, m$. *The neutral system (3.35) is said to be stable if its characteristic function (3.37) is stable, and is stable independent of delay if its characteristic function is stable independent of delay.*

It is worth emphasizing that in Definition 3.2, the stability concept has been strengthened to that of uniform (exponential) stability, while in studying retarded systems only the usual notion of (exponential) stability is required. The adoption of this stronger stability notion is a necessity; it is needed to insure that the characteristic roots of the system exhibit the same continuity property as that of retarded systems. **As was discussed in Chapter 1, although this condition seems to be more stringent than the one for retarded case, it is actually reduces to the usual definition when applied to systems with retarded delays. Another important point to keep in mind is that** More specifically, while the stability exponent of a retarded system is continuous with respect to all $r_k \geq 0$, $k = 1, 2, \cdots, m$, for a neutral system this continuity may not hold in general at $r_1 = \cdots = r_m = 0$, though it is continuous at other delay values. The requirement that the system be uniformly stable at $r_1 = \cdots = r_m = 0$ will guarantee that in the neutral case the stability exponent, defined in the same manner as for retarded systems, is also continuous at this point, and thus for all $r_k \geq 0$, $k = 1, 2, \cdots, m$. **{this is not true}**

Many stability criteria only considers the class of neutral time delay systems (sometimes known as Hale's type of neutral time delay systems) for which the following difference system is stable One additional requirement to insure stability of neutral systems concerns the difference equation

$$x(t) - \sum_{k=1}^{m} B_k \, x(t - r_k) = 0, \qquad r_k \geq 0. \tag{3.39}$$

It is therefore of interest to discuss the stability of this time of difference equations. For the system (3.35) to be stable, it is necessary that the difference equation (3.39) be stable, whereas the stability is defined in the same sense as in Definition 3.2. Otherwise, the solution of (3.39), and hence that of (3.35), may not be continuously differentiable. **{not sure whether this is correct. I know for systems with commensurate delays, this is necessary. I know for distributed delay this is not necessary. not sure about pointwise incommensurate delays. I remember Vladimir know this issue very well, and should be able to give some input in this issue}**

Definition 3.3 *The difference equation (3.39) is said to be stable if there exists an $\alpha < 0$ such that*

$$\det\left(I - \sum_{k=1}^{m} B_k \, e^{-r_k s}\right) \neq 0, \qquad \forall s \in \overline{\mathbb{C}}_{\alpha+}. \tag{3.40}$$

An important fact states that the stability of (3.39) is a global property with respect to r_k, $k = 1, 2, \cdots, m$; it will be stable independent of delay whenever it is stable for some delay values. **It is not difficult to see that if (3.39) is stable for some given r_k, $k = 1, 2, ..., m$, then it is also stable for any delays as long as the all the ratios between delays r_i/r_j, $1 \leq i \leq m$ remain unchanged. This, however, is different from the usually sense of delay-independent stability for incommensurate delays where such ratios are not required to be maintained, although it is similar to delay-independent stability for commensurate delays.** In the remainder of this section, when referring to stability, we mean the stability in the sense of Definition 3.2 and Definition 3.3.

With the above necessary modifications, we now proceed to derive stability conditions. We shall first consider the system (3.36), i.e., the class of neutral systems with commensurate delays. The following result gives a simple necessary and sufficient condition for the stability of the difference equation (3.39) in this case.

Lemma 3.18 *Let $r_k = k\tau$, $k = 1, \cdots, m$. Then the system (3.39) is stable for all $\tau \geq 0$ if and only if $\rho(B) < 1$, where*

$$B := \begin{bmatrix} B_1 & \cdots & B_{m-1} & B_m \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}.$$

Proof. Using Lemma 2.4 and (2.30), we obtain

$$\det\left(I - \sum_{k=1}^{m} B_k e^{-k\tau s}\right) = (-1)^{nm} \det\left(I - Be^{-\tau s}\right).$$

Hence, if $\rho(B) < 1$, we may find an $\alpha < 0$ such that (3.40) holds. Otherwise, assume that $\rho(B) \geq 1$. Then, B has an eigenvalue $\lambda(B) = \rho(B)e^{j\theta}$ for some $\theta \in [0,\ 2\pi]$. Consequently, for any $\tau > 0$, one may find $s = \alpha + j\omega \in \overline{\mathbb{C}}_+$, where $\alpha = (\log \rho(B)/\tau) \geq 0$ and $\omega = (\theta/\tau) \geq 0$, such that $\det(I - Be^{-\tau s}) = 0$, and hence

$$\det\left(I - \sum_{k=1}^{m} B_k e^{-k\tau s}\right) = 0.$$

Therefore, the system (3.39) is not stable. This completes the proof. ■

In much the same manner, we may then obtain similar delay-dependent and delay-independent stability conditions. With no loss of generality, we shall assume that the system (3.36) is stable at $\tau = 0$. This enforces the condition that

$$\det\left(I - \sum_{k=1}^{m} B_k\right) \neq 0.$$

The following results resemble to **{delete}** the frequency-sweeping criteria developed for retarded systems, presented in Theorem 2.5 and Theorem 2.6, respectively. The proofs are also similar and are left to the reader. We remind the reader the definition of delay margin, which can be generalized to neutral delay systems in the same manner.

Theorem 3.19 *Suppose that the system (3.36) is stable at $\tau = 0$, and that $\rho(B) < 1$. Let rank $(A_m) = q$. Furthermore, define*

$$\overline{\tau}_i := \begin{cases} \min\limits_{1\le k\le n} \frac{\theta_k^i}{\omega_k^i} & \text{if } \lambda_i\left(G(j\omega_k^i),\ H(j\omega_k^i)\right) = e^{-j\theta_k^i} \\ & \quad \text{for some } \omega_k^i \in (0,\ \infty),\ \theta_k^i \in [0,\ 2\pi] \\ \infty & \text{if } \underline{\rho}\left(G(j\omega),\ H(j\omega)\right) > 1,\ \forall \omega \in (0,\ \infty) \end{cases}$$

where

$$\begin{aligned} G(s) :&= \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -(sI - A_0) & F_1(s) & \cdots & F_{m-1}(s) \end{bmatrix}, \\ H(s) :&= diag\left(I,\ \cdots,\ I,\ -F_m(s)\right), \\ F_k(s) :&= sB_k + A_k, \qquad k = 1,\ \cdots,\ m. \end{aligned}$$

Then,

$$\overline{\tau} := \min_{1\le i\le q+n(m-1)} \overline{\tau}_i.$$

The system (3.36) is stable for all $\tau \in [0,\ \overline{\tau})$, but becomes unstable at $\tau = \overline{\tau}$.

Theorem 3.20 *The system (3.36) is stable independent of delay if and only if*

(i) $\rho(B) < 1$,
(ii) A_0 *is stable,*
(iii) $\left(I - \sum_{k=1}^{m} B_k\right)^{-1} \sum_{k=0}^{m} A_k$ *is stable, and*
(iv)

$$\rho\left(M_m(j\omega)\right) < 1, \qquad \forall \omega > 0, \tag{3.41}$$

where

$$M_m(s) := \begin{bmatrix} \Phi(s)F_1(s) & \cdots & \Phi(s)F_{m-1}(s) & \Phi(s)F_m(s) \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix},$$

with $\Phi(s) := (sI - A_0)^{-1}$.

Consider next systems described by the neutral differential-difference equation

$$y^{(n)}(t) + \sum_{k=1}^{q} b_k y^{(n)}(t - k\tau) + \sum_{i=0}^{n-1}\sum_{k=0}^{q} a_{ki} y^{(i)}(t - k\tau) = 0. \tag{3.42}$$

As with retarded systems, stability conditions in this case can be simplified. Define the polynomials

$$\begin{aligned} b(z) : &= \sum_{k=1}^{q} b_k z^k \\ p_k(s) : &= b_k s^n + a_k(s), \qquad k = 1, \ \cdots, \ q, \end{aligned}$$

where $a_0(s)$ and $a_k(s)$, $k = 1, \ \cdots, \ q$, are defined as in (3.18) and (3.19). Based on Theorem 3.19 and Theorem 3.20, it is not difficult to extend Corollary 2.7 and Collary 2.8 to the neutral system (3.42).

Corollary 3.21 *The system (3.42) is stable independent of delay if and only if*

(i) $b(z)$ *is Schur stable; that is,* $b(z)$ *has all its zeros in the open unit disc,*

(ii) $a_0(s)$ *is stable,*

(iii) $a_0(s) + \sum_{k=1}^{q} p_k(s)$ *is stable, and*

(iv) $\rho\left(M_a(j\omega)\right) < 1$, $\forall \omega > 0$,

where

$$M_a(s) := \begin{bmatrix} -\frac{p_1(s)}{a_0(s)} & \cdots & -\frac{p_{q-1}(s)}{a_0(s)} & -\frac{p_q(s)}{a_0(s)} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Corollary 3.22 *Suppose that the system (3.42) is stable at* $\tau = 0$*, and that* $b(z)$ *is Schur stable. Furthermore, define*

$$\overline{\tau}_i := \begin{cases} \min\limits_{1 \le k \le n} \frac{\theta_k^i}{\omega_k^i} & \text{if } \lambda_i\left(G_a(j\omega_k^i), \ H_a(j\omega_k^i)\right) = e^{-j\theta_k^i} \\ & \quad \text{for some } \omega_k^i \in (0, \ \infty), \ \theta_k^i \in [0, \ 2\pi] \\ \infty & \text{if } \underline{\rho}\left(G_a(j\omega), \ H_a(j\omega)\right) > 1, \ \forall \omega \in (0, \ \infty) \end{cases}$$

where

$$\begin{aligned} G_a(s) : &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0(s) & -p_1(s) & \cdots & -p_{q-1}(s) \end{bmatrix}, \\ H_a(s) : &= \mathit{diag}\left(1, \ \cdots, \ 1, \ p_q(s)\right). \end{aligned}$$

Then,

$$\overline{\tau} = \min_{1 \le i \le q} \overline{\tau}_i.$$

The system (3.42) is stable for all $\tau \in [0, \ \overline{\tau})$*, but becomes unstable at* $\tau = \overline{\tau}$.

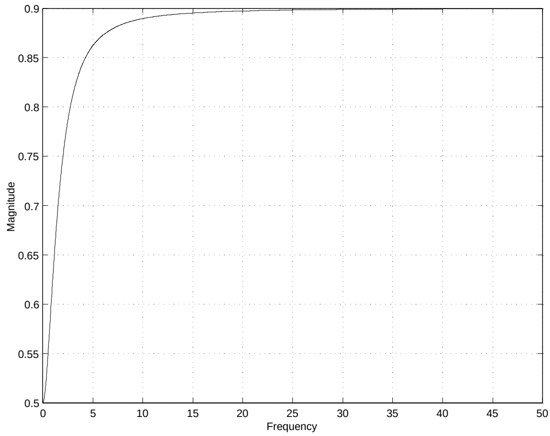

FIGURE 3.8. The spectral radius $\rho(M_{\mathsf{a}}(j\omega))$

Example 3.5 *A scalar neutral delay system is given as*

$$\dot{y}(t) + 0.8\dot{y}(t-\tau) - \beta\ \dot{y}(t-2\tau) + 2y(t) + y(t-\tau) = 0.$$

In light of Lemma 3.18 and Corollary 3.21,

$$B = \begin{bmatrix} -0.8 & \beta \\ 1 & 0 \end{bmatrix}$$

and

$$M_a(s) = \begin{bmatrix} -\frac{0.8s+1}{s+2} & \frac{\beta s}{s+2} \\ 1 & 0 \end{bmatrix}.$$

For $\beta = 0.2625$, $\rho(B) = 1.05$. Under this circumstance the system is unstable for any $\tau \geq 0$. For $\beta = 0.09$, $\rho(B) = 0.9$. Figure 3.8 plots the spectral radius $\rho(M_a(j\omega))$, which enables us to conclude that the system is stable independent of delay.

Finally, extensions may also be pursued for neutral systems with incommensurate delays. In particular, by mimicking the proof of Theorem 3.5, we obtain the following necessary and sufficient condition for delay-dependent **delay-independent** stability.

Theorem 3.23 *Let $r_k \geq 0$, $k = 1,\ 2,\ \cdots,\ m$ be independent, incommensurate delays. Then the system (3.35) is stable independent of delay if and only if*

(i) $\mu_{\mathsf{X}_m}(\widehat{B}) < 1$,

(ii) A_0 *is stable,*

(iii) $\left(I - \sum_{k=1}^{m} B_k\right)^{-1} \sum_{k=0}^{m} A_k$ *is stable, and*

(iv) $\mu_{\mathsf{X}_m}\left(M(j\omega)\right) < 1$, $\forall \omega > 0$,

where

$$\begin{aligned}\widehat{B}: &= \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix}, \\ M(s): &= \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} (sI - A_0)^{-1} \begin{bmatrix} sB_1 + A_1 & \cdots & sB_m + A_m \end{bmatrix}.\end{aligned}$$

Likewise, in light of Corollary 3.8 and its proof, it is straightforward to find a simplified condition for scalar differential-difference equations with incommensurate delays.

Corollary 3.24 *Let* $r_k \geq 0$, $k = 1, 2, \cdots, m$ *be independent, incommensurate delays. Then the scalar differential-difference equation*

$$y^{(n)}(t) + \sum_{k=1}^{q} b_k y^{(n)}(t - r_k) + \sum_{i=0}^{n-1} \sum_{k=0}^{q} a_{ki} y^{(i)}(t - r_k) = 0, \qquad r_k \geq 0$$

is stable independent of delay if and only if

(i) $\sum_{k=1}^{q} |b_k| < 1$,

(ii) $a_0(s)$ *is stable,*

(iii) $a_0(s) + \sum_{k=1}^{q} p_k(s)$ *is stable, and*

(iv)

$$\frac{\sum_{k=1}^{q} |p_k(j\omega)|}{|a_0(j\omega)|} < 1, \qquad \forall \omega > 0,$$

In conclusion, whether for computing the delay margin or to determine delay-independent stability, the frequency-sweeping conditions obtained herein bear no essential difference from their counterparts for retarded systems. Nevertheless, we point out that while it is possible to extend the preceding constant matrix tests also to neutral systems, such extensions will be more complex and computationally more intensive. We leave these extensions to the reader.

3.7 Summary

Our development of the frequency domain tests in this and the preceding chapter came under the heavy influence of robust control theory. Whether for systems with commensurate delays or those with incommensurate delays, the stability problem has been interpreted as one of robust stability. Indeed, from a conceptual standpoint, delay-independent and delay-dependent stability can both be held as notions of robust stability, whereas the delay values are uncertain within the specified intervals. The application of the continuity property of the stability exponent is reminiscent of the well-known *zero exclusion principle*, which has played a critical role in robust stability analysis and will continue to be an enabling tool in Chapter 4, in our study of robust stability of time-delay systems. The use of the structured singular values as stability criteria, as well as the use of the spectral radius for systems of commensurate delays, thus come more as a necessary outcome than a sheer coincidence. More broadly, both can be interpreted as generalized notions of gain, and the stability criteria as small gain conditions. Small gain type results can be suitably applied to time-varying systems; this has been demonstrated in the present chapter using examples involving time-varying delays, and will be further discussed in Chapter 8. They can also be extended to address uncertain time-delay systems, and used as a synthesis tool in control design for time-delay systems.

While it appears justified to contend that the stability problem is largely resolved in the case of commensurate delays, the problem remains open for systems with incommensurate delays. The proof of NP-hardness in this case leads to, on a firm theoretical basis, the somewhat pessimistic conclusion that it is unlikely to be tractable in general. One should therefore resort to computable, albeit approximate, stability tests. The structured singular values provide yet again relatively less conservative sufficient conditions, by means of its bounds. Various relaxations and derivatives, with varying degrees of conservatism versus computational ease, can be obtained from the bounds, among which the least conservative is the D-scaled upper bound. This bound can be computed rather efficiently using commercial software programs and yet may still poise to be an acceptable answer. Amid the generally pessimistic prospect, on the other hand, it should be understood that the NP-hardness is a worst-case, extreme measure of complexity. In other words, while the stability problem may be intractable in the worst case, the NP-hardness does not rule out the possibility that certain more specialized cases may still be benign enough to yield computable solutions; the rank-one case provides one such example.

In summary, frequency domain stability analysis of time-delay systems is a natural progression of classical stability results, both of which are built upon the fact that an LTI system, with or without delay, can be fully represented by its frequency domain description, and the system's stability be determined by the location of its characteristic roots. A frequency domain approach thus opens the door for many well-developed frequency domain tools. The continuity property of characteristic functions proves fundamental, and indeed constitutes the cornerstone of all frequency domain methods, both for systems with commensurate delays and for those with incommensurate delays; in both cases, it reduces effectively the delay-independent and delay-dependent stability problems into one of computing the zero-crossing frequencies of the characteristic function. The key thus lies in whether and how the zero-crossing frequencies can be computed.

Notes

3.7.1 Small gain theorem and μ

Robust control is an immensely rich subject of deep mathematical sophistication. There have been many well-written texts on the subject. The recent survey article by Chen and Tits [39] provides a concise summary on key progresses in robust stability analysis. The structured singular value is credited to Doyle [63] and Safonov [247], which has found widespread applications in robust control design. An in-depth study of this notion can be found in Packard and Doyle [229], and Zhou *et al.* [318]. A generalized μ with both complex- and real-valued uncertainties was suggested by Fan *et al.* [67], which can be useful in studying robust stability of time-delay systems. The small gain theorem is credited to Zames [314, 315], whose more recent treatise can be found in the texts by Desoer and Vidyasagar [60], Zhou *et al.* [318], and Chen and Gu [36]. The same references contain detailed discussions on norms and spaces relevant to the control context.

3.7.2 Stability of systems with incommensurate delays

Systems with incommensurate delays are an exclusive scene in Datko [54], Cooke and Ferreira [45], and Hale *et al.* [108], which all studied retarded as well as neutral delay systems. In fact, it was due to [54] and [45] that the continuity property was firmly established. Hale *et al.* [108] investigated in meticulous details the stability problem with the general state-space formulation, and gave the sweeping conditions for scalar differential-difference equations. The approach via the structured singular value, and the suggestion that the stability problem be treated as one of robust stability, were initiated by Chen and Latchman [38], which led to the μ characterization of conditions for stability independent of delay. The idea was later extended in development of delay-dependent conditions, by Niculescu and Chen [210], Huang and Zhou [128]. The results in Section 3.3 are mainly based on [38, 210] and [210]. Chen and Latchman [38], and Huang and

Zhou [127, 128] also addressed robust stability and stabilization of delay systems in this framework.

3.7.3 Complexity issues

Computational complexity theory is a routine course of study in computer science and operation research. Among numerous tests on the subject, Davis and Weyuker [58] offers rather thorough a classical treatment tuned to a typical theoretical computer science readership, while Blum *et al.* [17] focuses on the development of the theory in numerical analysis. Narratives of the theory aimed at a control readership are found in Vidyasagar [303], Chen and Gu [36]. The exposure in Section 3.4.1 is adapted from [36]. In the recent years, complexity studies have had a significant impact on control theoretical research, notably on problems found in robust stability analysis. A large number of control problems, both reasonably formulated and practically meaningful, have been found to be NP-hard. Pertinent cases include the computation of the so-called real, complex, and mixed real and complex μ [26, 274], the computation of bounds for μ [273, 74], and certainly, the stability problem for systems with incommensurate delays [276]. The NP-hardness proof given in Section 3.4.2 is essentially based on Toker and Ozbay [276].

3.7.4 Sufficient conditions and neutral systems

A wide variety of sufficient conditions exist for stability independent of delay. Some of the results most pretinent **pertinent** to the present chapter, to name a few, are obtained by Mori and co-workers [208, 203, 204, 205], Noldus [212], Wang *et al.* [307], Hmamed [120, 121], Bourlés [22], Luo and van den Bosch [191], Zhou [317], Gu and Lee [87], and Verriest *et al.* [291]. Among these results we note in particular the $\mathcal{H}_\infty$-norm, small gain type conditions in [317] and [87]. Section 3.5, however, is largely based on Chen *et al.* [40], which expands on the necessary and sufficient small gain conditions developed in [38], and provides a unified treatment for many of the aforementioned sufficient conditions. Many additional sufficient, delay-independent and delay-dependent conditions have been developed in time domain, based on an Lyapunov-Krasovskii-Razumikhin approach. These conditions are typically posed as solutions to LMI problems, a subject to be undertaken in the subsequent chapters.

While retarded delay systems have been extensively studied, fewer results are available for neutral systems, though several of the classical tests, such as [119] and [306], remain applicable. Section 3.6 is adapted from Chen [35], and draws additionally upon [108]. For more discussions on neutral delays systems and more references, we refer to Niculescu [215].

4

Frequency domain robust stability analysis

no change has been made except "notes and references" has been changed to "notes" to be consistent among all the chapters.

4.1 Uncertain systems

In the previous chapters the stability property of a given time-delay system has been studied. In engineering applications, it is now very common that one does not know exactly the system under investigation, that is, the system contains some elements (blocks) which are uncertain. Usually it is known that these uncertain elements belong to some specific admissible domains, which in turn depend on the nature of the elements and also on the information available about the system. In other words it is known only that the system belongs to the family of systems which arises when the uncertain elements (blocks) range over the admissible domains and therefore one may treat the family as a new object for analysis. This family is referred to as an *uncertain system*. When it is possible to show that all systems of the family are stable it guarantees the stability of the original system which is a particular member of the family.

In this chapter several basic results on stability of uncertain time-delay systems will be presented.

We start with linear time invariant delay systems whose description contains parametric uncertainty.

4.2 Characteristic quasipolynomial

It has been already mentioned that a linear time invariant system with concentrated delays

$$\sum_{k=0}^{N}\left[A_k\dot{x}(t-\tau_k)+B_kx(t-\tau_k)\right]=0,\quad \det(A_0)\neq 0, \tag{4.1}$$

where $0=\tau_0<\tau_1<...<\tau_N$, is exponentially stable if and only if there exists a positive ε such that the real part of every zero of the characteristic function,

$$f(s) = \det\left(\sum_{k=0}^{N}(A_k s + B_k)\, e^{-\tau_k s}\right) = \sum_{k=0}^{n}\sum_{i=0}^{m} a_{ki}s^{n-k}e^{-r_i s}, \qquad (4.2)$$

of the system is less than $-\varepsilon$. In this case $f(s)$ is called Hurwitz stable. In (4.2) coefficients a_{ki} are real or complex numbers and exponent coefficients are real and ordered as follows $0 = r_0 < r_1 < ... < r_m$. Function $f(s)$ is known as a *quasipolynomial* and may be written in different forms

$$f(s) = \sum_{i=0}^{m} p_i(s)e^{-r_i s} = \sum_{k=0}^{n} \psi_k(s)s^{n-k}, \qquad (4.3)$$

where polynomials

$$p_i(s) = a_{0i}s^n + a_{1i}s^{n-1} + ... + a_{ni}, \quad i = 0, 1, ..., m,$$

and quasipolynomial

$$\psi_k(s) = a_{k0}e^{-r_0 s} + a_{k1}e^{-r_1 s} + ... + a_{km}e^{-r_m s}, \quad k = 0, 1, ..., n.$$

Example 4.1 *Let us consider the scalar equation*

$$\dot{x}(t) + \dot{x}(t-\tau) + x(t) = 0. \qquad (4.4)$$

Its characteristic quasipolynomial is

$$f(s) = s + se^{-\tau s} + 1.$$

All zeros of the quasipolynomial have negative real part. To check this we assume first that it has a zero $s_0 = \alpha + j\beta$, with positive real part, $\alpha > 0$. Then, the real and the imaginary parts of $f(s_0)$ are equal to zero

$$\begin{cases} \alpha[1 + e^{-\tau\alpha}\cos(\tau\beta)] + \beta e^{-\tau\alpha}\sin(\tau\beta) + 1 = 0 \\ \beta[1 + e^{-\tau\alpha}\cos(\tau\beta)] - \alpha e^{-\tau\alpha}\sin(\tau\beta) = 0 \end{cases}.$$

From these equations we conclude that

$$1 + e^{-\tau\alpha}\cos(\tau\beta) = -\frac{\alpha}{\alpha^2 + \beta^2}.$$

Our assumption, $\alpha > 0$, implies that the left hand side of the last equality is positive while the right hand side is negative. Therefore $f(s)$ has no zeros with positive real part. Assume now that $s = j\beta$, then $\beta \neq 0$, because $f(0) = 1 \neq 0$. In this case

$$\beta\sin(\tau\beta) + 1 = 0 \qquad (4.5)$$

$$\beta[1 + \cos(\tau\beta)] = 0 \qquad (4.6)$$

From (4.6), we may conclude that

$$\cos(\tau\beta) = -1.$$

This implies that $\sin(\tau\beta) = 0$ *which contradicts (4.6). In other words* $f(s)$ *has no zeros in the closed right half-plane of the complex plane. Therefore all zeros of the quasipolynomial have negative real part. On the other hand equation (4.4) is not exponential stable because*

$$\sup \mathrm{Re}\{ \ s \mid f(s) = 0 \ \} = 0.$$

This statement can be easily checked if we observe that if s_0 *is a zero of* $f(s)$ *then*

$$\left| e^{-\tau s_0} \right| = \left| 1 + \frac{1}{s_0} \right|.$$

Therefore, as $s_0 \to \infty$*, it becomes asymptotically close to the imaginary axis.*

The above example shows that for a neutral time delay system to be exponentially stable it is not enough to check that all zeros of it's characteristic function lie on the open right half complex plane.

Remark 4.1 *There are also examples of neutral systems where all the zeros are on the open left half complex plane, but the system is not stable in the sense of Lyapunov in addition to not exponentially stable.*

4.3 Zeros of a quasipolynomial

In this subsection several important results about zeros of a quasipolynomial function will be presented.

In order to make our analysis a little bit more general we consider a quasipolynomial

$$f(s) = \sum_{k=0}^{n} \sum_{i=0}^{m} (a_{kj} + jb_{ki}) s^{n-k} e^{(\alpha_i + j\beta_i)s}, \tag{4.7}$$

where j is the imaginary unit and a_{ki}, b_{ki}, α_i, β_i are real numbers. This function may be written in two forms

$$f(s) = \sum_{j=0}^{m} p_i(s) e^{(\alpha_i + j\beta_i)s} = \sum_{k=0}^{n} \psi_k(s) s^{n-k},$$

where

$$p_i(s) = \sum_{k=0}^{n} (a_{ki} + jb_{ki}) s^{n-k}, \quad i = 0, 1, ..., m,$$

and

$$\psi_k(s) = \sum_{i=0}^{m} (a_{ki} + jb_{ki}) e^{(\alpha_i + j\beta_i)s}, \quad k = 0, 1, ..., n.$$

In the following we assume the that

- exponent coefficients $(\alpha_i + j\beta_i)$, $i = 0, 1, ..., m$, are distinct complex numbers;
- polynomials $p_i(s)$, $i = 0, 1, ..., m$, are not trivial.

Under these conditions $f(s)$ may have a finite number of zeros only in the case when $m = 0$. In the following we assume that $m > 0$, if not explicitly stated otherwise.

4.3.1 Exponential diagram

Let us mark on the complex plane points which are complex conjugate numbers to the exponent coefficients of $f(s)$, $(\alpha_i - j\beta_i)$, $i = 0, 1, ..., m$, and form then the convex hull of these points.

The convex hull is in general a convex polygon whose boundary consists of a finite number of segments, see Fig. 4.1.

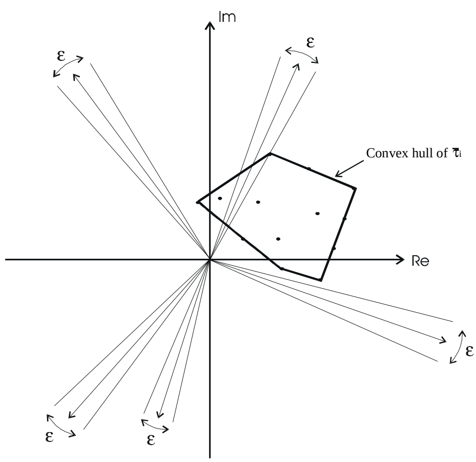

FIGURE 4.1. **Exponential diagram.**

The boundary polygon of the convex hull is known as *exponential diagram* of quasipolynomial $f(s)$. Now, with each boundary segment we associate a ray emanating from the origin and in the direction of the outer normal to the boundary segment. In the case when the exponential diagram degenerates into a segment of the complex plane, we consider the segment as a two-sided one, and therefore two opposite rays should be associated with the segment.

Let these rays be

$$\Gamma_i = \left\{ s = \rho e^{j\varphi_i} \mid \quad \rho \in [0, \infty) \right\}, \quad i = 1, 2, ..., N.$$

Then a sufficiently small $\varepsilon > 0$ may be chosen such that the ε-sectors

$$\begin{aligned} \Gamma_i(\varepsilon) &= \left\{ s = \rho e^{j\varphi} \,\middle|\, \rho \in [0, \infty),\ \varphi \in [\varphi_i - \frac{1}{2}\varepsilon, \varphi_i + \frac{1}{2}\varepsilon] \right\}, \\ i &= 1, 2, ..., N. \end{aligned}$$

have no common point except the origin.

Theorem 4.1 *For every $\varepsilon > 0$ there is $R(\varepsilon) > 0$, such that all zeros of $f(s)$ with magnitudes greater than $R(\varepsilon)$ lie inside one of the ε-sectors $\Gamma_i(\varepsilon)$, $i = 1, 2, ..., N$.*

Proof. Let us consider a vertex of the exponential diagram, for simplicity we assume that the vertex is $\alpha_0 - j\beta_0$. The vertex connects two consecutive boundary segments of the diagram. Let Γ_i, Γ_{i+1} be the rays associated with these boundary segments. Let Φ be the pointed sector of the complex plane between these rays. If the angular size of sector Φ is φ ($\varphi \in (0, \pi]$) then for any $\varepsilon \in (0, \varphi)$ sectors $\Gamma_i(\varepsilon)$ and $\Gamma_{i+1}(\varepsilon)$ have no common points except $s = 0$. Define by $\Phi(\varepsilon)$ the part of sector Φ which lie between sectors $\Gamma_i(\varepsilon)$ and $\Gamma_{i+1}(\varepsilon)$. Now we calculate

$$\rho = \min_{i \neq k} \left\{ \sqrt{(\alpha_i - \alpha_k)^2 + (\beta_i - \beta_k)^2} \right\} > 0.$$

Direct geometric calculations shows that for any $s \in \Phi(\varepsilon)$ we have

$$\operatorname{Re}\left\{ [(\alpha_i - \alpha_0) + j(\beta_i - \beta_0)]\, s \right\} \leq -\rho\, |s| \sin\left(\frac{\varepsilon}{2}\right), \text{ for } i = 1, 2, ..., m. \quad (4.8)$$

Now we select $R > 0$ such that all zeros of all polynomials $p_i(s)$, $i = 0, 1, ..., m$, lie inside the disc

$$D_R = \{ \ s \mid \ |s| < R \ \}.$$

Let $|s| \geq R$ then $f(s)$ can be factorized as

$$f(s) = p_0(s) e^{(\alpha_0 + j\beta_0)s} \left[1 + \sum_{i=1}^{m} \frac{p_i(s)}{p_0(s)} e^{[(\alpha_i - \alpha_0) + j(\beta_i - \beta_0)]s} \right]. \quad (4.9)$$

If additionally $s \in \Phi(\varepsilon)$ then by inequality (4.8)

$$\left| \frac{p_i(s)}{p_0(s)} e^{[(\alpha_i - \alpha_0) + j(\beta_i - \beta_0)]s} \right| \leq \left| \frac{p_i(s)}{p_0(s)} \right| e^{-\rho |s| \sin\left(\frac{\varepsilon}{2}\right)}, \text{ for } i = 1, 2, ..., m,$$

and we can conclude that

$$\left|\frac{p_i(s)}{p_0(s)}e^{[(\alpha_i-\alpha_0)+j(\beta_i-\beta_0)]s}\right| \to 0 \text{ as } |s| \to \infty$$

Hence there exists $R(\varepsilon) \geq R$ such that for all $s \in \Phi(\varepsilon)$ with magnitude greater than $R(\varepsilon)$ the expression in square brackets in (4.9) is not zero. The first two factors in (4.9) are also non-zero for such values of s. It means that $f(s)$ has no zeros in $\Phi(\varepsilon)$ with magnitude greater than $R(\varepsilon)$. Repeating these arguments for all other vertex points of the exponential diagram we arrive to the theorem statement. ■

Remark 4.2 *A more detailed analysis, based on the argument principle shows that each ε-sector, $\Gamma_i(\varepsilon)$, contains an infinite (countable) number of zeros of $f(s)$.*

From the above theorem one can easily derive the following consequence about zeros of $f(s)$.

Corollary 4.2 *If among exponential coefficients $(\alpha_i + j\beta_i)$, $i = 0, 1, ..., m$, there are at least two with distinct imaginary parts then $f(s)$ has zeros with arbitrarily large positive real parts.*

Proof. It follows from geometric consideration that in this case one of the rays points towards into the open right half complex plane. So, the corresponding ε-sector lies entirely in this half plane. According to Remark 4.2 inside the sector $f(s)$ has an infinite number of zeros. ■

This corollary implies that the only case when $f(s)$ may have all zeros in the open left half complex plane is when all exponent coefficients of the quasipolynomial have the same imaginary part. In this case without any loss of generality one may assume that all exponent coefficients are real numbers.

4.3.2 Potential diagram

As it has been shown in the previous subsection only quasipolynomial with real exponent coefficients may have all zeros in the open left half complex plane. Let

$$f(s) = \sum_{k=0}^{n}\sum_{i=0}^{m}(a_{ki} + jb_{ki})s^{n-k}e^{\alpha_i s}. \tag{4.10}$$

Here $\alpha_0 < \alpha_1 < ... < \alpha_m$. The quasipolynomial may also be written as

$$f(s) = \sum_{\nu=0}^{N} c_\nu s^{k_\nu} e^{\gamma_\nu s}, \tag{4.11}$$

where all complex coefficients, c_ν, are supposed to be nonzero, and additionally we assume that there are no two terms in the sum for which both

k_ν and γ_ν are the same, that is, the terms have either different potential factors or different exponential factors.

Let us superpose (γ, k)-plane with the complex plane in such a way that γ-axis coincides with the real axis of the complex plane and in turn k-axis coincides with the imaginary one. On this joint plane we mark points (γ_ν, k_ν) corresponding to all terms in (4.11).

Then we construct the upper part of the envelope of these points, see Fig. 4.2.

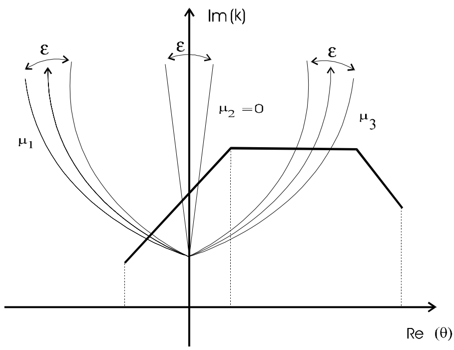

FIGURE 4.2. **Potential diagram.**

This upper part is known as *potential diagram* of $f(s)$, and it consists of a finite number of segments. Let M be the number of these segments. With each segment of the diagram we associate the logarithmic curve

$$\begin{aligned}\Lambda_\kappa &= \{s = x + iy \mid x = \mu_\kappa \ln(y), \quad y \in [1, \infty)\}, \\ \kappa &= 1, 2, ..., M.\end{aligned}$$

Here μ_κ is a real number such that vector $(\mu_\kappa, 1)$ is along the direction of the outer normal to the corresponding segment. If $\mu_\kappa > 0$, then Λ_κ lies in the right half complex plane, while the curve corresponding to $\mu_\kappa < 0$ belongs to the left half complex plane. For $\mu_k = 0$ the corresponding Λ_κ coincides with the imaginary axis.

For sufficiently small $\varepsilon > 0$ the logarithmic ε-sectors

$$\begin{aligned}\Lambda_\kappa(\varepsilon) &= \left\{s = x + iy \mid x \in [(\mu_\kappa - \frac{1}{2}\varepsilon)\ln(y), (\mu_\kappa + \frac{1}{2}\varepsilon)\ln(y)],\ y \in [1, \infty)\right\} \\ \kappa &= 1, 2, ..., M.\end{aligned}$$

have no common points except $(0, 1)$.

The following theorem describes distribution of zeros of quasipolynomial (4.11).

Theorem 4.3 *For every $\varepsilon > 0$ there exists $R_1(\varepsilon) > 0$, such that all zeros of $f(s)$ in the upper half complex plane with magnitudes greater than $R_1(\varepsilon)$ lie in the union of logarithmic ε-sectors $\Lambda_\kappa(\varepsilon)$, $\kappa = 1, 2, ..., M$.*

Proof. Let us consider an interior vertex of the potential diagram. For the sake of definiteness let this vertex be (γ_0, k_0). There are two neighboring segments of the diagram which have (γ_0, k_0) as the common vertex. Let Λ_κ and $\Lambda_{\kappa+1}$ be the logarithmic curves associated with these segments. Denote by Ψ the part of the upper half complex plane between the curves. Each point, s, of Ψ can be represented as

$$s = \mu \ln(y) + jy \tag{4.12}$$

for some $\mu \in (\mu_k, \mu_{k+1})$, and $y > 1$. Here μ_k and μ_{k+1} are coefficients of the corresponding logarithmic curves Λ_k and Λ_{k+1} ($\mu_k < \mu_{k+1}$). If ε is positive and

$$\varepsilon < \frac{\mu_{k+1} - \mu_k}{2},$$

then logarithmic sectors $\Lambda_k(\varepsilon)$ $\Lambda_{k+1}(\varepsilon)$ have no common points except $(0, 1)$. In this case we denote by $\Psi(\varepsilon)$ the part of the upper half complex plane between the sectors. In the set $\Psi(\varepsilon)$ the quasipolynomial (4.11) can be factorized as follows

$$f(s) = c_0 s^{k_0} e^{\gamma_0 s} \left[1 + \sum_{\nu=1}^{N} \frac{c_\nu}{c_0} s^{k_\nu - k_0} e^{(\gamma_\nu - \gamma_0)s} \right]. \tag{4.13}$$

Let us first define the value

$$\rho = \min_{\nu=1,\ldots,N} \sqrt{(k_\nu - k_0)^2 + (\gamma_\nu - \gamma_0)^2} > 0,$$

and the value

$$\mu^* = \max_{k=1,\ldots,M} |\mu_k|.$$

Then simple geometric manipulations show that for all $\mu \in [\mu_k + \varepsilon, \mu_k - \varepsilon]$ the following inequality holds

$$(k_\nu - k_0) + \mu(\gamma_\nu - \gamma_0) \leq -\varepsilon \frac{\rho}{\sqrt[2]{1 + (\mu^*)^2}}. \tag{4.14}$$

Now we can see that along the logarithmic line $s = \mu \ln(y) + jy$, $y \geq 1$, where $\mu \in [\mu_k + \varepsilon, \mu_k - \varepsilon]$,

$$\left| \frac{c_\nu}{c_0} s^{k_\nu - k_0} e^{(\gamma_\nu - \gamma_0)s} \right| \leq \left| \frac{c_\nu}{c_0} \right| 2^{|k_\nu - k_0|} y^{(k_\nu - k_0) + \mu(\gamma_\nu - \gamma_0)}.$$

Taking into account inequality (4.14) we arrive to conclusion that

$$\left| \frac{c_\nu}{c_0} s^{k_\nu - k_0} e^{(\gamma_\nu - \gamma_0)s} \right| \leq \left| \frac{c_\nu}{c_0} \right| 2^{|k_\nu - k_0|} y^{-\varepsilon \frac{\rho}{\sqrt[2]{1 + (\mu^*)^2}}}.$$

The right hand side of the last inequality approaches zero as $y \to \infty$. Therefore the sum in the square brackets in (4.13) also approaches zero as $y \to \infty$. In other words for sufficiently large values of y function $f(s)$ has no zeros along any logarithmic curve (4.12) where $\mu \in [\mu_k - \varepsilon, \mu_k + \varepsilon]$. This observation proves the fact that there exists $R > 0$ such that no zero of $f(s)$ with magnitude greater than R lies in the domain $\Psi(\varepsilon)$. These arguments can be applied for all other interior vertices of the potential diagram.

A slightly modified arguments can be also applied for the two extreme vertices of the diagram. ∎

Remark 4.3 *The zeros of $f(s)$ with large magnitudes in the lower half complex plane lie in the union of logarithmic ε-sectors obtained by the mirror image of $\Lambda_\kappa(\varepsilon)$, $\kappa = 1, 2, ..., M$, with respect to the real axis.*

Remark 4.4 *Applying the principle of argument it may be shown that $f(s)$ has infinite (countable) number of zeros in every logarithmic ε-sector $\Lambda_\kappa(\varepsilon)$.*

It follows from the theorem that $f(s)$ has zeros with arbitrarily large positive real parts when at least one of the values μ_κ is positive. Using the potential diagram of $f(s)$ one can conclude that such positive μ_κ exists only if the outer normal of one of the segments, forming the diagram, points towards the open right half complex plane. In order to guarantee the absence of zeros of $f(s)$ with arbitrary large positive real parts one has to assume that one of the terms in (4.11), say for example $c_0 s^{k_0} e^{\gamma_0 s}$, satisfies the following two conditions

- $k_0 \geq k_j$, $j = 1, 2, ..., N$;
- $\gamma_0 \geq \gamma_j$, $j = 1, 2, ..., N$.

When such a term exists it is called the *principal term* of quasipolynomial (4.11).

Corollary 4.4 *Quasipolynomial (4.11) may have all zeros in the open left half complex plane only if it has a principal term.*

4.4 Uncertain quasipolynomial

It has been shown in the previous section that quasipolynomial (4.2) may have all zeros in the open left half-plane only if it has a principal term. This condition may be expressed as follows:

$$\deg(p_0) \geq \deg(p_i), \quad i = 1, 2, ..., m. \tag{4.15}$$

Without any loss of generality one may assume that $a_{00} \neq 0$, then $a_{00} s^n e^{-r_0 s}$ is the principal term of $f(s)$.

With quasipolynomial (4.2) one may associate two vectors, the coefficient vector

$$\mathbf{a} = (a_{00}, ..., a_{0n}, a_{10}, ..., a_{1n}, ..., a_{mn}),$$

and the exponent coefficient vector

$$\mathbf{r} = (r_1, r_2, ..., r_m).$$

When it is convenient we will write $f(s, \mathbf{a}, \mathbf{r})$ instead of $f(s)$ in order to show explicitly these vectors. Now every quasipolynomial of the form (4.2) may be uniquely interpreted as the corresponding point $(\mathbf{a}, \mathbf{r})$ of the coefficient space $\mathbf{L}$. Assume now that there is a family F of quasipolynomials of the form (4.2). With this family one may associate the corresponding set

$$\mathbf{Q}_F = \{ \ (\mathbf{a}, \mathbf{r}) \ | \ f(s, \mathbf{a}, \mathbf{r}) \in F \ \} \subset \mathbf{L}.$$

in the coefficient space. In turn $\mathbf{Q}_F$ defines uniquely the family

$$F = \{ \ f(s, \mathbf{a}, \mathbf{r}) \ | \ (\mathbf{a}, \mathbf{r}) \in \mathbf{Q}_F \ \},$$

of quasipolynomials. Because of this one to one correspondence we will sometimes treat $\mathbf{Q}_F$ and F as the same object when it may not produce any ambiguity.

The family F is often referred to as an *uncertain quasipolynomial.*

We say that F is *robustly stable* if all members of the family are Hurwitz stable.

4.4.1 Value set

Let us introduce the concept of value set an uncertain quasipolynomial F. The *value set* of F is defined for a given complex number s_0 as the following set

$$V_F(s_0) = \{ \ f(s_0) \ | \ f \in F \ \},$$

of the complex plane.

4.4.2 Zero exclusion principle

There are several interesting results in the robust stability analysis which are based on the zero exclusion principle. In some sense this principle serves as a fundamental statement which allows us, under some mild assumptions on uncertain quasipolynomials, to obtain a very general robust stability criterion for F.

The assumptions one needs to formulate the principle are the following:

- **A**1: Every member of the uncertain quasipolynomial F has a non-zero principal term. According to our previous discussion this assumption may be stated as:

$$\deg(p_0) = n \geq \deg(p_i), \quad i = 1, 2, ..., m,$$

for all $f(s) \in F$.

- **A**2: The exponent coefficient vector of every member of the uncertain quasipolynomial has only positive components, that is, $r_i > 0$, $i = 1, 2, ..., m$, for every $f(s) \in F$.
- **A**3: There exist $R > 0$ and $\varepsilon > 0$ such that for every $f(s)$ in the uncertain quasipolynomial the corresponding quasipolynomial $\psi_0(s)$, see decomposition (4.3), has no zeros of magnitude greater than R (if any) with real part greater than $-\varepsilon$.
- **A**4: The set $\mathbf{Q}_F$ is compact and path-wise connected.

Remark 4.5 *The assumptions* A1-A3 *are necessary conditions for the robust stability of F.*

Remark 4.6 *Assumption* A*3 holds automatically when all quasipolynomials from F satisfy*

$$\deg(p_0) = n > \deg(p_i), \quad i = 1, 2, ..., m.$$

Now everything is ready to state the zero exclusion principle.

Theorem 4.5 *Let the uncertain quasipolynomial F satisfy assumptions* A*1*–A*4. Then all members of F are Hurwitz stable if and only if*

- *at least one member of F is Hurwitz stable,*
- *for every point $s = j\omega$ on the imaginary axis, the value set, $V_F(j\omega)$, computed at this point does not contain the origin of the complex plane.*

Proof. The necessity part is nearly obvious. In fact, the first condition of the theorem holds because all quasipolynomials in F are Hurwitz stable. To check the second condition let us assume that $V_F(j\omega_0)$ contains the origin, then there exists a quasipolynomial in the family that has $j\omega_0$ as a zero and therefore the quasipolynomial is not Hurwitz stable. This contradicts with Hurwitz stability of all members of F.

Now let us address the sufficiency part. It follows directly from assumption **A**3 that there exists a positive value R_0 such that no one member of F has zeros in the complex half-plane

$$\left\{ s \;\middle|\; \mathrm{Re}(s) \geq -\frac{1}{2}\varepsilon \right\},$$

with magnitude greater or equal than R_0. According to the second condition of the theorem there are no quasipolynomials in F which have zeros on the imaginary axis.

Assume by contradiction that there exists $f(s, \mathbf{a}_0, \mathbf{r}_0)$ in F that is not Hurwitz stable. It means that the quasipolynomial has at least one zero, s_0, with positive real part. By the first condition of theorem, there is $f(s, \mathbf{a}_1, \mathbf{r}_1) \in F$ which is Hurwitz stable, that is, all zeros of $f(s, \mathbf{a}_1, \mathbf{r}_1)$ have negative real parts. According to assumption **A**4 there is a path in $\mathbf{Q}_F$:

$$\mathbf{a} = \mathbf{a}_\mu, \ \mathbf{r} = \mathbf{r}_\mu, \ \mu \in [0, 1],$$

connecting points $(\mathbf{a}_0, \mathbf{r}_0)$ and $(\mathbf{a}_1, \mathbf{r}_1)$. Let us define

$$\mu_0 = \inf \{\mu > 0 \,|\text{such that } f(s, \mathbf{a}_\mu, \mathbf{r}_\mu) \text{ is Hurwitz stable}\}.$$

It follows directly from the definition of the value that $\mu_0 \in (0, 1)$. Polynomial $f(s, \mathbf{a}_{\mu_0}, \mathbf{r}_{\mu_0})$ can not be Hurwitz stable, otherwise by a continuity argument, assumptions **A**1-**A**3 imply that there should be a positive ν such that for every $\mu \in (\mu_0 - \nu, \mu_0 + \nu)$ quasipolynomial $f(s, \mathbf{a}_\mu, \mathbf{r}_\mu)$ is also Hurwitz stable which contradicts the definition of μ_0. On the other hand, $f(s, \mathbf{a}_{\mu_0}, \mathbf{r}_{\mu_0})$ can not have a zero with positive real part because otherwise for μ a little bit smaller than μ_0 quasipolynomial $f(s, \mathbf{a}_\mu, \mathbf{r}_\mu)$ will also have a zero in the open right half complex plane which contradicts again to the choice of μ_0. It also cannot have zeros on the imaginary axis by the second condition of the theorem. So, $f(s, \mathbf{a}_{\mu_0}, \mathbf{r}_{\mu_0})$ cannot be unstable, either. This contradiction has origin in our assumption that $f(s, \mathbf{a}_0, \mathbf{r}_0)$ is not a Hurwitz stable quasipolynomial. This conclusion completes the proof of the sufficiency part of the theorem. ■

The second condition of the theorem admits certain relaxation.

Remark 4.7 *The second condition of theorem 4.5 is equivalent to the following two conditions:*

- *at least for one point, $j\omega_0$, of the imaginary axis $0 \notin V_F(j\omega)$;*
- *$0 \notin \partial V_F(j\omega)$ for all other points of the imaginary axis. Here $\partial V_F(j\omega)$ stands for the boundary of the value set $V_F(j\omega)$.*

Proof. The statement follows directly from the property that the value set changes continuously with respect to ω. ■

It is a difficult task in some applications to check the second condition of the theorem because there are no effective algorithms for the exact construction of the value set and also in many cases one can not check this condition for all points on the imaginary axis but only for a finite set of such points. Having in mind this observation it seems quite natural to look for particular cases when this condition may be efficiently checked. Certainly in such cases some additional restrictions should be imposed on the set $\mathbf{Q}_F$. One of such particular cases is given by the edge theorem.

4.5 Edge theorem

In this section the class of polytopic families of quasipolynomials will be studied. For these families there is no uncertainty in the exponent coefficients, that is, all r_i , $i = 1, 2, ..., m$, are fixed numbers. And uncertainty in the coefficient vector has the polytopic nature. In order to define formally such a family one needs a finite set

$$f^{(\nu)}(s) = \sum_{k=0}^{n} \sum_{i=0}^{m} a_{ki}^{(\nu)} s^{n-k} e^{-r_i s}, \quad \nu = 1, 2, ..., N,$$

of quasipolynomials, where as before, $0 = \tau_0 < \tau_1 < ... < \tau_m$. The polytopic family is defined as the convex hull of the quasipolynomials

$$P = \left\{ f(s) = \sum_{\nu=1}^{N} \mu_\nu f^{(\nu)}(s) \;\middle|\; \mu_\nu \geqslant 0,\ \nu = 1, 2, ..., N;\ \sum_{\nu=1}^{N} \mu_\nu = 1 \right\}.$$

Quasipolynomials $f^{(\nu)}(s)$, $\nu = 1, 2, ..., N$, are called generators of the polytopic family. The set of generators is called the minimal one if after exclusion of any one of the generators the resulting polytopic family does not coincide with P. Let the coefficient vector of $f^{(\nu)}(s)$ be $\mathbf{a}^{(\nu)}$, $\nu = 1, 2, ..., N$.

The corresponding set $\mathbf{Q}_P$ for the family is the direct product of the polytope

$$\mathbf{A} = \left\{ \mathbf{a} = \sum_{\nu=1}^{N} \mu_\nu \mathbf{a}^{(\nu)} \;\middle|\; \mu_\nu \geqslant 0,\ \nu = 1, 2, ..., N;\ \sum_{\nu=1}^{N} \mu_\nu = 1 \right\},$$

and the point $\mathbf{r} = (r_1, r_2, ..., r_m)$:

$$\mathbf{Q}_P = \mathbf{A} \times \mathbf{r}.$$

It is clear that the vertices of $\mathbf{A}$ are coefficient vectors of generator quasipolynomials. When the set of generators is the minimal one then every $\mathbf{a}^{(\nu)}$ is a vertex of $\mathbf{A}$. It explains why quasipolynomials $f^{(\nu)}(s)$, $\nu = 1, 2, ..., N$, are called vertex quasipolynomials of P.

It is a necessary condition for the Hurwitz stability of a given quasipolynomial $f(s)$ that it has principal term. To prevent unnecessary technical complications associated with a possible "degree droping" the following technical condition is assumed to hold.

Condition 4.1 *The principal coefficient of every generator $f^{(\nu)}(s)$ is positive, $a_{00}^{(\nu)} > 0$, $\nu = 1, 2, ..., N$.*

This condition guarantees that all members of P have principal term, because now for arbitrary $f(s) \in P$ the coefficient

$$a_{00} = \sum_{\nu=1}^{N} \mu_\nu a_{00}^{(\nu)} > 0.$$

Generator $f^{(\nu)}(s)$ can be written in two different forms as

$$f^{(\nu)}(s) = \sum_{i=0}^{m} p_i^{(\nu)}(s) e^{-r_i s} = \sum_{k=0}^{n} \psi_k^{(\nu)}(s) s^{n-k},$$

where

$$p_i^{(v)}(s) = \sum_{k=0}^{n} a_{ki}^{(\nu)} s^{n-k}, \text{ and } \psi_k^{(v)}(s) = \sum_{i=0}^{m} a_{ki}^{(\nu)} e^{-r_i s}.$$

Now one may associate with P the auxiliary family

$$\Psi = \left\{ \psi(s) = \sum_{\nu=1}^{N} \mu_\nu \psi_0^{(\nu)}(s) \;\middle|\; \mu_\nu \geqslant 0,\ \nu = 1, 2, ..., N;\ \sum_{\nu=1}^{N} \mu_\nu = 1 \right\},$$

which also has the polytopic structure.

The following condition is needed to guarantee that all members of P are Hurwitz stable. Actually this condition is a consequence of assumption **A**3.

Condition 4.2 *There exist $\varepsilon > 0$ and $R > 0$ such that members of Ψ have no zeros in the half-plane*

$$\{\, s \mid \mathrm{Re}(s) \geq -\varepsilon \;\}$$

with magnitude greater than R.

Remark 4.8 *Condition 4.2 holds automatically when for all generators $\deg(p_0^{(\nu)}) = n > \deg(p_i^{(\nu)})$, $i = 1, 2, ..., m$.*

Remark 4.9 *Let $r_1, r_2, ..., r_m$ be such numbers that $r_i = k_i r$, where all k_i, $i = 1, 2, ..., m$, are natural numbers. Then condition 4.2 holds if and only if all polynomials from*

$$\Pi = \left\{ \sum_{\nu=1}^{N} \mu_\nu \pi^{(\nu)}(z) \;\middle|\; \mu_\nu \geqslant 0,\ \nu = 1, 2, ..., N;\ \sum_{\nu=1}^{N} \mu_\nu = 1 \right\},$$

are anti-Schur, that is, all zeros of every polynomial from Π lie outside the closed unit disc of the complex plane. Here

$$\pi^{(\nu)}(z) = \sum_{i=0}^{m} a_{0i}^{(\nu)} z^{k_i}, \;\; \nu = 1, 2, ..., N.$$

For the rest of the section a key role is played by the edges of **A**. An edge is a segment connecting two vertices of **A** such that the set **A** without the segment remains convex. In other words, an edge is a one dimensional face of polytope **A**. Every edge can be written as the convex hull of two vertices

$$\mu \mathbf{a}^{(\alpha)} + (1-\mu)\mathbf{a}^{(\beta)}, \text{ where } \mu \in [0, 1]\,.$$

One may associate the corresponding edge subfamily of quasipolynomials with the edge

$$\mu f^{(\alpha)}(s) + (1-\mu) f^{(\beta)}(s), \text{ where } \mu \in [0,1]\,. \tag{4.16}$$

Let us denote by E_P the union of all edge subfamilies of P. We can state the following Edge Theorem.

Theorem 4.6 *Let family P satisfy conditions 4.1, 4.2. All members of the family are Hurwitz stable if and only if all quasipolynomials of the edge subfamilies from E_P are Hurwitz stable.*

Proof. Necessity is obvious, because all quasipolynomials of the edge subfamilies are also members of P.

To check sufficiency, we observe first that the value set $V_P(s_0)$ computed in an arbitrary point s_0 of the complex plane is the polygon

$$V_P(s_0) = \left\{ \sum_{\nu=1}^{N} \mu_\nu f^{(\nu)}(s_0) \;\middle|\; \mu_\nu \geqslant 0,\ \nu = 1,2,...,N;\ \sum_{\nu=1}^{N} \mu_\nu = 1 \right\},$$

whose boundary segments are composed of images of some edge subfamilies. That is, for every point $z \in \partial V_P(s_0)$ there exists an edge subfamily (4.16) such that $z = \mu_0 f^{(\alpha)}(s_0) + (1-\mu_0) f^{(\beta)}(s_0)$ for some $\mu_0 \in [0,1]$.

Conditions 4.1- 4.2 imply that there exists $R > 0$ such that no one of quasipolynomials from P may have zeros in the closed complex half-plane

$$\left\{ s \;\middle|\; \operatorname{Re}(s) \geq -\frac{1}{2}\varepsilon \right\},$$

with magnitude greater than R. Assume by contradiction that there is an unstable quasipolynomial $f_1(s) \in P$. Then it should have a zero, say s_1, with $\operatorname{Re}(s_1) \geq 0$. It means that $0 \in V_P(s_1)$. Observe that

$$f(s) \to \infty, \text{ as } \operatorname{Re}(s) \to \infty$$

for every $f(s) \in P$. Hence, there exists an s_0 with sufficiently large real part such that no member of P has s_0 as a zero, in other words,. $0 \notin V_P(s_0)$. Define now $s_\gamma = (1-\gamma)s_0 + \gamma s_1$ and compute the value set $V_P(s_\gamma)$. For $\gamma = 0$ this set does not contain the origin, while for $\gamma = 1$ it does. By continuity arguments there exists $\gamma_0 \in (0,1]$, such that $0 \in \partial V_P(s_{\gamma_0})$. It is clear that $\operatorname{Re}(s_{\gamma_0}) \geq 0$. According to our previous observation there should be an edge family (4.16), such that $0 = \mu_0 f^{(\alpha)}(s_\gamma) + (1-\mu_0) f^{(\beta)}(s_\gamma)$ for some $\mu_0 \in [0,1]$. In other words, the edge quasipolynomial $f_0(s) = \mu_0 f^{(\alpha)}(s) + (1-\mu_0) f^{(\beta)}(s)$ is not stable. This conclusion contradicts the second condition of the theorem. The contradiction is a direct consequence of our assumption that family P has at least one unstable quasipolynomial.
∎

Theorem 4.6 reduces the robust stability analysis of a polytopic family to a simpler problem of Hurwitz stability analysis of a finite set of one parameter subfamilies of the type (4.16). Subfamilies (4.16) serve as a testing set for the robust stability analysis of P. The number of such subfamilies may in general be very high, the upper bound for this number is $C_N^2 = \frac{N(N-1)}{2}$. Analyzing the proof one can observe that not all edge subfamilies need to be taken into account in the analysis, we only need to check those edges whose images appear on the boundary of the value set for at least one point $j\omega$. The number of such subfamilies may be significantly smaller than that of all edge subfamilies. This observation sometimes allows one to obtain a simpler testing set. But one question remains open: How can one check the stability of an edge subfamily?

4.5.1 Stability of an edge subfamily

Here stability analysis of a simple one parameter family of the form

$$f_\mu(s) = (1-\mu)f_0(s) + \mu f_1(s), \text{ where } \mu \in [0,1], \tag{4.17}$$

is treated. This is a simple polytopic family with two generators: $f_0(s)$, $f_1(s)$. It is assumed that (4.17) satisfies the conditions 4.1-4.2.

Our principal goal now is to derive some tractable stability conditions for family (4.17).

Claim 4.1 *Let generator quasipolynomials, $f_0(s)$ and $f_1(s)$, be Hurwitz stable. Then all quasipolynomials $f_\mu(s)$, $\mu \in [0,1]$ are Hurwitz stable if and only if the complex curve*

$$z = \frac{f_0(j\omega)}{f_1(j\omega)}, \quad \omega \in (-\infty, +\infty),$$

does not touch the negative real semi-axis of the complex plane.

Proof. The statement follows directly from the fact that the instability of one of the quasipolynomials from (4.17) means that in (4.17) there is a quasipolynomial with at least one zero on the imaginary axis. ■

One interesting question is the following: Under what conditions Hurwitz stability of vertex quasipolynomials $f_0(s)$ and $f_1(s)$ implies that of $f_\mu(s)$ for all $\mu \in [0,1]$? This question leads us to the concept of convex directions. First we rewrite (4.17) as

$$f_\mu(s) = f_0(s) + \mu g(s), \; \mu \in [0,1],$$

where $g(s) = f_1(s) - f_0(s)$.

Definition 4.1 *A quasipolynomial*

$$g(s) = \sum_{k=0}^{n}\sum_{i=0}^{m} \alpha_{ki} s^{n-k} e^{-r_i s},$$

is called a convex direction for the set of Hurwitz stable quasipolynomials of the form (4.2) if for every Hurwitz stable quasipolynomial $f_0(s)$ of this form, Hurwitz stability of $f_0(s)+g(s)$ implies that of $f_\mu(s)$ for all $\mu \in [0,1]$.

Next theorem gives necessary and sufficient conditions for $g(s)$ be a convex direction.

Theorem 4.7 *A quasipolynomial $g(s)$ is a convex direction for the set of Hurwitz stable quasipolynomials of the form (4.2) if and only if for all $\omega > 0$ where $g(j\omega) \neq 0$ the following inequality holds*

$$\frac{\partial \arg(g(j\omega))}{\partial \omega} \leq -\frac{r_m}{2} + \left| \frac{\sin(2\arg(g(j\omega)) + r_m\omega)}{2\omega} \right|.$$

Example 4.2 *Quasipolynomial $q(s)e^{-rs}$, where $r > 0$ and $q(s)$ is a polynomial of degree $\leq n$, is a convex direction for the set of Hurwitz stable quasipolynomials of the form (4.2) if*

1. $2r \leq r_m$;
2. *for all $\omega > 0$ where $q(j\omega) \neq 0$ the following inequality holds*

$$\frac{\partial \arg(q(j\omega))}{\partial \omega} \leq \left| \frac{\sin(2\arg(g(j\omega)))}{2\omega} \right|.$$

4.5.2 Interval quasipolynomial

In this section a special polytope of quasipolynomials is treated. Namely, we consider an interval quasipolynomial which is a family of quasipolynomials of the form

$$I = \left\{ \sum_{k=0}^{n} \sum_{i=0}^{m} a_{ki} s^{n-k} e^{-r_i s} \,\middle|\, \begin{cases} a_{ki} \in [\underline{a}_{ki}, \overline{a}_{ki}], \\ k = 0, ..., n; \; i = 0, ..., m \end{cases} \right\},$$

where $0 = r_0 < r_1 < r_2 < ... < r_m$.

It can be also written as

$$I = \Pi_0 + \Pi_1 e^{-r_1 s} + ... + \Pi_m e^{-r_m s},$$

where

$$\Pi_i = \left\{ \sum_{k=0}^{n} a_{ki} s^{n-k} \mid a_{ki} \in [\underline{a}_{ki}, \overline{a}_{ki}], \; k = 0, 1, ..., n \right\}, \quad i = 0, ..., m,$$

are interval polynomials. It will be convenient sometimes to associate with Π_0 the exponential factor $e^{-r_0 s}$.

With each interval polynomial Π_i one may associate four special vertex polynomials

$$\begin{aligned}
p_i^{(1)}(s) &= \underline{a}_{ni} + \underline{a}_{n-1,i}s + \overline{a}_{n-2,i}s^2 + \overline{a}_{n-3,i}s^3 + \underline{a}_{n-4,i}s^4 + \underline{a}_{n-5,i}s^5 + \ldots\,, \\
p_i^{(2)}(s) &= \overline{a}_{ni} + \underline{a}_{n-1,i}s + \underline{a}_{n-2,i}s^2 + \overline{a}_{n-3,i}s^3 + \overline{a}_{n-4,i}s^4 + \underline{a}_{n-5,i}s^5 + \ldots\,, \\
p_i^{(3)}(s) &= \overline{a}_{ni} + \overline{a}_{n-1,i}s + \underline{a}_{n-2,i}s^2 + \underline{a}_{n-3,i}s^3 + \overline{a}_{n-4,i}s^4 + \overline{a}_{n-5,i}s^5 + \ldots\,, \\
p_i^{(4)}(s) &= \underline{a}_{ni} + \overline{a}_{n-1,i}s + \overline{a}_{n-2,i}s^2 + \underline{a}_{n-3,i}s^3 + \underline{a}_{n-4,i}s^4 + \overline{a}_{n-5,i}s^5 + \ldots\,.
\end{aligned}$$

These polynomials are such that the value set $V_{\Pi_j}(j\omega)$ is a rectangle whose corner points are $p_i^{(\nu)}(j\omega)$, $\nu = 1,2,3,4$:

$$V_{\Pi_i}(j\omega) = \left\{ \sum_{v=1}^{4} \mu_\nu p_i^{(\nu)}(j\omega) \;\middle|\; \mu_\nu \geq 0,\ \sum_{v=1}^{4} \mu_\nu = 1 \right\}.$$

The value set of $\Pi_i e^{-r_i s}$ at the point $j\omega$ can be obtained by rotation of $V_{\Pi_i}(j\omega)$ around the origin on the angle $-r_i\omega$. This observation allows us to conclude that the value set, $V_I(j\omega)$, of the interval quasipolynomial I is a sum of such rotated rectangles:

$$V_I(j\omega) = V_{\Pi_0}(j\omega) + V_{\Pi_1}(j\omega)e^{-jr_1\omega} + \ldots + V_{\Pi_m}(j\omega)e^{-jr_m\omega}.$$

In geometrical terms, the value set is a polygon. The number of corner points of the polygon cannot exceed 4^{m+1}, and all of them are images of vertex quasipolynomials of the form

$$\sum_{i=0}^{m} p_i^{(\nu_i)}(s)e^{-r_i s},$$

where $\nu_i \in \{1,2,3,4\}$, $i = 0,\ldots,m$. We remind the readers that here $r_0 = 0$.

On the other hand, at every frequency the boundary segments of the polygon are included into the union of value sets of the following one parameter families

$$\sum_{i\neq k} p_i^{(\nu_i)}(s)e^{-r_i s} + \left[\mu p_k^{(\alpha)}(s) + (1-\mu)p_k^{(\beta)}(s)\right] e^{-r_k s}, \tag{4.18}$$

where, as before, $k = 0,1,\ldots,m$, $\nu_i \in \{1,2,3,4\}$, and

$$(\alpha,\beta) \in \{(1,2),(2,3),(3,4),(4,1)\}.$$

Simple calculations show that there are exactly $(m+1)4^{m+1}$ families of this form.

Theorem 4.8 *The interval quasipolynomial I is Hurwitz stable if and only if*

- *there are* $\varepsilon > 0$ *and* $R > 0$ *such that in the interval family*

$$\left\{ \sum_{i=0}^{m} a_{0i} e^{-r_i s} \mid a_{0i} \in [\underline{a}_{0i}, \overline{a}_{0i}],\ i = 0, 1, ..., m \right\},$$

there is no quasipolynomial which has at least one zero with magnitude greater than R *and real part greater than* $-\varepsilon$*;*

- *all one parameter families of the form (4.18) with* $2r_k > r_m$ *are Hurwitz stable.*

Proof. This follows directly from the edge theorem and convex direction concept. ■

4.6 Multivariate polynomial approach

Let us consider the system

$$A_0 \dot{x}(t) + B_0 x(t) + A_1 \dot{x}(t - \tau_1) + B_1 x(t - \tau_1) = 0. \tag{4.19}$$

The characteristic function of the system

$$f(s) = \det\left(sA_0 + B_0 + se^{-\tau_1 s} A_1 + e^{-\tau_1 s} B_1\right), \tag{4.20}$$

may be written as a polynomial of two variables, s and $z = e^{-\tau_1 s}$:

$$p(s, z) = \det\left(sA_0 + B_0 + szA_1 + zB_1\right).$$

It is possible then to use the polynomial for stability analysis of system (4.19).

Lemma 4.9 *Quasipolynomial (4.20) has no zeros with nonnegative real parts if* $p(s, z)$ *has no zeros in the domain*

$$\Delta = \{\ (s, z) \mid \operatorname{Re}(s) \geq 0,\ |z| \leq 1 \}.$$

Proof. In fact, if $f(s)$ has a zero, s_0, for which $\operatorname{Re}(s_0) \geq 0$, then $p(s, z)$ has zero $(s_0, z_0) = (s_0, e^{-\tau_1 s_0}) \in \Delta$ ■

It is convenient to make the variables more uniform by means of the following change of variables

$$s_1 = s, \text{ and } s_2 = \frac{1 - z}{1 + z}.$$

Let n_2 be the partial degree of $p(s, z)$ with respect to z, then polynomial

$$q(s_1, s_2) = (1 + s_2)^{n_2} p\left(s_1, \frac{1 - s_2}{1 + s_2}\right)$$

has no zeros in the domain

$$\Gamma_2^{(0)} = \{ \ (s_1, s_2) \mid \operatorname{Re}(s_1) \geq 0, \ \operatorname{Re}(s_2) \geq 0 \ \}$$

if and only if $p(s, z)$ has no zeros in the domain Δ.

Having in mind this observation and lemma 4.9 one may apply known stability results for multivariate polynomials to the stability analysis of time-delay systems. Certainly one may consider also the case when the time delay system has more than one incommensurate delays. In this case polynomials with more than two variables will arise.

4.6.1 Multivariate polynomials

A multivariate polynomial is a finite sum of the following form

$$p(s_1, s_2, ..., s_m) = \sum a_{i_1 i_2 \ldots i_m} s_1^{i_1} s_2^{i_2} ... s_m^{i_m}. \tag{4.21}$$

Here $s_1, s_2, \ldots, s_m$ - are independent variables, and coefficients $a_{i_1 i_2 \ldots i_m}$ are real or complex numbers.

Two polynomials are called *relatively prime* if they have no common divisors except constant polynomials. Two primes are either coinciding up to a nonzero constant multiplier, or are relatively prime.

Vector $(s_1^{(0)}, s_2^{(0)}, \ldots, s_m^{(0)})$ with complex elements is called *zero of* (4.21), if $p(s_1^{(0)}, s_2^{(0)}, \ldots, s_m^{(0)}) = 0$.

In contrast to the case of univariate polynomials multivariate polynomials have infinite number of zeros. Let the partial degree n_m of $p(s_1, s_2, \ldots, s_m)$ with respect to s_m be positive. Consider the power expansion of the polynomial with respect to this variable

$$p(s_1, s_2, ..., s_m) = \sum_{i=0}^{n_m} a_i^{(m)}(s_1, s_2, ..., s_{m-1}) s_m^i. \tag{4.22}$$

Main coefficient $a_{n_m}^{(m)}(s_1, s_2, \ldots, s_{m-1})$ is a nonzero polynomial with respect to the rest of the variables, so one can always fix these variables such that the main coefficient is nonzero. Then, for the last variable there are exactly n_m possible values each one of them, together with the fixed variables, defines a zero of $p(s_1, s_2, \ldots, s_m)$. Roughly speaking, there are a finite number of $(m-1)$-dimensional zero manifolds in the m-dimensional complex space.

One more essential difference between univariate and multivariate polynomials is the following: Two multivariate polynomials may be relatively prime and still possess common zeros.

4.6.2 Stable polynomials

In this subsection we introduce a class of stable multivariate polynomials. To this end we first define the set of polynomials of a given degree:

Given a vector of partial degrees $(n_1, n_2, \ldots, n_m)$, one may define a set of polynomials

$$\mathbf{P}_{n_1,n_2,\ldots,n_m} = \{\ p(s_1, s_2, ..., s_m)\ ;\ \deg(p) = (n_1, n_2, ..., n_m)\ \}. \qquad (4.23)$$

With every polynomial $p(s_1, s_2, \ldots, s_m)$ from the set one may associate the real coefficient vector

$$\mathbf{a} = (a_{00\ldots0}, a_{10\ldots0}, \ldots, a_{n_1 n_2 \ldots n_m})$$

from the N-dimensional coefficient space, where $N = (n_1{+}1)(n_2{+}1)\cdots(n_m{+}1)$.

In the following we will often denote vector $(s_1, s_2, \ldots, s_m)$ by $\mathbf{s}$. The following lemma is the standard extension of the well-known continuity property of zeros of univariate polynomials with respect to small coefficient variations.

Lemma 4.10 *Let polynomial* $p_0(\mathbf{s}) \in \mathbf{P}_{n_1,n_2,\ldots,n_m}$ *have a zero* $\mathbf{s}^{(\mathbf{0})} = (s_1^{(0)}, s_2^{(0)}, \ldots, s_m^{(0)})$. *Denote by* $\mathbf{a}^{(\mathbf{0})}$ *the coefficient vector of this polynomial. For arbitrary* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that every polynomial* $p(\mathbf{s})$ *with the coefficient vector* $\mathbf{a}$ *in the* δ*-neighborhood of* $\mathbf{a}^{(\mathbf{0})}$*:*

$$||\mathbf{a} - \mathbf{a}^{(\mathbf{0})}|| < \delta,$$

has a zero in the ε*-neighborhood of* $\mathbf{s}^{(\mathbf{0})}$.

According to this lemma, the zeros of multivariate polynomials possess continuity property with respect to coefficient variations.

Define polydomain

$$\Gamma_m^{(0)} = \{\ (s_1, s_2, ..., s_m) \mid Re\{s_i\} \geq 0,\ i = 1, 2, ...m\ \}. \qquad (4.24)$$

Definition 4.2 *A polynomial* $p(\mathbf{s}) \in \mathbf{P}_{n_1,n_2,\ldots,n_m}$ *is said to be* strict sense stable *(SSS) if*

$$p(\mathbf{s}) \neq 0,\ \forall \mathbf{s} \in \Gamma_m^{(0)}.$$

The principal deficiency of this stability concept is that strict sense stable polynomials do not always preserve the stability property under arbitrary small coefficient variations. The property becomes very fragile when polynomials have large zeros close to the essential boundary

$$\Omega = \{\ (s_1, s_2, ..., s_m) \mid Re\{s_i\} = 0,\ \ i = 1, 2, ...m\ \}, \qquad (4.25)$$

of the polydomain $\Gamma_m^{(0)}$.

A new class of stable multivariate polynomials will be defined inductively with respect to the number of variables.

Definition 4.3 *1. For* $m = 1$*: given* $n_1 \geq 0$*, polynomial* $p(s_1)$ *of degree* n_1 *is called* stable *if it is Hurwitz stable;*

2. *for $m > 1$: given vector $(n_1, n_2, ..., n_m)$, polynomial*

$$p(s_1, s_2, ..., s_m)$$

of degree $(n_1, n_2, ..., n_m)$ is called stable *if it satisfies the following conditions:*

- *polynomial $p(s_1, s_2, ..., s_m)$ is strict sense stable;*
- *in decomposition (4.22) with respect to everyone of the variables the main coefficient (the coefficient of the highest order term) is a* stable *polynomial of $m-1$ variables whose partial degrees coincide with the corresponding partial degrees of $p(s_1, s_2, ..., s_m)$.*

The principle advantage of the stability concept is that the class of stable polynomials is the biggest one for which polynomials preserve the stability property under small coefficient variations.

Lemma 4.11 *Let a polynomial $p(\mathbf{s}) \in \mathbf{P}_{n_1,n_2,...,n_m}$ be stable, and assume that $n_m > 0$. Define the transformed polynomial*

$$\widehat{p}(\mathbf{s}) = (s_m)^{n_m} p(s_1, \ldots, s_{m-1}, \frac{1}{s_m}),$$

then $\widehat{p}(\mathbf{s})$ is stable and $\deg(\widehat{p}) = \deg(p)$.

It is worth noting that the statement is not true in general for strict sense stable polynomials , see the following example.

Example 4.3 *Polynomial $p(s_1, s_2) = s_1 s_2 + s_1 + 1 \in \mathbf{P}_{1,1}$ is strict sense stable, but the transformed one $\widehat{p}(s_1, s_2) = s_1 + s_1 s_2 + s_2$ has the trivial zero and therefore is not strict sense stable.*

Corollary 4.12 *Applying successively lemma 4.11 to several variables one can get from a stable polynomial $p(\mathbf{s})$ a set of stable polynomials of the same degree.*

Theorem 4.13 *Let $p(s_1, s_2, ..., s_m) \in \mathbf{P}_{n_1,n_2,...,n_m}$ be a stable polynomial. There exists $\varepsilon > 0$ such that every polynomial with a coefficient vector from the ε-neighborhood of the coefficient vector of $p(s_1, s_2, ..., s_m)$ is stable.*

One more useful property of stable polynomials is the invariance of the stability under differentiation.

Theorem 4.14 *Let $p(s_1, s_2, ..., s_m) \in \mathbf{P}_{n_1,n_2,...,n_m}$ be a stable polynomial. If $n_m > 0$, then derivative*

$$\begin{aligned}\frac{\partial\, p(s_1, s_2, ..., s_m)}{\partial s_m} &= \sum_{k=0}^{n_1} \frac{\partial\, a_k^{(1)}(s_2, ..., s_m)}{\partial s_m} s_1^k = ... \\ &= \sum_{i=0}^{n_{m-1}} \frac{\partial\, a_i^{(m-1)}(s_1, ..., s_m)}{\partial s_m} s_{m-1}^i \qquad (4.26) \\ &= \sum_{\ell=1}^{n_m} \ell a_\ell^{(m)}(s_1, ..., s_{m-1}) s_m^{\ell-1}, \qquad (4.27)\end{aligned}$$

is a stable polynomial from $\mathbf{P}_{n_1,n_2,...,n_m-1}$.

Corollary 4.15 *Let $p(s_1, s_2, ..., s_m) \in \mathbf{P}_{n_1,n_2,...,n_m}$ be a stable polynomial. Then all coefficients in decomposition (4.22) are stable $(m-1)$-variate polynomials from $\mathbf{P}_{n_1,n_2,...,n_{m-1}}$.*

As it follows from the example below for strict sense stable polynomials this statement is not true in general.

Example 4.4 *Polynomial $p(s_1, s_2) = s_1 s_2 + s_1 + 1$ belongs to $\mathbf{P}_{1,1}$, is strict sense stable, but the derivative with respect to the second argument is not a strict sense stable polynomial of $\mathbf{P}_{1,0}$.*

Theorem 4.16 *All coefficients of a real stable polynomial*

$$p(s_1, s_2, ..., s_m) \in \mathbf{P}_{n_1,n_2,...,n_m}$$

have the same sign.

It was already mentioned that the main reason of enormous sensitivity of the strict sense stability to small coefficient variations is the existence of zeros close to the essential boundary (4.25) of the set $\Gamma_m^{(0)}$. The following theorem shows that stable polynomials have no such zeros.

Theorem 4.17 *For every stable polynomial $p(\mathbf{s}) \in \mathbf{P}_{n_1,n_2,...,n_m}$ there exists $\varepsilon > 0$, such that it has no zeros in the ε-neighborhood of the essential boundary Ω.*

4.6.3 Stability of an interval multivariate polynomial

Consider a real multivariate interval polynomial

$$\mathcal{I} = \left\{ \sum_{i_1=0}^{n_1} ... \sum_{i_m=0}^{n_m} a_{i_1...i_m} s_1^{i_1} ... s_m^{i_m} \;\middle|\; \left\{ \begin{array}{l} a_{i_1...i_m} \in [\underline{a}_{i_1...i_m}, \overline{a}_{i_1...i_m}] \\ i_k = 0, ..., n_k;\; k = 1, ..., m \end{array} \right. \right\}. \qquad (4.28)$$

Of course we assume that $\mathcal{I} \subset \mathbf{P}_{n_1,n_2,...,n_m}$, that is, all members of the family have the same degree $\mathbf{n} = (n_1, n_2, ..., n_m)$. Moreover we assume that all lower bounds are positive, that is $\underline{a}_{i_1...i_m} > 0$.

Theorem 4.18 *The interval polynomial is stable if and only if* $4 \cdot 2^{m-1}$ *special members of the family with extreme coefficients are stable.*

Let us define explicitly these testing polynomials. To this end introduce the set of m-dimensional sign vectors

$$\Xi = \{(\nu_1, \nu_2, ..., \nu_m) \,|\nu_i \in \{-1, +1\}, \quad i = 1, 2, ..., m\}.$$

This set consists of 2^m elements. With every vector from the set one can associate 4 polynomials as follows:

Given $(\nu_1, \nu_2, ..., \nu_m)$, the coefficients of the first polynomial $p_1^{(\nu_1,\nu_2,...,\nu_m)}(s_1, s_2, ..., s_m$ are defined by the following rule:
When the index sum is even $(i_1 + i_2 + ... + i_m = 2l)$ then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \underline{a}_{i_1 i_2 ... i_m}, \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \overline{a}_{i_1 i_2 ... i_m}, \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases},$$

when the index sum is odd $(i_1 + i_2 + ... + i_m = 2l + 1)$, then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \underline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \overline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases}.$$

Coefficients of the second polynomial

$$p_2^{(\nu_1,\nu_2,...,\nu_m)}(s_1, s_2, ..., s_m)$$

are defined by the rule:
When the index sum is even $(i_1 + i_2 + ... + i_m = 2l)$, then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \overline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \underline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases},$$

when the index sum is odd $(i_1 + i_2 + ... + i_m = 2l + 1)$, then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \underline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \overline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases}.$$

Coefficients of the third polynomial

$$p_3^{(\nu_1,\nu_2,...,\nu_m)}(s_1, s_2, ..., s_m)$$

are defined by the rule:
When the index sum is even $(i_1 + i_2 + ... + i_m = 2l)$, then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \overline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \underline{a}_{i_1 i_2 ... i_m}, \text{ if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases},$$

when the index sum is odd ($i_1 + i_2 + ... + i_m = 2l + 1$), then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \overline{a}_{i_1 i_2 ... i_m}, & \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \underline{a}_{i_1 i_2 ... i_m}, & \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases}.$$

And the coefficients of the last one

$$p_4^{(\nu_1, \nu_2, ..., \nu_m)}(s_1, s_2, ..., s_m)$$

are defined by the rule:
When the index sum is even ($i_1 + i_2 + ... + i_m = 2l$) then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \underline{a}_{i_1 i_2 ... i_m}, & \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \overline{a}_{i_1 i_2 ... i_m}, & \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases},$$

when the index sum is odd ($i_1 + i_2 + ... + i_m = 2l + 1$) then

$$a_{i_1 i_2 ... i_m} = \begin{cases} \overline{a}_{i_1 i_2 ... i_m}, & \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} > 0 \\ \underline{a}_{i_1 i_2 ... i_m}, & \text{if } (-1)^l \nu_1^{i_1} \nu_2^{i_2} ... \nu_m^{i_m} < 0 \end{cases}.$$

Proceeding in a similar way for all the elements from Ξ one finally obtains $4 \cdot 2^m$ polynomials. Taking into account the fact that all polynomials in $\mathcal{I}$ have only real coefficients we may reduce the set by half and consider only those of them for which $\nu_m = 1$. After this reduction we obtain the desired testing set of $4 \cdot 2^{m-1}$ polynomials which is referred to in the theorem 4.18.

4.6.4 Stability of a diamond family of multivariate polynomials

Consider a m-variate polynomial whose real coefficient vector, $\mathbf{a}$, varies in the $N-$dimensional *diamond*

$$D = \left\{ \mathbf{a} \;\middle|\; \sum_{i_1 i_2 ... i_m} \left| a_{i_1 i_2 ... i_m} - a^{(0)}_{i_1 i_2 ... i_m} \right| \leq r \right\}, \tag{4.29}$$

where $\mathbf{a}^{(0)}$ denotes the center point and $p^{(0)}(\mathbf{s}) = p(\mathbf{s}, \mathbf{a}^{(0)})$ is the center polynomial; $r > 0$ is the radius of the diamond. From here on we denote a diamond family of multivariate polynomials by $\mathcal{D}_m = \{ \ p(\cdot, \mathbf{a}) \ | \ \mathbf{a} \in D \ \}$.

According to coefficient positivity assumption all coefficients of stable polynomials have the same sign. Therefore, without loss of generality, we may assume $a^{(0)}_{i_1 i_2 ... i_m} > 0$ and:

$$r < r_0 = \min_{i_1 i_2 ... i_m} \left\{ a^{(0)}_{i_1 i_2 ... i_m} \right\}. \tag{4.30}$$

It is clear that the last condition implies that $\mathcal{D}_m \subset \mathbf{P}_{n_1, n_2, ..., n_m}$.

Theorem 4.19 *The diamond family of multivariate polynomials, $\mathcal{D}_m$, is stable if and only if $(m+1)\,2^{m+1}$ extreme polynomials of the family are stable.*

As in the previous section let us define explicitly this testing set of polynomials. To this end we divide the essential boundary, $\Omega^{(m)}$, into 2^m subregions in such a way, that for every subregion there exist exactly $4m$ one parametric families which describe the boundary of the value set

$$\mathcal{V}_D\left(j\omega_1, j\omega_2, ..., j\omega_m\right) = \{\ p\left(j\boldsymbol{\omega}\right)\ \mid p \in \mathcal{D}_m\ \}$$

of the diamond family. Here $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_m)$ belongs to the selected subregion.

The first subregion

$$\Omega_0 = \{\ j\boldsymbol{\omega}\ \mid \omega_i \in (0,1)\,; i = 1, ..., m\ \ \}$$

Here the $4 \cdot m$ one parametric families covering the boundary of the value set, are the following ones:

$$\left\{\ p^{(0)}\left(\mathbf{s}\right) \pm r\left[\mu \pm (1-\mu)\, s_k\right]\ \middle|\ \mu \in [0,1]\right\},\ \ k = 1, ..., m\ . \tag{4.31}$$

Let us fix one of the indices, say α, then for subregion

$$\Omega_\alpha = \{\ \ j\boldsymbol{\omega} \mid \omega_\alpha \in (1,\infty)\,,\ \omega_k \in (0,1)\,;\ k \in \{1, ..., m\}\,,\ k \neq \alpha\ \}$$

Here the boundary of the value set of the diamond family is defined by the following one parametric families:

$$\left\{\ p^{(0)}\left(\mathbf{s}\right) \pm r\left[\mu \pm (1-\mu)\, s_k\right] s_\alpha^{l_\alpha}\ \middle|\ \mu \in [0,1]\right\},\ \ k = 1, ..., m\ . \tag{4.32}$$

Here

$$l_\alpha = \begin{cases} n_\alpha - 1, \ \text{if}\ \alpha = k \\ \ \ n_\alpha, \quad \text{if}\ \alpha \neq k \end{cases}, \tag{4.33}$$

where $\alpha \in \{1, ..., m\}$. There are exactly $C_m^1 = m$ subregions of this type. And for every one of them there are defined exactly $4m$ one parametric families.

Now fixing two of the indices, say α and β ($\alpha \neq \beta$), we define subregion

$$\Omega_{\alpha,\beta} = \left\{\ j\boldsymbol{\omega}\ \middle|\ \begin{cases} \omega_\alpha \in (1,\infty)\,,\ \omega_\beta \in (1,\infty)\,, \\ \omega_k \in (0,1)\,;\ k \in \{1, ..., m\}\,,\ k \neq \alpha,\ k \neq \beta; \end{cases}\ \right\}$$

Here we have the following one parametric families:

$$\left\{p^{(0)}\left(\mathbf{s}\right) \pm r\left[\mu \pm (1-\mu)\, s_k\right] s_\alpha^{l_\alpha} s_\beta^{l_\beta}\ |\mu \in [0,1]\right\}, \tag{4.34}$$
$$k = 1, ..., m;\ k \neq \alpha,\ k \neq \beta$$

where l_α and l_β are defined in (4.33), and $\alpha, \beta \in \{1, ..., m\}, \alpha \neq \beta$. The number of such subregions is $C_m^2 = m(m-1)/2$.

Now fixing $\nu < m$ of the indices, say $\alpha, \beta, ..., \gamma$ (they are all distinct), we can define subregion

$$\Omega_{\alpha,\beta,...,\gamma} = \left\{ j\boldsymbol{\omega} \left| \left\{ \begin{array}{c} \omega_\alpha \in (1,\infty), \ \omega_\beta \in (1,\infty), ..., \ \omega_\gamma \in (1,\infty) \\ \omega_k \in (0,1); \ k \in \{1,...,m\}, \ k \neq \alpha, \ k \neq \beta, ..., k \neq \gamma \end{array} \right. \right. \right\},$$

Here we have again $4m$ one parametric families:

$$\left\{ \ p^{(0)}(\mathbf{s}) \pm r \left[\mu \pm (1-\mu) s_k\right] s_\alpha^{l_\alpha} s_\beta^{l_\beta} ... s_\gamma^{l_\gamma} \ \middle| \ \mu \in [0,1] \right\}, \ k = 1, ..., m \tag{4.35}$$

where l_α, $l_\beta, ..., l_\gamma$ are defined as above. In this way we find C_m^ν subregions of this type.

Continuing this process, we define the last subregion

$$\Omega_{1,2,...,m} = \{ \ j\boldsymbol{\omega} \ | \ \omega_k \in (1,\infty); \ k = 1, ..., m \ \},$$

where the boundary of the value set is defined by the following one parametric families:

$$\left\{ \ p^{(0)}(\mathbf{s}) \pm r \left[\mu \pm (1-\mu) s_k\right] s_1^{l_1} s_2^{l_2} ... s_m^{l_m} \ \middle| \ \mu \in [0,1] \ \right\}, \ k = 1, ..., m. \tag{4.36}$$

Finally, we obtain $\sum_{i=0}^{m} C_m^i = 2^m$ subregions and associate $4m$ one parametric families with each one of them. Now, substituting in every one of these families $\mu = 0$ and $\mu = 1$ we obtain the testing polynomials.

4.7 Notes and References

For the case of time invariant systems there is a strong connection between the exponential stability of a system and location on the complex plane of the zeros of it's characteristic function. For the case of time delay systems of concentrated delays the characteristic function is a finite sum of potential and exponential factors.

The book [12] still remains one of the best reference for those who are interested in the study of zeros of such functions. It contains an exhaustive analysis of the location on the complex plane of zeros of quasipolynomials from several interesting classes. A much more general and profound study of zeros of quasipolynomial functions one may find in the Ph.D. dissertation of E. Schwengeler, see [250]. In our presentation of the exponential and potential diagrams we followed this dissertation. Example 4.1 is taken from [157].

The Edge theorem has been proved first for the case of retarded type quasipolynomials in [75], and then has been extended to the case of neutral type quasipolynomials in [76].

Stability analysis of an edge subfamily of quasipolynomials, including conditions for convex directions, Theorem 4.7, a reader may find in [145]. Stability conditions for an interval quasipolynomial are given in [145].

Basic results on applications of multivariate polynomials for stability analysis of quasipolynomials are given in [139]-[141].

In presentation of stability conditions for multivariate polynomials, we follow the recent publications [148], [149], [150].

5

Systems with Single Delay

Tile shortened, and "notes" is used in the last section title

5.1 Introduction

In this chapter, we will explore the time domain approaches of stability analysis. An advantage of time domain methods is the ease to handle non-linearity and time-varying uncertainties. However, in order to illustrate the basic ideas, in this chapter, we will concentrate on the stability problem of linear time invariant systems with single delay

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - r) \tag{5.1}$$

where A_0 and A_1 are given $n \times n$ real matrices. The usual initial condition is in the form of

$$x_t = \phi \tag{5.2}$$

We will defer the discussions on the uncertainties and systems with multiple-delays as well as distributed delays to later chapters.

We will use the Razumikhin Theorem (Theorem 1.4 in Chapter 1) and the Lyapunov-Krasovskii Stability Theorem (Theorem 1.3 in Chapter 1) to discuss the stability of the system. We will restrict ourselves to using bounded quadratic Lyapunov function or Lyapunov-Krasovskii functional, and aim at arriving at stability criteria which can be written in the form of Linear Matrix Inequalities (LMI) or closely related form. Efficient numerical methods are available to solve LMIs. The readers are encouraged to read Appendix B to get familiar with the basic concepts of LMI and a number of useful topics especially relevant to time-delay systems, notably the variable elimination technique in LMI and quadratic integral inequalities (Jensen Inequality). We will frequently refer to them in this Chapter.

The first part of this Chapter will mainly use the Razumikin Theorem. Section 5.2 discusses delay-independent stability criteria. The first part discusses systems with single delay expressed by (5.1). This can be derived using the simplest form of Lyapunov function. The delay-independent stability criteria test if a system is asymptotically stable for arbitrary delay. It is intuitively obvious that such a criterion would be very conservative if the delay r is already known and small. It turns out that delay-independent stability criteria effectively consider the delayed term, *i.e.*, second term on the right hand side of (5.1), to be always detrimental to the stability of the system. Therefore, such criteria are useful when the system behavior is dominated by the first term on the right hand side of (5.1) (*i.e.*, A_1 is rather small), and we want to obtain quick assurance that the delayed term will not destabilize the system. The later part of Section 5.2 discusses delay-independent stability of systems with distributed delays, which is useful for deriving delay-dependent stability criteria in Section 5.3.

Section 5.3 derives simple delay-dependent stability criteria using Razumikhin Theorem. In practice, as indicated in Example 1.2 in Chapter 1, there are systems which require feedback to improve the system performance, and the feedback channel involves delays. In these cases, the delay-independent stability criteria are clearly insufficient. In many cases, it is more desirable that the system does not have a delay, and the presence of delay is indeed detrimental to system stability and performance. It is therefore natural to transform a delayed system into a system without delay plus distributed delays, and treat the term with distributed delay as disturbance. This process is known as *model transformation*. We can use the delay-independent stability criterion for distributed delay discussed in Section 5.2 to obtain a stability criterion, which turns out to depend on the size of delay. This is due to the fact that the magnitude of the disturbance term is closely related to the size of time-delay. An important observation for such a criterion is that the delay is always considered as detrimental to stability. Indeed, a necessary condition for such a criterion to be satisfied is that it is satisfied for any smaller delay.

The delay-dependent stability criteria so arrived is still rather conservative. In addition to the potential conservatism caused by applying the delay-independent stability criterion to the system with distributed delays resulting from the model transformation, we will shows that the model transformation process introduces spurious poles which are not present in the original system. The dynamics represented by such spurious poles are known as *additional dynamics*. As delay increases from zero, it is possible for the additional dynamics to become unstable before the original system does. Therefore, the model transformation process itself may introduce conservatism.

We will also show that it is possible to derive a simple delay dependent stability criterion without explicit model transformation. It turns out that the resulting stability criterion include as special cases the delay-independent and delay-dependent stability criteria derived in the earlier part of Section 5.3.

Sections 5.4 and 5.5 uses Lyapunov-Krasovskii method to derive delay-independent and simple delay-dependent stability criteria, parallel to the results obtained by using Razumikhin Theorem. It is interesting to notice that some resulting stability criteria takes a very similar form as the ones obtained using Razumikhin Theorem: the Razumikhin results can be obtained from the Lyapunov-Krasovskii results by imposing additional constraints. While this may give the impression that the Razumikhin results are more conservative, it will be shown in Chapter 6 that the Razumikhin Theorem based results are also applicable to systems with uncertain time-varying delay. From this point of view, the results are not surprising: for the stability of a wider class of systems, the criteria need to be more stringent.

There are, however, many practical cases where both delay-independent and simple delay-dependent stability criteria discussed in Sections 5.2 to 5.5 are unsatisfactory. This is so not only from the quantitative point of view (i.e. they may be overly conservative), but also from qualitative point of view. Indeed, in both types of criteria, the delay is always regarded as detrimental to stability. Therefore, such criteria are clearly not satisfactory if we want to explore the possibility of using *time-delay* (as against the case of reluctantly accepting delay in the delayed feedback) to stabilize the system or to enhance the system performance. As is discussed in Chapter 1, time-delays are indeed introduced intentionally in practice. It is not uncommon that a system is unstable without delay, and is stable with certain delay. It is, therefore, natural to discuss the stability criteria which can handle this type of problems. In Section 5.6, we will show that the existence of a more general quadratic Lyapunov-Krasovskii functional is necessary and sufficient for stability.

Unfortunately, it is generally very difficult to check the existence of a quadratic Lyapunov-Krasovskii functional discussed in Section 5.6. For practical computation, it is shown in Section 5.7 that it is possible to choose a piecewise linear kernel to approximate the continuous one with resulting guaranteed stability limit approaching the continuous case. This method is known as the *discretized Lyapunov functional method.* Numerical examples show that in general, a rather coarse discretization is often sufficient to arrive at a very accurate stability limit in practice.

In this Chapter, due to the fact that only bounded quadratic Lyapunov function is used, we only need to use that following restricted form of Razumikhin Theorem:

Proposition 5.1 *A time-delay system with maximum time-delay r is asymptotically stable if there exists a bounded quadratic Lyapunov function V such that for some $\varepsilon > 0$, it satisfies*

$$V(x) \geq \varepsilon ||x||^2 \tag{5.3}$$

and its derivative along the system trajectory, $\dot{V}(x(t))$, satisfies

$$\dot{V}(x(t)) \leq -\varepsilon ||x(t)||^2 \tag{5.4}$$

whenever

$$V(x(t+\xi)) \leq pV(x(t)), \ -r \leq \xi \leq 0 \tag{5.5}$$

for some constant $p > 1$.

Proof. Compared with the Razumikhin Theorem (Theorem 1.4 in Chapter 1), the only necessary condition missing is that

$$V(x) \leq v(x)$$

Since $V(x)$ is bounded quadratic, for a sufficiently large $K > 0$, we have

$$V(x) \leq K||x||^2$$

Therefore, all the Razumikhin Theorem conditions are satisfied with

$$v(x) = K||x||^2$$

This completes the proof. ∎

We will refer to (5.3) as the Lyapunov function condition, and (5.4) subject to (5.5) as the Razumikhin derivative condition.

Similarly, we can state a restricted version of Lyapunov-Krasovskii Theorem:

Proposition 5.2 *A time-delay system is asymptotically stable if there exists a bounded quadratic Lyapunov-Krasovskii functional $V(\phi)$ such that for some $\varepsilon > 0$, it satisfies*

$$V(\phi) \geq \varepsilon||\phi(0)||^2 \tag{5.6}$$

and its derivative along the system trajectory,

$$\dot{V}(\phi) = \dot{V}(x_t)|_{x_t=\phi}$$

satisfies

$$\dot{V}(\phi) \leq -\varepsilon||\phi(0)||^2 \tag{5.7}$$

Proof. The only condition omitted from the Lyapunov-Krasovskii stability condition is

$$V(\phi) \leq v(||\phi||_c)$$

But since V is a bounded quadratic functional of ϕ, the above is satisfied for

$$v(||\phi||_c) = K||\phi||_c^2$$

when $K > 0$ is sufficiently large. ∎

We will refer to (5.6) as the Lyapunov-Krasovskii functional condition, and (5.7) as the Lyapunov-Krasovskii derivative condition.

5.2 Delay-independent stability criteria based on Razumikhin Theorem

5.2.1 Single delay case

Consider the system with single delay described by (5.1). We will use the Razumikhin Theorem to obtain a simple stability condition using the Lyapunov function

$$V(x) = x^T P x \tag{5.8}$$

Proposition 5.3 *The system described by (5.1) is asymptotically stable if there exist a scalar*

$$\alpha > 0$$

and a real symmetric matrix P such that

$$\begin{pmatrix} PA_0 + A_0^T P + \alpha P & PA_1 \\ A_1^T P & -\alpha P \end{pmatrix} < 0. \tag{5.9}$$

Proof. We will use Proposition 5.1. Choose Lyapunov function V as in (5.8). Since (5.9) implies $P > 0$, we can conclude that for some sufficiently small $\varepsilon > 0$, the Lyapunov function condition

$$V(x) \geq \varepsilon ||x||^2 \tag{5.10}$$

is satisfied. Now consider the derivative of V along the trajectory of system (5.1)

$$\dot{V}(x(t)) = \frac{d}{dt} V(x(t)) = 2x^T(t) P[A_0 x(t) + A_1 x(t-r)] \tag{5.11}$$

Whenever x_t satisfies

$$V(x(t+\theta)) < pV(x(t)) \text{ for all } -r \leq \theta \leq 0 \tag{5.12}$$

for some $p > 1$, we can conclude that for any $\alpha > 0$,

$$\begin{aligned} \dot{V}(x(t)) &\leq 2x^T(t) P[A_0 x(t) + A_1 x(t-r)] \\ &\quad + \alpha[px^T(t) P x(t) - x^T(t-r) P x(t-r)] \\ &= \phi_{0r}^T \begin{pmatrix} PA_0 + A_0^T P + \alpha p P & PA_1 \\ A_1^T P & -\alpha P \end{pmatrix} \phi_{0r} \end{aligned} \tag{5.13}$$

where

$$\phi_{0r} = \begin{pmatrix} x^T(t) & x^T(t-r) \end{pmatrix}^T.$$

The inequality (5.9) implies that for some sufficiently small $\delta > 0$, $p = 1 + \delta$,

$$\begin{pmatrix} PA_0 + A_0^T P + \alpha p P & PA_1 \\ A_1^T P & -\alpha P \end{pmatrix} < 0$$

which, according to (5.13), implies that the Razumikhin derivative condition

$$\dot{V}(x(t)) \leq -\varepsilon ||x(t)||^2 \tag{5.14}$$

is satisfied. According to Proposition 5.1, the system is asymptotically stable. ∎

The first important point to be noted is that the stability criterion stated in Proposition 5.3 does not depend on the delay r. Therefore, it is obvious that if the stability criterion is satisfied, then the system is asymptotically stable for arbitrary delay r. As also discussed in Chapter 2, this type of stability criteria are known as *delay-independent stability criteria*. If the delay of the system is rather small, such criteria often give very conservative stability assessment.

In essense, the criterion in Proposition 5.3 considers the term $A_1 x(t-r)$ as always detrimental to stability. To see this, using Schur complement, under the assumption $P > 0$, (5.9) is equivalent to

$$PA_0 + A_0^T P + \alpha P + \frac{1}{\alpha} P A_1 P^{-1} A_1^T P < 0$$

or, with $Q = P^{-1}$,

$$A_0 Q + Q A_0^T + \alpha Q + \frac{1}{\alpha} A_1 Q A_1^T < 0$$

Since the last two terms αQ and $\frac{1}{\alpha} A_1 Q A_1^T$ are both positive semi-definite, the above also implies that A_0 is Hurwitz. In fact, it can be said roughly that the larger A_1 is, the more difficult it is to satisfy the above criterion.

Although (5.9) is linear with respect to P for fixed α, and linear with respect to α for fixed P, it is not linear with the combined variables (P, α), and therefore, it is not an LMI. This makes computation more difficult. Let $\bar{\alpha}$ be the solution of the following *generalized eigenvalue problem* (in the LMI sense, often abbreviated as GEVP, see Appendix B):

$$\bar{\alpha} = \sup \alpha \tag{5.15}$$

subject to

$$PA_0 + A_0^T P + \alpha P < 0, \tag{5.16}$$
$$P > 0 \tag{5.17}$$

Then, since the $(1,1)$ entry of the matrix in (5.9) has to be negative definite, any α satisfying (5.9) must satisfy

$$0 < \alpha < \bar{\alpha}$$

For least conservative result using this formulation, it is necessary to solve (5.9) for all α in the interval $(0, \bar{\alpha})$. The sweeping of α may be rather costly computationally. Alternatively, we may choose to fix $\alpha = \bar{\alpha}/2$ and accept the additional conservatism. Then, the computation consists of solving the GEVP to obtain $\bar{\alpha}$ and solving LMI problem (5.9) with $\alpha = \bar{\alpha}/2$ fixed.

Example 5.1 *Consider the system*

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \beta \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r).$$

where $\beta \geq 0$ and $r > 0$. For each given β, we choose $\alpha = \bar{\alpha}/2$ and check the feasibility of (5.9). A bisection process is used to find $\beta_{\max}$, the greatest β for (5.9) to be feasible. It was found that $\beta_{\max} = 0.9$.

In feedback synthesis, sometimes it is preferred to use the following equivalent *dual form* of Proposition 5.3:

Proposition 5.4 *The system described by (5.1) is asymptotically stable if there exist a scalar*

$$\alpha > 0$$

and a symmetric matrix Q such that

$$\begin{pmatrix} A_0Q + QA_0^T + \alpha Q & A_1Q \\ QA_1^T & -\alpha Q \end{pmatrix} < 0. \tag{5.18}$$

Proof. Apply Proposition 5.3, the system is asymptotically stable if (5.9) is satisfied. Left multiply (5.9) by

$$\mathrm{diag}\begin{pmatrix} P^{-1} & P^{-1} \end{pmatrix}$$

and right multiply by its transpose. We can obtain (5.18) after a variable transformation

$$Q = P^{-1}$$

■

Based on the above dual form of stability criterion, state feedback stabilization design is more convenient. For example, given the system

$$\dot{x}(t) = A_0x(t) + A_1x(t-r) + Bu(t)$$

and we are to design a feedback control

$$u(t) = K_0x(t) + K_1x(t-r)$$

such that the closed loop system is stable independent of delay. Since the closed loop system is

$$\dot{x}(t) = (A_0 + BK_0)x(t) + (A_1 + BK_1)x(t-r)$$

using Proposition 5.4, the condition we need to satisfy is

$$\begin{pmatrix} (A_0 + BK_0)Q + Q(A_0 + BK_0)^T + \alpha Q & (A_1 + BK_1)Q \\ Q(A_1 + BK_1)^T & -\alpha Q \end{pmatrix} < 0.$$

With a variable transformation

$$\begin{aligned} V_0 &= K_0Q \\ V_1 &= K_1Q \end{aligned}$$

the above becomes

$$\begin{pmatrix} A_0Q + BV_0 + QA_0^T + V_0^T B^T + \alpha Q & A_1Q + BV_1 \\ (A_1Q + BV_1)^T & -\alpha Q \end{pmatrix} < 0.$$

which is an LMI if α is fixed. Such a process is not directly possible based on Proposition 5.3.

Most of the stability criteria discussed in this Chapter have a dual form, which can be obtained by a similar procedure. We will not discuss these dual forms individually to avoid repetition.

5.2.2 Distributed delay case

Now consider the system with distributed delays

$$\dot{x}(t) = A_0 x(t) + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta \tag{5.19}$$

where A_0 is a given constant matrix and $A(\theta)$ is a given matrix valued function of $\theta \in [-r, 0]$. We can again use $V(x)$ in (5.8) to study the stability of the system and conclude

Proposition 5.5 *The system with distributed delays described by (5.19) is asymptotically stable if there exist a symmetric matrix P, a scalar function*

$$\alpha(\theta) > 0, \textit{ for } 0 \le \theta \le r \tag{5.20}$$

and a symmetric matrix function $R(\theta)$ such that

$$PA_0 + A_0^T P + \int_{-r}^{0} R(\theta)d\theta \quad < \quad 0, \tag{5.21}$$

$$\begin{pmatrix} \alpha(\theta)P - R(\theta) & PA(\theta) \\ A^T(\theta)P & -\alpha(\theta)P \end{pmatrix} \quad < \quad 0 \textit{ for } 0 \le \theta \le r \tag{5.22}$$

Proof. Use Razumikhin Theorem in a similar way to the proof of Proposition 5.3. Since (5.22) implies $P > 0$, we can conclude that the Lyapunov function $V(x) = x^T Px$ satisfies

$$V(x) \ge \varepsilon ||x||^2$$

for sufficiently small $\varepsilon > 0$. Also, let $p > 1$. Whenever

$$V(x(t+\theta)) < pV(x(t)) \text{ for all } -r \le \theta \le 0$$

is satisfied, we can calculate

$$\begin{aligned}
\dot{V}(x(t)) &= 2x^T(t)P[A_0x(t) + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta] \\
&\leq 2x^T(t)P[A_0x(t) + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta] \\
&\quad + \int_{-r}^{0} \alpha(\theta)[px^T(t)Px(t) - x^T(t+\theta)Px(t+\theta)]d\theta \\
&= x^T(t)[PA_0 + A_0^TP + \int_{-r}^{0} R(\theta)d\theta]x(t) \\
&\quad + \int_{-r}^{0} \begin{pmatrix} x^T(t) & x^T(t+\theta) \end{pmatrix} \\
&\quad \begin{pmatrix} p\alpha(\theta)P - R(\theta) & PA(\theta) \\ A^T(\theta)P & -\alpha(\theta)P \end{pmatrix} \begin{pmatrix} x(t) \\ x(t+\theta) \end{pmatrix} d\theta
\end{aligned}$$

With the above expression, and the fact that $p > 1$ can be arbitrarily close to 1, (5.21) and (5.22) imply

$$\dot{V}(x(t)) \leq -\varepsilon||x(t)||^2$$

for some sufficiently small $\varepsilon > 0$. Therefore, the system is asymptotically stable according Proposition 5.1. ■

An especially interesting case is when $A(\theta)$ is piecewise constant, in which case, we may choose $\alpha(\theta)$ and $R(\theta)$ to be also piecewise constant, reducing (5.20), (5.21) and (5.22) to a finite set of matrix inequalities. This is the case when we apply Proposition 5.5 to the simple delay-dependent stability case discussed in the next Section.

The stability criterion described in Proposition 5.5 may be regarded as delay-independent in the following sense: if the distributed delay $-\theta$ in $x(t+\theta)$ in (5.19) is changed to $-(r_{\text{new}}/r)\theta$, thus the maximum delay of the system is changed from r to r_{new}, the stability condition in Proposition 5.5 remains the same:

Proposition 5.6 *The conditions in Proposition 5.5 is satisfied for system (5.19) if and only if they are also satisfied for the following system*

$$\dot{x}(t) = A_0x(t) + \int_{-r}^{0} A(\theta)x(t + \frac{r_{new}}{r}\theta)d\theta. \tag{5.23}$$

Proof. With a transformation of integration variable, the system (5.23) can be written as

$$\dot{x}(t) = A_0x(t) + \int_{-r_{\text{new}}}^{0} \frac{r}{r_{\text{new}}}A(\frac{r}{r_{\text{new}}}\theta)x(t+\theta)d\theta \tag{5.24}$$

The stability conditions in Proposition 5.5 for this new system is therefore

$$
\begin{aligned}
\alpha_{\text{new}}(\theta) &> 0, \text{ for } 0 \leq \theta \leq r_{\text{new}} \\
PA_0 + A_0^T P + \int_{-r_{\text{new}}}^{0} R_{\text{new}}(\theta) d\theta &< 0 \\
\begin{pmatrix} \alpha_{\text{new}}(\theta)P - R_{\text{new}}(\theta) & \frac{r}{r_{\text{new}}} PA(\frac{r}{r_{\text{new}}}\theta) \\ \frac{r}{r_{\text{new}}} A^T(\frac{r}{r_{\text{new}}}\theta)P & -\alpha_{\text{new}}(\theta)P \end{pmatrix} &< 0, \quad \theta \in [-r_{\text{new}}, 0]
\end{aligned}
$$

It is then obvious that the satisfaction of (5.20) to (5.22) is equivalent to the satisfaction of the above three inequalities with the following correspondence

$$
\begin{aligned}
\alpha_{\text{new}}(\theta) &= \frac{r}{r_{\text{new}}} \alpha(\frac{r}{r_{\text{new}}}\theta) \\
R_{\text{new}}(\theta) &= \frac{r}{r_{\text{new}}} R(\frac{r}{r_{\text{new}}}\theta)
\end{aligned}
$$

This completes the proof. ∎

5.3 Simple delay-dependent stability criteria based on Razumikhin Theorem

The delay-independent stability problem of the system (5.1) discussed in the last section takes this point of view: consider an asymptotically stable system without time-delay

$$\dot{x}(t) = A_0 x(t) \tag{5.25}$$

With arbitrary time-delay, as the coefficient matrix A_1 of the delayed term grows from zero, the system performance deteriorates, eventually the system is in danger of losing stability.

However, in practice, one often encounter nominal systems of the form (5.25) which is unstable. Feedback with small delay (due either to measurement or control delay) is needed to stabilize the system. It is therefore useful to take another point of view: rewrite the system (5.1) in the following form

$$\dot{x}(t) = (A_0 + A_1)x(t) + A_1(x(t-r) - x(t))$$

We consider the term

$$A_1(x(t-r) - x(t))$$

as a disturbance to the nominal stable system

$$\dot{x}(t) = (A_0 + A_1)x(t)$$

The nominal system is stable with satisfactory performance. As the delay r increases from zero, the performance of the system may deteriorate, and eventually is in danger of losing stability. We need some way to reflect the fact that the disturbance grows from zero as r increases from zero. This can be achieved by *model transformation*, which, in combination to the delay-independent stability for distributed delay problem, allows us to derive a rather simple delay-dependent stability sufficient condition. While this formulation still gives rather conservative stability estimate, it is still very useful due to its simplicity.

5.3.1 Model transformation

Consider again system (5.1) with initial condition (5.2), where the initial function $\phi \in \mathcal{C}([-r,0],\mathsf{R}^n)$. With the observation that

$$\begin{aligned} x(t-r) &= x(t) - \int_{-r}^{0} \dot{x}(t+\theta)d\theta \\ &= x(t) - \int_{-r}^{0} [A_0 x(t+\theta) + A_1 x(t-r+\theta)]d\theta \end{aligned}$$

for $t \geq r$, we can write the system (5.1) as

$$\begin{aligned} \dot{x}(t) &= [A_0 + A_1]x(t) \\ &\quad + \int_{-r}^{0} [-A_1 A_0 x(t+\theta) - A_1 A_1 x(t-r+\theta)]d\theta, \end{aligned} \tag{5.26}$$

with initial condition

$$x(\theta) = \psi(\theta), \quad -r \leq \theta \leq r \tag{5.27}$$

where

$$\psi(\theta) = \begin{cases} \phi(\theta), & -r \leq \theta \leq 0 \\ \text{solution of (5.1) with initial condition (5.2)}, & 0 < \theta \leq r \end{cases} \tag{5.28}$$

Therefore, the system described by (5.1) and (5.2) is embeded in the system described by (5.26) and (5.27) without the initial condition constraint (5.28). Since this is a time-invariant system, and therefore, in studying the stability problem, we can shift the initial time to write it in a more standard form

$$\dot{y}(t) = \bar{A}_0 y(t) + \int_{-2r}^{0} \bar{A}(\theta) y(t+\theta) d\theta \tag{5.29}$$

where

$$\begin{cases} \bar{A}_0 = A_0 + A_1 \\ \bar{A}(\theta) = -A_1 A_0,\ \theta \in [-r,0] \\ \bar{A}(\theta) = -A_1 A_1,\ \theta \in [-2r,-r) \end{cases} \tag{5.30}$$

with initial condition

$$y(\theta) = \psi(\theta), \ -2r \le \theta \le 0 \tag{5.31}$$

Here, we have used y instead of x to represent the state variable to emphasize that these two are indeed different systems.

The process of transforming a system represented by (5.1) to one represented by (5.29) and (5.30) is known as a *model transformation.* Clearly, the stability of the system represented by (5.29) to (5.31) implies the stability of the original system. However, the reverse is not necessarily true due to the lifting of the initial condition constraint (5.28). As will be seen later in this Section, the effect of lifting the initial condition constraint results in *additional dynamics* characterized by spurious poles of the transformed systems which are not present in the original system.

5.3.2 Simple delay-dependent stability using explicit model transformation

With the model transformation, we arrived at a system described by (5.29) to (5.31), which includes the original system described by (5.1) and (5.2) as a special case. Since the transformed system is one with distributed delay, we can use Proposition 5.5 to derive the stability condition, which, of course, is sufficient for the stability of the original system.

Proposition 5.7 *The system described by (5.1) is asymptotically stable if there exist real scalars $\alpha_0 > 0$, $\alpha_1 > 0$ and real symmetric matrices $P > 0$, R_0, R_1, such that*

$$P(A_0 + A_1) + (A_0 + A_1)^T P + r(R_0 + R_1) \ < \ 0 \tag{5.32}$$

$$\begin{pmatrix} \alpha_k P - R_k & -PA_1A_k \\ -A_k^T A_1^T P & -\alpha_k P \end{pmatrix} \ < \ 0, \ k = 0, 1. \tag{5.33}$$

Proof. We only need to prove the stability of the transformed system described by (5.29) and (5.30). Using Proposition 5.5, it can be concluded that the system is asymptotically stable if there exist $\alpha(\theta)$, P and $R(\theta)$ to satisfy

$$P\bar{A}_0 + \bar{A}_0^T P + \int_{-2r}^{0} R(\theta) d\theta \ < \ 0 \tag{5.34}$$

$$\begin{pmatrix} \alpha(\theta)P - R(\theta) & P\bar{A}(\theta) \\ \bar{A}^T(\theta)P & -\alpha(\theta)P \end{pmatrix} \ < \ 0, \qquad -2r \le \theta < 0 \tag{5.35}$$

Choosing the following piecewise constant (matrix) functions

$$\alpha(\theta) = \begin{cases} \alpha_0, & -r \le \theta < 0 \\ \alpha_1, & -2r \le \theta < -r \end{cases}$$

$$R(\theta) = \begin{cases} R_0, & -r \le \theta < 0 \\ R_1, & -2r \le \theta < -r \end{cases}$$

completes the proof. ■

The stability criterion in Proposition 5.7 depends on the time-delay r, and is therefore, delay-dependent. We can eliminate the arbitrary matrices R_0 and R_1 among the three matrix inequalities (5.32) to (5.33) to arrive at the following equivalent form

Corollary 5.8 *The system described by (5.1) is asymptotically stable if there exist real symmetric matrix $P > 0$ and real scalars $\alpha_0 > 0$, $\alpha_1 > 0$ such that*

$$\begin{pmatrix} M & -PA_1A_0 & -PA_1^2 \\ -A_0^TA_1^TP & -\alpha_0 P & 0 \\ -(A_1^2)^TP & 0 & -\alpha_1 P \end{pmatrix} < 0 \tag{5.36}$$

where

$$M = \frac{1}{r}[P(A_0 + A_1) + (A_0 + A_1)^T P] + (\alpha_0 + \alpha_1)P \tag{5.37}$$

Proof. Divide (5.32) by r, and use Proposition B.6 in Appendix B to eliminate R_0 among the resulting matrix inequality and (5.33) for $k = 0$, then eliminate R_1 among the resulting matrix inequality and (5.33) for $k = 1$. ■

Again, due to the multiplication of α_0 and α_1, the stability condition described by (5.32) and (5.33) or (5.36) is not LMI. Let $\bar{\alpha}$ be the solution of the GEVP

$$\bar{\alpha} = \sup \alpha \tag{5.38}$$

subject to the constraint

$$\begin{aligned} P &> 0 & (5.39)\\ \frac{1}{r}[P(A_0 + A_1) + (A_0 + A_1)^T P] + \alpha P &< 0 & (5.40) \end{aligned}$$

Then, $\alpha_1 + \alpha_2 < \bar{\alpha}$ since $M < 0$. Also, it is clear that $\bar{\alpha}$ is proportional to $1/r$. In order to check the satisfaction of the conditions, it is necessary to search through the triangular region

$$\begin{aligned} \alpha_0 &> 0 \\ \alpha_1 &> 0 \\ \alpha_0 + \alpha_1 &< \bar{\alpha} \end{aligned}$$

for the existence of fixed α_0 and α_1 such that the LMI (5.36) is feasible. We can, of course, choose α_0 and α_1 *a priori* within this region and accept the conservatism. A reasonable choice seems to be

$$\alpha_0 = \frac{\bar{\alpha}||A_1A_0||}{2(||A_1A_0|| + ||A_1||^2)} \tag{5.41}$$

$$\alpha_1 = \frac{\bar{\alpha}||A_1||^2}{2(||A_1A_0|| + ||A_1||^2)} \tag{5.42}$$

Example 5.2 *Consider the system*

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r).$$

Applying Proposition 5.7, and using fixed α_0 and α_1 calculated by (5.41) and (5.42), we use a bisection process to calculate $r_{\max}$ such that (5.36) is feasible for $r < r_{\max}$. The calculation results in $r_{\max} = 0.9041$. For comparison, the analytical delay limit for stability is also calculated to be $r_{\max}^{analytical} = 6.1725$. The analytical delay limit is calculated by the smallest r such that there exists an imaginary pole. It can be seen that the criterion presented here can be very conservative as compared to the analytical limit.

It should be pointed out that although delay-independent stability implies delay-dependent stability for any given delay, due to conservatism of the above criterion, there indeed exist delay-independent systems which do not satisfy the above delay-dependent stability criterion, as will be shown in Example 5.4 later in this Chapter.

5.3.3 Additional dynamics

In this Subsection, we will try to present the basic idea of additional dynamics phenomena. Most of the proofs are omitted. Interested readers are referred to references discussed in the Notes and references Section at the end of this Chapter. It is important to realize that model transformation may introduce significant conservatism which is independent of the method used to judge the stability of the transformed system.

In order to more explicitly express the additional dynamics and their effects on the system behaviors, let us look at the transformed system (5.29) from a slightly different angle.

Theorem 5.9 *The transformed system described by (5.29) and (5.30) with initial condition (5.31) is equivalent to the following system*

$$\begin{cases} \dot{y}(t) = A_0 y(t) + A_1 y(t-r) + z(t) \\ z(t) = A_1 \int_{-r}^{0} z(t+\theta) d\theta. \end{cases} \tag{5.43}$$

and the initial condition

$$\begin{cases} y(\theta) = \varphi(\theta) \\ z(\theta) = \phi(\theta) \end{cases} \qquad \text{for } -r \le \theta \le 0$$

where

$$\phi(\theta) = A_1 \left[\varphi(\theta) - \varphi(\theta - r) - \int_{-r}^{0} (A_0 \varphi(\theta+\xi) + A_1 \varphi(\theta - r + \xi)) d\xi \right]$$

Notice, instead of $\psi(\theta)$, $-2r \le \theta \le 0$ we may consider $(\varphi(\theta), \phi(\theta))$, $-r \le \theta \le 0$ as independent initial condition.

Proof. We will show that given any one system, we can derive the other. From the transformed system (5.29) and (5.30), we can write

$$\dot{y}(t) = A_0 y(t) + A_1 y(t-r) + z(t)$$

where

$$\begin{aligned} z(t) &= A_1 \left[y(t) - y(t-r) - \int_{-r}^{0} (A_0 y(t+\theta) + A_1 y(t-r+\theta)) d\theta \right] \\ &= A_1 \int_{-r}^{0} \left[\dot{y}(t+\theta) - A_0 y(t+\theta) + A_1 y(t-r+\theta) \right] d\theta \\ &= A_1 \int_{-r}^{0} z(t+\theta) d\theta. \end{aligned}$$

We have therefore obtained (5.43).

On the other hand, from (5.43), we can also obtain (5.29) and (5.30). From the first equation of (5.43), we obtain

$$z(t) = \dot{y}(t) - A_0 y(t) - A_1 y(t-r)$$

Using the above in the second equation of (5.43) we arrive at

$$\begin{aligned} &\dot{y}(t) - A_0 y(t) - A_1 y(t-r) \\ = \; & A_1 \int_{-r}^{0} \left[\dot{y}(t+\theta) - A_0 y(t+\theta) + A_1 y(t-r+\theta) \right] d\theta. \end{aligned}$$

Integrating the $\dot{y}(t+\theta)$ term, we obtain

$$\dot{y}(t) = (A_0 + A_1)\, y(t) - A_1 \int_{-r}^{0} \left[A_0 y(t+\theta) + A_1 y(t-r+\theta) \right] d\theta.$$

which is (5.29) and (5.30).

The initial condition relation can also be easily verified. ∎

The Theorem above explicitly expressed the *additional dynamics*, *i.e.*, the dynamics of the variable $z(t)$, which is governed by the second equation of (5.43).

In order to further characterize the additional dynamics, notice the characteristic quasipolynomial of (5.43) is

$$\begin{aligned} \Delta_t(s) &= \det \begin{pmatrix} sI - A_0 - e^{-rs} A_1 & I \\ 0 & I - \frac{1-e^{-rs}}{s} A_1 \end{pmatrix} \\ &= \Delta_a(s) \Delta_o(s) \end{aligned}$$

where

$$\Delta_o(s) = \det(sI - A_0 - e^{-rs} A_1) \tag{5.44}$$

is the characteristic quasipolynomial of the original system (5.1), and

$$\Delta_a(s) = \det\left(I - \frac{1-e^{-rs}}{s}A_1\right)$$

Therefore, the poles of the transformed system consist of the poles of the original system satisfying $\Delta_o(s) = 0$, and additional poles, which are the solutions of

$$\Delta_a(s) = \det\left(I - \frac{1-e^{-rs}}{s}A_1\right) = 0 \tag{5.45}$$

It is therefore clear that although the stability of the transformed system implies the stability of the original system, the reverse is not necessarily true. As the delay r increases, it is possible that one of the additional poles may cross the imaginary axis before any of the poles of the original system do. Therefore, it is indeed possible that the original system is stable but the transformed system is unstable. This is the stability implication of the additional dynamics.

To further characterize the additional dynamics, we can conclude from (5.45) that

$$\Delta_a(s) = \det\left(I - \frac{1-e^{-rs}}{s}A_1\right) = \prod_{i=1}^{n}\left(1 - \lambda_i\frac{1-e^{-rs}}{s}\right),$$

where λ_i, is the ith eigenvalue of matrix A_1. Let $s = s_{ik}$, $k = 1, 2, 3, ...$ be all the solutions of the equation

$$1 - \lambda_i\frac{1-e^{-rs}}{s} = 0,$$

Then s_{ik}, $i = 1, 2, ..., n$; $k = 1, 2, 3, ...$ are all the additional poles of system described by (5.29) and (5.30). We will refer to s_{ik} as the k-th additional pole corresponding to λ_i (or corresponding to the ith eigenvalue of A_1).

It is of interest to consider the trend of the additional poles s_{ik} as the delay $r \to 0^+$. We will state the following result without proof:

Proposition 5.10 *For any given A_1, all the additional poles satisfy*

$$\lim_{r\to 0^+} \mathrm{Re}(s_{ik}) = -\infty.$$

As a result of this Proposition, all the additional poles have negative real parts for sufficiently small r. As r increases, some of the additional poles s_{ik} may cross the imaginary axis. It turns out that the exact value when that happens can be analytically calculated. We will state the follow Theorem, again without proof.

Theorem 5.11 *Corresponding to an eigenvalue λ_i of A_1, $\mathrm{Im}(\lambda_i) \neq 0$, there is an additional pole s_{ik} on the imaginary axis if and only if the time-delay satisfies*

$$r = r_{ik} = \frac{k\pi + \angle(\lambda_i)}{\mathrm{Im}(\lambda_i)} > 0, \quad k = 0, \pm 1, \pm 2, ...$$

Corresponding to a positive real eigenvalue λ_i of A_1, there is an additional pole on the imaginary axis if and only if

$$r = \frac{1}{\lambda_i}.$$

No additional poles corresponding to a negative real eigenvalue λ_i of A_1 will reach the imaginary axis for any finite delay.

Given a time-delay r, we can use the above Theorem to find the region Γ_r, such that all the additional poles (zeros of $\Delta_a(s)$) have strictly negative real parts if and if all the eigenvalues of A_1 lies in Γ_r:

$$\Gamma_r = \{ z = x + iy \ \mid x < y/\tan(ry), \ -\pi/r < y < \pi/r \}.$$

The region Γ_r is shown in Figure 5.1. It is worth mentioning that the set Γ_r satisfies the following relations:

$$\Gamma_r = \frac{1}{r}\Gamma_1 = \{z/r \mid z \in \Gamma_1\}.$$

In other words, the shape of the set Γ_1 remains the same for any $r > 0$, but its linear dimension is inversely proportional to the delay r. Obviously, as $r \to +0$, the region Γ_r approaches the whole complex plane C.

The following examples are presented to illustrate a number of possibilities.

Example 5.3 *Consider the system*

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r).$$

Since all the eigenvalues of A_1 are real *and* negative*, all the additional poles of the transformed system are on the left half plane for any r. Therefore, the stability of this system and the transformed system are equivalent. The conservatism shown in Example 5.2 is due to the application of the Razumikhin Theorem, not due to the model transformation.*

Example 5.4 *Consider*

$$\dot{x}(t) = \begin{pmatrix} -6 & 0 \\ 0.2 & -5.8 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 4 \\ -8 & -8 \end{pmatrix} x(t-r).$$

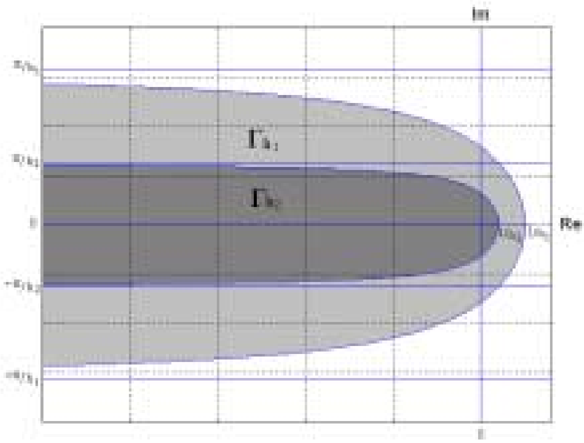

FIGURE 5.1. Regions Γ_{h_1} and Γ_{h_2}, with $h_2 > h_1$

A_1 has a pair of complex conjugate eigenvalues, $\lambda_{1,2} = -4 \pm j4$. The smallest positive delay for an additional pole to reach imaginary axis can be calculated as

$$r = \frac{3\pi}{16} = 0.589.$$

Therefore, the stability of the system and that of the transformed system is equivalent for $r < \frac{3\pi}{16}$. It can be checked using Proposition 5.14 in the next section that system is stable independent of delay. However, the delay $r = \frac{3\pi}{16}$ destabilizes the transformed system. Therefore, the transformation introduces considerable conservatism.

Example 5.5 *Consider*

$$\dot{x}(t) = \begin{pmatrix} -3 & -2.5 \\ 1 & 0.5 \end{pmatrix} x(t) + \begin{pmatrix} 1.5 & 2.5 \\ -0.5 & -1.5 \end{pmatrix} x(t-r).$$

A_1 has two real eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = -1$, respectively. The smallest positive delay for an additional pole to reach imaginary axis can be calculated as

$$r = 1.$$

Therefore, the stability of the system and that of the transformed system are equivalent for $r < 1$. However, it can be calculated that the minimum delay to destabilize the original system is known to be $r = 2.4184$ (using, for example, the method covered in Chapter 2), while the minimum delay to destabilize the corresponding transformed system is only $r = 1$. Again, as in the previous example, significant conservatism will be introduced by the transformation.

5.3.4 Simple delay-dependent stability using implicit model transformation

Consider again the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-r) \tag{5.46}$$

before model transformation. It is possible not to use model transformation directly and still obtain a relatively simple stability criterion. During the bounding in deriving stability criteria, information $x(t+\theta)$, $-2r \le \theta \le 0$ is used. The constraint

$$\dot{x}(t+\theta) = A_0 x(t+\theta) + A_1 x(t-r+\theta), \ -r \le \theta \le 0 \tag{5.47}$$

is partially but not fully accounted for. We will call such methods as using *implicit model transformation* based methods. We will show that such a scheme can indeed obtain stability results which include as special case both delay-independent and delay-dependent stability criteria obtained earlier in this chapter, and therefore is less conservative.

To illustrate the idea, for the Lyapunov function $V(x) = x^T P x$ and real matrices $X^T = X$, Y, $Z^T = Z$ satisfying

$$\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} > 0 \tag{5.48}$$

we can write

$$\begin{aligned} \dot{V}(x) &= 2x^T(t)P[A_0 x(t) + A_1 x(t-r)] \\ &\le 2x^T(t)P[A_0 x(t) + A_1 x(t-r)] \\ &\quad + \int_{-r}^{0} \begin{pmatrix} x^T(t) & \frac{d}{d\xi}x^T(t+\xi) \end{pmatrix} \\ &\quad \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \begin{pmatrix} x(t) \\ \frac{d}{d\xi}x(t+\xi) \end{pmatrix} d\xi \end{aligned} \tag{5.49}$$

Using (5.47), we can write the term involving Z above as

$$\begin{aligned} \dot{V}_Z &= \int_{-r}^{0} [\frac{d}{d\xi}x(t+\xi)]^T Z[\frac{d}{d\xi}x(t+\xi)]d\xi \\ &= \int_{-r}^{0} x_{tr}^T(\xi) \begin{pmatrix} A_0^T Z A_0 & A_0^T Z A_1 \\ A_1^T Z A_0 & A_1^T Z A_1 \end{pmatrix} x_{tr}(\xi) d\xi \end{aligned} \tag{5.50}$$

where

$$x_{tr}(\xi) = \begin{pmatrix} x(t+\xi) \\ x(t+\xi-r) \end{pmatrix} \tag{5.51}$$

When for a $p > 1$,

$$V(x(t+\theta)) < pV(x(t)) \text{ for all } -2r \le \theta \le 0$$

we have

$$\begin{aligned}
&\alpha[px^T(t)Px(t) - x^T(t-r)Px(t-r)] \\
&+\alpha_0 \int_{-r}^{0} [px^T(t)Px(t) - x^T(t+\xi)Px(t+\xi)]d\xi \\
&+\alpha_1 \int_{-r}^{0} [px^T(t)Px(t) - x^T(t+\xi-r)Px(t+\xi-r)]d\xi \geq 0 \quad (5.52) \\
&\qquad \text{for } \alpha \geq 0, \alpha_0 \geq 0, \alpha_1 \geq 0.
\end{aligned}$$

Using (5.50) and (5.52) in (5.49), carrying out integration for terms in (5.49) whenever possible, we arrive at

$$\begin{aligned}
&\dot{V}(x(t)) \\
\leq\ & x_{tr}^T(0) \begin{pmatrix} \Delta + (\alpha + \alpha_0 r + \alpha_1 r)pP & PA_1 - Y \\ (PA_1 - Y)^T & -\alpha P \end{pmatrix} x_{tr}(0) \\
&+ \int_{t-r}^{t} x_{tr}^T(\xi) \begin{pmatrix} A_0^T Z A_0 - \alpha_0 P & A_0^T Z A_1 \\ A_1^T Z A_0 & A_1^T Z A_1 - \alpha_1 P \end{pmatrix} x_{tr}(\xi) d\xi
\end{aligned}$$

where

$$\Delta = PA_0 + A_0^T P + rX + Y + Y^T$$

In view of the fact that $p > 1$ can be arbitrary close to 1, the above expression implies that the system is asymptotically stable if there exist real scalars $\alpha \geq 0, \alpha_0 \geq 0, \alpha_1 \geq 0$, real symmetric matrices $P^T = P > 0$, $X^T = X$, Y, $Z^T = Z$ such that (5.48) and

$$\begin{pmatrix} \Delta + (\alpha + \alpha_0 r + \alpha_1 r)P & PA_1 - Y \\ (PA_1 - Y)^T & -\alpha P \end{pmatrix} < 0 \quad (5.53)$$

$$\begin{pmatrix} A_0^T Z A_0 - \alpha_0 P & A_0^T Z A_1 \\ A_1^T Z A_0 & A_1^T Z A_1 - \alpha_1 P \end{pmatrix} < 0 \quad (5.54)$$

are satisfied.

We can eliminate the matrix variable Z in (5.48) and (5.54), then eliminate X in the resulting matrix inequality and (5.53) to obtain the following equivalent form:

Theorem 5.12 *The system described by (5.46) is asymptotically stable if there exist real scalars $\alpha > 0$, $\alpha_0 > 0$, $\alpha_1 > 0$, and real matrix Y and real symmetric matrix P such that*

$$\begin{pmatrix} N & -YA_0 & -YA_1 & PA_1 - Y \\ -A_0^T Y & -\alpha_0 P & 0 & 0 \\ -A_1^T Y & 0 & -\alpha_1 P & 0 \\ (PA_1 - Y)^T & 0 & 0 & -\alpha r P \end{pmatrix} < 0 \quad (5.55)$$

where

$$N = \frac{1}{r}(PA_0 + A_0^T P + Y + Y^T) + (\frac{\alpha}{r} + \alpha_0 + \alpha_1)P$$

It is interesting to examine closely (5.55). If we set $Y = 0$, then obviously the optimal selection is $\alpha_0 \to 0^+$, $\alpha_1 \to 0^+$, we recover the delay-independent stability criterion discussed in Proposition 5.3. On the other hand, if we set $Y = PA_1$, then the optimal selection is $\alpha \to 0^+$, we recover the delay-dependent stability criterion derived by explicit model transformation discussed in Corollary 5.8. It is possible to eliminate the matrix variable Y using Corollary B.5 in Appendix B to obtain

Corollary 5.13 *The system described by (5.46) is asymptotically stable if there exist real scalars $\alpha > 0$, $\alpha_0 > 0$, $\alpha_1 > 0$, and real symmetric matrix P such that*

$$\begin{pmatrix} M & P(\alpha I - A_1)A_0 & P(\alpha I - A_1)A_1 \\ [P(\alpha I - A_1)A_0]^T & -\alpha_0 P - \alpha r A_0^T P A_0 & -\alpha r A_0^T P A_1 \\ [P(\alpha I - A_1)A_1]^T & -\alpha r A_1^T P A_0 & -\alpha_1 P - \alpha r A_1^T P A_1 \end{pmatrix} < 0 \tag{5.56}$$

where

$$M = \frac{1}{r}[P(A_0 + A_1) + (A_0 + A_1)^T P] + (\alpha_0 + \alpha_1)P \tag{5.57}$$

To discuss the region of the scaling factors α, α_0 and α_1, first notice the (1,1) entry of (5.56) defined in (5.57) implies that α_0 and α_1 must lie in the triangular region $\alpha_0 > 0$, $\alpha_1 > 0$, $\alpha_0 + \alpha_1 < \bar{\alpha}$, where $\bar{\alpha}$ is defined in (5.38) to (5.40).

To find a bound for α, assume the matrix $\omega = \begin{pmatrix} A_0 & A_1 \end{pmatrix}$ has full row rank, let $\hat{\omega} = \begin{pmatrix} \hat{A}_0^T & \hat{A}_1^T \end{pmatrix}^T$ be a right inverse of ω. We can left multiply (5.56) by the matrix

$$\begin{pmatrix} I & \frac{1}{r}\hat{\omega}^T \\ 0 & I \end{pmatrix}$$

and right multiply by its transpose to obtain

$$\begin{pmatrix} \hat{M} & -PA_1\omega - \frac{1}{r}\hat{\omega}^T R \\ -\omega^T A_1^T P - \frac{1}{r}R\hat{\omega} & -R - \alpha r \omega^T P \omega \end{pmatrix} < 0$$

where

$$R = \mathrm{diag}\begin{pmatrix} \alpha_0 P & \alpha_1 P \end{pmatrix}$$

and

$$\hat{M} = \frac{1}{r}(PA_0 + A_0^T P) + (\frac{\alpha}{r} + \alpha_0 + \alpha_1)P - \frac{1}{r^2}(\alpha_0 \hat{A}_0^T P \hat{A}_0 + \alpha_1 \hat{A}_1^T P \hat{A}_1) \tag{5.58}$$

Therefore, given α_0, α_1, let $\hat{\alpha}(\alpha_0, \alpha_1)$ be the solution of the following GEVP

$$\hat{\alpha}(\alpha_0, \alpha_1) = \max \alpha \tag{5.59}$$

$$\text{subject to } \hat{M} < 0, \ P > 0 \tag{5.60}$$

Then, α must satisfy $0 < \alpha < \hat{\alpha}(\alpha_0, \alpha_1)$ if $\hat{\alpha}(\alpha_0, \alpha_1) > 0$.

With the above discussions, a reasonable choice for the scaling factors is to choose α_0 and α_1 is

$$\alpha_0 = \frac{\bar{\alpha}||(\alpha I - A_1)A_0||}{3(||(\alpha I - A_1)A_0|| + ||(\alpha I - A_1)A_1||)} \tag{5.61}$$

$$\alpha_1 = \frac{\bar{\alpha}||(\alpha I - A_1)A_1||}{3(||(\alpha I - A_1)A_0|| + ||(\alpha I - A_1)A_1||)} \tag{5.62}$$

and a reasonable choice of α is

$$\alpha = \hat{\alpha}(\alpha_0, \alpha_1)/6 \tag{5.63}$$

We can first set $\alpha = 0$ in calculating α_0 and α_1 using (5.61) and (5.62), then calculate α using α_0 and α_1 just calculated in (5.63), and updating α_0 and $\alpha_1$using the newly calculated α (we can, of course, also update α using the newly calculated α_0 and α_1).

Example 5.6 *Consider the same system discussed in Example 5.2. Using the method discussed here with α, α_0 and α_1 calculated by (5.61) to (5.63), with α_0, α_1 and α all updated, the calculation indicates that the the system is asymptotically stable for $r < r_{\max} = 0.9476$. For $r = r_{\max}$, the scaling factors are $\alpha = 0.4813$, $\alpha_0 = 0.7160$ and $\alpha_1 = 0.6208$. This is an improvement over the method using explicit model transformation.*

5.4 Delay-independent stability based on Lyapunov-Krasovskii stability theorem

In this Section, we will discuss the stability of the same system (5.1) using some simple Lyapunov-Krasovskii functional method. The results parallel those obtained by Razumikhin Theorem.

5.4.1 Systems with single delay

Consider again system (5.1). Probably the simpliest stability criterion can be obtained by using the following Lyapunov-Krasovskii functional

$$V(x_t) = x^T(t)Px(t) + \int_{t-r}^{t} x^T(\xi)Sx(\xi)d\xi$$

where the matrices P and S are symmetric and positive definite. We will write the above as

$$V(\phi) = \phi^T(0)Px(0) + \int_{-r}^{0} \phi^T(\theta)S\phi(\theta)d\theta \tag{5.64}$$

It can be easily calculated that the derivative of V along the system trajectory is

$$\dot{V}(x_t) = \begin{pmatrix} x^T(t) & x^T(t-r) \end{pmatrix} \begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-r) \end{pmatrix}$$

or we can write x_t as ϕ to obtain

$$\dot{V}(\phi) = \begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix} \begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(-r) \end{pmatrix} \tag{5.65}$$

It is clear that $\dot{V}(x_t) \leq -\varepsilon ||x(t)||^2$ for some sufficiently small $\varepsilon > 0$ if the matrix in expression (5.65) is negative definite. Thus we can conclude that

Proposition 5.14 *System (5.1) is asymptotically stable if there exist real symmetric matrices*

$$P > 0 \tag{5.66}$$

and S, such that

$$\begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} < 0 \tag{5.67}$$

is satisfied.

Proof. Use Proposition 5.2, and choose Lyapunov-Krasovskii functional (5.64). Notice that (5.67) implies

$$S > 0 \tag{5.68}$$

which together with (5.66) implies

$$V(\phi) \geq \varepsilon ||\phi(0)||^2$$

for some sufficiently small $\varepsilon > 0$. The Lyapunov-Krasovskii functional condition is satisfied. Also, (5.67) implies that

$$\dot{V}(\phi) \leq -\varepsilon ||\phi(0)||^2$$

in view of (5.65), the Lyapunov-Krasovskii derivative condition is also satisfied. Therefore, the system is asymptotically stable according to Proposition 5.2. ■

It is interesting to compare the stability criterion in Proposition 5.16 obtained using Lyapunov-Krasovkii Theorem with the corresponding criterion in Proposition 5.7 obtained using Razumikhin Theorem. It can be seen that Proposition 5.7 can be obtained from Proposition 5.16 by introducing the additional constraint

$$S = \alpha P \tag{5.69}$$

This constraint, of course, makes the criterion in Proposition 5.7 obtained using Razumikhin Theorem more conservative than the corresponding criterion in Proposition 5.16 obtained using Lyapunov-Krasovskii Theorem. Also notice that the criterion in Proposition 5.16 is in the form of LMI. Therefore, it is easier to check using the readily available LMI solver. Therefore, for this type of systems, it is better to use Lyapunov-Krasovskii Theorem. On the other hand, if the system has a time-varying delay, then Proposition 5.16 is no longer applicable, but the criterion in Proposition 5.7 is still applicable as will be shown in Chapter 6.

To illustrate the criterion, we will present the following numerical example:

Example 5.7 *Consider the system*

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \beta \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r).$$

where $\beta \geq 0$ and $r > 0$. The parameter β in some extent measures the size of the delayed term. A bisection scheme is used to calculate $\beta_{\max}$, the maximum value of β the system can tolerate without violating the delay-independent stability criterion in Proposition 5.14. The calculation results in $\beta_{\max} = 0.9$. This is the same as the result obtained using Razumikhin Theorem as in Example 5.1. However, this is not always the case. For example, for the system discussed in Example 5.5, Razumikhin Theorem based method yields $\beta_{\max} = 1.0425$, while the Lyapunov-Krasovskii functional based method yields a slightly large bound $\beta_{\max} = 1.0428$.

5.4.2 Systems with distributed delays

Consider the system with distributed delay

$$\dot{x}(t) = A_0 x(t) + \int_{-r}^{0} A(\theta) x(t+\theta) d\theta, \tag{5.70}$$

where A_0, is a given constant matrix and $A(\theta)$ is a continuous matrix valued function given for $\theta \in [-r, 0]$.

The Lyapunov-Krasovskii functional for the system can be chosen as

$$V(\phi) = \phi^T(0) P \phi(0) + \int_{-r}^{0} \left[\int_{\theta}^{0} \phi^T(\xi) S(\theta) \phi(\xi) d\xi \right] d\theta \tag{5.71}$$

The derivative of V can be calculated as

$$\begin{aligned}\dot{V}(\phi) &= \phi^T(0)[PA_0 + A_0^T P + \int_{-r}^{0} S(\theta)d\theta]\phi(0) \\ &\quad +2\phi^T(0)\int_{-r}^{0} PA(\theta)\phi(\theta)d\theta \\ &\quad -\int_{-r}^{0} \phi^T(\theta)S(\theta)\phi(\theta)d\theta\end{aligned}$$

To facilitate further development, add and subtract a relaxation matrix function $R(\theta)$ term in the above, resulting in

$$\begin{aligned}\dot{V}(\phi) &= \phi^T(0)[PA_0 + A_0^T P + \int_{-r}^{0} R(\theta)d\theta]\phi(0) \\ &\quad +\int_{-r}^{0} \begin{pmatrix} \phi^T(0) & \phi^T(\theta) \end{pmatrix} \begin{pmatrix} S(\theta)-R(\theta) & PA(\theta) \\ A^T(\theta)P & -S(\theta) \end{pmatrix} \\ &\quad \begin{pmatrix} \phi(0) \\ \phi(\theta) \end{pmatrix} d\theta \end{aligned} \tag{5.72}$$

The derivative condition will be satisfied if the two matrices in the above are negative definite.

Proposition 5.15 *The system described by (5.70) is asymptotically stable if there exist real symmetric matrix*

$$P > 0 \tag{5.73}$$

and real symmetric matrix functions $R(\theta)$ and $S(\theta)$ such that

$$PA_0 + A_0^T P + \int_{-r}^{0} R(\theta)d\theta < 0 \tag{5.74}$$

and

$$\begin{pmatrix} S(\theta)-R(\theta) & PA(\theta) \\ A^T(\theta)P & -S(\theta) \end{pmatrix} < 0, \quad \theta \in [-r, 0]. \tag{5.75}$$

Proof. Use Proposition 5.2 and use Lyapunov-Krasovskii functional (5.71). Since (5.75) implies $S(\theta) > 0$, it is clear that $V(\phi) \geq \varepsilon||\phi(0)||^2$ for some sufficiently small $\varepsilon > 0$, the Lyapunov-Krasovskii functional condition (5.6) is satisfied. In expression (5.72), the second term is always less than or equal to zeros due to (5.75). Also, (5.74) implies the existence of a sufficiently small $\varepsilon > 0$ such that

$$PA_0 + A_0^T P + \int_{-r}^{0} R(\theta)d\theta \leq -\varepsilon I.$$

Therefore, $\dot{V}(\phi) \leq -\varepsilon||\phi(0)||^2$, the Lyapunov-Krasovskii derivative condition is also satisfied. Therefore, the system is asymptotically stable according to Proposition 5.2. ■

Similar to the Proposition 5.6, we can show that the above stability criterion is delay-independent. Also, the corresponding result using Razumikhin Theorem, Proposition 5.5, can be obtained from the above result by introducing additional constraint

$$S(\theta) = \alpha(\theta)P.$$

5.5 Delay-dependent stability criteria using simple Lyapunov-Krasovskii functional

5.5.1 Stability criteria using explicit model transformation

Simple delay-dependent stability criterion can also be derived with Lyapunov-Krasovskii Theorem, in parallel to the Razumikhin Theorem formulation. Consider a system described by (5.1). Recall that we can use a model transformation to obtain a system with distributed delays represented by (5.29) and (5.30). The stability of the system described by (5.29) and (5.30) implies that of (5.1). We can apply Proposition 5.15 to obtain

Proposition 5.16 *The system described by (5.1) is asymptotically stable if there exist real symmetric matrices P, R_0, R_1, S_0 and S_1 such that*

$$P > 0 \tag{5.76}$$

$$\begin{pmatrix} M & -PA_1A_0 & -PA_1^2 \\ -A_0^TA_1^TP & -S_0 & 0 \\ -(A_1^2)^TP & 0 & -S_1 \end{pmatrix} < 0 \tag{5.77}$$

where

$$M = \frac{1}{r}[P(A_0 + A_1) + (A_0 + A_1)^TP] + S_0 + S_1 \tag{5.78}$$

Proof. As is discussed above, it is sufficient to prove the stability of the transformed system described by (5.29) and (5.30). Apply Proposition 5.15 and choose

$$R(\theta) = \begin{cases} R_0, & -r \leq \theta < 0 \\ R_1, & -2r \leq \theta < -r \end{cases}$$

$$S(\theta) = \begin{cases} S_0, & -r \leq \theta < 0 \\ S_1, & -2r \leq \theta < -r \end{cases}$$

we obtain the sufficient condition consisting of (5.76) and

$$[P(A_0 + A_1) + (A_0 + A_1)^TP] + r(R_0 + R_1) < 0 \tag{5.79}$$

$$\begin{pmatrix} S_k - R_k & -PA_1A_k \\ -A_k^TA_1^TP & -S_k \end{pmatrix} < 0, \ k = 0, 1 \tag{5.80}$$

Divide (5.79) by r, and eliminate R_0 and R_1 from the resulting matrix inequality and (5.80) to obtain (5.77). ∎

Once again, the corresponding result in Corollary 5.8 can be obtained by applying additional constraints

$$S_k = \alpha_k P$$

If one is to consider the stability of only system (5.1), the above criterion is clearly better than Proposition 5.8 since it is less conservative and computationally more convenient due to the linearity of parameters. The main value of Corollary 5.8 is again due to the fact that it is also valid for time-varying delay, as will be discussed in Chapter 6.

Example 5.8 *Consider the same system discussed in Example 5.2. Using Proposition 5.16, we can find that* $r_{\max}$*, the maximum delay for the system to satisfy the condition, is* $r_{\max} = 1.000$*. This is less conservative than the result in Example 5.2 obtained using the Razumikhin Theorem. However, it is still very far from the true delay limit for stability* $r_{\max}^{analytic} = 6.1725$*.*

5.5.2 Stability criteria using implicit model transformation

Similar to the case of Razumikhin Theorem based methods, it is also possible to obtain less conservative stability criteria using implicit model transformation. Consider again the system with single delay described by (5.1).

For real matrices $X^T = X$, Y and $Z^T = Z$ satisfying

$$\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} > 0 \tag{5.81}$$

Consider quadratic Lyapunov Krasovskii functional

$$V(\phi) = \phi^T(0)P\phi(0) + \int_{-r}^{0}\int_{\theta}^{0} f^T(\phi_\xi)Zf(\phi_\xi)d\xi d\theta + \int_{-r}^{0} \phi^T(\theta)S\phi(\theta)d\theta$$

where

$$\begin{aligned} \phi_\xi(\theta) &= \phi(\xi+\theta), \ -r \le \theta \le 0, \ -r \le \xi \le 0 \\ f(\phi) &= A_0\phi(0) + A_1\phi(-r) \end{aligned}$$

Realizing

$$\frac{d}{dt}\int_{\theta}^{0} f^T(\phi_\xi)Zf(\phi_\xi)d\xi = f^T(\phi)Zf(\phi) - f^T(\phi_\theta)Zf(\phi_\theta)$$

We can calculate

$$\dot{V}(\phi) = \phi_{0r}^T \begin{pmatrix} M & PA_1 + rA_0^T ZA_1 \\ \text{symmetric} & rA_1^T ZA_1 - S \end{pmatrix} \phi_{0r} - \int_{-r}^{0} f^T(\phi_\theta)Zf(\phi_\theta) \tag{5.82}$$

where

$$\begin{aligned} M &= PA_0 + A_0^T P + rA_0^T Z A_0 + S \\ \phi_{0r}^T &= \begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix} \end{aligned}$$

Let

$$\phi_{0\dot{\theta}} = \begin{pmatrix} \phi^T(0) & \dot{\phi}^T(\theta) \end{pmatrix}$$

we have

$$\begin{aligned} 0 &\le \int_{-r}^{0} \phi_{0\dot{\theta}}^T \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \phi_{0\dot{\theta}} d\theta \\ &= r\phi^T(0)X\phi(0) + 2\phi^T(0)Y(\phi(0) - \phi(-r)) \\ &\quad + \int_{-r}^{0} \dot{\phi}^T(\theta)Z\dot{\phi}(\theta)d\theta \end{aligned} \tag{5.83}$$

Add (5.83) and (5.82) using the constraint

$$\dot{\phi}(\theta) = f(\phi_\theta) \tag{5.84}$$

we obtain

$$\dot{V}(\phi) \le \phi_{0r}^T \begin{pmatrix} N & PA_1 + rA_0^T ZA_1 - Y \\ \text{symmetric} & -S + rA_1^T ZA_1 \end{pmatrix} \phi_{0r}$$

where

$$N = PA_0 + A_0^T P + rA_0^T ZA_0 + S + rX + Y + Y^T$$

from which we conclude that system (5.1) is asymptotically stable if there exist real matrices $P = P^T > 0$, $X^T = X$, Y and $Z^T = Z$ satisfying (5.81) and

$$\begin{pmatrix} N & PA_1 + rA_0^T ZA_1 - Y \\ \text{symmetric} & -S + rA_1^T ZA_1 \end{pmatrix} < 0 \tag{5.85}$$

To illustrate the connection between this result and the previous results based on Lyapunov-Krasovskii functional methods, we eliminate the matrix variable Z in (5.85) and (5.81) using Proposition B.6 in Appendix B to obtain:

Proposition 5.17 *The system described by (5.1) is asymptotically stable if there exist real matrices $X^T = X$, Y and*

$$P > 0$$

such that

$$\begin{pmatrix} \hat{N} & PA_1 - Y & -A_0^T Y^T \\ A_1^T P - Y^T & -S & -A_1^T Y^T \\ -YA_0 & -YA_1 & -\frac{1}{r}X \end{pmatrix} < 0 \tag{5.86}$$

where

$$\hat{N} = PA_0 + A_0 P + S + rX + Y + Y^T$$

If we set $Y = 0$, we obtain the delay-independent stability criterion discussed in Proposition 5.14.

On the other hand, if we choose $Y = PA_1$, it can be easily shown that the resulting criterion is equivalent to the delay-dependent stability criterion discussed in Proposition 5.16 with the following variable transformation from (S, X) to (S_0, S_1):

$$\begin{aligned} S &= rS_1 \\ X &= -\frac{1}{r}[P(A_0 + A_1) + (A_0 + A_1)^T P] - S_0 - S_1 \end{aligned}$$

Example 5.9 *Consider the system discussed in Example 5.8. Using the above stability criterion, it can be concluded that the maximum delay for stability is* $r_{\max} = 4.3588$. *This is much less conservative as compared to the corresponding result obtained using explicit model transformation.*

5.6 Complete quadratic Lyapunov-Krasovskii functional

5.6.1 Introduction

From the numerical examples presented so far in this Chapter, it is obvious that there is still substantial conservatism for the simple delay-dependent stability criteria discussed so far. Another serious limitation of these delay-dependent stability criteria is that the system is always regarded as "less stable" than the corresponding system with the delay set to zero. As is discussed in Chapter 1, there are indeed practical cases where time-delay is intentionally introduced to stabilize the system or enhance the system performance. A striking example is the following system

$$\ddot{x}(t) - 0.1\dot{x}(t) + x(t) = u(t) \tag{5.87}$$

If the control $u(t)$ is set to zero, the system is clearly unstable due to "negative damping". Classic control theory would suggest using a derivative feedback $u(t) = -k\dot{x}(t)$. Indeed, a control gain $k > 0.1$ will make the system stable by achieving positive damping. As is well known that derivative feedback is usually not easy to achieve. It seems reasonable to use the following finite difference

$$u(t) = x(t-r) - x(t) = -r\frac{x(t) - x(t-r)}{r} \tag{5.88}$$

to approximate $-r\dot{x}(t)$. Such an approximation seems reasonable as long as r is sufficiently small. However, none of the methods discussed so far in this chapter will be able to affirm the stability of such a system, since setting the delay $r = 0$ voids the effect of the control $u(t)$. It will be shown later in the next Section of this Chapter that there is indeed a large interval $[r_{\min}, r_{\max}]$ with $r_{\min} \approx 0.1$ such that the system is asymptotically stable if r falls into this interval.

In this Section, we will enlarge the class of Lyapunov-Krasovskii functional to complete quadratic form. It will be shown that the existence of such a functional, satisfying the Lyapunov-Krasovskii functional condition and derivative condition, is indeed necessary and sufficient condition for asymptotic stability of the system.

The development in this Section, while much more complicated, has some similarity to the Lyapunov equation

$$PA + A^T P = -W \tag{5.89}$$

corresponding to the ordinary differential equation

$$\dot{x}(t) = Ax(t) \tag{5.90}$$

Recall that Equation (5.89) has positive definite solution for any given positive definite matrix W if and only if system (5.90) is asymptotically stable. Indeed, if both P and W are positive definite, we can choose Lyapunov function

$$V(x) = x^T P x$$

and verify that its derivative along the trajectory of the system (5.90) is

$$\begin{aligned} \dot{V}(x) &= x^T (PA + A^T P) x \\ &= -x^T W x \end{aligned}$$

Furthermore, the solution in this case can be explicitly expressed as

$$P = \int_0^\infty \Phi^T(t) W \Phi(t) dt$$

where

$$\Phi(t) = e^{tA}$$

is the fundamental solution of system (5.90), which satisfies the equation

$$\dot{\Phi}(t) = A\Phi(t)$$

and the initial condition

$$\Phi(0) = I$$

Indeed, it will be shown that this idea can be extended to time-delay systems in the sense that it is still possible to prescribe the derivative expression, and "solve for" the Lyapunov-Krasovskii functional.

5.6.2 Fundamental solution and matrix $U_W(\tau)$

We will return to the discussion of the system with single delay

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-r), \tag{5.91}$$

where A_0 and A_1 are constant $n \times n$ matrices and r is a positive delay. We will assume that the system is asymptotically stable, and try to obtain a quadratic Lyapunov-Krasovskii functional. The solution of the system under the initial condition

$$x_0 = \phi \tag{5.92}$$

will be denoted as $x(t, \phi)$. Recall from Chapter 1 that the fundamental solution $\Phi(t)$ satisfies the equation

$$\frac{d}{dt}\Phi(t) = A_0\Phi(t) + A_1\Phi(t-r), \; t \geq 0, \tag{5.93}$$

and the initial condition

$$\Phi(0) = I, \text{ and } \Phi(t) = 0 \text{ for } t < 0. \tag{5.94}$$

Any solution of (5.91) under initial condition (5.92), $x(t, \phi)$, can be expressed in terms of $\Phi(t)$

$$x(t, \phi) = \Phi(t)\phi(0) + \int_{-r}^{0} \Phi(t-r-\theta)A_1\phi(\theta)d\theta, \text{ for } t \geq 0. \tag{5.95}$$

which is a special case of (1.33) in Chapter 1. The fundamental solution $\Phi(t)$ can also be expressed as the inverse Laplace transform of $\Delta^{-1}(s)$,

$$\Phi(t) = \mathcal{L}^{-1}[\Delta^{-1}(s)] = \mathcal{L}^{-1}[(sI - A_0 - e^{-rs}A_1)^{-1}] \tag{5.96}$$

From (5.96), it can be easily concluded that $\Phi(t)$ also satisfies the equation

$$\frac{d}{dt}\Phi(t) = \Phi(t)A_0 + \Phi(t-r)A_1, \; t \geq 0 \tag{5.97}$$

and initial condition (5.94). Since system (5.91) is asymptotically stable, the stability exponent α_0 is negative. Therefore, we can find an α, $\alpha_0 < \alpha < 0$, such that

$$||\Phi(t)|| \leq Ke^{\alpha t}, \text{ for all } t \geq 0 \tag{5.98}$$

for some $K > 1$. In other words, the fundamental solution approaches zero exponentially.

With the fundamental solution $\Phi(t)$, for a symmetric matrix W and real scalar τ, we can define the matrix $U_W(\tau)$ as

$$U_W(\tau) = \int_0^\infty \Phi^T(t)W\Phi(t+\tau)dt \tag{5.99}$$

This integration is well defined due to the fact that $\Phi(t)$ vanishes for $t < 0$ and approaches zero exponentially as $t \to +\infty$. We will omit the subscript W when no confusion may arise. For $\tau \geq 0$, with a transformation of integration variable $\xi = t + \tau$, and considering the symmetry of W, we may write

$$\begin{aligned} U^T(\tau) &= \int_0^\infty \Phi^T(t+\tau)W\Phi(t)dt \\ &= \int_\tau^\infty \Phi^T(\xi)W\Phi(\xi-\tau)d\xi \\ &= \int_0^\infty \Phi^T(\xi)W\Phi(\xi-\tau)d\xi \end{aligned}$$

In the last step, we have used the fact that $\Phi(\xi-\tau)$ vanishes when $0 < \xi \leq \tau$. Thus we have arrived at the useful fact

$$U(-\tau) = U^T(\tau) \tag{5.100}$$

In particular $U(0)$ is a symmetric matrix.

Differentiating $\Phi^T(t)W\Phi(t)$ and using (5.97), we can write

$$\begin{aligned} &\frac{d}{dt}\left(\Phi^T(t)W\Phi(t)\right) \\ = \ &\Phi^T(t)W\Phi(t)A_0 + \Phi^T(t)W\Phi(t-r)A_1 \\ &+A_0^T\Phi^T(t)W\Phi(t) + A_1^T\Phi^T(t-r)W\Phi(t) \end{aligned}$$

Integrating both sides of the above from 0 to ∞, we arrive at another useful fact

$$W + U(0)A_0 + A_0^TU(0) + U^T(r)A_1 + A_1^TU(r) = 0 \tag{5.101}$$

Differentiating (5.99) with respect to τ using (5.97) and (5.100), we can also write

$$\frac{d}{d\tau}U(\tau) = U(\tau)A_0 + U^T(r-\tau)A_1, \ \ \tau \in [0, r]. \tag{5.102}$$

With the above discussions, we can show that $U(\tau)$, $\tau \in [0, r]$ can be written as the solution of a two point boundary value problem as stated in the following Theorem.

Theorem 5.18 *Matrix $U(\tau)$ is the solution of the second order ordinary differential equation*

$$\frac{d^2U(\tau)}{d\tau^2} = \frac{dU(\tau)}{d\tau}A_0 - A_0^T\frac{dU(\tau)}{d\tau} + A_0^TU(\tau)A_0 - A_1^TU(\tau)A_1. \tag{5.103}$$

with the boundary conditions

$$\left.\frac{dU(\tau)}{d\tau}\right|_{\tau=0} + \left.\frac{dU^T(\tau)}{d\tau}\right|_{\tau=0} = -W \tag{5.104}$$

$$\left.\frac{dU(\tau)}{d\tau}\right|_{\tau=0} = U(0)A_0 + U^T(r)A_1 \tag{5.105}$$

Proof. Differentiate (5.102) and use (5.102) in the transposed form, we can write

$$\begin{aligned}\frac{d^2U(\tau)}{d\tau^2} &= \frac{d}{d\tau}U(\tau)A_0 + \frac{d}{d\tau}U^T(r-\tau)A_1 \\ &= \frac{d}{d\tau}U(\tau)A_0 - A_0^T U^T(r-\tau)A_1 - A_1^T U(\tau)A_1\end{aligned}$$

Substituting $U^T(r-\tau)A_1$ in the second term on the right by (5.102), we reach (5.103). (5.105) is (5.102) for $\tau = 0$. (5.104) is obtained by using (5.105) in (5.101). ■

It is well known that such a two point boundary value problem has unique solution. Indeed, the definition of $U(\tau)$ can be extended to unstable systems by considering the solution as the definition.

5.6.3 Lyapunov-Krasovskii functionals

For a stable system (5.91), we will first try to find a quadractic Lyapunov-Krasovskii functional $V(\phi) = v_W(\phi)$ such that

$$\dot{v}_W(x_t) = -x^T(t)Wx(t) \tag{5.106}$$

where

$$\dot{v}_W(x_t) = \frac{d}{dt}v_W(x_t) \tag{5.107}$$

Let $x(t,\phi)$ be the solution of (5.91) with initial condition $x_0 = \phi$, then, it can be easily confirmed that

$$v_W(\phi) = \int_0^\infty x^T(t,\phi)Wx(t,\phi)dt \tag{5.108}$$

satisfies (5.106). We can express $v_W(\phi)$ as an explicit quadratic functional of ϕ using (5.95) and (5.99)

$$\begin{aligned}&v_W(\phi) \\ =\ &\phi^T(0)U_W(0)\phi(0) \\ &+2\phi^T(0)\int_{-r}^0 U_W(-r-\theta)A_1\phi(\theta)d\theta \\ &+\int_{-r}^0\int_{-r}^0 \phi^T(\theta_1)A_1^T U_W(\theta_1-\theta_2)A_1\phi(\theta_2)d\theta_1 d\theta_2\end{aligned} \tag{5.109}$$

Based on $v_W(\phi)$, we can easily construct a Lyapunov-Krasovskii functional to satisfy

$$\dot{v}(x_t) = -x^T(t+\tau)Wx(t+\tau) \tag{5.110}$$

for $-r \leq \tau \leq 0$. This is accomplished by

$$v(\phi) = v_W(\phi) + \int_{\tau}^{0} \phi^T(\theta) W \phi(\theta) d\theta \tag{5.111}$$

With (5.106) and (5.109) to (5.111), we can take a linear combination to achieve

$$\begin{aligned} \dot{v}(x_t) &= -x^T(t) W_1 x(t) - x^T(t-r) W_3 x(t-r) \\ &\quad - \int_{-r}^{0} x^T(t+\tau) W_2 x(t+\tau) d\tau \end{aligned} \tag{5.112}$$

by

$$\begin{aligned} & v(x_t) \\ = \; & v_{W_1 + rW_2 + W_3}(x_t) + \int_{-r}^{0} \int_{\tau}^{0} x^T(t+\theta) W_2 x(t+\theta) d\theta d\tau \\ & + \int_{-r}^{0} x^T(t+\theta) W_3 x(t+\theta) d\theta \\ = \; & v_{W_1 + rW_2 + W_3}(x_t) + \int_{-r}^{0} x^T(t+\theta)[(r+\theta) W_2 + W_3] x(t+\theta) d\theta \end{aligned}$$

Or more explicitly

$$\begin{aligned} v(\phi) &= \phi^T(0) U(0) \phi(0) \\ &\quad + 2\phi^T(0) \int_{-r}^{0} U(-r-\theta) A_1 \phi(\theta) d\theta \\ &\quad + \int_{-r}^{0} \int_{-r}^{0} \phi^T(\theta_1) A_1^T U(\theta_1 - \theta_2) A_1 \phi(\theta_2) d\theta_1 d\theta_2 \\ &\quad + \int_{-r}^{0} \phi^T(\theta)[(r+\theta) W_2 + W_3] \phi(\theta) d\theta \end{aligned} \tag{5.113}$$

where

$$U(\theta) = U_{W_1 + rW_2 + W_3}(\theta)$$

Indeed we can prove:

Theorem 5.19 *If the system described by (5.91) is asymptotically stable, then, for any real symmetric matrices $W_1 > 0$, $W_2 \geq 0$ and $W_3 > 0$, the Lyapunov-Krasovskii functional $V(\phi) = v(\phi)$, with $v(\phi)$ defined in (5.113), satisfies both the Lyapunov-Krasovskii functional condition*

$$v(\phi) \geq \varepsilon ||\phi(0)||^2 \tag{5.114}$$

and the Lyapunov-Krasovskii derivative condition

$$\dot{v}(\phi) \leq -\varepsilon ||\phi(0)||^2 \tag{5.115}$$

Proof. Condition (5.115) is obviously satisfied. To show (5.114), consider, for sufficiently small $\varepsilon > 0$ such that

$$H = \begin{pmatrix} W_1 + \varepsilon(A_0 + A_0^T) & \varepsilon A_1 \\ \varepsilon A_1^T & W_3 \end{pmatrix} > 0$$

and let

$$\tilde{v}(\phi) = v(\phi) - \varepsilon\phi^T(0)\phi(0)$$

We have

$$\begin{aligned} \frac{d}{dt}\tilde{v}(x_t) &= \dot{v}(x_t) - 2\varepsilon x^T(t)[A_0 x(t) + A_1 x(t-r)] \\ &= -x^T(t)W_1 x(t) - \int_{-r}^{0} x^T(t+\xi)W_2 x(t+\xi)d\xi \\ &\quad -x^T(t-r)W_3 x(t-r) - 2\varepsilon x^T(t)[A_0 x(t) + A_1 x(t-r)] \\ &= -\begin{pmatrix} x^T(t) & x^T(t-r) \end{pmatrix} H \begin{pmatrix} x(t) \\ x(t-r) \end{pmatrix} \\ &\quad -\int_{-r}^{0} x^T(t+\xi)W_2 x(t+\xi)d\xi \\ &\leq -\delta||x(t)||^2 \end{aligned}$$

for $0 < \delta < \lambda_{\min}(H)$. Integrate from 0 to ∞, noticing $\lim_{t\to\infty} x(t) = 0$, we obtain

$$0 - \tilde{v}(x_t) \leq -\delta\int_t^{\infty} ||x(t+\tau)||^2 d\tau \leq 0$$

Therefore

$$v(x_t) \geq \varepsilon||x(t)||^2$$

This completes the proof. ∎

5.7 Discretized Lyapunov functional method for systems with single delay

5.7.1 *Introduction*

From the discussions in the last section, in order to minimize the conservatism in the stability problem of the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-r) \tag{5.116}$$

we should consider a complete quadratic Lyapunov-Krasovskii functional

$$
\begin{aligned}
V(\phi) &= \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\
&\quad + \int_{-r}^{0}[\int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta]d\xi \\
&\quad + \int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \qquad (5.117)
\end{aligned}
$$

where

$$P = P^T \in \mathsf{R}^{n\times n} \qquad (5.118)$$

and for all $-r \le \xi \le 0$ and $-r \le \eta \le 0$,

$$
\begin{aligned}
Q(\xi) &\in \mathsf{R}^{n\times n} \qquad (5.119)\\
R(\xi,\eta) &= R^T(\eta,\xi) \in \mathsf{R}^{n\times n} \qquad (5.120)\\
S(\xi) &= S^T(\xi) \in \mathsf{R}^{n\times n} \qquad (5.121)
\end{aligned}
$$

and Q, R and S are continuous. In practice, to numerically check the existence of such a quadratic functional is not an easy task. In this section, we will choose Q, R and S to be piecewise linear functions. This will allow us to write both the Lyapunov-Krasovskii functional condition and Lyapunov-Krasovskii derivative condition in LMI form, allowing convenient and efficient numerical implementation. This method is known as the *discretized Lyapunov functional* (DLF) method.

Considering (5.120) and (5.121), the derivative of the Lyapunov-Krasovskii functional along the trajectory of system (5.116) may be calculated as

$$
\begin{aligned}
\dot{V}(\phi) &= 2\phi^T(0)P[A_0\phi(0) + A_1\phi(-r)] \\
&\quad +2[A_0\phi(0) + A_1\phi(-r)]^T \int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\
&\quad +2\phi^T(0)\int_{-r}^{0} Q(\xi)\dot{\phi}(\xi)d\xi \\
&\quad +2\int_{-r}^{0} d\xi \int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\dot{\phi}(\eta)d\eta \\
&\quad +2\int_{-r}^{0} \phi^T(\xi)S(\xi)\dot{\phi}(\xi)d\xi.
\end{aligned}
$$

An integration by parts, considering (5.120) and (5.121), allows us to write

$\dot{V}(\phi)$ in a standard quadratic form as follows

$$\begin{aligned}
\dot{V}(\phi) &= -\phi^T(0)[-PA_0 - A_0^T P - Q(0) - Q^T(0) - S(0)]\phi(0) \\
&\quad -\phi^T(-r)S(-r)\phi(-r) \\
&\quad -\int_{-r}^{0} \phi^T(\xi)\dot{S}(\xi)\phi(\xi)d\xi \\
&\quad -\int_{-r}^{0} d\xi \int_{-r}^{0} \phi^T(\xi)[\frac{\partial}{\partial \xi}R(\xi,\eta) + \frac{\partial}{\partial \eta}R(\xi,\eta)]\phi(\eta)d\eta \\
&\quad +2\phi^T(0)[PA_1 - Q(-r)]\phi(-r) \\
&\quad +2\phi^T(0)\int_{-r}^{0}[A_0^T Q(\xi) - \dot{Q}(\xi) + R(0,\xi)]\phi(\xi)d\xi \\
&\quad +2\phi^T(-r)\int_{-r}^{0}[A_1^T Q(\xi) - R(-r,\xi)]\phi(\xi)d\xi. \qquad (5.122)
\end{aligned}$$

Recall that the system is asymptotically stable if and only if there exists a Lyapunov-Krasovskii functional (5.117) such that the *Lyapunov-Krasovskii functional condition*

$$V(\phi) \geq \varepsilon ||\phi(0)||^2 \qquad (5.123)$$

and its derivative (5.122) satisfies the *Lyapunov-Krasovskii derivative condition*

$$\dot{V}(\phi) \leq -\varepsilon ||\phi(0)||^2 \qquad (5.124)$$

for some $\varepsilon > 0$.

5.7.2 Discretization

The basic idea of *discretized Lyapunov functional* (DLF) method is to divide the domain of definition of matrix functions Q, R and S into smaller regions, and choose these matrix functions to be continuous piecewise linear, thus reducing the choice of Lyapunov-Krasovskii functional V into choosing a finite number of parameters. Furthermore, it is possible to write the Lyapunov-Krasovskii functional condition and Lyapunov-Krasovskii derivative condition in the form of LMIs. The techniques discussed in the Appendix B, especially variable elimination and quadratic integral inequality, are especially useful.

Divide the delay interval $\mathcal{I} = [-r, 0]$ into N segments $\mathcal{I}_p = [\theta_p, \theta_{p-1}]$, $p = 1, 2, ..., N$ of equal length

$$h = r/N$$

Then

$$\theta_p = -ph = -\frac{pr}{N}, \qquad p = 0, 1, 2, ..., N$$

This also divides the square $\mathcal{S} = [-r, 0] \times [-r, 0]$ into $N \times N$ small squares $\mathcal{S}_{pq} = [\theta_p, \theta_{p-1}] \times [\theta_q, \theta_{q-1}]$. Each square is further divided into two triangles

$$\begin{aligned}
\mathcal{T}^u_{pq} &= \left\{ (\theta_p + \alpha h, \theta_q + \beta h) \;\middle|\; \begin{array}{c} 0 \le \beta \le 1, \\ 0 \le \alpha \le \beta \end{array} \right\}, \\
\mathcal{T}^l_{pq} &= \left\{ (\theta_p + \alpha h, \theta_q + \beta h) \;\middle|\; \begin{array}{c} 0 \le \alpha \le 1, \\ 0 \le \beta \le \alpha \end{array} \right\}.
\end{aligned}$$

The continuous matrix functions $Q(\xi)$ and $S(\xi)$ are chosen to be linear within each segment $\mathcal{I}_p$, and the continuous matrix function $R(\xi, \eta)$ is chosen to be linear within each triangular region $\mathcal{T}^u_{pq}$ or $\mathcal{T}^l_{pq}$. Let

$$\begin{aligned}
Q_p &= Q(\theta_p) \\
S_p &= S(\theta_p) \\
R_{pq} &= R(\theta_p, \theta_q)
\end{aligned}$$

Then, since these functions are piecewise linear, they can be expressed in terms of their values at the dividing points using a linear interpolation formula, *i.e.*, for $0 \le \alpha \le 1$, $p = 1, 2, ..., N$

$$Q(\theta_p + \alpha h) = Q^{(p)}(\alpha) = (1 - \alpha) Q_p + \alpha Q_{p-1} \tag{5.125}$$

$$S(\theta_p + \alpha h) = S^{(p)}(\alpha) = (1 - \alpha) S_p + \alpha S_{p-1} \tag{5.126}$$

and for $0 \le \alpha \le 1, 0 \le \beta \le 1$, $p = 1, 2, ..., N$, $q = 1, 2, ..., N$

$$\begin{aligned}
& R(\theta_p + \alpha h, \theta_q + \beta h) \\
= \; & R^{(pq)}(\alpha, \beta) \\
= \; & \begin{cases} (1 - \alpha) R_{pq} + \beta R_{p-1,q-1} + (\alpha - \beta) R_{p-1,q}, & \alpha \ge \beta \\ (1 - \beta) R_{pq} + \alpha R_{p-1,q-1} + (\beta - \alpha) R_{p,q-1}, & \alpha < \beta \end{cases}
\end{aligned} \tag{5.127}$$

Thus, the Lyapunov-Krasovskii functional is completely determined by P, Q_p, S_p, R_{pq}, $p, q = 0, 1, ..., N$.

5.7.3 Lyapunov-Krasovskii functional condition

With the choice of piecewise linear Q, R and S as (5.125) to (5.127), we may divide the integration in (5.117) into segments $[\theta_p, \theta_{p-1}]$, thus rewriting $V(\phi)$ as

$$V(\phi) = \phi^T(0) P \phi(0) + 2\phi^T(0) \sum_{p=1}^{N} V_{Q^p} + \sum_{p=1}^{N} \sum_{q=1}^{N} V_{R^{pq}} + \sum_{p=1}^{N} V_{S^p} \tag{5.128}$$

where

$$V_{Q^p} = \int_0^1 Q^{(p)}(\alpha)\phi^{(p)}(\alpha)hd\alpha \tag{5.129}$$

$$V_{R^{pq}} = \int_0^1 [\int_0^1 \phi^{(p)T}(\alpha)R^{(pq)}(\alpha,\beta)\phi^{(q)}(\beta)hd\beta]hd\alpha \tag{5.130}$$

$$V_{S^p} = \int_0^1 \phi^{(p)T}(\alpha)S^{(p)}(\alpha)\phi^{(q)}(\alpha)hd\alpha \tag{5.131}$$

and

$$\phi^{(p)}(\alpha) = \phi(\theta_p + \alpha h) \tag{5.132}$$

We can show that

Proposition 5.20 *The Lyapunov-Krasovskii functional $V(\phi)$ in (5.117) to (5.121), with Q, S and R piecewise linear as in (5.125) to (5.127), satisfies*

$$V(\phi) = \int_0^1 \begin{pmatrix} \phi^T(0) & \Psi^T(\alpha) \end{pmatrix} \begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} \phi(0) \\ \Psi(\alpha) \end{pmatrix} d\alpha + \sum_{p=1}^{N} V_{S^p} \tag{5.133}$$

Furthermore, it also satisfies

$$V(\phi) \geq \int_0^1 \begin{pmatrix} \phi^T(0) & \Psi^T(\alpha) \end{pmatrix} \begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R}+\tilde{S} \end{pmatrix} \begin{pmatrix} \phi(0) \\ \Psi(\alpha) \end{pmatrix} d\alpha \tag{5.134}$$

if

$$S_p > 0, \qquad p = 0, 1, ..., N. \tag{5.135}$$

Therefore the Lyapunov-Krasovskii functional condition (5.123) is satisfied if (5.135) and

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R}+\tilde{S} \end{pmatrix} > 0 \tag{5.136}$$

are satisfied. In the above

$$\tilde{Q} = \begin{pmatrix} Q_0 & Q_2 & ... & Q_N \end{pmatrix} \tag{5.137}$$

$$\tilde{R} = \begin{pmatrix} R_{00} & R_{01} & ... & R_{0N} \\ R_{10} & R_{11} & ... & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & ... & R_{NN} \end{pmatrix} \tag{5.138}$$

$$\tilde{S} = \mathrm{diag}\begin{pmatrix} \frac{1}{h}S_0 & \frac{1}{h}S_1 & ... & \frac{1}{h}S_N \end{pmatrix} \tag{5.139}$$

and

$$\Psi(\alpha) = \begin{pmatrix} \Psi_{(0)}(\alpha) \\ \Psi_{(1)}(\alpha) \\ \Psi_{(2)}(\alpha) \\ \vdots \\ \Psi_{(N-1)}(\alpha) \\ \Psi_{(N)}(\alpha) \end{pmatrix} = \begin{pmatrix} \psi_{(1)}(\alpha) \\ \psi_{(2)}(\alpha) + \psi^{(1)}(\alpha) \\ \psi_{(3)}(\alpha) + \psi^{(2)}(\alpha) \\ \vdots \\ \psi_{(N)}(\alpha) + \psi^{(N-1)}(\alpha) \\ \psi^{(N)}(\alpha) \end{pmatrix} \tag{5.140}$$

$$\psi^{(p)}(\alpha) = h\int_0^\alpha \phi^{(p)}(\beta)d\beta \tag{5.141}$$

$$\psi_{(p)}(\alpha) = h\int_\alpha^1 \phi^{(p)}(\beta)d\beta \tag{5.142}$$

$$p = 1, 2, ..., N$$

Proof. Once (5.134) are established, the sufficiency of (5.136) is easily seen. Therefore, we will proceed to prove (5.134). With the expression of V in (5.128) to (5.132), for V_{Q^p} we may carry out integration by parts to obtain

$$\begin{aligned} V_{Q^p} &= \int_0^1 Q^{(p)}(\alpha)\phi^{(p)}(\alpha)hd\alpha \\ &= Q^{(p)}(1)\psi^{(p)}(1) - \int_0^1 [\frac{d}{d\alpha}Q^{(p)}(\alpha)]\psi^{(p)}(\alpha)d\alpha \\ &= Q_{p-1}\psi^{(p)}(1) - \int_0^1 (Q_{p-1} - Q_p)\psi^{(p)}(\alpha)d\alpha \\ &= \int_0^1 [Q_{p-1}\psi_{(p)}(\alpha) + Q_p\psi^{(p)}(\alpha)]d\alpha \end{aligned} \tag{5.143}$$

Similarly, for $V_{R^{pq}}$, integration by parts with respect to β yields

$$\begin{aligned} V_{R^{pq}} &= \int_0^1 [\int_0^1 \phi^{(p)T}(\alpha)R^{(pq)}(\alpha,\beta)\phi^{(q)}(\beta)hd\beta]hd\alpha \\ &= \int_0^1 \phi^{(p)T}(\alpha)[R^{(pq)}(\alpha,1)\psi^{(q)}(1) - \int_0^1 \frac{\partial R^{(pq)}(\alpha,\beta)}{\partial\beta}\psi^{(q)}(\beta)d\beta]hd\alpha \\ &= \int_0^1 \phi^{(p)T}(\alpha)\Big([\alpha R_{p-1,q-1} + (1-\alpha)R_{p,q-1}]\psi^{(q)}(1) \\ &\quad - \int_0^\alpha (R_{p-1,q-1} - R_{p-1,q})\psi^{(q)}(\beta)d\beta \\ &\quad - \int_\alpha^1 (R_{p,q-1} - R_{pq})\psi^{(q)}(\beta)d\beta\Big) hd\alpha \end{aligned}$$

Further carry out integration by parts with respect to α

$$\begin{aligned}
V_{R^{pq}} &= \psi^{(p)T}(1)[R_{p-1,q-1}\psi^{(q)}(1) - \int_0^1 (R_{p-1,q-1} - R_{p-1,q})\psi^{(q)}(\beta)d\beta] \\
&\quad - \int_0^1 \psi^{(p)T}(\alpha)[(R_{p-1,q-1} - R_{p,q-1})\psi^{(q)}(1) \\
&\qquad -(R_{p-1,q-1} - R_{p-1,q})\psi^{(q)}(\alpha) \\
&\qquad +(R_{p,q-1} - R_{pq})\psi^{(q)}(\alpha)]d\alpha \\
&= \int_0^1 [\psi^T_{(p)}(\alpha)R_{p-1,q-1}\psi_{(q)}(\alpha) + \psi^T_{(p)}(\alpha)R_{p-1,q}\psi^{(q)}(\alpha) \\
&\quad +\psi^{(p)T}(\alpha)R_{p,q-1}\psi_{(q)}(\alpha) + \psi^{(p)T}(\alpha)R_{pq}\psi^{(q)}(\alpha)]d\alpha
\end{aligned} \tag{5.144}$$

Using (5.143), (5.144) in (5.128) yields (5.133). Also, given (5.135), we have

$$\begin{aligned}
&\sum_{p=1}^{N} V_{S^p} \\
= &\sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha)S^{(p)}(\alpha)\phi^{(p)}(\alpha)hd\alpha \\
= &\sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha)[(1-\alpha)S_p + \alpha S_{p-1}]\phi^{(p)}(\alpha)hd\alpha \\
= &\int_0^1 \alpha\phi^{(1)T}(\alpha)S_0\phi^{(1)}(\alpha)hd\alpha + \int_0^1 (1-\alpha)\phi^{(N)T}(\alpha)S_N\phi^{(N)}(\alpha)hd\alpha \\
&+ \sum_{p=1}^{N-1} \int_0^1 [\alpha\phi^{(p+1)T}(\alpha)S_p\phi^{(p+1)}(\alpha) + (1-\alpha)\phi^{(p)T}(\alpha)S_p\phi^{(p)}(\alpha)]hd\alpha
\end{aligned}$$

Using quadratic integral inequality (or Jensen inequality, Corollary B.9 in Appendix B) in each term above we obtain

$$
\begin{aligned}
\sum_{p=1}^{N} V_{S^p} \geq \;& \frac{1}{h}\int_0^1 [(\int_\alpha^1 \phi^{(1)}(\beta)hd\beta)^T S_0(\int_\alpha^1 \phi^{(1)}(\beta)hd\beta) \\
& +(\int_0^\alpha \phi^{(N)T}(\beta)hd\beta)^T S_N(\int_0^\alpha \phi^{(N)T}(\beta)hd\beta)]d\alpha \\
& +\frac{1}{h}\sum_{p=1}^{N-1}\int_0^1 [\int_\alpha^1 \phi^{(p+1)}(\alpha)hd\alpha + \int_0^\alpha \phi^{(p)}(\alpha)hd\alpha]^T \\
& S_p[\int_\alpha^1 \phi^{(p+1)}(\alpha)hd\alpha + \int_0^\alpha \phi^{(p)}(\alpha)hd\alpha]d\alpha \\
= \;& \frac{1}{h}\int_0^1 [\psi_{(1)}^T(\alpha)S_0\psi_{(1)}(\alpha) + \psi^{(N)T}(\alpha)S_N\psi^{(N)}(\alpha) \\
& +\frac{1}{h}\sum_{p=1}^{N-1}(\psi_{(p+1)}(\alpha)+\psi^{(p)}(\alpha))^T S_p(\psi_{(p+1)}(\alpha)+\psi^{(p)}(\alpha))]d\alpha \\
= \;& \int_0^1 \Psi^T(\alpha)\tilde{S}\Psi(\alpha)d\alpha
\end{aligned}
\tag{5.145}
$$

Using (5.145) in (5.133) yields (5.134). ■

5.7.4 Lyapunov-Krasovskii derivative condition

With piecewise linear Q, R and S as (5.125) to (5.127), we have, for $\theta_p < \xi < \theta_{p-1}$, $\theta_q < \eta < \theta_{q-1}$,

$$
\begin{aligned}
\dot{S}(\xi) &= \frac{1}{h}(S_{p-1} - S_p) \\
\dot{Q}(\xi) &= \frac{1}{h}(Q_{p-1} - Q_p) \\
\frac{\partial R(\xi,\eta)}{\partial \xi} + \frac{\partial R(\xi,\eta)}{\partial \eta} &= \frac{1}{h}(R_{p-1,q-1} - R_{pq})
\end{aligned}
$$

Therefore, after dividing the integration interval $[-r, 0]$ into discretization segments $[\theta_p, \theta_{p-1}]$, $p = 1, 2, ..., N$, through tedious calculation, we can verify that $\dot{V}(\phi)$ in (5.122) may be written as

$$\begin{aligned}
\dot{V}(\phi) &= -\phi_{0r}^T \Delta \phi_{0r} - \sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha) S_{dp} \phi^{(p)}(\alpha) d\alpha \\
&\quad - \sum_{p=1}^{N} \sum_{q=1}^{N} \int_0^1 \int_0^1 \phi^{(p)T}(\alpha) R_{dpq} \phi^{(q)}(\beta) d\alpha d\beta \\
&\quad + 2\phi_{0r}^T \sum_{p=1}^{N} \int_0^1 [(1-\alpha)(D_p^s + D_p^a) + \alpha(D_p^s - D_p^a)] \phi^{(p)}(\alpha) d\alpha
\end{aligned}$$

or

$$\begin{aligned}
\dot{V}(\phi) &= -\phi_{0r}^T \Delta \phi_{0r} - \int_0^1 \tilde{\phi}^T(\alpha) S_d \tilde{\phi}(\alpha) d\alpha \\
&\quad - \int_0^1 [\int_0^1 \tilde{\phi}^T(\alpha) R_d \tilde{\phi}(\beta) d\alpha] d\beta \\
&\quad + 2\phi_{0r}^T \int_0^1 [D^s + (1-2\alpha) D^a] \tilde{\phi}(\alpha) h d\alpha
\end{aligned} \tag{5.146}$$

where

$$\phi_{0r} = \begin{pmatrix} \phi(0) \\ \phi(-r) \end{pmatrix} \tag{5.147}$$

$$\tilde{\phi}(\alpha) = \begin{pmatrix} \phi^1(\alpha) \\ \phi^2(\alpha) \\ \vdots \\ \phi^N(\alpha) \end{pmatrix} \tag{5.148}$$

$$\Delta = \begin{pmatrix} \Delta_{00} & \Delta_{01} \\ \Delta_{01}^T & \Delta_{11} \end{pmatrix} \tag{5.149}$$

$$\Delta_{00} = -PA_0 - A_0^T P - Q_0 - Q_0^T - S_0 \tag{5.150}$$

$$\Delta_{01} = Q_N - PA_1 \tag{5.151}$$

$$\Delta_{11} = S_N \tag{5.152}$$

$$S_d = \text{diag}\begin{pmatrix} S_{d1} & S_{d2} & ... & S_{dN} \end{pmatrix} \tag{5.153}$$

$$S_{dp} = S_{p-1} - S_p \tag{5.154}$$

$$R_d = \begin{pmatrix} R_{d11} & R_{d12} & ... & R_{d1N} \\ R_{d21} & R_{d22} & ... & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & ... & R_{dNN} \end{pmatrix} \tag{5.155}$$

$$R_{dpq} = h(R_{p-1,q-1} - R_{pq}) \tag{5.156}$$

$$D^s = \begin{pmatrix} D_1^s & D_2^s & ... & D_N^s \end{pmatrix} \tag{5.157}$$

$$D_p^s = \begin{pmatrix} D_{0p}^s \\ D_{1p}^s \end{pmatrix} \tag{5.158}$$

$$\begin{aligned} D_{0p}^s &= \frac{h}{2}A_0^T(Q_{p-1}+Q_p) + \frac{h}{2}(R_{0,p-1}+R_{0p}) \\ &\quad -(Q_{p-1}-Q_p) \end{aligned} \tag{5.159}$$

$$D_{1p}^s = \frac{h}{2}A_1^T(Q_{p-1}+Q_p) - \frac{h}{2}(R_{N,p-1}+R_{Np}) \tag{5.160}$$

$$D^a = \begin{pmatrix} D_1^a & D_2^a & ... & D_N^a \end{pmatrix} \tag{5.161}$$

$$D_p^a = \begin{pmatrix} D_{0p}^a \\ D_{1p}^a \end{pmatrix} \tag{5.162}$$

$$D_{0p}^a = -\frac{h}{2}A_0^T(Q_{p-1}-Q_p) - \frac{h}{2}(R_{0,p-1}-R_{0p}) \tag{5.163}$$

$$D_{1p}^a = -\frac{h}{2}A_1^T(Q_{p-1}-Q_p) + \frac{h}{2}(R_{N,p-1}-R_{Np}) \tag{5.164}$$

Proposition 5.21 *For the Lyapunov-Krasovskii functional $V(\phi)$ in (5.117) to (5.121), with Q, S and R piecewise linear as in (5.125) to (5.127), its derivative $\dot{V}(\phi)$ along the trajectory of the system (5.116), satisfies*

$$\begin{aligned} \dot{V}(\phi) \leq & -\begin{pmatrix} \phi_{0r}^T & \int_0^1 \tilde{\phi}^T(\alpha)d\alpha \end{pmatrix} \\ & \begin{pmatrix} \Delta - \frac{1}{3}D^aUD^{aT} & -D^s \\ -D^{sT} & R+S_d \end{pmatrix} \begin{pmatrix} \phi_{0r} \\ \int_0^1 \tilde{\phi}(\alpha)d\alpha \end{pmatrix} \end{aligned} \tag{5.165}$$

for arbitrary matrix U satisfying

$$\begin{pmatrix} U & -I \\ -I & S_d \end{pmatrix} > 0, \tag{5.166}$$

As a result, the Lyapunov-Krasovskii derivative condition (5.124) is satisfied if

$$\begin{pmatrix} \Delta & -D^s & -D^a \\ -D^{sT} & R_d+S_d & 0 \\ -D^{aT} & 0 & 3S_d \end{pmatrix} > 0 \tag{5.167}$$

Proof. From (5.146), it is easy to verify that

$$\begin{aligned} \dot{V}(\phi) = & -\int_0^1 \begin{pmatrix} \phi_{0r}^T[D^s+(1-2\alpha)D^a] & \tilde{\phi}^T(\alpha) \end{pmatrix} \\ & \begin{pmatrix} U & -I \\ -I & S_d \end{pmatrix} \begin{pmatrix} [D^s+(1-2\alpha)D^a]^T\phi_{0r} \\ \tilde{\phi}(\alpha) \end{pmatrix} d\alpha \\ & -\phi_{0r}^T[\Delta - D^sUD^{sT} - \frac{1}{3}D^aUD^{aT}]\phi_{0r} \\ & -(\int_0^1 \tilde{\phi}(\alpha)d\alpha)^T R_d (\int_0^1 \tilde{\phi}(\alpha)d\alpha) \end{aligned} \tag{5.168}$$

for arbitrary U. If (5.166) is satisfied, then use Jensen inequality (Proposition B.8 in Appendix B) to the first term, we have

$$\int_0^1 \begin{pmatrix} \phi_{0r}^T[D^s + (1-2\alpha)D^a] & \tilde{\phi}^T(\alpha) \end{pmatrix}$$
$$\begin{pmatrix} U & -I \\ -I & S_d \end{pmatrix} \begin{pmatrix} [D^s + (1-2\alpha)D^a]^T \phi_{0r} \\ \tilde{\phi}(\alpha) \end{pmatrix} d\alpha$$
$$\geq \int_0^1 \begin{pmatrix} \phi_{0r}^T[D^s + (1-2\alpha)D^a] & \tilde{\phi}^T(\alpha) \end{pmatrix} d\alpha$$
$$\begin{pmatrix} U & -I \\ -I & S_d \end{pmatrix} \int_0^1 \begin{pmatrix} [D^s + (1-2\alpha)D^a]^T \phi_{0r} \\ \tilde{\phi}(\alpha) \end{pmatrix} d\alpha$$
$$= \begin{pmatrix} \phi_{0r}^T & \int_0^1 \tilde{\phi}^T(\alpha)d\alpha \end{pmatrix} \begin{pmatrix} D^s U D^{sT} & -D^s \\ -D^{sT} & S_d \end{pmatrix} \begin{pmatrix} \phi_{0r} \\ \int_0^1 \tilde{\phi}(\alpha)d\alpha \end{pmatrix}$$

Using above in (5.168) yields (5.165). Therefore, the Lyapunov-Krasovskii derivative condition (5.124) is satisfied if (5.166) and

$$\begin{pmatrix} \Delta - \frac{1}{3}D^a U D^{aT} & -D^s \\ -D^{sT} & R + S_d \end{pmatrix} > 0$$

are satisfied. The arbitrary matrix function U can be eliminated from the above inequality and (5.166) using Proposition B.6 in Appendix B to yield (5.167). ■

5.7.5 Stability criterion and examples

We can summarize from the above discussion that

Proposition 5.22 *The system with single time-delay described by (5.116) is asymptotically stable if there exist $n \times n$ matrices $P = P^T$; Q_p, $S_p = S_p^T$, $p = 0, 1, ..., N$; $R_{pq} = R_{qp}^T$, $p = 0, 1, ..., N$, $q = 0, 1, ..., N$ such that (5.136) and (5.167) are satisfied, with notations defined in (5.137) to (5.139) and (5.149) to (5.164).*

Proof. Choose Lyapunov-Krasovskii functional (5.117) to (5.121), with Q, S and R piecewise linear as in (5.125) to (5.127). Then (5.167) implies that the Lyapunov-Krasovskii derivative condition is satisfied according to Proposition 5.21.

To show that the Lyapunov-Krasovskii functional condition is satisfied, we only need to show that (5.135) is satisfied, which is the only condition needed in addition to (5.136) according to Proposition 5.20. Since $S_N = \Delta_{11}$, $S_{p-1} - S_p = S_{dp}$, $p = 1, 2, ..., N$ are all principal minors of the left hand side of (5.167), we can conclude

$$\begin{aligned} S_N &> 0 \\ S_{p-1} - S_p &> 0,\ p = 1, 2, ..., N \end{aligned}$$

This implies

$$S_0 > S_1 > ... > S_N > 0$$

which implies (5.135).

Since both Lyapunov-Krasovskii functional condition and Lyapunov-Krasovskii derivative condition are satisfied, the systems is asymptotically stable. ■

Clearly, the criterion is in the form of LMI since all the system parameters appear linearly. The following examples are provided to show the effectiveness of DLF method. The first example illustrate the convergence to the analytical limit as N increases for a system without uncertainty.

Example 5.10 *Consider the system*

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r).$$

It is desired to calculate the maximum time-delay $r_{\max}$ the system can tolerate and still retain stability. Proposition 5.22 can be used to check the stability of the system for any given r. A bisection process is used to estimate $r_{\max}$ for different N. The analytical stability limit is also listed for comparison. The results are very close to analytical limit. Even the result for $N = 1$ is sufficient for most practical applications, and is much better than the ones obtained in earlier sections. The convergent trend to the analytical limit is obvious.

N	1	2	3	*analytical*
$r_{\max}$	*6.059*	*6.165*	*6.171*	*6.17258*

Example 5.11 *Consider the system*

$$\ddot{x}(t) - 0.1\dot{x}(t) + 2x(t) - x(t-r) = 0$$

This is obtained from system (5.87) by applying feedback control (5.88). This system is clearly unstable for $r = 0$. To apply Proposition 5.22, the system is written in the standard form

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x(t-r) \\ \dot{x}(t-r) \end{pmatrix} \end{aligned}$$

The system is stable for $r \in (r_{\min}, r_{\max})$. Proposition 5.22 with different N is used to estimate $r_{\min}$ and $r_{\max}$. The computation consists of sweeping through a range of r with relatively large step size, and a bisection process near the lower limit $r_{\min}$ and the upper limit $r_{\max}$. The result is listed in the following table along with the analytical limit.

N	*1*	*2*	*3*	*Analytical*
$r_{\min}$	*0.1006*	*0.1003*	*0.1003*	*0.1002*
$r_{\max}$	*1.4272*	*1.6921*	*1.7161*	*1.7178*

5.8 Notes

5.8.1 Results based on Razumikhin Theorem

An example of the results prior to modern LMI approach is by Thowsen [266]. There is a large amount of work done on this topic. The delay-independent formulation in Proposition 5.3 and the dual form is equivalent to Proposition 18 in Niculescu, et. al. [224]. The delay-dependent formulation in Proposition 5.7 is equivalent to Proposition 19 in Niculescu, et. al. [224].

Some studies try to avoid the scaling factor and write the stability criteria in LMI form. For example, it can be shown using variable elimination of LMI that Li and de Souza [180] effectively chooses the scaling factors $\alpha_1 = \alpha_2 = 1$ when specialized to the case of Proposition 5.7. It seems clear that fixing the scaling factors are essential if one is to restrict the formulation to LMI form. However, it is also essential that these scaling factors should be proportional to the magnitude of the system matrices A_0 and A_1. For example, if A_0 and A_1 are multiplied by 1000 and the delay r is divided by 1000 (such as choosing a different unit, say from second versus millisecond), which accelerate the evolution of the system without affecting the stability of the system. A stability criterion with the scaling factor not proportional to A_0 and A_1 would conclude that the stability of these two system may be different. The scaling scheme proposed here is from Gu and Han [102].

The Razumikhin Theorem based formulation with implicit model transformation described in Equations (5.53) and (5.54) was first proposed in Park [241] for the special case of $\alpha = \alpha_0 = \alpha_1$. The forms presented here with connections to other criteria is from Gu [100].

5.8.2 Model transformation and additional dynamics

A number of other model transformations are used in practice, see Kolmanovskii and Richard [163].

The nonequivalence of transformed system and the original system was pointed out by Verriest [288] in the context of reducing a class of systems with distributed delays to pointwise delays. The concept of additional dynamics was proposed by Gu and Niculescu [105] based on decomposition of characteristic quasipolynomial, where the proofs not included here can be found. Kharitonov and Melchor-Aguilar [147] extended the additional dynamics to systems with multiple delays. Additional dynamics in other types of model transformations was discussed in Gu and Niculescu [106]. Kharitonov and Melchor-Aguilar [151] wrote the additional dynamics as an explicit dynamic equation as in (5.43) and extended to systems with time-varying delays and coefficients.

5.8.3 Lyapunov-Krasovskii method

It should be noticed that the delay-independent criterion using Lyapunov-Krasovskii functional method presented here is not necessary and sufficient condition, see Bliman [16]. There are also a large amount of work done on delay-independent results and delay-dependent stability based on simple Lyapunov-Krasovskii functional. See Niculescu, *et. al.* [224]. The delay-dependent stability result here can be found in Niculescu, *et. al.* [222]. The particular derivation for simple delay-dependent stability based on explicit model transformation is from Gu and Han [102].

A stability criterion based on implicit model transformation was proposed by Park [240], which is the special case of (5.81) and (5.85) with

$$\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} = \begin{pmatrix} P + W^T \\ \frac{1}{r} A_1^T V \end{pmatrix} rV^{-1} \begin{pmatrix} P + W & \frac{1}{r} V A_1 \end{pmatrix}$$

The formulation without such a constraint was given in Park [241] with extensions to considering uncertainty and cost function by Lee, *et. al.* [172]. The elimination of variable with connections to other delay-independent and delay-dependent stability criteria were given by Gu [100].

The idea of complete quadratic Lyapunov-Krasovskii functional was proposed by Rupin [244]. The special case of x being scalar was solved by Datko [53]. The case of $W_2 = W_3 = 0$ was solved by Infante and Castelan [130]. Kharitonov proposed the case $W_2 \neq 0$ and $W_3 \neq 0$ in studying the robust stability.

The complete quadratic Lyapunov-Krasovskii functional is also used in, *e.g.*, Fridman and Shaked [68, 69].

The idea discretized Lyapunov functional method (DLF) was proposed by Gu [91]. Refinements using variable elimination [95] and integral inequality (Jensen inequality) [96] were proposed to improve the method. The formulation presented is based on [97], where an alternative condition for Lyapunov-Krasovskii functional condition was also discussed in addition to the one discussed here.

6

Robust Stability Analysis

title shortened, "notes" used for the last section. reference to appendix B mismatch corrected.

6.1 Introduction

In this chapter, we will discuss the robust stability problem of time-delay systems using time-domain approach. Robust stability problem considers the stability problems of systems which contain some uncertainties.

As is well known, it is usually impossible to describe a practical system exactly. First, there are often parameters or parasitic processes which are not completely known. Second, due to the limitation of mathematical tools available, we usually try to use a relatively simple model to approximate a practical system. As a result, some aspects of the system dynamics (known as unmodeled dynamics) are ignored. Third, some control systems are required to operate within a range of different operating conditions. To capture these uncertain factors, it is often possible to identify a bounding set such that all the possible uncertainties fall within this set and yet it is not too difficult to analyze mathematically.

In this chapter, we will first discuss some useful ways of characterizing uncertainties. This includes subpolytopic uncertainty and norm bounded uncertainty. It is often desirable to refine the norm bounded uncertainty to be of block-diagonal structure to reduce conservatism. Then, we will discuss the treatment of robust stability problem with either generic uncertainty form, or specific uncertainty form. Razumikhin Theorem, simple Lyapunov-Krasovskii functional, or complete quadratic Lyapunov-Krasovskii functional will be used to derive delay-independent or delay-dependent stability results.

6.2 Uncertainty characterization

We will discuss the uncertainty characterization for the systems with single delay. Consider the system

$$\dot{x}(t) = A_0(x_t, t)x(t) + A_1(x_t, t)x(t - r)$$

where $A_0 \in \mathsf{R}^{n\times n}$, $A_1 \in \mathsf{R}^{n\times n}$ are uncertain coefficient matrices not known completely, except that they are within a compact set Ω which we will refer to as the uncertainty set,

$$(A_0(x_t, t), A_1(x_t, t)) \in \Omega \text{ for all } t \geq 0$$

The uncertainty set characterizes the uncertainties, and serves as one of the basic information needed to carry out robust stability analysis. Notice also that the coefficients may depend on the time t as well as the current and previous state variable $x(t+\xi)$, $-r \leq \xi \leq 0$. For the same of convenience, we will not explicitly show these dependences or only show the dependence on time t when no confusion may arise.

A good choice of uncertainty set is a compromise between minimizing conservatism (and therefore, it is desirable to make the uncertainty set "small"), and the mathematical tractibility (and therefore, it is desirable to make the uncertainty set structurally simple). In this section, we will discuss some common uncertainty structures.

In addition to uncertainties in the coefficient matrices, the time-delay r may also involve uncertainties. In this chapter, we will mainly discuss the case where $r = r(t)$ may be time-varying, and is known to be within the interval $r(t) \in (0, r_{\max}]$. However, we will reserve the term "uncertainty" to coefficient uncertainty.

6.2.1 Polytopic uncertainty

The first class of uncertainty frequently encountered in practice is the polytopic uncertainty. In this case, there exist, say, n_v elements of the uncertainty set Ω,

$$\omega^{(k)} = (A_0^{(k)}, A_1^{(k)}), \quad k = 1, 2, ..., n_v$$

known as *vertices*, such that Ω can be expressed as the convex hull of these vertices

$$\Omega = \text{co}\{\omega^{(k)} \mid k = 1, 2, ..., n_v\}$$

In other words, the elements of the uncertainty set Ω consists of all the convex linear combination of the vertices

$$\Omega = \left\{ \sum_{k=1}^{n_v} \alpha_k \omega^{(k)} \;\middle|\; \alpha_k \geq 0, k = 1, 2, ..., n_v; \sum_{k=1}^{n_v} \alpha_k = 1 \right\}$$

For example, in practice, there are often some *uncertain parameters* in the system. Each uncertain parameter may vary between a lower limit and an upper limit, and these uncertain parameters often appear linearly in the system matrices. In this case, the collection of all the possible system matrices form a polytopic set. The vertices of this set can be calculated by setting the parameters to either lower or upper limit. If there are n_p uncertain parameters, it is easy to see that there are $n_v = 2^{n_p}$ vertices.

To see this idea more clearly, assume that there are two uncertain parameters, α and β, varying between lower and upper bounds

$$\alpha_{\min} \le \alpha \le \alpha_{\max}, \qquad \beta_{\min} \le \beta \le \beta_{\max}$$

They appear linearly in the system matrices

$$\begin{aligned} A_0 &= A_{0n} + \alpha A_{0\alpha} + \beta A_{0\beta} \\ A_1 &= A_{1n} + \alpha A_{1\alpha} + \beta A_{1\beta} \end{aligned}$$

or in the abbreviated form

$$\omega = \omega_n + \alpha\omega_\alpha + \beta\omega_\beta$$

Then, the uncertainty set possesses 4 vertices

$$\begin{aligned} \omega^{(1)} &= \omega_n + \alpha_{\min}\omega_\alpha + \beta_{\min}\omega_\beta \\ \omega^{(2)} &= \omega_n + \alpha_{\max}\omega_\alpha + \beta_{\min}\omega_\beta \\ \omega^{(3)} &= \omega_n + \alpha_{\min}\omega_\alpha + \beta_{\max}\omega_\beta \\ \omega^{(4)} &= \omega_n + \alpha_{\max}\omega_\alpha + \beta_{\max}\omega_\beta \end{aligned}$$

6.2.2 *Subpolytopic uncertainty*

The class of subpolytopic uncertainty is more general than polytopic uncertainty. In this case, the uncertainty set also possesses, say, n_v vertices, and the uncertainty set Ω is *contained* in (but not necessarily equal to) the convex hull of the vertices

$$\Omega \subset \mathrm{co}\{\omega^{(i)},\ i = 1, 2, ..., n_v\}$$

In other words, any element $\omega \in \Omega$ can be expressed as a convex linear combination of the vertices, *i.e.*,

$$\omega = \sum_{i=1}^{n_v} \beta_i \omega^{(i)}$$

for some scalars β_i satisfying

$$\beta_i \ge 0,\ \sum_{i=1}^{n_v} \beta_i = 1$$

Clearly, polytopic uncertainty is a special case of subpolytopic uncertainty.

An important class of subpolytopic uncertainties arise from uncertain parameters which appear multilinearly in the system matrices as the following Proposition shows. In such cases, the products of different parameters may appear in the system matrices. Each parameter appears linearly if all the other parameters are held constant.

Proposition 6.1 *The set Ω formed by n_p parameters, α_i, $i = 1, 2, ..., n_p$, appearing multilinearly with each parameter varying between a lower limit and an upper limit*

$$\Omega = \{\omega_n + \sum_{k=1}^{n_p} \sum_{\substack{i_1, i_2, ..., i_k \\ distinct}} \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_k}\omega_{i_1 i_2 \cdots i_k} \,|\, \alpha_{i\min} \leq \alpha_i \leq a_{i\max}\}$$

is subpolytopic. The $n_v = 2^{n_p}$ vertices can be calculated by setting each parameter α_i to either the lower limit $\alpha_{i\min}$ or the upper limit $\alpha_{i\max}$ for $i = 1, 2, ..., n_p$.

Proof. We will use induction. For $n_p = 1$, it is obvious. Assume the Proposition is valid for $n_p = m$. Then for $n_p = m + 1$,

$$\Omega = \left\{ \omega_{m,n} + \alpha_{m+1}\omega_{m,v} \;\middle|\; \begin{array}{l} \alpha_{m+1,\min} \leq \alpha_{m+1} \leq \alpha_{m+1,\max}, \\ (\omega_{m,n}, \omega_{m,v}) \in \Omega_m \end{array} \right\}$$

where Ω_m is a set of multilinear expression of m parameters α_i, $i = 1, 2, ..., m$ with each parameter varying between the lower limit and the upper limit. According to the inductive assumption, Ω_m is subpolytopic, and all the vertices can be obtained by letting each of the m parameters assume the lower and the upper limit. Denote these vertices as $(\omega_{m,n}^{(i)}, \omega_{m,v}^{(i)})$, $i = 1, 2, 3, ..., 2^m$. Then, for any $(\omega_{m,n}, \omega_{m,v}) \in \Omega_m$, we can write

$$\begin{aligned} \omega_{m,n} &= \sum_{i=1}^{2^m} \beta_i \omega_{m,n}^{(i)} \\ \omega_{m,v} &= \sum_{i=1}^{2^m} \beta_i \omega_{m,v}^{(i)} \end{aligned}$$

for some

$$\beta_i \geq 0, \; \sum_{i=1}^{2^m} \beta_i = 1$$

This indicates that any element ω of Ω can be expressed as

$$\omega = \sum_{i=1}^{2^m} \beta_i (\omega_{m,n}^{(i)} + \alpha_{m+1}\omega_{m,v}^{(i)})$$

But $\alpha_{m+1,\min} \leq \alpha_{m+1} \leq \alpha_{m+1,\max}$ can be written as

$$\alpha_{m+1} = (1-\gamma)\alpha_{m+1,\min} + \gamma\alpha_{m+1,\max}, \; 0 \leq \gamma \leq 1$$

Therefore,

$$\begin{aligned}\omega &= \sum_{i=1}^{2^m} \beta_i(1-\gamma)(\omega_{m,n}^{(i)} + \alpha_{m+1,\min}\omega_{m,v}^{(i)}) \\ &\quad + \sum_{i=1}^{2^m} \beta_i\gamma(\omega_{m,n}^{(i)} + \alpha_{m+1,\max}\omega_{m,v}^{(i)})\end{aligned}$$

Since the 2^{m+1} coefficients $\delta_i = \beta_i\gamma \geq 0, \delta_{2^m+i} = \beta_i(1-\gamma) \geq 0$, and the sum

$$\sum_{i=1}^{2^{m+1}} \delta_i = \sum_{i=1}^{2^m} \beta_i(1-\gamma) + \sum_{i=1}^{2^m} \beta_i\gamma = \sum_{i=1}^{2^m} \beta_i = 1$$

we can conclude that Ω is subpolytopic with 2^{m+1} vertices

$$(\omega_{m,n}^{(i)} + \alpha_{m+1,\min}\omega_{m,v}^{(i)}), (\omega_{m,n}^{(i)} + \alpha_{m+1,\max}\omega_{m,v}^{(i)}), i = 1, 2, ..., 2^m$$

which can be obtained by letting each parameter assuming its lower and upper limit. ■

As will be seen later, subpolytopic uncertainties are rather easy to treat in most of the time domain stability analysis, since most stability criteria are in the form of matrix inequalities with system matrices appearing linearly. Such matrix inequalities are satisfied for all the possible system matrices in the uncertainty set if and only if they are satisfied at the vertices.

There are, however, still practical difficulties. For example, for the case of multilinear uncertain parameters, there are $n_v = 2^{n_p}$ vertices of uncertainty set if there are n_p uncertain parameters. The number n_v can be very large for a rather modest n_p. For example, $n_p = 10$ uncertain parameters will result in $n_v = 1024$. This phenomenon, known as *combinatorial explosion*, limits the practical applications of subpolytopic uncertainty model to only a few uncertain parameters. There are also other uncertainties which cannot be reasonably modelled by a subpolytopic uncertainty set with a reasonable number of vertices. In such cases, norm bounded uncertainties are often used.

6.2.3 Norm bounded uncertainty

In terms of *norm bounded uncertainty*, we decompose the system matrices $\omega = (A_0, A_1)$ into two parts: the nominal part $\omega_n = (A_{0n}, A_{1n})$, and the uncertain part $\Delta\omega = (\Delta A_0, \Delta A_1)$:

$$\omega = \omega_n + \Delta\omega$$

i.e.,

$$\begin{aligned}A_0 &= A_{0n} + \Delta A_0 \\ A_1 &= A_{1n} + \Delta A_1\end{aligned}$$

The uncertain part is written as

$$\begin{pmatrix} \Delta A_0 & \Delta A_1 \end{pmatrix} = EF \begin{pmatrix} G_0 & G_1 \end{pmatrix} \tag{6.1}$$

where E, G_0 and G_1 are known constant matrices. F is an uncertain matrix satisfying

$$||F|| \leq 1 \tag{6.2}$$

In other words, the uncertainty set Ω can be expressed as

$$\Omega = \{(A_{0n} + EFG_0, A_{1n} + EFG_1) \mid ||F|| \leq 1\}$$

A slightly more general uncertainty is the *linear fractional norm bounded uncertainty* (or LF norm bounded uncertainty), in which, the uncertain matrices in (6.1) is replaced by

$$\begin{pmatrix} \Delta A_0 & \Delta A_1 \end{pmatrix} = E(I - FD)^{-1}F \begin{pmatrix} G_0 & G_1 \end{pmatrix} \tag{6.3}$$

and F satisfies (6.2). Clearly, the norm bounded uncertainty is a special case of linear fractional norm bounded uncertainty when $D = 0$. Of course, for the LF norm bounded uncertainty to be well defined, the matrix $I - FD$ has to be invertible for arbitrary F satisfying (6.2). It is not difficult to see that this is equivalent to requiring

$$I - D^T D > 0 \tag{6.4}$$

We will say that the uncertainty description (6.3) is *well posed* if (6.4) is satisfied. The LF norm bounded uncertainty can be motivated by the following system

$$\dot{x}(t) = A_{0n}x(t) + A_{1n}x(t-r) + Eu(t) \tag{6.5}$$

$$y(t) = G_0x(t) + G_1x(t-r) + Du(t) \tag{6.6}$$

subject to uncertain feedback

$$u(t) = F(t)y(t) \tag{6.7}$$

or equivalently, in view of (6.2),

$$||u(t)||_2 \leq ||y(t)||_2$$

From this setting, it may appear at the first sight that the bounding (6.2) is rather restrictive. For example, it is often desirable to model the uncertain matrix F in (6.3) to satisfy

$$FPF^T \leq Q \tag{6.8}$$

in stead of (6.2) where P and Q are symmetric positive definite matrices introduced to reflect different scales of the uncertainty components. However, this can be easily transformed into the standard setting as follows: Let U and V be matrices satisfying

$$\begin{aligned} VV^T &= P \\ U^TU &= Q^{-1} \end{aligned}$$

and let

$$F^* = UFV \tag{6.9}$$
$$E^* = EU^{-1} \tag{6.10}$$
$$G_0^* = V^{-1}G_0 \tag{6.11}$$
$$G_1^* = V^{-1}G_1 \tag{6.12}$$
$$D^* = V^{-1}DU^{-1} \tag{6.13}$$

then (6.8) and (6.3) are equivalent to

$$||F^*|| = ||UFV|| \leq 1$$

and

$$\begin{pmatrix} \Delta A_0 & \Delta A_1 \end{pmatrix} = E^*(I - F^*D^*)^{-1}F^* \begin{pmatrix} G_0^* & G_1^* \end{pmatrix}$$

On the other hand, we can, of course, consider another possible situation where the uncertainties ΔA_0 and ΔA_1 are independent of each other, *i.e.*,

$$\Delta A_0 = E_0F_0G_0 \tag{6.14}$$
$$\Delta A_1 = E_1F_1G_1 \tag{6.15}$$

with

$$||F_0|| \leq 1 \tag{6.16}$$
$$||F_1|| \leq 1 \tag{6.17}$$

In handling (LF) norm bounded uncertainty, the following Lemmas prove very useful.

Lemma 6.2 *For any real matrices E, G and real symmetric positive definite matrix P, with compatible dimensions,*

$$EG + G^TE^T \leq EPE^T + G^TP^{-1}G$$

Proof. Since

$$\begin{aligned} & (EPE^T + G^TP^{-1}G) - (EG + G^TE^T) \\ = & (EP - G^T)P^{-1}(EP - G^T)^T \\ \geq & 0 \end{aligned}$$

the conclusion is easy to see. ■

Lemma 6.3 *For $D \in \mathsf{R}^{p\times q}$, $E \in \mathsf{R}^{n\times m}$, $G \in \mathsf{R}^{p\times n}$ and $P = P^T \in \mathsf{R}^{n\times n}$ and*

$$I - D^T D > 0 \tag{6.18}$$

The inequality

$$-P + E(I - XD)^{-1}XG + [E(I - XD)^{-1}XG]^T < 0 \tag{6.19}$$

is satisfied for any real matrix $X \in \mathsf{R}^{q\times p}$ satisfying

$$||X|| \leq 1 \tag{6.20}$$

if and only if there exists a real scalar $\lambda > 0$ such that

$$\begin{pmatrix} -P + \lambda G^T G & E + \lambda G^T D \\ E^T + \lambda D^T G & -\lambda(I - D^T D) \end{pmatrix} < 0 \tag{6.21}$$

or equivalently,

$$-P + \lambda G^T G + \frac{1}{\lambda}(E + \lambda G^T D)(I - D^T D)^{-1}(E^T + \lambda D^T G) < 0 \tag{6.22}$$

Proof. In view of (6.18), the inequality (6.19) is well posed, and is equivalent to

$$-\xi^T P\xi + 2\xi^T E(I - XD)^{-1}XG\xi < 0 \tag{6.23}$$

for all $\xi \neq 0$. The inequality (6.23) can be written as

$$-\xi^T P\xi + 2\xi^T E\eta < 0 \tag{6.24}$$

where

$$\eta = (I - XD)^{-1}XG\xi \tag{6.25}$$

or

$$\eta = X(D\eta + G\xi) \tag{6.26}$$

Equation (6.26) is valid for all X satisfying (6.20) if and only if

$$\eta^T\eta \leq (D\eta + G\xi)^T(D\eta + G\xi) \tag{6.27}$$

Therefore, (6.19) subject to (6.20) is equivalent to (6.24) subject to (6.27). Using $\mathcal{S}$-procedure (Proposition B.3 in Appendix B), this is equivalent to the existence of a $\lambda > 0$ such that (6.21) is satisfied. This is equivalent to (6.22) according to the Schur complement. ■

For the special case of $D = 0$, we have

Lemma 6.4 *For $E \in \mathsf{R}^{n\times m}$, $G \in \mathsf{R}^{p\times n}$ and $P = P^T \in \mathsf{R}^{n\times n}$, the inequality*

$$-P + EXG + [EXG]^T < 0 \tag{6.28}$$

is satisfied for all the real matrices $X \in \mathsf{R}^{q\times p}$ *satisfying*

$$||X|| \leq 1 \tag{6.29}$$

if and only if there exists a real scalar $\lambda > 0$ *such that*

$$\begin{pmatrix} -P + \lambda G^T G & E \\ E^T & -\lambda I \end{pmatrix} < 0 \tag{6.30}$$

or equivalently

$$-P + \lambda G^T G + \frac{1}{\lambda} E E^T < 0 \tag{6.31}$$

In the above two Lemmas, if the matrices P, E, G and D depend on a parameter ρ, then λ also depends on ρ. We may restrict λ to be a constant independent of ρ, in which case, (6.21), (6.22) or (6.31) become a set of sufficient but not necessary conditions.

6.2.4 Block-diagonal uncertainty

Block-diagonal uncertainty represents a very general class of uncertainty. In our context, a system with block-diagonal uncertainty can be expressed as

$$\dot{x}(t) = A_{0n}x(t) + A_{1n}x(t-r) + \sum_{j=1}^{m} E_j u_j(t) \tag{6.32}$$

$$y_i(t) = G_{0i}x(t) + G_{1i}x(t-r) + \sum_{j=1}^{m} D_{ij}u_{ij}(t),\ i = 1, 2, ..., m \tag{6.33}$$

subject to uncertain feedback

$$u_i = F_i y_i,\ i = 1, 2, ..., m \tag{6.34}$$

We can properly scale the uncertain matrices F_i, $i = 1, 2, ..., m$ similar to (6.9) to (6.13) such that they satisfy

$$||F_i|| \leq 1,\ i = 1, 2, ..., m \tag{6.35}$$

Indeed, it is often possible to extract all the uncertain elements from the system, and consider them as feedbacks. Then the system with all the uncertain elements removed can be regarded as a multi-input-multi-output system. This process allows us to write the uncertain system in the standard block-diagonal form (6.32) to (6.35), and is known as "pulling out uncertainties".

Equation (6.32) to (6.34) can be written as (6.5) to (6.7), with

$$F = \mathrm{diag}\begin{pmatrix} F_1 & F_2 & \cdots & F_m \end{pmatrix} \tag{6.36}$$

which justifies the name "block-diagonal uncertainty". Notice also that the block-diagonal uncertainty include as a special case the independent uncertainty case described by (6.14) to (6.17), since (6.14) and (6.15) can be written as

$$\begin{pmatrix} \Delta A_0 & \Delta A_1 \end{pmatrix} = \begin{pmatrix} E_0 & E_1 \end{pmatrix} \begin{pmatrix} F_0(t) & 0 \\ 0 & F_1(t) \end{pmatrix} \begin{pmatrix} G_0 & 0 \\ 0 & G_1 \end{pmatrix}$$

Due to (6.35), F clearly satisfies (6.2). However, the analysis based on (6.2) does not take advantage of special structure of the uncertainty, and may lead to overly conservative stability estimate. Let

$$K_l = \operatorname{diag}\begin{pmatrix} k_1 I_{1l} & k_2 I_{2l} & \cdots & k_m I_{ml} \end{pmatrix} \tag{6.37}$$

$$K_r = \operatorname{diag}\begin{pmatrix} k_1 I_{1r} & k_2 I_{2r} & \cdots & k_m I_{mr} \end{pmatrix} \tag{6.38}$$

where I_{il} and I_{ir} are the identity matrices with the dimensions the same as the number of rows and columns of F_i, respectively. Then an important observation is that

$$K_l F K_r^{-1} = F$$

for arbitrary k_i, $i = 1, 2, ..., m$. Therefore, we can rewrite (6.7) as

$$u = K_l F K_r^{-1} y$$

or

$$u_K = F y_K \tag{6.39}$$

where

$$\begin{aligned} u &= K_l u_K \\ y &= K_r y_K \end{aligned}$$

Therefore, the system can be written as

$$\dot{x}(t) = A_{0n} x(t) + A_{1n} x(t-r) + E K_l u_K(t) \tag{6.40}$$

$$y_K(t) = K_r^{-1} G_0 x(t) + K_r^{-1} G_1 x(t-r) + K_r^{-1} D K_l u_K(t) \tag{6.41}$$

and (6.39). We can therefore apply the (LF) norm bounded uncertainty criteria to the system described by (6.40), (6.41) and (6.39) and

$$||F|| \leq 1$$

If the stability criterion is satisfied for some k_i, $i = 1, 2, ..., m$, then, the system is stable. Many stability criteria can be written as LMI with either k_i^2 or $1/k_i^2$ appearing linearly. The possibility of searching through all the possible scaling factors significantly reduces the conservatism. Indeed, it has been argued that this scaling process allows one to reach "very close to the optimum" in terms of reducing conservatism.

6.3 Robust stability based on Razumikhin Theorem

In this section, we will discuss the robust stability problem using Razumikhin Theorem. Similar to the systems without uncertainty, the conditions will be closely related to the conditions obtained using Lyapunov-Krasovskii functional discussed later in this Chapter. To avoid repetition, we will only discuss the generic uncertainty set. The development on specific uncertainty form (such as polytopic uncertainty and norm bounded uncertainty) will be carried out in the context of Lyapunov-Krasovskii functional method. The corresponding results based on the Razumikhin Theorem can be obtained by a simple substitution from the corresponding results based on the Lyapunov-Krasovskii functional method.

6.3.1 Delay-independent stability for systems with single delay

Consider the system

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t - r(t)) \tag{6.42}$$

where

$$(A_0(t), A_1(t)) \in \Omega \text{ for all } t > 0 \tag{6.43}$$

and the time-delay is uncertain and possibly time-varying, within a known interval

$$0 < r(t) \leq r_{\max} \tag{6.44}$$

The matrices A_0, A_1 and delay r may also depend on the state x_t. However, we will suppress the notation for the dependence on x_t for the sake of convenience. Sometimes, we will also suppress the notation for the dependence on time t. The uncertainty set Ω is compact (which is equivalent to closed and bounded in this finite dimensional context). We will use the Razumikhin Theorem to study the stability of the system. Let

$$V(x) = x^T P x \tag{6.45}$$

then

$$\dot{V}(x(t)) = 2x^T(t)P[A_0 x(t) + A_1 x(t - r)]$$

Razumikhin Theorem indicates that the system is asymptotically stable if there exists a scalar $p > 1$ and matrix $P > 0$ such that

$$\dot{V}(x(t)) \leq -\varepsilon||x(t)||^2 \tag{6.46}$$

whenever

$$V(x(t + \theta)) < pV(x(t)) \text{ for all } - r_{\max} \leq \theta \leq 0. \tag{6.47}$$

With (6.47), then for any $\alpha \geq 0$,

$$\begin{aligned}\dot{V}(x(t)) &\leq 2x^T(t)P[A_0x(t) + A_1x(t-r)] \\ &\quad +\alpha[px^T(t)Px(t) - x^T(t-r)Px(t-r)] \\ &= \phi_{0r}^T \begin{pmatrix} PA_0 + A_0^TP + \alpha pP & PA_1 \\ A_1^TP & -\alpha P \end{pmatrix} \phi_{0r} \end{aligned} \tag{6.48}$$

where

$$\phi_{0r} = \begin{pmatrix} x^T(t) & x^T(t-r) \end{pmatrix}^T$$

We can therefore conclude

Proposition 6.5 *The system described by (6.42) to (6.44) is asymptotically stable if there exist a scalar*

$$\alpha > 0$$

and a symmetric matrix P such that

$$\begin{pmatrix} PA_0 + A_0^TP + \alpha P & PA_1 \\ A_1^TP & -\alpha P \end{pmatrix} < 0 \tag{6.49}$$

for all

$$(A_0, A_1) \in \Omega$$

Proof. Similar to the case without uncertainty discussed in Chapter 5. ■

The stability condition above is delay-independent since the stability criterion is independent of the delay bound $r_{\max}$. Similar to the case of systems without uncertainty, we will see that the condition (6.49) can be obtained from the corresponding condition using Lyapunov-Krasovskii functional by imposing a constraint. Notice that (6.49) generally represents an infinite number of matrix inequalities since Ω contains an infinite number of points. It is therefore unrealistic to use Proposition 6.5 directly to check stability. There are many cases, such as polytopic uncertainty and norm bounded uncertainty, where we can reduce to a finite number of matrix inequalities. We will illustrate these ideas in the section dealing with the Lyapunov-Krasovskii functional methods.

6.3.2 Delay independent stability for systems with distributed delays

Consider a system with distributed delays

$$\dot{x}(t) = A_0(t)x(t) + \int_{-r(t)}^{0} A(t,\theta)x(t+\tau(t,\theta))d\theta \tag{6.50}$$

with

$$(A_0(t), A(t,\cdot)) \in \Omega_r \text{ for all } t \geq 0 \tag{6.51}$$

$$0 < \tau(t,\theta) \leq r_{\max} \text{ for } -r \leq \theta < 0 \tag{6.52}$$

where Ω_r is compact. For the sake of convenience, we will often suppress the notation of the explicit dependence of time t when no confusion may arise. It is possible to write the system in a simpler form without sacrificing the generality. For example, by reparameterize θ, one may easily change the integration limit in (6.50) to $[-1, 0]$. However, since the main purpose of studying such a system with distributed delay here is to derive the delay-dependent stability criteria for systems with a single delay, we choose to keep the formulation as close to the original form as possible. The study of distributed delay in its own right will be conducted in Chapter 7.

We will study this system again using Lyapunov function $V(x)$ as (6.45). The system is asymptotically stable if (6.46) is satisfied whenever (6.47) is satisfied. For any $\alpha(t,\theta) \geq 0$, $-r(t) \leq \theta \leq 0$, when (6.47) is satisfied,

$$\begin{aligned}
\dot{V}(x(t)) &= 2x^T(t)P[A_0x(t) + \int_{-r(t)}^{0} A(\theta)x(t+\tau(t,\theta))d\theta] \\
&\leq 2x^T(t)P[A_0x(t) + \int_{-r(t)}^{0} A(\theta)x(t+\tau(t,\theta))d\theta] \\
&\quad + \int_{-r(t)}^{0} \alpha(t,\theta)[px^T(t)Px(t) - x^T(t+\tau(t,\theta))Px(t+\tau(t,\theta))]d\theta \\
&= \int_{-r(t)}^{0} \phi_{0\theta}^T \begin{pmatrix} M(t,\theta) & PA(t,\theta) \\ A^T(t,\theta)P & -\alpha(t,\theta)P \end{pmatrix} \phi_{0\theta} d\theta \qquad (6.53)
\end{aligned}$$

where

$$M(t,\theta) = \frac{1}{r}[PA_0(t) + A_0^T(t)P] + \alpha(t,\theta)pP + \hat{R}(\theta, A_0(t), A(t,\cdot)) \tag{6.54}$$

and

$$\phi_{0\theta} = \begin{pmatrix} x(t) \\ x(t+\tau(t,\theta)) \end{pmatrix}$$

The matrix function $\hat{R}(\theta, A_0(t), A(t,\cdot))$ (which we will write $\hat{R}(\theta)$ for the sake of convenience) is constrained to satisfy

$$\int_{-r(t)}^{0} \hat{R}(\theta)d\theta = 0 \tag{6.55}$$

From the expression of $\dot{V}(x(t))$ we may conclude

Proposition 6.6 *The system with distributed delays described by (6.50) to (6.52) is asymptotically stable if there exist a symmetric matrix P, scalar function*

$$\alpha(t,\theta) > 0, \ -r(t) \leq \theta \leq 0$$

and symmetric matrix function $\hat{R}(\theta) = \hat{R}(\theta, A_0(t), A(t,\cdot))$ satisfying the constraint (6.55), such that

$$\begin{pmatrix} N(\theta) & PA(\theta) \\ A^T(\theta)P & -\alpha(t,\theta)P \end{pmatrix} < 0 \tag{6.56}$$

for all $-r(t) \leq \theta \leq 0$ and $(A_0, A(\cdot)) \in \Omega_r$, where

$$N(\theta) = \frac{1}{r_{\max}}[PA_0 + A_0^T P] + \alpha(t,\theta)P + \hat{R}(\theta)$$

Proof. If (6.56) is satisfied, the $(2,2)$ entry of the matrix in (6.56) implies

$$P > 0 \tag{6.57}$$

which in turn implies

$$V(x) \geq \varepsilon ||x||^2 \tag{6.58}$$

Also, the (1,1) entry implies

$$N(\theta) < 0$$

Using (6.55), the above implies

$$\begin{aligned} & \int_{-r(t)}^{0} N(\theta) d\theta \\ = & \frac{r(t)}{r_{\max}}[PA_0 + A_0^T P] + r(t)\alpha(t,\theta)P \\ < & \ 0 \end{aligned}$$

In view of (6.57), the above implies

$$PA_0 + A_0^T P < 0$$

Therefore, if we set $p = 1$, we have

$$N(\theta) \geq M(\theta)$$

Therefore,

$$\begin{pmatrix} M(\theta) & PA(t,\theta) \\ A^T(t,\theta)P & -\alpha(\theta)P \end{pmatrix} < 0$$

for $p = 1$. Due to continuity and the compactness of Ω_r, it is also valid for $p = 1 + \delta > 0$ if $\delta > 0$ is sufficiently small. Therefore,

$$\dot{V}(\phi(0)) \leq -\varepsilon ||\phi(0)||^2 \tag{6.59}$$

in view of (6.53). Therefore, the system is asymptotically stable according the Razumikhin Theorem in view of (6.58) and (6.59). ■

It is not difficult to see that condition (6.55) can be relaxed to

$$\int_{-r(t)}^{0} \hat{R}(\theta)d\theta \geq 0 \tag{6.60}$$

and $\hat{R}$ can be chosen to be independent of the uncertain matrices $A_0(t)$ and matrix function $A(t,\cdot)$. However, it will be seen that keeping $\hat{R}$ as a general matrix function of the uncertainty is essential in deriving a simple delay-dependent stability criterion.

6.3.3 Delay-dependent stability with explicit model transformation

Consider again an uncertain system described by (6.42) to (6.52). Here, we will attempt to derive a simple delay-dependent stability criterion using the Razumikhin Theorem. Using

$$x(t-r(t)) = x(t) - \int_{-r(t)}^{0} \dot{x}(t+\theta)d\theta$$

with $\dot{x}(t+\theta)$ replaced by the right hand side of the system equation (6.42) with appropriate time shift, we can obtain the transformed system

$$\begin{aligned}\dot{x}(t) &= [A_0(t)+A_1(t)]x(t) \\ &\quad -A_1(t)\int_{-r(t)}^{0}[A_0(t+\theta)x(t+\theta)+A_1(t+\theta)x(t+\theta-r(t+\theta))]d\theta\end{aligned}$$

which can be written as

$$\dot{x}(t) = \bar{A}_0(t)x(t) + \int_{-2r(t)}^{0} \bar{A}(t,\theta)x(t+\tau(t,\theta))d\theta \tag{6.61}$$

$$(\bar{A}_0(t), \bar{A}(t,\cdot)) \in \bar{\Omega}_{2r},\ t \geq 0$$

where

$$\bar{\Omega}_{2r} = \left\{ (\bar{A}_0, \bar{A}(\cdot)) \;\middle|\; \begin{array}{l} \bar{A}_0 = A_0 + A_1 \\ \bar{A}(\theta) = -A_1 A_{0\theta}, \bar{A}(-r+\theta) = -A_1 A_{1\theta}, -r \leq \theta < 0 \\ \text{for } (A_0, A_1) \in \Omega \text{ and } (A_{0\theta}, A_{1\theta}) \in \Omega,\ \theta \in [-r, 0] \end{array} \right\} \tag{6.62}$$

and

$$\tau(t,\theta) = \begin{cases} \theta, & -r(t) \leq \theta < 0 \\ \theta - r(t+\theta), & -2r(t) \leq \theta < r(t) \end{cases}$$

This model transformaiton, of course, also introduces additional dynamics. This additional dynamics is more complicated, and will not be discussed here. Applying Proposition 6.6, we can obtain

Lemma 6.7 *The system described by (6.42) to (6.52) is asymptotically stable if there exist scalars* α_k, $k = 0, 1$, *and symmetric matrix* P *and symmetric matrix function* $R = R(A_0, A_1, A_{0\theta}, A_{1\theta})$ *such that*

$$\begin{pmatrix} N_k & -PA_1A_{k\theta} \\ -(PA_1A_{k\theta})^T & -\alpha_k P \end{pmatrix} < 0, \; k = 0, 1 \tag{6.63}$$

for $(A_0, A_1) \in \Omega$ *and* $(A_{0\theta}, A_{1\theta}) \in \Omega$, *where*

$$\begin{aligned} N_0 &= \frac{1}{2r_{\max}}[P(A_0 + A_1) + (A_0 + A_1)^T P] + \alpha_0 P + R \\ N_1 &= \frac{1}{2r_{\max}}[P(A_0 + A_1) + (A_0 + A_1)^T P] + \alpha_1 P - R \end{aligned}$$

Proof. Apply Proposition 6.6 to the transformed system (6.61) and uncertainty set (6.62) with the selection

$$\alpha(t, \theta) = \begin{cases} \alpha_0 & -r(t) < \theta \leq 0 \\ \alpha_1 & -2r(t) < \theta \leq -r(t) \end{cases}$$

and for $-r < \theta \leq 0$

$$\begin{aligned} \hat{R}(\theta) &= R(A_0, A_1, A_{0\theta}, A_{1\theta}) \\ \hat{R}(\theta - r) &= -R(A_0, A_1, A_{0\theta}, A_{1\theta}) \end{aligned}$$

where $A_{k\theta} = A_k(t + \theta)$, $k = 0, 1$. ∎

Proposition 6.8 *The system described by (6.42) to (6.52) is asymptotically stable if there exist symmetric matrix* P, *and scalars* α_0 *and* α_1 *such that*

$$\begin{pmatrix} M & -PA_1A_{0\theta} & -PA_1A_{1\theta} \\ -A_{0\theta}^T A_1^T P & -\alpha_0 P & 0 \\ -A_{1\theta}^T A_1^T P & 0 & -\alpha_1 P \end{pmatrix} < 0 \tag{6.64}$$

for all $(A_0, A_1) \in \Omega$ *and* $(A_{0\theta}, A_{1\theta}) \in \Omega$, *where*

$$M = \frac{1}{r_{\max}}[P(A_0 + A_1) + (A_0 + A_1)^T P] + (\alpha_0 + \alpha_1)P$$

Proof. Apply Lemma 6.7, use Proposition **??** in the Appendix to eliminate the matrix function R in (6.63) for $k = 0$ and 1. ∎

Notice, it is essential to allow R to be an arbitrary function of $A_0, A_1, A_{0\theta}$ and $A_{1\theta}$ in Lemma 6.7 in order for its elimination to be valid in the above proof. Of course, we may choose R to be independent of the uncertain matrices $A_0, A_1, A_{0\theta}$ and $A_{1\theta}$ in Lemma 6.7 and accept it as the final stability criterion. This will obviously be more conservative than Proposition 6.8 due to the additional restriction on R.

6.4 Delay independent stability using Lyapunov-Krasovskii functional

6.4.1 Systems with single delay

Consider the system

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-r) \tag{6.65}$$

The system matrices A_0 and A_1 may be dependent on time, and not known, except that they are within a compact set Ω:

$$(A_0(t), A_1(t)) \in \Omega \text{ for all } t \geq 0 \tag{6.66}$$

The time-delay r is assumed to be constant.

Similar to the systems without uncertainty, we may choose the Lyapunov-Krasovskii functional

$$V(\phi) = \phi^T(0)Px(0) + \int_{-r}^{0} \phi^T(\theta)S\phi(\theta)d\theta \tag{6.67}$$

The derivative along the system trajectory is

$$\dot{V}(\phi) = \begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix} \begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(-r) \end{pmatrix} \tag{6.68}$$

We can therefore arrive at

Proposition 6.9 *The system described by (6.65) and (6.66) is asymptotically stable if there exist symmetric matrices*

$$P > 0 \tag{6.69}$$

and S, such that

$$\Pi_1 = \begin{pmatrix} PA_0 + A_0^T P + S & PA_1 \\ A_1^T P & -S \end{pmatrix} < 0 \tag{6.70}$$

is satisfied for all the possible system matrices $(A_0, A_1) \in \Omega$.

It is clear that the stability criterion in Proposition 6.9 is independent of delay.

Since (6.70) is applicable to all the points of the uncertainty set Ω, one needs to check an infinite number of LMIs, which is unrealistic. The first simple case where the problem can be reduced to a finite number of LMIs is subpolytopic uncertainty.

Proposition 6.10 *If the uncertainty set Ω is subpolytopic with vertices*

$$\omega^{(i)} = (A_0^{(i)}, A_1^{(i)}), \ i = 1, 2, ..., n_v$$

Then system described by (6.65) and (6.66) is asymptotically stable if there exist symmetric matrices P and S, such that (6.69) is satisfied and (6.70) is satisfied for all

$$(A_0, A_1) = (A_0^{(i)}, A_1^{(i)}), \ i = 1, 2, ..., n_v$$

Proof. Since

$$(A_0^{(i)}, A_1^{(i)}) \in \Omega, \ i = 1, 2, ..., n_v$$

we only need to prove that if (6.70) is satisfied at all the vertices, it is also sastisfied at arbitrary $(A_0, A_1) \in \Omega$. To see this, since for any $(A_0, A_1) \in \Omega$, we can write

$$\begin{aligned} A_i &= \sum_{j=1}^{n_v} \beta_j A_i^{(j)}, \quad i = 0, 1 \\ & \quad \text{for some } \beta_j \geq 0, \ \sum_{j=1}^{n_v} \beta_j = 1 \end{aligned}$$

(6.70) can be written as

$$\sum_{j=1}^{n_v} \beta_j \begin{pmatrix} PA_0^{(j)} + A_0^{(j)T} P + S & PA_1^{(j)} \\ A_1^{(j)T} P & -S \end{pmatrix} < 0$$

Since $\beta_j \geq 0$, the above is implied by

$$\begin{pmatrix} PA_0^{(j)} + A_0^{(j)T} P + S & PA_1^{(j)} \\ A_1^{(j)T} P & -S \end{pmatrix} < 0, \quad j = 1, 2, ..., n_v \tag{6.71}$$

■

Example 6.1 *Consider the following uncertain system,*

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -2 + \rho(t) & \rho(t) \\ \rho(t) & -0.9 + \rho(t) \end{pmatrix} x(t) \\ & \quad + \alpha \begin{pmatrix} -1 + \rho(t) & 0 \\ -1 & -1 - \rho(t) \end{pmatrix} x(t - r), \end{aligned}$$

where

$$|\rho(t)| \leq 0.1.$$

This is a system with subpolytopic uncertainty. The uncertainty set Ω *has two vertices* $(A_0^{(i)}, A_1^{(i)})$, $i = 1, 2$, *where*

$$\begin{aligned} A_0^{(1)} &= \begin{pmatrix} -2.1 & -0.1 \\ -0.1 & -1.0 \end{pmatrix}, \qquad A_1^{(1)} = \alpha \begin{pmatrix} -1.1 & 0 \\ -1 & -0.9 \end{pmatrix} \\ A_0^{(2)} &= \begin{pmatrix} -1.9 & 0.1 \\ 0.1 & -0.8 \end{pmatrix}, \qquad A_1^{(2)} = \alpha \begin{pmatrix} -0.9 & 0 \\ -1.0 & -1.1 \end{pmatrix} \end{aligned}$$

The calculation using Proposition 6.10 indicates that the system is stable independent of delays for $0 \le \alpha < \alpha_{\max} = 0.6730$.

Next, we will discuss LF norm bounded uncertainty case. In this case,

$$A_0(t) = A_{0n} + \Delta A_0(t) \tag{6.72}$$
$$A_1(t) = A_{1n} + \Delta A_1(t) \tag{6.73}$$

where

$$\begin{pmatrix} \Delta A_0(t) & \Delta A_1(t) \end{pmatrix} = E(I - F(t)D)^{-1} F(t) \begin{pmatrix} G_0 & G_1 \end{pmatrix} \tag{6.74}$$

and

$$||F|| \le 1 \tag{6.75}$$

We have

Proposition 6.11 *The system described by (6.65) and (6.72) to (6.75) is asymptotically stable if there exist symmetric matrices*

$$P > 0$$

and S *such that*

$$\begin{pmatrix} PA_{0n} + A_{0n}^T P + S + G_0^T G_0 & PA_{1n} + G_0^T G_1 & PE + G_0^T D \\ A_{1n}^T P + G_1^T G_0 & -S + G_1^T G_1 & G_1^T D \\ E^T P + D^T G_0 & D^T G_1 & -I + D^T D \end{pmatrix} < 0 \tag{6.76}$$

Proof. Use proposition 6.9, the system is stable if there exists a $P' > 0$ and S' such that

$$\begin{pmatrix} P'A_0 + A_0^T P' + S' & P'A_1 \\ A_1^T P' & -S' \end{pmatrix} < 0$$

In view of (6.72) to (6.75), the above can be written as

$$-\bar{P} + \bar{E}(I - FD)^{-1} F\bar{G} + (\bar{E}(I - FD)^{-1} F\bar{G})^T < 0 \tag{6.77}$$

where

$$\bar{P} = -\begin{pmatrix} P'A_{0n} + A_{0n}^T P' + S' & P'A_{1n} \\ A_{1n}^T P' & -S' \end{pmatrix} \tag{6.78}$$

$$\bar{E} = \begin{pmatrix} P'E \\ 0 \end{pmatrix} \tag{6.79}$$

$$\bar{G} = \begin{pmatrix} G_0 & G_1 \end{pmatrix} \tag{6.80}$$

Use Lemma 6.3, (6.77) is satisfied for all $||F|| \leq 1$ if and only if there exists a $\lambda > 0$ such that

$$\begin{pmatrix} -\bar{P} + \lambda \bar{G}^T \bar{G} & \bar{E} + \lambda \bar{G}^T D \\ \bar{E}^T + \lambda D^T \bar{G} & -\lambda(I - D^T D) \end{pmatrix} < 0$$

which is (6.76) with

$$P = \frac{1}{\lambda} P'$$
$$S = \frac{1}{\lambda} S'$$

■

At this point, it is interesting to revisit the Razumikhin Theorem method to study a closely related system

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t - r(t)) \tag{6.81}$$

with $r(t)$ uncertain and possibly time-varying. The uncertain coefficient matrices are described by (6.72) to (6.75). First notice that with a generic uncertainty, the Razumikhin result in Proposition 6.5 is very similar to the result in Proposition 6.9. In fact, the conditions in Proposition 6.5 can be obtained by restricting $S = \alpha P$ for $\alpha > 0$ (and $P > 0$ is no longer needed since it is already implied by the other LMI). With this in mind, we can immediately obtain from Proposition 6.76

Corollary 6.12 *The system described by (6.81) with the coefficient uncertainty described by (6.72) to (6.75) is asymptotically stable if there exists a symmetric matrix P and scalar $\alpha > 0$ such that (6.76) is satisfied with $S = \alpha P$.*

Proof. Since Proposition 6.76 is derived from Proposition 6.9, the result is immediately clear with the observation of the connection between Proposition 6.5 and Proposition 6.9, and the fact that P and S in Proposition 6.76 is proportional to P and S in Proposition 6.9. ■

There are a number of other cases in this Chapter that the Razumikhin Theorem based results can be easily read out from the corresponding Lyapunov-Krasovskii functional based results by a simple substitution. The readers should not have difficulty identifying these cases.

Finally, we consider the system with block-diagonal uncertainty discribed by (6.65) and (6.72) to (6.75), with F having a diagonal structure

$$F = \mathrm{diag}\begin{pmatrix} F_1 & F_2 & \cdots & F_m \end{pmatrix} \tag{6.82}$$

To be specific, let

$$F_i \in \mathsf{R}^{p_i \times q_i},\ p = \sum_{i=1}^{m} p_i,\ q = \sum_{i=1}^{m} q_i \tag{6.83}$$

With this structure, (6.75) is equivalent to $||F_i|| \leq 1$, $i = 1, 2, ..., m$. Then, with the scaling discussed in Subsection 6.2.4, we can arrive at the following stability criterion:

Corollary 6.13 *The uncertain system described by (6.65), (6.72) to (6.75), and (6.82) to (6.83) is asymptotically stable if there exist real symmetric matrices $P > 0$, S and scalars $\lambda_i > 0$, $i = 1, 2, ..., m$ such that*

$$\begin{pmatrix} PA_{0n} + A_{0n}^T P + S + G_0^T \Lambda_r G_0 & PA_{1n} + G_0^T \Lambda_r G_1 & PE + G_0^T \Lambda_r D \\ A_{1n}^T P + G_1^T \Lambda_r G_0 & -S + G_1^T \Lambda_r G_1 & G_1^T \Lambda_r D \\ E^T P + D^T \Lambda_r G_0 & D^T \Lambda_r G_1 & -\Lambda_l + D^T \Lambda_r D \end{pmatrix} < 0$$

where

$$\begin{aligned} \Lambda_r &= \mathrm{diag}\begin{pmatrix} \lambda_1 I_{q_1} & \lambda_2 I_{q_2} & \cdots & \lambda_m I_{q_m} \end{pmatrix} \\ \Lambda_l &= \mathrm{diag}\begin{pmatrix} \lambda_1 I_{p_1} & \lambda_2 I_{p_2} & \cdots & \lambda_m I_{p_m} \end{pmatrix} \end{aligned}$$

where I_k is the $k \times k$ identity matrix.

Proof. As discussed in Subsection 6.2.4, it is sufficient to consider the stability of the system by the one discribed by (6.40), (6.41), (6.39), (6.75), and (6.37) to (6.38), with k_i, $i = 1, 2, ..., m$ arbitrary constants. Using Proposition 6.76 to this system, and left-multiply the third row of (6.76) by K_l^{-1} and right multiply the third column by K_l^{-1}, define $\lambda_i = 1/k_i^2$. ■

Notice, this stability criterion is still in the form of LMI. It is reduced to Proposition 6.76 if we restrict $\lambda_i = 1$, $i = 1, 2, ..., m$.

Example 6.2 *Consider the system with LF norm bounded uncertainty described by (6.65) and (6.72) to (6.75) with*

$$\begin{aligned} A_{0n} &= \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} \\ A_{1n} &= \begin{pmatrix} -0.5 & 0 \\ -0.5 & -0.5 \end{pmatrix} \end{aligned}$$

$$E = \beta I,\ D = 0.5I,\ G_0 = G_1 = I$$

Using Proposition 6.11, we can conclude that the system is asymptotically stable independent of delays for $\beta < \beta_{\max} = 0.3378$.

If $D = 0$ and all the other parameters remain the same, then the uncertainty is norm bounded. We can calculate $\beta_{\max} = 0.6073$.

6.4.2 Systems with distributed delays

Consider the system with distributed delays

$$\dot{x}(t) = A_0(t)x(t) + \int_{-r}^{0} A(t,\theta)x(t+\theta)d\theta \tag{6.84}$$

with

$$(A_0(t), A(t,\cdot)) \in \Omega \text{ for all } t \geq 0 \tag{6.85}$$

where Ω is compact, and

$$\Omega \subset R^{n\times n} \times \mathcal{C}([-r,0], R^{n\times n}).$$

We will supress the explicit notation for the dependence of A_0 and A_1 on time t when no confusion may arise. Lyapunov-Krasovskii functional for the system can be chosen as

$$V(\phi) = \phi^T(0)P\phi(0) + \int_{-r}^{0} \left[\int_{\theta}^{0} \phi^T(\xi)S(\theta)\phi(\xi)d\xi \right] d\theta$$

The derivative of V can be calculated as

$$\begin{aligned}
\dot{V}(\phi) \quad = \quad & \phi^T(0)[PA_0 + A_0^T P + \int_{-r}^{0} S(\theta)d\theta]\phi(0) \\
& +2\phi^T(0)\int_{-r}^{0} PA(\theta)\phi(\theta)d\theta \\
& -\int_{-r}^{0} \phi^T(\theta)S(\theta)\phi(\theta)d\theta
\end{aligned}$$

or for arbitrary $R(\theta, A_0, A(\theta))$ satisfying

$$\int_{-r}^{0} R(\theta, A_0, A(\cdot))d\theta = 0 \tag{6.86}$$

we have

$$\dot{V}(\phi) = \int_{-r}^{0} \begin{pmatrix} \phi^T(0) & \phi^T(\theta) \end{pmatrix} \begin{pmatrix} M(\theta) & PA(\theta) \\ A^T(\theta)P & -S(\theta) \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(\theta) \end{pmatrix} d\theta \tag{6.87}$$

where

$$M(\theta) = \frac{1}{r}(PA_0 + A_0^T P) + S(\theta) + R(\theta, A_0, A(\cdot)) \tag{6.88}$$

We can therefore conclude that

Proposition 6.14 *The distributed delay system described by (6.84) and (6.85) is asymptotically stable if there exists a symmetric matrix*

$$P > 0$$

symmetric matrix function $R(\theta, A_0, A(\cdot))$ satisfying (6.86) and symmetric matrix function $S(\theta)$ such that

$$\begin{pmatrix} M(\theta) & PA(\theta) \\ A^T(\theta)P & -S(\theta) \end{pmatrix} < 0, \ -r \le \theta \le 0$$

for all

$$(A_0, A(\cdot)) \in \Omega.$$

where $M(\theta)$ is defined in (6.88).

Notice in the above that $M(\theta)$ also depends on the uncertain system matrices A_0 and $A(\theta)$ due to R. In practice, we may choose R to be independent of the uncertain matrices and accept the additional conservatism. As in the case of Razumikhin Theorem based formulation, an important application of this delay independent stability criterion is to derive a delay dependent stability criteria for systems with discrete delays. In such case, it is essential to allow R to be dependent on the uncertainty such that it can be analytically eliminated to result in a simple formulation.

6.5 Delay-dependent stability using simple Lyapunov-Krasovskii functional

Consider again the uncertain system with single delay

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-r) \tag{6.89}$$

and uncertainty expression

$$(A_0(t), A_1(t)) \in \Omega \text{ for all } t \ge 0 \tag{6.90}$$

Here, we assume the delay r to be a known constant. Similar to the case of systems without uncertainty, a model transformation can be carried out to transform the system to one with distributed delays

$$\dot{x}(t) = \bar{A}_0(t)x(t) + \int_{-2r}^{0} \bar{A}(t,\theta)x(t+\theta)d\theta \tag{6.91}$$

where

$$\begin{array}{rcl} \bar{A}_0(t) & = & A_0(t) + A_1(t) \\ \bar{A}(t,\theta) & = & -A_1(t)A_0(t+\theta) \\ \bar{A}(-r+\theta) & = & -A_1(t)A_1(t+\theta) \\ -r < & \theta \le 0 & \end{array}$$

Therefore, the uncertainty set

$$(\bar{A}_0(t), \bar{A}(t,\cdot)) \in \bar{\Omega},\ t \geq 0$$

can be described as

$$\bar{\Omega} = \left\{ \bar{A}_0, \bar{A}(\cdot) \;\middle|\; \begin{array}{l} \bar{A}_0 = A_0 + A_1 \\ \bar{A}(\theta) = -A_1 A_{0\theta}, \bar{A}(-r+\theta) = -A_1 A_{1\theta}, -r \leq \theta < 0 \\ \text{for } (A_0, A_1) \in \Omega \text{ and } (A_{0\theta}, A_{1\theta}) \in \Omega,\ \theta \in [-r, 0] \end{array} \right\} \tag{6.92}$$

As discussed earlier, the stability of the transformed system implies the stability of the original system. However, the reverse is not necessarily true due to the presence of additional dynamics.

Applying Proposition 6.14 to the transformed system, we can conclude that

Lemma 6.15 *The system described by (6.89) and (6.90) is asymptotically stable if there exist symmetric matrices P, S_0, S_1, and matrix function $R(A_0, A_1, A_{0\theta}, A_{1\theta})$ such that*

$$\begin{pmatrix} N_k & -PA_1A_{k\theta} \\ -(PA_1A_{k\theta})^T & -S_k \end{pmatrix} < 0,\ k = 0, 1 \tag{6.93}$$

for all $(A_0, A_1) \in \Omega$ and $(A_{0\theta}, A_{1\theta}) \in \Omega$, where

$$\begin{aligned} N_0 &= \frac{1}{2r}(P(A_0 + A_1) + (A_0 + A_1)^T P) + S_0 + R \\ N_1 &= \frac{1}{2r}(P(A_0 + A_1) + (A_0 + A_1)^T P) + S_1 - R \end{aligned}$$

Proof. Apply Proposition 6.14 to the transformed system described by (6.91) to (6.92), and choose

$$S(\theta) = \begin{cases} S_0, & -r < \theta \leq 0 \\ S_1, & -2r < \theta \leq -r \end{cases}$$

and for $-r < \theta \leq 0$

$$\begin{aligned} R(\theta) &= R(A_0, A_1, A_{0\theta}, A_{1\theta}) \\ R(\theta - r) &= -R(A_0, A_1, A_{0\theta}, A_{1\theta}) \end{aligned}$$

■

We can eliminate the arbitrary matrix function R in (6.93) for $k = 0$ and 1 using Proposition **??** of Appendix B to obtain

Proposition 6.16 *The system described by (6.89) and (6.90) is asymptotically stable if there exist symmetric matrices*

$$P > 0$$

S_0 and S_1 such that

$$\begin{pmatrix} M & -PA_1A_{0\theta} & -PA_1A_{1\theta} \\ -A_{0\theta}^T A_1^T P & -S_0 & 0 \\ -A_{1\theta} A_1 P & 0 & -S_1 \end{pmatrix} < 0 \tag{6.94}$$

for all $(A_0, A_1) \in \Omega$ and $(A_{0\theta}, A_{1\theta}) \in \Omega$, where

$$M = \frac{1}{r}[P(A_0 + A_1) + (A_0 + A_1)^T P] + S_0 + S_1$$

Similar to the delay-independent stability case, it is interesting to compare Proposition 6.16 with Proposition 6.8. The result in Proposition 6.8 can be obtained from Proposition 6.16 by the following substitutions

$$\begin{aligned} S_k &\Leftarrow \alpha_k P, \; \alpha_k > 0 \text{ for } k = 0, 1 \\ r &\Leftarrow r_{\max} \end{aligned}$$

Recall that Proposition 6.8 is applicable to time-varying delay, and is derived using Razumikhin Theorem.

As compared to delay independent stability discussed earlier and the discretized Lyapunov functional method to be discussed later, a significant complicating factor in using this formulation for robust stability analysis is that some entries in (6.94) involve products of system matrices. This is the result of model transformation. For subpolytopic uncertainty, let

$$\omega^{(i)} = (A_0^{(i)}, A_1^{(i)}), \; i = 1, 2, ..., n_v$$

be the vertices of the uncertainty set Ω. Since (A_0, A_1) and $(A_{0\theta}, A_{1\theta})$ appear bilinearly in (6.94), an argument similar to Proposition 6.1 allows us to conclude that the set of the matrices on the left hand side of (6.94) for $(A_0, A_1) \in \Omega$ and $(A_{0\theta}, A_{1\theta}) \in \Omega$ is a subpolytopic set. We can therefore conclude that it is only necessary to check (6.94) at n_v^2 points:

$$\begin{aligned} (A_0, A_1) &= (A_0^{(i)}, A_1^{(i)}) \\ (A_{0\theta}, A_{1\theta}) &= (A_0^{(j)}, A_1^{(j)}) \\ i &= 1, 2, ..., n_v; j = 1, 2, ..., n_v \end{aligned}$$

We will next discuss the norm bounded uncertainty described by

$$A_0(t) = A_{0n} + \Delta A_0(t) \tag{6.95}$$

$$A_1(t) = A_{1n} + \Delta A_1(t) \tag{6.96}$$

where

$$\begin{pmatrix} \Delta A_0(t) & \Delta A_1(t) \end{pmatrix} = EF(t) \begin{pmatrix} G_0 & G_1 \end{pmatrix} \tag{6.97}$$

and

$$||F(t)|| \leq 1 \tag{6.98}$$

We can state

Proposition 6.17 *The system described by (6.89), with uncertainty described by (6.95) to (6.98) is asymptotically stable if there exist symmetric matrices*

$$P > 0$$

S_0 *and* S_1 *and scalar* μ *such that*

$$\begin{pmatrix} M_n & -PA_{1n}A_{0n} & -PA_{1n}A_{1n} & PE & \frac{1}{r}(G_0+G_1)^T & -PA_{1n}E \\ -A_{0n}^T A_{1n}^T P & -S_0+\mu G_0^T G_0 & \mu G_0^T G_1 & 0 & -(G_1A_{0n})^T & 0 \\ -A_{1n}A_{1n}P & \mu G_1^T G_0 & -S_1+\mu G_1^T G_1 & 0 & -(G_1A_{1n})^T & 0 \\ E^T P & 0 & 0 & -I & 0 & 0 \\ \frac{1}{r}(G_0+G_1) & -G_1A_{0n} & -G_1A_{1n} & 0 & -I & -G_1E \\ -(PA_{1n}E)^T & 0 & 0 & 0 & -(G_1E)^T & -\mu I \end{pmatrix} < 0 \quad (6.99)$$

where

$$M_n = \frac{1}{r}[P(A_{0n}+A_{1n}) + (A_{0n}+A_{1n})^T P] + S_0 + S_1$$

Proof. According to Proposition 6.16, the system is asymptotically stable if (6.94) is satisfied for all (A_0, A_1) satisfying (6.95) to (6.97), and all $(A_{0\theta}, A_{1\theta})$ satisfying

$$\begin{aligned} A_{0\theta}(t) &= A_{0n} + \Delta A_{0\theta}(t) &(6.100)\\ A_{1\theta}(t) &= A_{1n} + \Delta A_{1\theta}(t) &(6.101)\end{aligned}$$

where

$$\begin{pmatrix} \Delta A_{0\theta}(t) & \Delta A_{1\theta}(t) \end{pmatrix} = EF_\theta(t) \begin{pmatrix} G_0 & G_1 \end{pmatrix} \quad (6.102)$$

and

$$||F_\theta|| \leq 1 \quad (6.103)$$

Using (6.95) to (6.97) in (6.94), we obtain

$$\bar{P} + \bar{E}F(t)\bar{G} < 0 \quad (6.104)$$

where

$$\begin{aligned} \bar{P} &= -\begin{pmatrix} M_n & -PA_{1n}A_{0\theta} & -PA_{1n}A_{1\theta} \\ -A_{0\theta}^T A_{1n}^T P & -S_0 & 0 \\ -A_{1\theta}A_{1n}P & 0 & -S_1 \end{pmatrix} \\ \bar{E} &= \begin{pmatrix} PE \\ 0 \\ 0 \end{pmatrix} \\ \bar{G} &= \begin{pmatrix} \frac{1}{r}(G_0+G_1) & -G_1A_{0\theta} & -G_1A_{1\theta} \end{pmatrix} \end{aligned}$$

According to Lemma 6.4, a sufficient condition for (6.104) is

$$\begin{pmatrix} -\bar{P} + \lambda \bar{G}^T \bar{G} & \bar{E} \\ \bar{E} & -\lambda I \end{pmatrix} < 0$$

for some $\lambda > 0$. Dividing λ and using Schur complement, the above is equivalent to

$$\begin{pmatrix} -\frac{1}{\lambda}\bar{P} & \frac{1}{\lambda}\bar{E} & \bar{G}^T \\ \frac{1}{\lambda}\bar{E}^T & -I & 0 \\ \bar{G} & 0 & -I \end{pmatrix} < 0$$

Redefine

$$\begin{aligned} &\frac{1}{\lambda} P \text{ as } P \\ &\frac{1}{\lambda} S_0 \text{ as } S_0 \\ &\frac{1}{\lambda} S_1 \text{ as } S_1 \end{aligned}$$

Then

$$\begin{pmatrix} M_n & -PA_{1n}A_{0\theta} & -PA_{1n}A_{1\theta} & PE & \frac{1}{r}(G_0+G_1)^T \\ -A_{0\theta}^T A_{1n}^T P & -S_0 & 0 & 0 & -(G_1 A_{0\theta})^T \\ -A_{1\theta} A_{1n} P & 0 & -S_1 & 0 & -(G_1 A_{1\theta})^T \\ E^T P & 0 & 0 & -I & 0 \\ \frac{1}{r}(G_0+G_1) & -G_1 A_{0\theta} & -G_1 A_{1\theta} & 0 & -I \end{pmatrix} < 0$$

Using (6.100) to (6.103) in the above, we obtain

$$-\bar{P} + \bar{E} F_\theta(t) \bar{G} < 0$$

where

$$\begin{aligned} \bar{P} &= -\begin{pmatrix} M_n & -PA_{1n}A_{0n} & -PA_{1n}A_{1n} & PE & \frac{1}{r}(G_0+G_1)^T \\ -A_{0n}^T A_{1n}^T P & -S_0 & 0 & 0 & -(G_1 A_{0n})^T \\ -A_{1n} A_{1n} P & 0 & -S_1 & 0 & -(G_1 A_{1n})^T \\ E^T P & 0 & 0 & -I & 0 \\ \frac{1}{r}(G_0+G_1) & -G_1 A_{0n} & -G_1 A_{1n} & 0 & -I \end{pmatrix} \\ \bar{E} &= \begin{pmatrix} -PA_{1n}E \\ 0 \\ 0 \\ 0 \\ -G_1 E \end{pmatrix} \\ \bar{G} &= \begin{pmatrix} 0 & G_0 & G_1 & 0 & 0 \end{pmatrix} \end{aligned}$$

Using Lemma 6.4, the above is equivalent to the existence of a real scalar $\mu > 0$ such that

$$\begin{pmatrix} \bar{P} + \mu \bar{G}^T \bar{G} & \bar{E} \\ \bar{E}^T & -\mu I \end{pmatrix} < 0$$

which is (6.99). Notice, $\mu > 0$ is implied by (6.99). ■

Example 6.3 *Consider the system (6.89) with norm bounded uncertainty described by (6.95) to (6.98) with*

$$A_{0n} = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix}, \quad A_{1n} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

and

$$E = 0.2I,\ G_0 = G_1 = I$$

Using Proposition 6.17, we can obtain that the system is asymptotically stable for delays $r < r_{\max} = 0.5935$. We will see later that this result is still substantially more conservative than the one obtained by discretized Lyapunov functional method.

6.6 Complete quadratic Lyapunov-Krasovskii functional approach

Recall from Chapter 5 that for a system without uncertainty

$$\dot{x}(t) = A_{0n}x(t) + A_{1n}x(t-r) \tag{6.105}$$

we may prescribe the derivative of a quadratic Lyapunov-Krasovskii functional as

$$\dot{V}(\phi)|_{(6.105)} = -\phi^T(0)W_1\phi(0) - \int_{-r}^{0} \phi^T(\tau)W_2\phi(\tau)d\tau - \phi^T(-r)W_3\phi(-r) \tag{6.106}$$

The corresponding Lyapunov-Krasovskii functional is

$$\begin{aligned} V(\phi) &= \phi^T(0)U(0)\phi(0) \\ &\quad +2\phi^T(0)\int_{-r}^{0} U(-r-\theta)A_{1n}\phi(\theta)d\theta \\ &\quad +\int_{-r}^{0}\int_{-r}^{0} \phi^\top(\theta_1)A_{1n}^\top U(\theta_1-\theta_2)A_{1n}\phi(\theta_2)d\theta_1 d\theta_2 \\ &\quad +\int_{-r}^{0} \phi^T(\theta)W(\theta)\phi(\theta)d\theta \end{aligned} \tag{6.107}$$

where

$$\begin{aligned} W(\theta) &= W_3 + (r+\theta)W_2 \\ U(\theta) &= U_{W_1}(\theta) + \int_{-r}^{0} U_{W_2}(\theta)d\tau + U_{W_3}(\theta) \\ &= \int_{0}^{\infty} \Phi^T(t)[W_1 + rW_2 + W_3]\Phi(t+\theta)dt \end{aligned}$$

and $\Phi(t)$ is the fundamental solution of the system (6.105). When $W_1 > 0$, $W_2 \geq 0$, $W_3 \geq 0$, we may easily conclude that $\dot{V}(\phi) \leq -\varepsilon||\phi(0)||^2$ and $V(\phi) \geq 0$. A slight modification of $V(\phi)$ allows us to satisfy the condition $V(\phi) \geq \varepsilon||\phi(0)||^2$.

In this Section we will explore applying the Lyapunov-Krasovskii functional (6.107) to the uncertain system

$$\dot{x}(t) = [A_{0n} + \Delta A_0(t)]x(t) + [A_{1n} + \Delta A_1(t)]x(t - r) \qquad (6.108)$$

Especially, we will illustrate that although we may set $W_2 = W_3 = 0$ in systems without uncertainty, these two matrices play important roles in robust analysis. The purpose is to illustrate the idea rather than obtaining the least conservative condition. We will also assume the uncertainty satisfies

$$\Delta A_0^T R_0 \Delta A_0 \leq \rho_0 S, \ \Delta A_1^T R_1 \Delta A_1 \leq \rho_1 S \qquad (6.109)$$

where $\rho_0 > 0$ and $\rho_1 > 0$ are real scalars, R_0, R_1 and S are positive definite matrices. As indicated earlier, this can also be reformulated in the standard norm bound uncertainty. The uncertainties ΔA_0 and ΔA_1 are assumed to be independent.

We still use $\Phi(t)$ to represent the fundamental solution of the nominal system without uncertainty (6.105). Taking derivative of the Lyapunov-Krasovskii functional along the trajectory of the uncertain system (6.108) yields

$$\begin{aligned} \dot{V}(\phi)|_{(6.108)} &= \dot{V}(\phi)|_{(6.105)} + 2[\Delta A_0\phi(0) + \Delta A_1\phi(-r)]^T \cdot \\ &\quad \cdot [U(0)\phi(0) + \int_{-r}^{0} U(-r-\theta)A_{1n}\phi(\theta)d\theta] \end{aligned} \qquad (6.110)$$

Using Lemma 6.4, we can obtain

$$\begin{aligned}
& 2[\Delta A_0\phi(0)]^T U(0)\phi(0) \\
\leq\ & \phi^T(0)\Delta A_0^T \frac{R_0}{\mu}\Delta A_0\phi(0) + \phi^T(0)U^T(0)(\frac{R_0}{\mu})^{-1}U(0)\phi(0) \\
& 2[\Delta A_1\phi(-r)]^T[U(0)\phi(0)] \\
\leq\ & \phi^T(-r)\Delta A_1^T \frac{R_1}{\mu}\Delta A_1\phi(-r) + \phi^T(0)U^T(0)(\frac{R_1}{\mu})^{-1}U(0)\phi(0) \\
& 2[\Delta A_0\phi(0)]^T \int_{-r}^0 U(-r-\theta)A_{1n}\phi(\theta)d\theta \\
\leq\ & r\phi^T(0)\Delta A_0^T \frac{R_0}{\mu}\Delta A_0\phi(0) \\
& + \int_{-r}^0 [U(-r-\theta)A_{1n}\phi(\theta)]^T(\frac{R_0}{\mu})^{-1}[U(-r-\theta)A_{1n}\phi(\theta)]d\theta \\
& 2[\Delta A_1\phi(-r)]^T \int_{-r}^0 U(-r-\theta)A_{1n}\phi(\theta)d\theta \\
\leq\ & r\phi^T(-r)\Delta A_1^T \frac{R_1}{\mu}\Delta A_1\phi(-r) \\
& + \int_{-r}^0 [U(-r-\theta)A_{1n}\phi(\theta)]^T(\frac{R_1}{\mu})^{-1}[U(-r-\theta)A_{1n}\phi(\theta)]d\theta
\end{aligned}$$

Using the above and (6.106) in (6.110), considering (6.109), we obtain

$$\begin{aligned}
& \dot{V}(\phi)|_{(6.108)} \\
\leq\ & -\phi^T(0)[W_1 - (1+r)\frac{\rho_0}{\mu}S - \mu U^T(0)(R_0^{-1}+R_1^{-1})U(0)]\phi(0) \\
& -\int_{-r}^0 \phi^T(\tau)[W_2 - \mu A_{1n}^T U^T(-r-\theta)(R_0^{-1}+R_1^{-1})U(-r-\theta)A_{1n}]\phi(\tau)d\tau \\
& -\phi^T(-r)[W_3 - (1+r)\frac{\rho_1}{\mu}S]\phi(-r)
\end{aligned}$$

From which we can conclude that

Theorem 6.18 *If the nominal system (6.105) is stable, then the uncertain system (6.108) to (6.109) is asymptotically stable if we can find a scalar* $\mu > 0$, *matrices* $W_1 > 0$, $W_2 \geq 0$ *and* $W_3 \geq 0$ *such that*

$$\begin{aligned}
W_1 &> (1+r)\frac{\rho_0}{\mu}S + \mu U^T(0)(R_0^{-1}+R_1^{-1})U(0) \\
W_2 &\geq \mu A_{1n}^T U^T(-r-\theta)(R_0^{-1}+R_1^{-1})U(-r-\theta)A_{1n} \\
W_3 &\geq (1+r)\frac{\rho_1}{\mu}S
\end{aligned}$$

Proof. From the above discussion, the Lyapunov-Krasovskii functional $V(\phi) \geq 0$, and $\dot{V}(\phi) \leq -\varepsilon||\phi(0)||^2$ for some sufficiently small $\varepsilon > 0$. A slight modification similar to the case without uncertainty allows us to conclude $V(\phi) \geq \varepsilon||\phi(0)||^2$. Therefore, the stability of the system is established. ■

6.7 Discretized Lyapunov functional method for systems with single delay

6.7.1 General case

Consider again the uncertain system with single delay,

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-r) \tag{6.111}$$

where the system matrices are bounded by a known compact set Ω in the following manner,

$$(A_0(t), A_1(t)) \in \Omega \text{ for all } t \geq 0. \tag{6.112}$$

Use the process almost identical to the case with systems without uncertainty described in Section 5.7 of Chapter 5: use Lyapunov functional

$$\begin{aligned} V(\phi) &= \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\quad + \int_{-r}^{0}[\int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta]d\xi \\ &\quad + \int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \end{aligned} \tag{6.113}$$

with discretization process for the Lyapunov functional and derivative, it can be concluded that

Proposition 6.19 *The system described by (6.111) and (6.112) is asymptotically stable if there exist $n \times n$ matrices $P = P^T$; Q_p, $S_p = S_p^T$, $p = 0, 1, ..., N$; $R_{pq} = R_{qp}^T$, $p = 0, 1, ..., N$, $q = 0, 1, ..., N$ such that*

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} + \tilde{S} \end{pmatrix} > 0 \tag{6.114}$$

and

$$\begin{pmatrix} \Delta & -D^s & -D^a \\ -D^{sT} & R_d + S_d & 0 \\ -D^{aT} & 0 & 3S_d \end{pmatrix} > 0 \tag{6.115}$$

are satisfied for all $(A_0, A_1) \in \Omega$, *where*

$$\tilde{Q} = \begin{pmatrix} Q_0 & Q_2 & ... & Q_N \end{pmatrix} \tag{6.116}$$

$$\tilde{R} = \begin{pmatrix} R_{00} & R_{01} & ... & R_{0N} \\ R_{10} & R_{11} & ... & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & ... & R_{NN} \end{pmatrix} \tag{6.117}$$

$$\tilde{S} = \operatorname{diag}\begin{pmatrix} \frac{1}{h}S_0 & \frac{1}{h}S_1 & ... & \frac{1}{h}S_N \end{pmatrix} \tag{6.118}$$

$$\Delta = \begin{pmatrix} \Delta_{00} & \Delta_{01} \\ \Delta_{01}^T & \Delta_{11} \end{pmatrix} \tag{6.119}$$

$$\Delta_{00} = -PA_0 - A_0^T P - Q_0 - Q_0^T - S_0 \tag{6.120}$$

$$\Delta_{01} = Q_N - PA_1 \tag{6.121}$$

$$\Delta_{11} = S_N \tag{6.122}$$

$$S_d = \operatorname{diag}\begin{pmatrix} S_{d1} & S_{d2} & ... & S_{dN} \end{pmatrix} \tag{6.123}$$

$$S_{dp} = S_{p-1} - S_p \tag{6.124}$$

$$R_d = \begin{pmatrix} R_{d11} & R_{d12} & ... & R_{d1N} \\ R_{d21} & R_{d22} & ... & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & ... & R_{dNN} \end{pmatrix} \tag{6.125}$$

$$R_{dpq} = h(R_{p-1,q-1} - R_{pq}) \tag{6.126}$$

$$D^s = \begin{pmatrix} D_1^s & D_2^s & ... & D_N^s \end{pmatrix} \tag{6.127}$$

$$D_p^s = \begin{pmatrix} D_{0p}^s \\ D_{1p}^s \end{pmatrix} \tag{6.128}$$

$$D_{0p}^s = \frac{h}{2}A_0^T(Q_{p-1} + Q_p) + \frac{h}{2}(R_{0,p-1} + R_{0p}) - (Q_{p-1} - Q_p) \tag{6.129}$$

$$D_{1p}^s = \frac{h}{2}A_1^T(Q_{p-1} + Q_p) - \frac{h}{2}(R_{N,p-1} + R_{Np}) \tag{6.130}$$

$$D^a = \begin{pmatrix} D_1^a & D_2^a & ... & D_N^a \end{pmatrix} \tag{6.131}$$

$$D_p^a = \begin{pmatrix} D_{0p}^a \\ D_{1p}^a \end{pmatrix} \tag{6.132}$$

$$D_{0p}^a = -\frac{h}{2}A_0^T(Q_{p-1} - Q_p) - \frac{h}{2}(R_{0,p-1} - R_{0p}) \tag{6.133}$$

$$D_{1p}^a = -\frac{h}{2}A_1^T(Q_{p-1} - Q_p) + \frac{h}{2}(R_{N,p-1} - R_{Np}) \tag{6.134}$$

Proof. The process is almost identical to the case of without uncertainty discussed in Section 5.7 of Chapter 5. The only subtlety is that the U matrix in deriving Lyapunov-Krasovskii derivative condition needs to be replaced by an arbitrary matrix function of the uncertain system matrices A_0 and A_1 in order for the elimination of U to be valid. ■

For subpolytopic uncertainty, let the vertices of the uncertainty set Ω be

$$(A_0^{(i)}, A_1^{(i)}), \ i = 1, 2, ..., n_v$$

Then the inequality (6.115) only needs to be checked for

$$(A_0, A_1) = (A_0^{(i)}, A_1^{(i)})$$

reducing the problem into a finite set of LMIs.

6.7.2 *Norm bounded uncertainty*

We will consider again the norm bounded uncertainty

$$\begin{aligned} A_0(t) &= A_{0n} + \Delta A_0(t) && (6.135) \\ A_1(t) &= A_{1n} + \Delta A_1(t) && (6.136) \end{aligned}$$

where

$$\begin{pmatrix} \Delta A_0(t) & \Delta A_1(t) \end{pmatrix} = EF(t) \begin{pmatrix} G_0 & G_1 \end{pmatrix} \tag{6.137}$$

and

$$||F|| \leq 1 \tag{6.138}$$

In spite of the apparent complexity, it can be seen that (6.114) does not involve system parameters, and (6.115) has a structure very similar to the delay independent stability case, and much simpler than the delay dependent case using model transformation. Taking advantage of this structure, we can write (6.115) as

$$\bar{P} + \bar{E}F(t)\bar{G} + (\bar{E}F(t)\bar{G})^T < 0 \tag{6.139}$$

where

$$\begin{aligned}
\bar{E} &= \begin{pmatrix} E_P \\ E_s \\ E_a \end{pmatrix} \\
E_P &= PE && (6.140) \\
E_s &= \begin{pmatrix} \frac{h}{2}(Q_0+Q_1)^T E \\ \frac{h}{2}(Q_1+Q_2)^T E \\ \vdots \\ \frac{h}{2}(Q_{N-1}+Q_N)^T E \end{pmatrix} && (6.141) \\
E_a &= \begin{pmatrix} \frac{h}{2}(Q_0-Q_1)^T E \\ \frac{h}{2}(Q_1-Q_2)^T E \\ \vdots \\ \frac{h}{2}(Q_{N-1}-Q_N)^T E \end{pmatrix} && (6.142) \\
\bar{G} &= \begin{pmatrix} G_0 & G_1 & 0 & 0 & ... & 0 \end{pmatrix}
\end{aligned}$$

$$\bar{P} = -\begin{pmatrix} \Delta_n & -D_n^s & -D_n^a \\ -D_n^{sT} & R_d+S_d & 0 \\ -D_n^{aT} & 0 & 3S_d \end{pmatrix}$$

where the subscript n indicates that in the corresponding expressions, the matrices A_0 and A_1 should be replaced by the corresponding nominal value A_{0n} and A_{1n}, respectively. Use Lemma 6.4, we can conclude that (6.139) is satisfied for all $||F|| \leq 1$ if there is a $\lambda > 0$ such that

$$\begin{pmatrix} \bar{P}+\frac{1}{\lambda}\bar{G}^T\bar{G} & \bar{E} \\ \bar{E}^T & -\frac{1}{\lambda}I \end{pmatrix} < 0 \tag{6.143}$$

To summarize, we have

Proposition 6.20 *The system described by (6.111) and (6.135) to (6.138) is asymptotically stable if there exist* $n \times n$ *matrices* $P = P^T$*;* Q_p*,* $S_p = S_p^T$*,* $p = 0, 1, ..., N$*;* $R_{pq} = R_{qp}^T$*,* $p = 0, 1, ..., N$*,* $q = 0, 1, ..., N$ *such that*

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R}+\tilde{S} \end{pmatrix} > 0 \tag{6.144}$$

and

$$\begin{pmatrix} \Delta_n - \Delta_G & -D_n^s & -D_n^a & E_P \\ -D_n^{sT} & R_d+S_d & 0 & E_s \\ -D_n^{aT} & 0 & 3S_d & E_a \\ E_P^T & E_s^T & E_a^T & I \end{pmatrix} > 0 \tag{6.145}$$

are satisfied. In the above,

$$\Delta_G = \begin{pmatrix} G_0^T G_0 & G_0^T G_1 \\ G_1^T G_0 & G_1^T G_1 \end{pmatrix}$$

and other notations are defined in (6.116) to (6.134) and (6.140) to (6.142), with the subscript "n" indicating that A_0 and A_1 should be replaced by the nominal values A_{0n} and A_{1n}, respectively, in the corresponding expressions.

Proof. It has already been shown above that (6.143) and (6.144) are sufficient for stability. Divide both by λ, and redefine $\frac{1}{\lambda}P$, $\frac{1}{\lambda}Q_p$, $\frac{1}{\lambda}S_p$ and $\frac{1}{\lambda}R_{pq}$ as P, Q_p, S_p and R_{pq}, respectively. ■

Example 6.4 *Consider the system (6.111) with norm bounded uncertainty described by (6.135) to (6.138) with*

$$A_{0n} = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix}, \quad A_{1n} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

and

$$E = 0.2I, \; G_0 = G_1 = I$$

Using Proposition 6.20, we estimate the maximum delay $r_{\max}$ such that the system remains asymptotically stable for $r < r_{\max}$. The computational results using different discretization N are listed in the following table. Similar to the simple delay case, even for the coarsest discretization of $N = 1$, the result is much less conservative than the corresponding result obtained using simple Lyapunov-Krasovskii functional method in Example 6.3 obtained using Proposition 6.17.

N	1	2	3
$r_{\max}$	3.098	3.132	3.133

6.8 Notes

6.8.1 Uncertainty characterization

The uncertainty characterization has been extensively studied in the literature on robust stability and control. Earlier literature include the study by Yakubovich [311][312][313], which is equivalent to the norm bounded uncertainty discussed here. Indeed, an equivalent of Lemma 6.3 was obtained in these papers. The special case, Lemma 6.4 was obtained by Petersen and Hollot [230]. The generality of block-diagonal uncertainty was shown in the "pulling out the uncertainty" process by Doyle, *et. al.* [64]. The polytopic uncertainty and subpolytopic uncertainty were discussed in Horisberger and Belanger [123], Boyd and Yang [25] and Gu [90]. Most of the results on manipulating these uncertainties in the context of LMI can be found in the book by Boyd, *et. al.*[24].

6.8.2 Stability results based on Razumikhin Theorem and Lyapunov-Krasovskii functional

Due to the lack of effective computational methods available at that time, the earlier works are limited to estimation of bounds of uncertainty and delay-coefficient using simple matrix norm or matrix measures. For examples using Razumikhin Theorem, see Wu and Mizukami [309] for a delay-independent result and Su and Huang [264] for delay-dependent result. For examples using Lyapunov-Krasovskii functional method, see Kim [153] and Sun, *et. al.* [265] for delay-independent results.

Later on, due to advancement of numerical techniques for solving Riccati equations, most attempts are made to formulate such problems in the form of Riccati equations or Riccati inequality, see, Kwon and Pearson [170], Lee, *et. al.* [175] and Shen, *et. al.* [253] for delay-independent results. In fact, some of the results are indeed equivalent to the LMI formulations in view of Schur complement.

The particular derivation used here is based on Gu and Han [102]. The variable elimiation (or introduction) makes it easier to compare different formulation. It is also possible to pursue results using implicit model transformation, see, for example, Lee, *et. al.* [172], which can be improved based on the same principle as done in Chapter 5.

As the advancement of computational methods to solve LMI, interests are shifting to much more flexible formulation of LMI. See, for example, Kokame, *et. al.* [156] an example of delay-independent stability using Lyapunov-Krasovskii functional method. For delay-dependent robust stability with norm bounded uncertainty, see Li and de Souza [181] for Lyapunov-Krasovskii functional approach, and Li and de Souza [180] for Razumikhin Theorem approach. As mentioned in Chapter 5, scaling factors are usually not included in the LMI formulation, which represents a major shortcoming in applications.

The additional dynamics of systems with time-varying delay and coefficients are treated in Kharitonov and Melchor-Aguilar [151]. The treatment of norm bounded uncertainty for discretized Lyapunov functional is based on the idea of Han and Gu [115], but using the more recent formulation in discretized Lyapunov functional method proposed in Gu [97]. The importance of using W_2 and W_3 in the complete quadratic Lyapunov-Kraosvskii functional for robust stability was suggested by Kharitonov.

For time-varying delays with bounded derivative, it also possible to use Lyapunov-Krasovskii functional approach, see Chapter 8. In many situations, it is also possible to directly estimate a bound of exponential decay rate without using Razumikhin Theorem, see Wu and Mizukami [310], Hou and Qian [124], Lehman and Shujaee [176].

7

Systems with Multiple and distributed delays

7.1 Introduction

In this Chapter, we will discuss the stability and robust stability of systems with multiple delays and distributed delays using time domain methods. We will not treat specific uncertainty type, and only give results in the form of "matrix inequalities need to be satisfied for every point in the uncertainty set". Of course, such a form cannot be directly implemented numerically. However, readers should not have difficulty applying the same principle discussed in Chapter 6 to reduce the problem to a finite number of matrix inequalities for some special type of uncertainties, such as subpolytopic uncertainty and norm bounded uncertainty. Also, in view of the similarities between Razumikhin Theorem based results and the ones based on simple Lyapunov-Krasovskii functional results, we will not discuss Razumikhin Theorem based methods in this Chapter.

Section 7.2 discusses the delay-independent stability criteria for systems with multiple delays. Section 7.3 discusses delay-dependent stability using simple Lyapunov-Krasovskii functional. Similar to the single delay case, these results are very conservative. Section 7.4 discusses the complete quadratic Lyapunov-Krasovskii functional for the general case including systems with multiple delays and distributed delays. Similar to the single delay case, the existence of a complete quadratic Lyapunov-Krasovskii functional is necessary and sufficient for asymptotic stability. Section 7.5 discusses the discretized Lyapunov functional method for systems with multiple delays. Section 7.6 discusses the discretized Lyapunov functional method for systems with distributed delays. Based on the principles discussed, the readers should not have difficulty obtaining stability results for systems with both discrete and distributed delays.

The delay-independent stability problem for systems with distributed delays has already been discussed in Chapter 5. Delay-dependent stability criteria using simple Lyapunov-Krasovskii functional can be derived using model transformation with a similar procedure as discrete delays and will not be pursued here.

7.2 Delay-independent stability of systems with multiple delays

Consider the uncertain system with multiple delays

$$\dot{x}(t) = \sum_{i=0}^{K} A_i(t)x(t - r_i), \tag{7.1}$$

where

$$0 = r_0 < r_1 < ... < r_K = r, \tag{7.2}$$

and

$$\omega(t) \in \Omega, \quad \text{for all } t \geq 0, \tag{7.3}$$

and ω represents all the coefficient matrices:

$$\omega(t) = \begin{pmatrix} A_0(t) & A_1(t) & ... & A_K(t) \end{pmatrix}, \tag{7.4}$$

and Ω is a compact set. Using Lyapunov-Krasovskii functional

$$V(\phi) = \phi^T(0)P\phi(0) + \sum_{i=1}^{K} \int_{-r_i}^{0} \phi^T(\theta)S_i\phi(\theta)d\theta,$$

where

$$\begin{aligned} P &= P^T > 0, && (7.5) \\ S_i &= S_i^T > 0, \quad i = 1, 2, ..., K, && (7.6) \end{aligned}$$

then we can easily calculate

$$\begin{aligned} \dot{V}(\phi) = {} & \phi^T(0)[PA_0 + A_0^T P + \sum_{i=1}^{K} S_i]\phi(0) \\ & +2\phi^T(0)P\sum_{i=1}^{K} A_i\phi(-r_i) \\ & -\sum_{i=1}^{K} \phi^T(-r_i)S_i\phi(-r_i). \end{aligned}$$

Let

$$\tilde{\phi}_K = \begin{pmatrix} \phi^T(0) & \phi^T(-r_1) & \phi^T(-r_2) & ... & \phi^T(-r_K) \end{pmatrix}^T,$$

and

$$\Pi_K = \begin{pmatrix} PA_0 + A_0^T P + \sum_{i=1}^{K} S_i & PA_1 & PA_2 & ... & PA_K \\ A_1^T P & -S_1 & 0 & ... & 0 \\ A_2^T P & 0 & -S_2 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_K^T P & 0 & 0 & ... & -S_K \end{pmatrix}. \tag{7.7}$$

Then

$$\dot{V}(\phi) = \tilde{\phi}_K^T \Pi_K \tilde{\phi}_K.$$

We can therefore conclude that

Proposition 7.1 *The system with multiple delays described by (7.1) to (7.4) is asymptotically stable if there exist symmetric matrices P, S_1, S_2, ..., S_K such that*

$$P > 0, \tag{7.8}$$

and

$$\Pi_K < 0 \textit{ for all } \omega \in \Omega, \tag{7.9}$$

where Π_K is defined in (7.7) and ω is defined in (7.4).

Notice that (7.6) is already implied by (7.9). The stability conditions are clearly independent of delays. Therefore, Proposition 7.1 is a delay-independent stability criterion.

The conditions in the above Proposition is equivalent to the following statement: there exist symmetric matrices P, S_1, S_2, ..., S_K and matrix functions $R_i(A_0, ..., A_K)$, $i = 1, 2, ..., K$, such that (7.8) and the following LMIs are satisfied for all possible uncertainties $\omega \in \Omega$

$$PA_0 + A_0^T P + \sum_{i=1}^{K}(S_i + R_i) \quad < \quad 0, \tag{7.10}$$

$$\begin{pmatrix} -R_i & PA_i \\ A_i^T & -S_i \end{pmatrix} \quad < \quad 0, \; i = 1, 2, ..., K. \tag{7.11}$$

The equivalence can be established by eliminating R_i, $i = 1, 2, ..., K$ in (7.10) and (7.11). We can, of course, restrict R_i to constant matrices and accept the additional conservatism.

7.3 Simple delay-dependent stability of systems with multiple delays

To derive a simple delay-dependent stability criteria, we apply model transformation. Utilizing

$$x(t - r_i) = x(t) - \int_{-r_i}^{0} \dot{x}(t + \theta) d\theta.$$

We can transform (7.1) to

$$\dot{x}(t) = \sum_{i=0}^{K} A_i(t)x(t) - \sum_{i=1}^{K}\sum_{j=0}^{K} \int_{-r_i}^{0} A_i(t)A_j(t + \theta)x(t + \theta - r_j)d\theta.$$

Of course, this transformation introduces additional dynamics similar to the case of systems with single delay. Using the simple Lyapunov-Krasovskii functional $V(x_t)$ of the following form:

$$V(\phi) = \phi^T(0)P\phi(0) + \sum_{i=1}^{K}\sum_{j=0}^{K}\int_{-r_j-r_i}^{-r_j}[\int_{\theta}^{0}\phi^T(\xi)S_{ij}\phi(\xi)d\xi]d\theta,$$

where P and S_{ij}, $i = 1,2,...,K$; $j = 0,1,...,K$ are symmetric positive definite matrices. We can easily obtain

$$\begin{aligned}
\dot{V}(\phi) &= \phi^T(0)[P\sum_{i=0}^{K}A_i(t) + \sum_{i=0}^{K}A_i^T(t)P + \sum_{i=1}^{K}\sum_{j=0}^{K}r_iS_{ij}]\phi(0) \\
&\quad -2\phi^T(0)P\sum_{i=1}^{K}\sum_{j=0}^{K}\int_{-r_i}^{0}A_i(t)A_j(t+\xi)\phi(\xi-r_j)d\xi \\
&\quad -\sum_{i=1}^{K}\sum_{j=0}^{K}\int_{-r_i}^{0}\phi^T(\xi-r_j)S_{ij}\phi(\xi-r_j)d\xi,
\end{aligned}$$

or for arbitrary matrix functions $R_{ij}(\omega(t),\omega(t+\xi))$, $i = 1,2,...,K$; $j = 1,2,...,K$ and $U_i(\omega(t),\omega(t+\xi))$, $i = 2,3,...,K$

$$\begin{aligned}
\dot{V}(\phi) &= -\sum_{i=1}^{K}\sum_{j=0}^{K}\int_{-r_i}^{0}\begin{pmatrix}\phi^T(0) & \phi^T(\xi-r_j)\end{pmatrix} \\
&\quad \begin{pmatrix} W_{ij} & PA_i(t)A_j(t+\xi) \\ [PA_i(t)A_j(t+\xi)]^T & S_{ij}\end{pmatrix}\begin{pmatrix}\phi(0) \\ \phi(\xi-r_j)\end{pmatrix}d\xi,
\end{aligned}$$

where

$$W_{ij} = R_{ij},\ i = 1,2,...,K;\ j = 1,2,...,K, \tag{7.12}$$

$$W_{i0} = -\sum_{j=1}^{K}R_{ij} + \frac{1}{r_i}U_i,\ i = 2,3,...,K, \tag{7.13}$$

$$W_{10} = -\sum_{j=1}^{K}R_{1j} - \frac{1}{r_1}\sum_{i=2}^{K}U_i + M, \tag{7.14}$$

and

$$M = -\frac{1}{r_1}(P\sum_{i=0}^{K}A_i + \sum_{i=0}^{K}A_i^TP + \sum_{i=1}^{K}\sum_{j=0}^{K}r_iS_{ij}). \tag{7.15}$$

From the above, we can obtain the following simple delay-dependent stability criterion.

Proposition 7.2 *The system with multiple delays described by (7.1) to (7.4) is asymptotically stable if there exist symmetric matrices P, S_{ij}, $i = 1, 2, ..., K$, and symmetric matrix functions $R_{ij}(\omega, \omega_\theta)$, $i = 1, 2, ..., K$; $j = 1, 2, ..., K$; $U_i(\omega, \omega_\theta)$, $i = 2, 3, ..., K$, such that*

$$P > 0, \tag{7.16}$$

and

$$\begin{pmatrix} W_{ij} & PA_iA_{j\theta} \\ [PA_iA_{j\theta}]^T & S_{ij} \end{pmatrix} > 0 \tag{7.17}$$

for all $\omega \in \Omega$, $\omega_\theta \in \Omega$, where

$$\omega = \begin{pmatrix} A_0 & A_1 & ... & A_K \end{pmatrix}, \tag{7.18}$$

$$\omega_\theta = \begin{pmatrix} A_{0\theta} & A_{1\theta} & ... & A_{K\theta} \end{pmatrix}, \tag{7.19}$$

$W_{ij} = W_{ij}(\omega, \omega_\theta)$ are defined in (7.12) to (7.15).

Notice, $S_{ij} > 0$ has already been implied by (7.17). We can restrict R_{ij} and U_i to be constant matrices and accept the additional conservatism. We can also eliminate these matrix functions to arrive at the following equivalent stability condition

Corollary 7.3 *The system with multiple delays described by (7.1) and (7.4) is asymptotically stable if there exist symmetric matrices P, S_{ij}, $i = 1, 2, ..., K$; $j = 0, 1, 2, ..., K$, such that (7.16) is satisfied, and*

$$\begin{pmatrix} M & PA_1\omega_\theta & PA_2\omega_\theta & ... & PA_K\omega_\theta \\ (PA_1\omega_\theta)^T & \frac{1}{r_1}S_1 & 0 & ... & 0 \\ (PA_2\omega_\theta)^T & 0 & \frac{1}{r_2}S_2 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (PA_K\omega_\theta)^T & 0 & 0 & ... & \frac{1}{r_K}S_K \end{pmatrix} > 0$$

is satisfied for all $\omega \in \Omega$ and $\omega_\theta \in \Omega$, where

$$S_i = \mathrm{diag}\begin{pmatrix} S_{i0} & S_{i1} & ... & S_{iK} \end{pmatrix}$$

and ω and ω_θ are defined in (7.18) and (7.19), respectively.

Proof. For a given i, eliminate R_{ij} in (7.17) for $j = 0, 1, ..., K$ to obtain

$$\begin{pmatrix} W_i & PA_i\omega_\theta \\ [PA_i\omega_\theta]^T & S_i \end{pmatrix} > 0,$$

where

$$W_i = \frac{1}{r_i}U_i, \; i > 1,$$

$$W_1 = M - \frac{1}{r_1}\sum_{i=2}^{K} U_i.$$

This is equivalent to

$$\begin{pmatrix} r_i W_i & P A_i \omega_\theta \\ [P A_i \omega_\theta]^T & \frac{1}{r_i} S_i \end{pmatrix} > 0.$$

Then eliminate U_i, $i = 2, 3, ..., K$ from the above. ■

It it important to realize that the conditions in Proposition 7.2 or Corollary 7.3, albeit structurally appealing due to its simplicity, are very conservative, not only because of the additional dynamics introduced in the model transformation and the usage of the particular Lyapunov-Krasovskii functional, but also due to the fact that we ignored the overlapping of distributed delay intervals of the transformed system.

7.4 Complete quadratic functional for general linear systems

To overcome the conservatism of the stability criteria discussed so far in this chapter, we will in the following discuss complete quadratic Lyapunov-Krasovskii functional. Similar to the case of single delay, the existence of such a Lyapunov-Krasovskii functional is necessary and sufficient for stability. Consider a general linear time-invariant time-delay system

$$\dot{x}(t) = \int_{-r}^{0} d[F(\theta)]x(t+\theta). \tag{7.20}$$

This system include, as special cases, the systems with multiple discrete delays and distributed delays. The complete quadratic Lyapunov-Krasovskii functional is very similar to the case with single delay. However, the derivation is far more sophisticated. In this section, we will present the main idea. Interested readers are referred to the literature discussed in the Notes section for technical details.

Let $\Phi(t)$ denotes the fundamental solution of the system (7.20), *i.e.*, it satisfies

$$\dot{\Phi}(t) = \int_{-r}^{0} d[F(\theta)]\Phi(t+\theta)$$

with initial condition

$$\Phi(t) = \begin{cases} I, & t = 0 \\ 0, & t < 0 \end{cases}$$

For a real positive definite symmetric matrix W, let

$$U_W(\tau) = \int_{0}^{\infty} \Phi^T(t) W \Phi(t+\tau) dt.$$

Using Parseval Theorem, we can alternatively write

$$U_W(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\Delta^T(-i\omega)]^{-1} W [\Delta(i\omega)]^{-1} e^{-i\omega\tau} d\omega,$$

where

$$\Delta(s) = sI - \int_{-r}^{0} e^{\theta s} dF(\theta).$$

Obviously,

$$U_W^T(\tau) = U_W(-\tau).$$

Based on $U_W(\tau)$, for a asymptotically stable system, we can again prescribe

$$\dot{v}_W(\phi) = -\phi^T(0) W \phi(0) \tag{7.21}$$

using the Lyapunov-Krasovskii functional

$$v_W(\phi) = \int_0^{\infty} x^T(t,\phi) W x(t,\phi) dt,$$

where $x(t,\phi)$ is the solution of (7.20) with initial condition

$$x_0 = \phi.$$

As shown by Huang, $v_W(\phi)$ can be explicitly expressed in terms of ϕ,

$$\begin{aligned} v_W(\phi) \quad = \quad & \phi^T(0) U_W(0) \phi(0) + 2\phi^T(0) \int_{-r}^{0} \int_{u}^{0} U_W(u-\theta) d_u[F(u)] \phi(\theta) d\theta \\ & + \int_{-r}^{0} \int_{\eta}^{0} dv \phi^T(v) d_\eta [F(\eta)]^T \\ & [\int_{-r}^{0} \int_{u}^{0} U_W(v-\eta+u-\xi) d_u[F(u)] \phi(\xi) d\xi]. \end{aligned} \tag{7.22}$$

Also notice that, for $-r \le \tau < 0$, we can achieve

$$\dot{v}(\phi) = -\phi^T(\tau) W \phi(\tau),$$

by the Lyapunov-Krasovskii functional

$$v(\phi) = v_W(\phi) + \int_{\tau}^{0} \phi^T(\theta) W \phi(\theta) d\theta.$$

With the above discussion, we can construct a complete quadratic Lyapunov-Krasovskii functional. We will, however, state the following Proposition for the following special class of systems

$$\dot{x}(t) = \sum_{i=0}^{K} A_i x(t-r_i) + \int_{-r}^{0} A(\theta) x(t+\theta) d\theta \tag{7.23}$$

which contains only pointwise and distributed delays.

Proposition 7.4 *If the system described by (7.23) is asymptotically stable, then there exists a Lyapunov-Krasovskii functional*

$$\begin{aligned} V(\phi) &= \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\quad + \int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \\ &\quad + \int_{-r}^{0}[\int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta]d\xi, \end{aligned} \tag{7.24}$$

where $P^T = P \in \mathsf{R}^{n\times n}$, and the matrix functions $Q(\xi) \in \mathsf{R}^{n\times n}$, $S(\xi) = S^T(\xi) \in \mathsf{R}^{n\times n}$, $R^T(\xi,\eta) = R(\eta,\xi) \in \mathsf{R}^{n\times n}$, such that

$$V(\phi) \geq \varepsilon||\phi||^2, \tag{7.25}$$

and

$$\dot{V}(\phi) \leq -\varepsilon||\phi||^2, \tag{7.26}$$

are satisfied for some $\varepsilon > 0$. Furthermore, the matrix functions $Q(\xi)$, $S(\xi)$, and $R(\xi,\eta)$ are continuous everywhere except at pointwise delays $\xi,\eta = r_i$, $i = 1,2,...,K-1$.

Proof. We will construct a complete quadratic Lyapunov-Krasovskii functional for the general system described by (7.20), which reduces to the form (7.24) with appropriate continuity for the special type of system (7.23). For arbitrary continuous function $\beta(\theta) \geq 0$, $-r \leq \theta \leq 0$, denote

$$\int_{-r}^{0} \beta(\theta)|dF(\theta)| \triangleq \sup_{\substack{K>0 \\ -r=\theta_0<\theta_2<\cdots<\theta_K=0}} \sum_{i=1}^{K} \beta(\theta_i)||F(\theta_i) - F(\theta_{i-1})||$$

Since F has bounded variation, the above is well defined. Especially, for $\beta(\theta) \equiv 1$, the above becomes the total variation, which we denote as μ

$$\mu = \int_{-r}^{0} |dF(\theta)|.$$

Consider for some small $\delta > 0$,

$$V(\phi) = v_{(1+\delta\mu)I}(\phi) + \delta\int_{-r}^{0}[\int_{\tau}^{0} \phi^T(\theta)\phi(\theta)d\theta]|dF(\tau)| \tag{7.27}$$

Since

$$\begin{aligned} \dot{V}(\phi) &= -(1+\delta\mu)||\phi(0)||^2 + \delta\int_{-r}^{0}[||\phi(0)||^2 - ||\phi(\tau)||^2]|dF(\tau)| \\ &= -||\phi(0)||^2 - \delta\int_{-r}^{0} ||\phi(\tau)||^2|dF(\tau)| \\ &\leq -||\phi(0)||^2, \end{aligned}$$

inequality (7.26) is satisfied. To show (7.25), consider, for $0 < \varepsilon < \sqrt{\delta/\mu}$,

$$v(\phi) = V(\phi) - \varepsilon||\phi(0)||^2$$

Direct calculation, using (7.21) yields

$$\begin{aligned}
\dot{v}(\phi) &= -||\phi(0)||^2 - \delta \int_{-r}^{0} ||\phi(\tau)||^2 |dF(\tau)| - 2\varepsilon\phi^T(0) \int_{-r}^{0} dF(\theta)\phi(\theta) \\
&\leq -||\phi(0)||^2 - \delta \int_{-r}^{0} ||\phi(\tau)||^2 |dF(\tau)| + 2\varepsilon||\phi(0)|| \int_{-r}^{0} ||\phi(\theta)||\, |dF(\theta)| \\
&= -\int_{-r}^{0} (\frac{\varepsilon}{\sqrt{\delta}}||\phi(0)|| - \sqrt{\delta}||\phi(\tau)||)^2 |dF(\tau)| - (1 - \frac{\varepsilon^2\mu}{\delta})||\phi(0)||^2 \\
&\leq 0.
\end{aligned}$$

Since the above is valid for any ϕ, it is also valid for x_t, we therefore have

$$\dot{v}(x_t) \leq 0.$$

Integrating the above from $t = 0$ to ∞, using the fact that $\lim_{t\to\infty} x(t) = 0$, we have

$$0 - v(x_0) \leq 0$$

or

$$V(\phi) = V(x_0) \geq \varepsilon||\phi(0)||^2$$

which is (7.25). Finally, we notice that when the system is of the specially case of (7.23), $F(\theta)$ is continuously differentiable everywhere except at $\theta = r_i$, $i = 0, 1, ..., K$, in which case, the constructed Lyapunov-Krasovskii functional (7.27) can be written as (7.24) with appropriate continuity. ■

It should be noted that the proof above actually constructed a complete quadratic Lyapunov-Krasovskii functional for the general form of system (7.20).

7.5 Discretized Lyapunov functional method for systems with multiple delays

The basic ideas of discretized Lyapunov functional method for systems with multiple delays are very similar to the single delay case. The formulations are much more involved. The main new technical issues unique to the multiple delay case are:

1. The discretization mesh needs to be nonuniform in general to accommodate the possibility of incommensurate delays or commensurate delay with small common factor in order to keep computational cost reasonable;

2. Many quantities involve two indices: one related to the delays, which we usually use i or j; the other related to the discretization, which we usually use p or q. We often need to manipulate these two indices such as change order of summation.
3. The appropriate application and quadratic integral inequality for the S terms in order to render as small as possible the conservatism due to overlapping of integration interval.

7.5.1 Problem setup

Consider the system with multiple delays

$$\dot{x}(t) = \sum_{i=0}^{K} A_i(t)x(t - r_i), \tag{7.28}$$

where

$$0 < r_0 < r_1 < r_2 < ... < r_K = r$$

are the constant delays. It is possible that the delays are incommensurate. The system matrices may be uncertain, a compact bounding set Ω is known:

$$\omega(t) = (A_0(t), A_1(t), ..., A_K(t)) \in \Omega, \text{ for all } t \geq 0. \tag{7.29}$$

As is discussed in the last Section, the existence of a quadratic Lyapunov-Krasovskii functional

$$\begin{aligned} V(\phi) &= \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\quad + \int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \\ &\quad + \int_{-r}^{0} [\int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta]d\xi \end{aligned}$$

is a necessary and sufficient condition for the stability of such a system without uncertainty. Generally, the matrix functions Q, S and R have discontinuities for ξ and η at $-r_i$, $i = 1, 2, ..., K-1$. To avoid dealing with discontinuous functions, we choose an equivalent form of Lyapunov-Krasovskii functional involving only continuous functions

$$\begin{aligned} V(\phi) &= \phi^T(0)P\phi(0) + 2\sum_{i=1}^{K}\phi^T(0)\int_{-r_i}^{0} Q^i(\xi)\phi(\xi)d\xi \\ &\quad + \sum_{i=1}^{K}\int_{-r_i}^{0} \phi^T(\xi)S^i(\xi)\phi(\xi)d\xi \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-r_i}^{0} [\int_{-r_j}^{0} \phi^T(\xi)R^{ij}(\xi,\eta)\phi(\eta)d\eta]d\xi, \end{aligned} \tag{7.30}$$

where Q^i, S^i and R^{ij}, are continuous matrix functions, and

$$\begin{aligned} P &= P^T \in \mathsf{R}^{n\times n}, & (7.31)\\ Q^i(\xi) &\in \mathsf{R}^{n\times n}, & (7.32)\\ S^{iT}(\xi) &= S^i(\xi) \in \mathsf{R}^{n\times n}, & (7.33)\\ R^{ijT}(\xi,\eta) &= R^{ji}(\eta,\xi) \in \mathsf{R}^{n\times n}, & (7.34)\\ & \text{for } i = 1,2,...,K;\ j = 1,2,...,K. \end{aligned}$$

Since the additional functions $Q^i(\xi)$, $S^i(\xi)$, $R^{ij}(\xi,\eta)$ and $R^{iK}(\xi,\eta)$, $i,j = 1,2,...,K-1$ are introduced only to account for discontinuities, we will constrain these functions to the following special forms without loss of generality

$$\begin{aligned} Q^i(\xi) &= Q^i = \text{constant}, & (7.35)\\ S^i(\xi) &= S^i = \text{constant}, & (7.36)\\ R^{ij}(\xi,\eta) &= R^{ij} = \text{constant}, & (7.37)\\ R^{iK}(\xi,\eta) &= R^{iK}(\eta) = R^{KiT}(\eta) \text{ independent of } \xi, & (7.38)\\ & \text{for } i = 1,2,...,K-1;\ j = 1,2,...,K-1. \end{aligned}$$

Taking derivative of V in (7.30) with respect to time along the trajectory of system (7.28), carrying out integration by parts similar to the single delay case, we can again write $\dot{V}(\phi)$ as a quadratic expression of ϕ:

$$\begin{aligned} \dot{V}(\phi) = & -\sum_{i=0}^{K}\sum_{j=0}^{K}\phi^T(-r_i)\Delta_{ij}\phi(-r_j)\\ & +2\sum_{i=0}^{K}\sum_{j=1}^{K}\phi^T(-r_i)\int_{-r_i}^{0}\Pi^{ij}(\xi)\phi(\xi)d\xi\\ & -\int_{-r}^{0}\phi^T(\xi)\dot{S}^K(\xi)\phi(\xi)d\xi\\ & -2\sum_{i=1}^{K-1}\int_{-r_i}^{0}\phi^T(\xi)[\int_{-r}^{0}\dot{R}^{iK}(\eta)\phi(\eta)d\eta]d\xi\\ & -\int_{-r}^{0}\phi^T(\xi)[\int_{-r}^{0}(\frac{\partial R^{KK}(\xi,\eta)}{\partial\xi}\\ & \quad +\frac{\partial R^{KK}(\xi,\eta)}{\partial\eta})\phi(\eta)d\eta]d\xi, \end{aligned} \tag{7.39}$$

where

$$
\begin{aligned}
\Delta_{00} &= -[PA_0 + A_0^T P + \sum_{i=1}^{K-1} (Q^i + Q^{iT} + S^i) \\
&\quad + Q^K(0) + Q^{KT}(0) + S^K(0)], \qquad (7.40) \\
\Delta_{0i} &= Q^i - PA_i, \ \Delta_{ii} = S^i, \ 1 \leq i \leq K-1, \\
\Delta_{KK} &= S^K(-r), \ \Delta_{0K} = Q^K(-r) - PA_K, \\
\Delta_{ij} &= 0, \ i \neq j, \ 1 \leq i \leq K, \ 1 \leq j \leq K, \\
\Delta_{ij} &= \Delta_{ji}^T, \ 0 \leq i \leq K, \ 0 \leq j \leq K, \qquad (7.41)
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi^{0j}(\xi) &= A_0^T Q^j + \sum_{i=1}^{K-1} R^{ij} + R^{Kj}(0), \qquad (7.42) \\
\Pi^{0K}(\xi) &= A_0^T Q^K(\xi) + \sum_{i=1}^{K-1} R^{iK}(\xi) + R^{KK}(0,\xi) - \dot{Q}^K(\xi), \\
\Pi^{ij}(\xi) &= A_i^T Q^j - R^{ij}, \ 1 \leq i \leq K-1, \ 1 \leq j \leq K-1, \\
\Pi^{iK}(\xi) &= A_i^T Q^K(\xi) - R^{iK}(\xi), \ 1 \leq i \leq K-1, \\
\Pi^{Kj}(\xi) &= A_K^T Q^j - R^{Kj}(-r), \ 1 \leq j \leq K-1, \\
\Pi^{KK}(\xi) &= A_K^T Q^K(\xi) - R^{KK}(-r,\xi). \qquad (7.43)
\end{aligned}
$$

Recall that the system is asymptotically stable if there exist an $\varepsilon > 0$ and a Lyapunov-Krasovskii functional (7.30) such that the Lyapunov-Krasovskii functional condition

$$V(\phi) \geq \varepsilon ||\phi(0)||^2, \qquad (7.44)$$

and the Lyapunov-Krasovskii derivative condition

$$\dot{V}(\phi) \leq -\varepsilon ||\phi(0)||^2 \qquad (7.45)$$

are satisfied. The conditions are necessary and sufficient if there is no uncertainty.

7.5.2 Discretization

For systems with multiple delays, it is essential that the division of the delay interval $[-r, 0]$ is compatible with the delays such that $-r_i$, $i = 1, 2, ..., K-1$ are among the division points. In other words, let θ_p, $p = 0, 1, ..., N$ be the dividing points,

$$0 = \theta_0 > \theta_1 > \theta_2 > ... > \theta_N = -r,$$

then

$$-r_i = \theta_{N_i}, \ i = 1, 2, ..., K.$$

Thus each interval $[-r_i, 0]$ is divided into N_i intervals. Let the length of pth segment be h_p

$$h_p = \theta_{p-1} - \theta_p.$$

For the sake of convenience, define

$$\begin{aligned} N_0 &= 0, & (7.46)\\ h_0 &= 0, & (7.47)\\ h_{N+1} &= 0. & (7.48) \end{aligned}$$

We have

$$0 = N_0 < N_1 < ... < N_K = N,$$

and

$$r_i = \sum_{p=1}^{N_i} h_p,\ i = 1, 2, ..., K.$$

The matrix functions $Q^K(\xi)$, $S^K(\xi)$, $R^{iK}(\xi)$ and $R^{KK}(\xi, \eta)$ are again chosen to be piecewise linear as follows: for $0 \le \alpha \le 1$, $p = 1, 2, ..., N$,

$$\begin{aligned} Q^K(\theta_p + \alpha h_p) &= Q^{K(p)}(\alpha) = (1-\alpha)Q_p^K + \alpha Q_{p-1}^K, & (7.49)\\ S^K(\theta_p + \alpha h_p) &= S^{K(p)}(\alpha) = (1-\alpha)S_p^K + \alpha S_{p-1}^K, & (7.50)\\ R^{iK}(\theta_p + \alpha h_p) &= R^{iK(p)}(\alpha) = (1-\alpha)R_p^{iK} + \alpha R_{p-1}^{iK}, & (7.51)\\ & \qquad i = 1, 2, ..., K-1, \end{aligned}$$

and for $0 \le \alpha \le 1$, $0 \le \beta \le 1$, $p = 1, 2, ..., N$, $q = 1, 2, ..., N$,

$$\begin{aligned} & R^{KK}(\theta_p + \alpha h_p, \theta_q + \beta h_q)\\ = \ & R^{KK(pq)}(\alpha, \beta)\\ = \ & \begin{cases} (1-\alpha)R_{pq}^{KK} + \beta R_{p-1,q-1}^{KK} + (\alpha - \beta)R_{p-1,q}^{KK}, \\ \qquad\qquad\qquad\qquad \alpha \ge \beta; \\ (1-\beta)R_{pq}^{KK} + \alpha R_{p-1,q-1}^{KK} + (\beta - \alpha)R_{p,q-1}^{KK}, \\ \qquad\qquad\qquad\qquad \alpha < \beta. \end{cases} \qquad (7.52) \end{aligned}$$

Thus, the Lyapunov-Krasovskii functional V is completely determined by P, Q^i, Q_p^K, S^i, S_p^K, R^{ij}, R_p^{iK}, R_{pq}^{KK}, $i, j = 1, 2, ..., K-1$; $p, q = 0, 1, ..., N$.

As will be seen later, we often need to manipulate terms involving two types of indices: one related to the delay, which we usually use i or j, the other related to the discretization, which we usually use p or q. For this purpose, it is useful to introduce the notation

$$M_p = \min\{i \mid N_i \ge p\}. \qquad (7.53)$$

In other words, M_p may assume $K+1$ distinct values $0, 1, ..., K$ according to the following rule

$$M_p = \begin{cases} i & \text{if } N_{i-1} < p \leq N_i, \\ 0 & \text{if } p = 0. \end{cases}$$

It is not difficult to see that

$$M_{N_i} = i, \tag{7.54}$$
$$N_{M_p} \geq p > N_{M_p - 1}. \tag{7.55}$$

Therefore, M_p may be regarded as the inverse of N_i in some sense.

For the sake of convenience, we will also adopt the convention that that for any U_k,

$$\sum_{k=i}^{j} U_k = 0, \qquad \text{whenever } i > j. \tag{7.56}$$

Then, for any indexed expressions U_p^i, $i = 1, 2, ..., K$; $p = 1, 2, ..., N$, it is often useful to change the order of summation as follows

$$\sum_{i=1}^{K} \sum_{p=1}^{N_i} U_p^i = \sum_{p=1}^{N} \sum_{i=M_p}^{K} U_p^i, \tag{7.57}$$

$$\sum_{i=1}^{K-1} \sum_{p=1}^{N_i} U_p^i = \sum_{p=1}^{N_{K-1}} \sum_{i=M_p}^{K-1} U_p^i = \sum_{p=1}^{N} \sum_{i=M_p}^{K-1} U_p^i, \tag{7.58}$$

$$\sum_{i=1}^{K-1} \sum_{p=1}^{N_i - 1} U_p^i = \sum_{p=1}^{N_{K-1}-1} \sum_{i=M_{p+1}}^{K-1} U_p^i = \sum_{p=1}^{N} \sum_{i=M_{p+1}}^{K-1} U_p^i. \tag{7.59}$$

In the last step of (7.58) and (7.59), we have used the convention (7.56) and the fact that $M_p = K > K - 1$ for $p > N_{K-1}$.

7.5.3 Lyapunov-Krasovskii functional condition

With the discretization, the integrations over $[-r, 0]$ in the expression of V in (7.30) may be divided into integrations over $[\theta_p, \theta_{p-1}]$, $p = 1, 2, ..., N$. With some manipulations, all the terms except those involving S^i, $i = 1, 2, ..., K$ can be collected to arrived at the following result:

Lemma 7.5 *The Lyapunov-Krasovskii $V(\phi)$ as expressed in (7.30) to (7.38), with Q^K, S^K, R^{iK} and R^{KK} piecewise linear as expressed in (7.49) to*

(7.52), may be written as

$$
\begin{aligned}
& V(\phi) \\
= & \int_0^1 \left(\begin{array}{ccc} \phi^T(0) & \Phi^T & \Psi^T(\alpha) \end{array} \right) \\
& \left(\begin{array}{ccc} P & \bar{Q} & \tilde{Q}^K \\ \bar{Q}^T & \bar{R} & \hat{R}^K \\ \tilde{Q}^{KT} & \hat{R}^{KT} & \tilde{R}^{KK} \end{array} \right) \left(\begin{array}{c} \phi(0) \\ \Phi \\ \Psi(\alpha) \end{array} \right) d\alpha \\
& + \sum_{i=1}^{K-1} \int_{-r_i}^0 \phi^T(\xi) S^i \phi(\xi) d\xi \\
& + \sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha) S^{K(p)}(\alpha) \phi^{(p)}(\alpha) h_p d\alpha, \qquad (7.60)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{Q} &= \left(\begin{array}{cccc} Q^1 & Q^2 & ... & Q^{K-1} \end{array} \right), \qquad (7.61) \\
\tilde{Q}^K &= \left(\begin{array}{cccc} Q_0^K & Q_1^K & ... & Q_N^K \end{array} \right),
\end{aligned}
$$

$$
\begin{aligned}
\bar{R} &= \left(\begin{array}{cccc} R^{11} & R^{12} & ... & R^{1,K-1} \\ R^{21} & R^{22} & ... & R^{2,K-1} \\ \vdots & \vdots & \ddots & \vdots \\ R^{K-1,1} & R^{K-1,2} & ... & R^{K-1,K-1} \end{array} \right), \\
\hat{R}^K &= \left(\begin{array}{cccc} R_0^{1K} & R_1^{1K} & ... & R_N^{1K} \\ R_0^{2K} & R_1^{2K} & ... & R_N^{2K} \\ \vdots & \vdots & \ddots & \vdots \\ R_0^{K-1,K} & R_1^{K-1,K} & ... & R_N^{K-1,K} \end{array} \right), \\
\tilde{R}^{KK} &= \left(\begin{array}{cccc} R_{00}^{KK} & R_{01}^{KK} & ... & R_{0N}^{KK} \\ R_{10}^{KK} & R_{11}^{KK} & ... & R_{1N}^{KK} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0}^{KK} & R_{N1}^{KK} & ... & R_{NN}^{KK} \end{array} \right), \qquad (7.62)
\end{aligned}
$$

and

$$\Phi = \begin{pmatrix} \int_{-r_1}^{0} \phi(\xi)d\xi \\ \int_{-r_2}^{0} \phi(\xi)d\xi \\ \vdots \\ \int_{-r_{K-1}}^{0} \phi(\xi)d\xi \end{pmatrix}, \tag{7.63}$$

$$\Psi(\alpha) = \begin{pmatrix} \Psi_{(0)}(\alpha) \\ \Psi_{(1)}(\alpha) \\ \Psi_{(2)}(\alpha) \\ \vdots \\ \Psi_{(N-1)}(\alpha) \\ \Psi_{(N)}(\alpha) \end{pmatrix} = \begin{pmatrix} \psi_{(1)}(\alpha) \\ \psi_{(2)}(\alpha) + \psi^{(1)}(\alpha) \\ \psi_{(3)}(\alpha) + \psi^{(2)}(\alpha) \\ \vdots \\ \psi_{(N)}(\alpha) + \psi^{(N-1)}(\alpha) \\ \psi^{(N)}(\alpha) \end{pmatrix}, \tag{7.64}$$

$$\psi^{(p)}(\alpha) = h_p \int_0^{\alpha} \phi^{(p)}(\beta)d\beta, \qquad p = 1, 2, ..., N, \tag{7.65}$$

$$\psi_{(p)}(\alpha) = h_p \int_{\alpha}^{1} \phi^{(p)}(\beta)d\beta, \qquad p = 1, 2, ..., N, \tag{7.66}$$

$$\phi^{(p)}(\alpha) = \phi(\theta_p + \alpha h_p), \qquad p = 1, 2, ..., N. \tag{7.67}$$

Proof. Similar to the single delay case, divide the integration interval $[-r, 0]$ to the segments $[\theta_p,\theta_{p-1}]$, we have

$$\begin{aligned} V(\phi) &= \phi^T(0)P\phi(0) + 2\sum_{i=1}^{K-1} \phi^T(0)Q^i \int_{-r_i}^{0} \phi(\xi)d\xi + 2\sum_{p=1}^{N} \phi^T(0)V_{Q^{K(p)}} \\ &\quad + \sum_{i=1}^{K-1} V_{S^i} + \sum_{p=1}^{N} V_{S^{K(p)}} + \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} \int_{-r_i}^{0} [\int_{-r_j}^{0} \phi^T(\xi)R^{ij}\phi(\eta)d\eta]d\xi \\ &\quad +2\sum_{i=1}^{K-1}\sum_{p=1}^{N} \int_{-r_i}^{0} \phi^T(\xi)V_{R^{iK(p)}}d\xi + \sum_{p=1}^{N}\sum_{q=1}^{N} V_{R^{KK(pq)}}, \end{aligned}$$

where

$$\begin{aligned} V_{Q^{K(p)}} &= \int_0^1 Q^{K(p)}(\alpha)\phi^{(p)}(\alpha)h_p d\alpha, \\ V_{S^i} &= \int_{-r_i}^{0} \phi^T(\xi)S^i\phi(\xi)d\xi, \\ V_{S^{K(p)}} &= \int_0^1 \phi^{(p)T}(\alpha)S^{K(p)}(\alpha)\phi^{(p)}(\alpha)h_p d\alpha, \\ V_{R^{iK(p)}} &= \int_0^1 R^{iK(p)}(\alpha)\phi^{(p)}(\alpha)h_p d\alpha, \\ V_{R^{KK(pq)}} &= \int_0^1 [\int_0^1 \phi^{(p)T}(\alpha)R^{KK(pq)}(\alpha,\eta)\phi^{(q)}(\beta)h_q d\beta]h_p d\alpha. \end{aligned}$$

Integration by parts for $V_{Q^{K(p)}}$, $V_{R^{iK(p)}}$, $V_{R^{KK(pq)}}$ similar to the single delay case, and collect terms, we may reach (7.60). ∎

If the last two terms involving S^i and $S^{K(p)}$ in (7.30) do not exist, the positive definiteness of the matrix in the first term of (7.60) is clearly a necessary and sufficient condition for the Lyapunov-Krasovksii functional condition (7.44). The extent of conservatism depends on how the terms involving S^i and $S^{K(p)}$ are incorporated to the rest of the quadratic expression.

Proposition 7.6 *For the Lyapunov-Krasovskii functional $V(\phi)$ as expressed in (7.30) to (7.38), with Q^K, S^K, R^{iK} and R^{KK} piecewise linear as expressed in (7.49) to (7.52), the Lyapunov-Krasovskii functional condition (7.44) is satisfied for some sufficiently small $\varepsilon > 0$ if*

$$\begin{aligned} S^i &> 0, \ i = 1, 2, ..., K-1, && (7.68) \\ S_p^K &> 0, \ p = 0, 1, ..., N, && (7.69) \end{aligned}$$

and

$$\begin{pmatrix} P & \bar{Q} & \tilde{Q}^K \\ \bar{Q}^T & \bar{R} + \bar{S} & \hat{R}^K - \bar{S}F \\ \tilde{Q}^{KT} & (\hat{R}^K - \bar{S}F)^T & \tilde{R}^{KK} + \tilde{S}' + F^T \bar{S} F \end{pmatrix} > 0 \tag{7.70}$$

are satisfied, where

$$\bar{S} = \mathrm{diag}\left(\frac{1}{h_{N_1}} S^1 \quad \frac{1}{h_{N_2}} S^2 \quad \cdots \quad \frac{1}{h_{N_{K-1}}} S^{K-1} \right), \tag{7.71}$$

$$\begin{aligned} F &= \begin{pmatrix} f_0^1 & f_1^1 & \cdots & f_N^1 \\ f_0^2 & f_1^2 & \cdots & f_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{K-1} & f_1^{K-1} & \cdots & f_N^{K-1} \end{pmatrix}, \\ f_p^i &= \begin{cases} I, & p \le N_i - 1 \ \textit{(or equivalently } i \ge M_{p+1}\textit{)}, \\ 0, & \textit{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} \tilde{S}' &= \mathrm{diag}\left(\frac{1}{\tilde{h}_0} S_0' \quad \frac{1}{\tilde{h}_1} S_1' \quad \cdots \quad \frac{1}{\tilde{h}_N} S_N' \right), \\ \tilde{h}_p &= \max\{h_p, h_{p+1}\}, \ p = 1, 2, ..., N-1, \\ \tilde{h}_0 &= h_1, \qquad \tilde{h}_N = h_N, \\ S_p' &= S_p^K + \sum_{i=M_{p+1}}^{K-1} S^i, \qquad p = 0, 1, ..., N. && (7.72) \end{aligned}$$

Proof. Let

$$\begin{aligned} V_S &= \sum_{i=1}^{K-1} \int_{-r_i}^{0} \phi^T(\xi) S^i \phi(\xi) d\xi \\ &+ \sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha) S^{K(p)}(\alpha) \phi^{(p)}(\alpha) h_p d\alpha. \end{aligned} \tag{7.73}$$

Since the terms in the first summation of (7.73) can be written as

$$\begin{aligned} &\int_{-r_i}^{0} \phi^T(\xi) S^i \phi(\xi) d\xi \\ = &\sum_{p=1}^{N_i-1} \int_0^1 \phi^{(p)T}(\alpha)[\alpha S^i + (1-\alpha)S^i]\phi^{(p)}(\alpha) h_p d\alpha \\ &+ \int_0^1 \phi^{(N_i)T}(\alpha)[\alpha S^i + (1-\alpha)0]\phi^{(N_i)}(\alpha) h_{N_i} d\alpha \\ &+ \int_0^1 \phi^{(N_i)T}(\alpha)[(1-\alpha)S^i]\phi^{(N_i)}(\alpha) h_{N_i} d\alpha, \end{aligned}$$

where the first two terms are of similar form to the expression of $S^{K(p)}$, we can combine the first two terms of the above with the last term in (7.73) to arrive at

$$V_S = V_{S'} + V_{\bar{S}}, \tag{7.74}$$

where

$$\begin{aligned} V_{S'} &= \sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha)[\alpha S'_{p-1} + (1-\alpha)S'_p]\phi^{(p)}(\alpha) h_p d\alpha \\ V_{\bar{S}} &= \sum_{i=1}^{K-1} \int_0^1 \phi^{(N_1)T}(\alpha)(1-\alpha) S^i \phi^{(N_i)}(\alpha) h_{N_i} d\alpha. \end{aligned}$$

In view of (7.68) and (7.69), we can use quadratic integral inequality (Corollary **??** of the Appendix B) to obtain

$$
\begin{aligned}
V_{S'} &= \sum_{p=0}^{N} \int_0^1 [\alpha \phi^{(p+1)T}(\alpha) S'_p \phi^{(p+1)}(\alpha) h_{p+1} \\
&\qquad + (1-\alpha)\phi^{(p)T}(\alpha) S'_p \phi^{(p)}(\alpha) h_p] d\alpha \\
&\geq \sum_{p=0}^{N} \int_0^1 [\alpha [h_{p+1}\phi^{(p+1)}(\alpha)]^T \frac{1}{\tilde{h}_p} S'_p [h_{p+1}\phi^{(p+1)}(\alpha)] \\
&\qquad + (1-\alpha)[h_p \phi^{(p)}(\alpha)]^T \frac{1}{\tilde{h}_p} S'_p [h_p \phi^{(p)}(\alpha)]] d\alpha \\
&\geq \sum_{p=0}^{N} \int_0^1 [\psi_{(p+1)}(\alpha) + \psi^{(p)}(\alpha)]^T \frac{1}{\tilde{h}_p} S'_p [\psi_{(p+1)}(\alpha) + \psi^{(p)}(\alpha)] d\alpha \\
&= \int_0^1 \Psi^T(\alpha) \tilde{S} \Psi(\alpha) d\alpha. \qquad (7.75)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
V_{\bar{S}} &\geq \sum_{i=1}^{K-1} \int_0^1 \psi^{(N_i)T}(\alpha) \frac{1}{h_{N_i}} S^i \psi^{(N_i)} d\alpha \\
&= \sum_{i=1}^{K-1} \int_0^1 [\Phi_i - \sum_{p=0}^{N_i - 1} \Psi_p(\alpha)]^T \frac{1}{h_{N_i}} S^i [\Phi_i - \sum_{p=0}^{N_i-1} \Psi_p(\alpha)] d\alpha \\
&= \int_0^1 [\Phi - F\Psi(\alpha)]^T \bar{S} [\Phi - F\Psi(\alpha)] d\alpha. \qquad (7.76)
\end{aligned}
$$

Use (7.73), (7.74), (7.75) and (7.76) in (7.60), we have

$$
\begin{aligned}
& V(\phi) \\
\geq & \int_0^1 \begin{pmatrix} \phi^T(0) & \Phi^T & \Psi^T(\alpha) \end{pmatrix} \\
& \begin{pmatrix} P & \bar{Q} & \tilde{Q}^K \\ \bar{Q}^T & \bar{R} + \bar{S} & \hat{R}^K - \bar{S}F \\ \tilde{Q}^{KT} & (\hat{R}^K - \bar{S}F)^T & \tilde{R}^{KK} + \tilde{S}' + F^T \bar{S} F \end{pmatrix} \begin{pmatrix} \phi(0) \\ \Phi \\ \Psi(\alpha) \end{pmatrix} d\alpha.
\end{aligned}
$$

In view of the above and (7.70), the Lyapunov-Krasovskii functional condition (7.44) is satisfied. ■

It is easily seen that if Proposition 7.6 is applied to the case of single delay with uniform mesh ($h_p = h$, $p = 1, 2, ..., N$), it reduces to Proposition 5.20 in Chapter 5. Although the Lyapunov-Krasovskii functional condition for the nonuniform mesh case is no more complicated than the uniform mesh case, it is not the case for the Lyapunov-Krasovskii derivative condition as will be seen in the next subsection.

7.5.4 Lyapunov-Krasovskii derivative condition

After discretization, the derivative $\dot{V}$ expression is rather complicated. However, after tedious algebra, it is possible to consolidate to the form stated in the following Lemma:

Lemma 7.7 *Let Lyapunov-Krasovskii functional V be expressed in (7.30) to (7.38), with Q^K, S^K, R^{iK} and R^{KK} piecewise linear as expressed in (7.49) to (7.52). The derivative $\dot{V}$ along the trajectory of the system described by (7.28) to (7.29) satisfies*

$$\begin{aligned}
\dot{V}(\phi) &= -\hat{\phi}^T \Delta \hat{\phi} + 2\hat{\phi}^T \int_0^1 [D^s + (1-2\alpha)D^a]\tilde{\phi}(\alpha)d\alpha \\
&\quad - \int_0^1 \tilde{\phi}^T(\alpha) S_d^K \tilde{\phi}(\alpha) d\alpha \\
&\quad -[\int_0^1 \tilde{\phi}(\alpha)d\alpha]^T (R_{ds}^{KK} + R_{ds}^{\cdot K})[\int_0^1 \tilde{\phi}(\alpha)d\alpha] \\
&\quad - \int_0^1 [\int_0^\alpha \begin{pmatrix} \tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta) \end{pmatrix} \\
&\quad \begin{pmatrix} 0 & R_{da}^{KK} \\ R_{da}^{KK} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\phi}(\alpha) \\ \tilde{\phi}(\beta) \end{pmatrix} d\beta] d\alpha,
\end{aligned} \tag{7.77}$$

where

$$\Delta = \begin{pmatrix} \Delta_{00} & \Delta_{01} & \dots & \Delta_{0K} \\ \Delta_{10} & \Delta_{11} & \dots & \Delta_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{K0} & \Delta_{K1} & \dots & \Delta_{KK} \end{pmatrix}, \tag{7.78}$$

$$\begin{aligned}
\Delta_{00} &= -[PA_0 + A_0^T P + \sum_{i=1}^{K-1}(Q^i + Q^{iT} + S^i) \\
&\quad + Q_0^K + Q_0^{KT} + S_0^K],
\end{aligned} \tag{7.79}$$

$$\begin{aligned}
\Delta_{0i} &= Q^i - PA_i, \ \Delta_{ii} = S^i, \qquad 1 \le i \le K-1, \\
\Delta_{KK} &= S_N^K, \ \Delta_{0K} = Q_N^K - PA_K, \\
\Delta_{ij} &= 0, \qquad i \ne j, \ 1 \le i \le K, \ 1 \le j \le K, \\
\Delta_{ij} &= \Delta_{ji}^T, \qquad 0 \le i \le K, \, 0 \le j \le K,
\end{aligned} \tag{7.80}$$

$$
\begin{aligned}
D^s &= \begin{pmatrix} D^s_{01} & D^s_{02} & \dots & D^s_{0N} \\ D^s_{11} & D^s_{12} & \dots & D^s_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ D^s_{K1} & D^s_{K2} & \dots & D^s_{KN} \end{pmatrix}, \\
D^s_{0p} &= \sum_{j=M_p}^{K-1} h_p(A_0^T Q^j + \sum_{k=1}^{K-1} R^{kj} + R_0^{jKT}) \\
&\quad + \frac{h_p}{2}[A_0^T(Q_p^K + Q_{p-1}^K) + \sum_{k=1}^{K-1}(R_p^{kK} + R_{p-1}^{kK}) + (R_{0p}^{KK} + R_{0,p-1}^{KK})] \\
&\quad -(Q_{p-1}^K - Q_p^K), \\
D^s_{ip} &= \sum_{j=M_p}^{K-1} h_p(A_i^T Q^j - R^{ij}) + \frac{h_p}{2}[A_i^T(Q_p^K + Q_{p-1}^K) - (R_p^{iK} + R_{p-1}^{iK})], \\
D^s_{Kp} &= \sum_{j=M_p}^{K-1} h_p(A_K^T Q^j - R_N^{jKT}) \\
&\quad + \frac{h_p}{2}[A_K^T(Q_p^K + Q_{p-1}^K) - (R_{Np}^{KK} + R_{N,p-1}^{KK})], \\
&\qquad \textit{for } 1 \le i \le K-1,\ 1 \le p \le N,
\end{aligned}
$$

$$
\begin{aligned}
D^a &= \begin{pmatrix} D^a_{01} & D^a_{02} & \dots & D^a_{0N} \\ D^a_{11} & D^a_{12} & \dots & D^a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ D^a_{K1} & D^a_{K2} & \dots & D^a_{KN} \end{pmatrix}, \\
D^a_{0p} &= \frac{h_p}{2}[A_0^T(Q_p^K - Q_{p-1}^K) + \sum_{k=1}^{K-1}(R_p^{kK} - R_{p-1}^{kK}) + R_{0p}^{KK} - R_{0,p-1}^{KK}], \\
D^a_{ip} &= \frac{h_p}{2}[A_i^T(Q_p^K - Q_{p-1}^K) - (R_p^{iK} - R_{p-1}^{iK})], \\
D^a_{Kp} &= \frac{h_p}{2}[A_K^T(Q_p^K - Q_{p-1}^K) - (R_{Np}^{KK} - R_{N,p-1}^{KK})], \\
&\qquad \textit{for } 1 \le i \le K-1,\ 1 \le p \le N,
\end{aligned}
$$

$$
\begin{aligned}
S_d^K &= \operatorname{diag}\begin{pmatrix} S_{d1}^K & S_{d2}^K & \dots & S_{dN}^K \end{pmatrix}, \\
S_{dp}^K &= S_{p-1}^K - S_p^K, \qquad 1 \le p \le N,
\end{aligned}
$$

$$
\begin{aligned}
R_{ds}^{\cdot K} &= \begin{pmatrix} R_{ds11}^{\cdot K} & R_{ds12}^{\cdot K} & \dots & R_{ds1N}^{\cdot K} \\ R_{ds21}^{\cdot K} & R_{ds22}^{\cdot K} & \dots & R_{ds2N}^{\cdot K} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dsN1}^{\cdot K} & R_{dsN2}^{\cdot K} & \dots & R_{dsNN}^{\cdot K} \end{pmatrix}, \\
R_{dspq}^{\cdot K} &= \sum_{i=M_p}^{K-1} h_p(R_{q-1}^{iK} - R_q^{iK}) + \sum_{i=M_q}^{K-1} h_q(R_{p-1}^{iKT} - R_p^{iKT}),
\end{aligned}
$$

$$
\begin{aligned}
R_{ds}^{KK} &= \begin{pmatrix} R_{ds11}^{KK} & R_{ds12}^{KK} & \dots & R_{ds1N}^{KK} \\ R_{ds21}^{KK} & R_{ds22}^{KK} & \dots & R_{ds2N}^{KK} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dsN1}^{KK} & R_{dsN2}^{KK} & \dots & R_{dsNN}^{KK} \end{pmatrix}, \\
R_{dspq}^{KK} &= \frac{1}{2}[(h_p + h_q)(R_{p-1,q-1}^{KK} - R_{pq}^{KK}) + (h_p - h_q)(R_{p,q-1}^{KK} - R_{p-1,q}^{KK})],
\end{aligned}
$$

$$
\begin{aligned}
R_{da}^{KK} &= \begin{pmatrix} R_{da11}^{KK} & R_{da12}^{KK} & \dots & R_{da1N}^{KK} \\ R_{da21}^{KK} & R_{da22}^{KK} & \dots & R_{da2N}^{KK} \\ \vdots & \vdots & \ddots & \vdots \\ R_{daN1}^{KK} & R_{daN2}^{KK} & \dots & R_{daNN}^{KK} \end{pmatrix}, \\
R_{dapq}^{KK} &= \frac{1}{2}(h_p - h_q)(R_{p-1,q-1}^{KK} - R_{p-1,q}^{KK} - R_{p,q-1}^{KK} + R_{pq}^{KK}), \quad (7.81)
\end{aligned}
$$

$$
\hat{\phi} = \begin{pmatrix} \phi(-r_0) \\ \phi(-r_1) \\ \vdots \\ \phi(-r_K) \end{pmatrix}, \qquad \tilde{\phi}(\alpha) = \begin{pmatrix} \phi^{(1)}(\alpha) \\ \phi^{(2)}(\alpha) \\ \vdots \\ \phi^{(N)}(\alpha) \end{pmatrix} = \begin{pmatrix} \phi(\theta_1 + \alpha h_1) \\ \phi(\theta_2 + \alpha h_2) \\ \vdots \\ \phi(\theta_N + \alpha h_N) \end{pmatrix}.
$$

Proof. Divide all integrations on the delay intervals $[-r_i, 0]$ into integration on segments $[\theta_p, \theta_{p-1}]$, with a change of variable such as

$$
\xi = \theta_p + \alpha h_p, \qquad p = 1, 2, ..., N,
$$

within each segment, one obtains

$$
\dot{V}(\phi) = -\dot{V}_\Delta + 2\dot{V}_\Pi - \dot{V}_{S^K} - 2\dot{V}_{R^{\cdot K}} - \dot{V}_{R^{KK}}, \qquad (7.82)
$$

where

$$\dot{V}_{\Delta} = \hat{\phi}^T \Delta \hat{\phi}, \tag{7.83}$$

$$\dot{V}_{\Pi} = \sum_{i=0}^{K} \phi^T(-r_i) \sum_{j=1}^{K} \sum_{p=1}^{N_j} \int_0^1 \Pi^{ij}(\theta_p + \alpha h_p)\phi^{(p)}(\alpha) h_p d\alpha, \tag{7.84}$$

$$\dot{V}_{S^K} = \sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha) S_{dp}^K \phi^{(p)}(\alpha) d\alpha, \tag{7.85}$$

$$\begin{aligned} \dot{V}_{R^{\cdot K}} = & \sum_{i=1}^{K-1} \sum_{p=1}^{N_i} \int_0^1 \phi^{(p)T}(\alpha) \\ & [\sum_{q=1}^{N} \int_0^1 (R_{q-1}^{iK} - R_q^{iK})\phi^{(q)}(\beta) d\beta] h_p d\alpha, \end{aligned} \tag{7.86}$$

$$\begin{aligned} \dot{V}_{R^{KK}} = & \sum_{p=1}^{N} \sum_{q=1}^{N} \int_0^1 \phi^{(p)T}(\alpha) \\ & [\int_0^1 (\frac{1}{h_p}\frac{\partial}{\partial \alpha} + \frac{1}{h_q}\frac{\partial}{\partial \beta}) R^{KK}(\theta_p + \alpha h_p, \theta_q + \beta h_q) \\ & \phi^{(q)}(\beta) h_q d\beta] h_p d\alpha. \end{aligned} \tag{7.87}$$

Exchange the order of summation between indices j and p, then direct calculation gives

$$\begin{aligned} \dot{V}_{\Pi} &= \sum_{i=0}^{K} \sum_{p=1}^{N} \phi^T(-r_i) \int_0^1 [\sum_{j=M_p}^{K} h_p \Pi^{ij}(\theta_p + \alpha h_p)] \phi^{(p)}(\alpha) d\alpha \\ &= \sum_{i=0}^{K} \sum_{p=1}^{N} \phi^T(-r_i) \int_0^1 [D_{ip}^s + (1-2\alpha) D_{ip}^a] \phi^{(p)}(\alpha) d\alpha. \end{aligned} \tag{7.88}$$

Similarly, exchange the order of summation between indices i and p

$$\begin{aligned} \dot{V}_{R^{\cdot K}} &= \sum_{p=1}^{N} \sum_{i=M_p}^{K-1} \int_0^1 \phi^{(p)T}(\alpha) [\sum_{q=1}^{N} \int_0^1 (R_{q-1}^{iK} - R_q^{iK}) \phi^{(q)}(\beta) d\beta] h_p d\alpha \\ &= \sum_{p=1}^{N} \sum_{q=1}^{N} [\int_0^1 \phi^{(p)}(\alpha) d\alpha]^T [\sum_{i=M_p}^{K-1} h_p (R_{q-1}^{iK} - R_q^{iK})] [\int_0^1 \phi^{(q)}(\alpha) d\alpha] \\ &= [\int_0^1 \hat{\phi}(\alpha) d\alpha]^T R_d^{\cdot K} [\int_0^1 \hat{\phi}(\alpha) d\alpha], \end{aligned}$$

where

$$R_d^{\cdot K} = \begin{pmatrix} R_{d11}^{\cdot K} & R_{d12}^{\cdot K} & & R_{d1N}^{\cdot K} \\ R_{d21}^{\cdot K} & R_{d22}^{\cdot K} & & R_{d2N}^{\cdot K} \\ & & & \\ R_{dN1}^{\cdot K} & R_{dN2}^{\cdot K} & & R_{dNN}^{\cdot K} \end{pmatrix},$$

$$R_{dpq}^{\cdot K} = \sum_{i=M_p}^{K-1} h_p(R_{q-1}^{iK} - R_q^{iK}).$$

Using $R_{ds}^{\cdot K} = \frac{1}{2}[R_d^{\cdot K} + R_d^{\cdot KT}]$, We can further write

$$\dot{V}_{R^{\cdot K}} = \frac{1}{2}[\int_0^1 \hat{\phi}(\alpha)d\alpha]^T R_{ds}^{\cdot K}[\int_0^1 \hat{\phi}(\alpha)d\alpha]. \tag{7.89}$$

For $\dot{V}_{R^{KK}}$, we can break the integration over

$$\{(\alpha,\beta) \mid 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1\}$$

into two triangular regions

$$\begin{aligned} \{(\alpha,\beta) \mid 0 &\leq \alpha \leq 1, 0 \leq \beta \leq \alpha\}, \\ \{(\alpha,\beta) \mid 0 &\leq \alpha \leq 1, \alpha \leq \beta \leq 1\}, \end{aligned}$$

to write

$$\begin{aligned} \dot{V}_{R^{KK}} &= 2\sum_{p=1}^N\sum_{q=1}^N \int_0^1 \phi^{(p)T}(\alpha)\int_0^\alpha [h_q(R_{p-1,q}^{KK} - R_{pq}^{KK}) \\ &\quad + h_p(R_{p-1,q-1}^{KK} - R_{p-1,q}^{KK})]\phi^{(q)}(\beta)d\beta d\alpha \\ &= 2\int_0^1 \tilde{\phi}^T(\alpha)[\int_0^\alpha (R_{ds}^{KK} + R_{da}^{KK})\tilde{\phi}(\beta)d\beta]d\alpha \\ &= \dot{V}_{R_s^{KK}} + \dot{V}_{R_a^{KK}}, \end{aligned} \tag{7.90}$$

where

$$\dot{V}_{R_s^{KK}} = 2\int_0^1 [\int_0^\alpha \hat{\phi}^T(\alpha)R_{ds}^{KK}\hat{\phi}(\beta)d\beta]d\alpha, \tag{7.91}$$

$$\dot{V}_{R_a^{KK}} = 2\int_0^1 [\int_0^\alpha \hat{\phi}^T(\alpha)R_{da}^{KK}\hat{\phi}(\beta)d\beta]d\alpha.$$

Exchange the order of integration

$$\dot{V}_{R_s^{KK}} = 2\int_0^1 [\int_{1-\beta}^1 \hat{\phi}^T(\alpha)R_{ds}^{KK}\hat{\phi}(\beta)d\alpha]d\beta.$$

Since R_{ds}^{KK} is symmetric, take transpose yields

$$\begin{aligned}\dot{V}_{R_s^{KK}} &= 2\int_0^1 [\int_{1-\beta}^1 \hat{\phi}^T(\beta) R_{ds}^{KK} \hat{\phi}(\alpha) d\alpha] d\beta \\ &= 2\int_0^1 [\int_{1-\alpha}^1 \hat{\phi}^T(\alpha) R_{ds}^{KK} \hat{\phi}(\beta) d\beta] d\alpha. \end{aligned} \tag{7.92}$$

Adding (7.91) and (7.92) yields

$$\dot{V}_{R_s^{KK}} = \int_0^1 [\int_0^1 \hat{\phi}^T(\alpha) R_{ds}^{KK} \hat{\phi}(\beta) d\beta] d\alpha.$$

For $\dot{V}_{R_a^{KK}}$, one can write

$$\begin{aligned} &\dot{V}_{R_a^{KK}} \\ =& \int_0^1 [\int_0^\alpha \hat{\phi}^T(\alpha) R_{da}^{KK} \hat{\phi}(\beta) d\beta] d\alpha \\ &+ \int_0^1 [\int_0^\alpha \hat{\phi}^T(\beta) R_{da}^{KKT} \hat{\phi}(\alpha) d\beta] d\alpha \\ =& \int_0^1 [\int_0^\alpha \begin{pmatrix} \hat{\phi}^T(\alpha) & \hat{\phi}^T(\beta) \end{pmatrix} \\ & \begin{pmatrix} 0 & R_{da}^{KK} \\ R_{da}^{KKT} & 0 \end{pmatrix} \begin{pmatrix} \hat{\phi}(\alpha) \\ \hat{\phi}(\beta) \end{pmatrix} d\beta] d\alpha. \end{aligned} \tag{7.93}$$

Use (7.83), (7.88), (7.85), (7.90) and (7.93) in (7.82). ■

Notice, the last term in (7.77) arises due to the nonuniform mesh. If uniform mesh is used (which is not possible for the incommensurate delay case, and is not practical in the case of commensurate delays with small common factor), it is easily seen that $R_{da}^{KK} = 0$, and the last term in (7.77) vanishes.

With the above Lemma, we can obtain the Lyapunov-Krasovskii derivative condition in a similar manner to the single delay case.

Proposition 7.8 *Let Lyapunov-Krasovskii functional V be expressed in (7.30) to (7.38), with Q^K, S^K, R^{iK} and R^{KK} piecewise linear as expressed in (7.49) to (7.52). The derivative $\dot{V}$ along the trajectory of system described by (7.28) to (7.29) satisfies the Lyapunov-Krasovskii derivative condition (7.45) if there exists a real matrix $W = W^T$ such that*

$$\begin{pmatrix} \Delta & -D^s & -D^a \\ -D^{sT} & S_d^K - W + R_{ds}^{KK} + R_{ds}^{\cdot K} & 0 \\ -D^{aT} & 0 & 3(S_d^K - W) \end{pmatrix} > 0 \tag{7.94}$$

$$\textit{for all } (A_0, A_1, ..., A_K) \in \Omega$$

and

$$\begin{pmatrix} W & R_{da}^{KK} \\ R_{da}^{KK} & W \end{pmatrix} > 0 \tag{7.95}$$

Proof. From the Lemma 7.7, it is not difficult to verify

$$\begin{aligned}
&\dot{V}(\phi) \\
= \ & -\int_0^1 \begin{pmatrix} \hat{\phi}^T[D^s + (1-2\alpha)D^a] & \tilde{\phi}^T(\alpha) \end{pmatrix} \\
&\begin{pmatrix} U & -I \\ -I & S_d^K - W \end{pmatrix} \begin{pmatrix} [D^s + (1-2\alpha)D^a]^T\hat{\phi} \\ \tilde{\phi}(\alpha) \end{pmatrix} d\alpha \\
&+\hat{\phi}^T(-\Delta + D^sUD^{sT} + \frac{1}{3}D^aUD^{aT})\hat{\phi} \\
&-[\int_0^1 \tilde{\phi}(\alpha)d\alpha]^T(R_{ds}^{KK} + R_{ds}^{\cdot K})[\int_0^1 \tilde{\phi}(\alpha)d\alpha] \\
&-\int_0^1 [\int_0^\alpha \begin{pmatrix} \tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta) \end{pmatrix} \begin{pmatrix} W & R_{da}^{KK} \\ R_{da}^{KK} & W \end{pmatrix} \begin{pmatrix} \tilde{\phi}(\alpha) \\ \tilde{\phi}(\beta) \end{pmatrix} d\beta]d\alpha
\end{aligned}$$

Given

$$\begin{pmatrix} U & -I \\ -I & S_d^K - W \end{pmatrix} > 0 \tag{7.96}$$

one may use the Jensen inequality (Proposition **??** of the Appendix B) to obtain

$$\begin{aligned}
&\dot{V}(\phi) \\
\leq \ & -\int_0^1 \begin{pmatrix} \hat{\phi}^T[D^s + (1-2\alpha)D^a] & \tilde{\phi}^T(\alpha) \end{pmatrix} d\alpha \\
&\begin{pmatrix} U & -I \\ -I & S_d^K - W \end{pmatrix} \int_0^1 \begin{pmatrix} [D^s + (1-2\alpha)D^a]^T\hat{\phi} \\ \tilde{\phi}(\alpha) \end{pmatrix} d\alpha \\
&+\hat{\phi}^T(-\Delta + D^sUD^{sT} + \frac{1}{3}D^aUD^{aT})\hat{\phi} \\
&-[\int_0^1 \tilde{\phi}(\alpha)d\alpha]^T(R_{ds}^{KK} + R_{ds}^{\cdot K})[\int_0^1 \tilde{\phi}(\alpha)d\alpha] \\
&-\int_0^1 [\int_0^\alpha \begin{pmatrix} \tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta) \end{pmatrix} \begin{pmatrix} W & R_{da}^{KK} \\ R_{da}^{KK} & W \end{pmatrix} \begin{pmatrix} \tilde{\phi}(\alpha) \\ \tilde{\phi}(\beta) \end{pmatrix} d\beta]d\alpha \\
= \ & -\begin{pmatrix} \hat{\phi}^T & \int_0^1 \tilde{\phi}^T(\alpha)d\alpha \end{pmatrix} \\
&\begin{pmatrix} \Delta - \frac{1}{3}D^aUD^{aT} & -D^s \\ -D^{sT} & S_d^K - W + R_{ds}^{KK} + R_{ds}^{\cdot K} \end{pmatrix} \begin{pmatrix} \hat{\phi} \\ \int_0^1 \tilde{\phi}(\alpha)d\alpha \end{pmatrix} \\
&-\int_0^1 [\int_0^\alpha \begin{pmatrix} \tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta) \end{pmatrix} \begin{pmatrix} W & R_{da}^{KK} \\ R_{da}^{KK} & W \end{pmatrix} \begin{pmatrix} \tilde{\phi}(\alpha) \\ \tilde{\phi}(\beta) \end{pmatrix} d\beta]d\alpha
\end{aligned}$$

Therefore, the Lyapunov-Krasovskii derivative condition is satisfied if (7.96), (7.95) and

$$\begin{pmatrix} \Delta - \frac{1}{3}D^a U D^{aT} & -D^s \\ -D^{sT} & S_d^K - W + R_{ds}^{KK} + R_{ds}^{\cdot K} \end{pmatrix} > 0 \tag{7.97}$$

are satisfied. We may eliminate the variable U from (7.95) and (7.97) using Proposition **??** of the Appendix B to obtain (7.94). ■

For the case of uniform mesh, since $R_{da}^{KK} = 0$, one can choose $W = 0$, and (7.95) can be omitted, in which case, (7.94) reduces to (5.167) if applied to the single delay case.

7.5.5 Stability condition and examples

Combining Proposition 7.6 and Proposition 7.8, we can obtain the following stability condition:

Proposition 7.9 *The system described by (7.28) to (7.29) is asymptotically stable if there exist n by n real matrices $P = P^T$, Q^i, Q_p^K, $S^i = S^{iT}$, $S_p^K = S_p^K$, $R^{ij} = R^{jiT}$, R_p^{iK}, $R_{pq}^{KK} = R_{qp}^{KKT}$; $i, j = 1, 2, ..., K-1$; $p, q = 0, 1, ..., N$; and Nn by Nn real matrix $W = W^T$, such that (7.70) and (7.95) are satisfied and (7.94) is satisfied for all $(A_0, A_1, ..., A_K) \in \Omega$, with the notations defined in (7.61) to (7.62), (7.71) to (7.72) and (7.78) to (7.81).*

Proof. According to Propositions 7.6 and 7.8, the system satisfies both Lyapunov-Krasovskii funtional condition and Lyapunov-Krasovskii derivative condition, and therefore is asymptotically stable, if (7.68), (7.69), (7.70), (7.94) and (7.95) are satisfied. It remains to be shown that (7.68) and (7.69) are not needed.

From (7.95), one concludes that $W > 0$, which together with the (3,3) entries of the matrix in (7.94) implies (7.69). Examining the diagonal entries of Δ of the matrix in (7.94), one may conclude that (**??**) is implied by (7.94). ■

To illustrate the effectiveness of the method, the following examples are presented. The first example gives a sense of how close to analytical limit can be reached with rather small N.

Example 7.1 *Consider the following uncertainty-free time-delay system*

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} [0.05x(t-0.5r) + 0.95x(t-r)] .$$

The maximum time-delay for stability can be analytically calculated as

$$r_{\max} = 8.5976.$$

Using Proposition 7.9, the maximum time-delay for stability is estimated using different N_{d1} and N_{d2} as follows

	N_{d2}	
N_{d1}	*1*	*2*
1	8.585	8.586
2	8.594	8.596

It is clear that the results are very close to the analytical limit even for rather small N.

The next example illustrates the case of polytopic uncertainty and incommensurate delays.

Example 7.2 *This example consider the uncertain system*

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -2+\rho(t) & \rho(t) \\ \rho(t) & -0.9+\rho(t) \end{pmatrix} x(t) \\ &+0.15 \begin{pmatrix} -1+\rho(t) & 0 \\ -1 & -1-\rho(t) \end{pmatrix} x(t-\frac{r}{\sqrt{3}}) \\ &+0.85 \begin{pmatrix} -1-\rho(t) & \rho(t) \\ -1-\rho(t) & -1-\rho(t) \end{pmatrix} x(t-r),\end{aligned}$$

where ρ is a time-varying parameter satisfying

$$|\rho(t)| \leq 0.05, \text{ for all } t.$$

Using Proposition 7.9 with $N_{d1} = N_{d2} = 1$, it is concluded that the system is stable for $r \leq r_{\max} = 5.70$. For $N_{d1} = N_{d2} = 2$, the result is improved to $r_{\max} = 5.75$.

The next example considers the case where the system is unstable when the delay vanishes.

Example 7.3 *Consider the system*

$$\ddot{x}(t) - 0.1\dot{x}(t) + x(t) + x(t-r/2) - x(t-r) = 0.$$

This system is unstable for $r = 0$. For small r, the last two terms can be viewed as a finite difference, approximating $\frac{r\dot{x}(t)}{2}$, and therefore, as r increases, it may improve stability. However, this approximation may not be valid for large r. Write the system in the state space form

$$\begin{aligned}\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0.1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-r/2) \\ x_2(t-r/2) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-r) \\ x_2(t-r) \end{pmatrix}.\end{aligned}$$

The stability criterion in Corollary 7.9 is used to find an interval $[r_{\min}, r_{\max}]$ *of* r *for which the system is guaranteed to be stable. A simple search with uniform increment of* 0.1 *is conducted. Then, a bisection near the upper and lower bound is conducted to find* $r_{\min}$ *and* $r_{\max}$ *to within* 0.001 *for a given* N_{d1} *and* N_{d2}. *Using* $N_{d1} = N_{d2} = 1$, *it was estimated that* $[r_{\min}, r_{\max}] = [0.204, 1.350]$. *For* $N_{d1} = N_{d2} = 2$, *the interval can be enlarged to* $[0.203, 1.372]$. *It can be verified that for* $r = 0.2025$, *the system has an imaginary pole at* $1.0077i$, *and for* $r = 1.3723$, *the system has an imaginary pole at* $1.3786i$. *Therefore, the exact stability interval is* $(0.2025, 1.3723)$. *Therefore, these estimates are very accurate.*

7.6 Discretized Lyapunov functional method for systems with distributed delays

We will only consider the case where the coefficient matrix for the distributed delay is piecewise constant. The Lyapunov functional used is identical to the case of single delay. However, similar to the case of multiple delays, it is important that the discretization mesh is compatible to the delay, *i.e.*, the points where the coefficient matrix changes have to be selected as the dividing points. This again often dictates the usage of nonuniform mesh.

7.6.1 Problem statement

Consider the system with distributed delays

$$\dot{x}(t) = A_0(t)x(t) + \int_{-r}^{0} A(t,\theta)x(t+\theta)d\theta, \tag{7.98}$$

where the coefficient matrix A is piecewise independent of θ

$$A(t,\theta) = A^i(t), \ -r_i \le \theta < -r_{i-1}, \ i = 1, 2, ..., K, \tag{7.99}$$

with r_i's satisfy

$$0 = r_0 < r_1 < r_2 < ... < r_K = r. \tag{7.100}$$

In other words,

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{K} A^i(t) \int_{-r_i}^{-r_{i-1}} x(t+\theta)d\theta. \tag{7.101}$$

The system matrices are uncertain, and are bounded by a known compact set

$$(A_0(t), A^1(t), A^2(t), ..., A^K(t)) \in \Omega \text{ for all } t \ge 0. \tag{7.102}$$

For the sake of simplicity, we will often suppress the time dependence of the sysmtem matrices, and write A_0, A^i, and $A(\theta)$ instead of $A_0(t)$, $A^i(t)$ and $A(t,\theta)$.

A general quadratic Lyapunov functional will again be used

$$\begin{aligned} V(\phi) &= \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\quad + \int_{-r}^{0}[\int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta]d\xi \\ &\quad + \int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi, \end{aligned} \tag{7.103}$$

where

$$P = P^T \in \mathsf{R}^{n\times n}, \tag{7.104}$$

and for all $-r \le \xi \le 0$ and $-r \le \eta \le 0$,

$$Q(\xi) \in \mathsf{R}^{n\times n}, \tag{7.105}$$

$$S(\xi) = S^T(\xi) \in \mathsf{R}^{n\times n}, \tag{7.106}$$

$$R(\xi,\eta) = R^T(\eta,\xi) \in \mathsf{R}^{n\times n}. \tag{7.107}$$

Take derivative of V with respect to time along the trajectory of system (7.98), integration by parts when necessary similar to the single delay case, one can write $\dot{V}$ as a quadratic functional of ϕ as follows:

$$\begin{aligned} &\dot{V}(\phi) \\ = &-\begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix}\begin{pmatrix} \Delta_{00} & Q(-r) \\ Q^T(-r) & S(-r) \end{pmatrix}\begin{pmatrix} \phi(0) \\ \phi(-r) \end{pmatrix} \\ &+2\begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix}\int_{-r}^{0}\begin{pmatrix} \Gamma^0(\xi) \\ \Gamma^1(\xi) \end{pmatrix}\phi(\xi)d\xi \\ &-\int_{-r}^{0}\phi^T(\xi)\frac{dS(\xi)}{d\xi}\phi(\xi)d\xi \\ &-\int_{-r}^{0}d\xi\int_{-r}^{0}\phi^T(\xi)[\frac{\partial R(\xi,\eta)}{\partial\xi}+\frac{\partial R(\xi,\eta)}{\partial\eta} \\ &-A^T(\xi)Q(\eta)-Q^T(\xi)A(\eta)]\phi(\eta)d\eta, \end{aligned} \tag{7.108}$$

where

$$\begin{aligned} \Delta_{00} &= -PA_0 - A_0^TP - Q(0) - Q^T(0) - S(0), \\ \Gamma^0(\xi) &= A_0^TQ(\xi) + PA(\xi) - \frac{dQ(\xi)}{d\xi} + R(0,\xi), \\ \Gamma^1(\xi) &= -R(-r,\xi). \end{aligned}$$

Recall that the system is asymptotically stable if there exists a Lyapunov-Krasovskii functional (7.103) which satisfies

$$V(\phi) \geq \varepsilon ||\phi(0)||^2, \tag{7.109}$$

and its derivative satisfies

$$\dot{V}(\phi) \leq -\varepsilon ||\phi(0)||^2, \tag{7.110}$$

for some $\varepsilon > 0$. The conditions are necessary and sufficient if there is no uncertainty in the system.

7.6.2 Discretization

Similar to the case of multiple delays, it is again important to allow nonuniform mesh in general, such that $A(\xi)$ is independent of ξ in each segment. Specifically, let $0 = \theta_0 > \theta_1 > ... > \theta_N = -r$ be chosen such that

$$-r_i = \theta_{N_i},\ i = 0, 1, 2, ..., K,$$

thus the intervals $[-r_i, 0]$ is divided into N_i segments $[\theta_p, \theta_{p-1}]$ of length

$$h_p = \theta_{p-1} - \theta_p,$$

$p = 1, 2, ..., N_i$. Adopt the convention

$$N_0 = 0.$$

Then

$$0 = N_0 < N_1 < N_2 < ... < N_K = N.$$

The parameters of Lyapunov-Krasovskii functional are again chosen to be piecewise linear, *i.e.*, for $0 \leq \alpha \leq 1$ and $p = 1, 2, ..., N$

$$\begin{aligned} Q(\theta_p + \alpha h_p) &= Q^{(p)}(\alpha) = (1-\alpha)Q_p + \alpha Q_{p-1}, && (7.111) \\ S(\theta_p + \alpha h_p) &= S^{(p)}(\alpha) = (1-\alpha)S_p + \alpha S_{p-1}, && (7.112) \end{aligned}$$

and for $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $p = 1, 2, ..., N$, $q = 1, 2, ..., N$,

$$\begin{aligned} & R(\theta_p + \alpha h_p, \theta_q + \beta h_q) \\ = \ & R^{(pq)}(\alpha, \beta) \\ = \ & \begin{cases} (1-\alpha)R_{pq} + \beta R_{p-1,q-1} + (\alpha - \beta)R_{p-1,q}, & \alpha \geq \beta; \\ (1-\beta)R_{pq} + \alpha R_{p-1,q-1} + (\beta - \alpha)R_{p,q-1}, & \alpha < \beta. \end{cases} \end{aligned} \tag{7.113}$$

7.6.3 Lyapunov-Krasovskii functional condition

Even though the mesh is not uniform, the Lyapunov-Krasovskii functional condition has a similar form as the uniform mesh case presented in the single delay case. Specifically, let

$$\phi^{(p)}(\alpha) = \phi(\theta_p + \alpha h_p). \tag{7.114}$$

We have

Proposition 7.10 *The Lyapunov-Krasovskii functional $V(\phi)$ as expressed in (7.103) to (7.107), with Q, S and R piecewise linear as expressed in (7.111), to (7.113), satisfies*

$$\begin{aligned} & V(\phi) \\ = & \int_0^1 \begin{pmatrix} \phi^T(0) & \Psi^T(\alpha) \end{pmatrix} \begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} \phi(0) \\ \Psi(\alpha) \end{pmatrix} d\alpha \\ & + \sum_{p=1}^{N} \int_0^1 \phi^{(p)T}(\alpha) S^{(p)}(\alpha) \phi^{(p)}(\alpha) h_p d\alpha \qquad (7.115) \\ \geq & \int_0^1 \begin{pmatrix} \phi^T(0) & \Psi^T(\alpha) \end{pmatrix} \begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R}+\tilde{S} \end{pmatrix} \begin{pmatrix} \phi(0) \\ \Psi(\alpha) \end{pmatrix} d\alpha \qquad (7.116) \end{aligned}$$

if

$$S_p > 0, \qquad p = 0, 1, ..., N. \tag{7.117}$$

Therefore the Lyapunov-Krasovskii functional condition (7.109) is satisfied if (7.117) and

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R}+\tilde{S} \end{pmatrix} > 0 \tag{7.118}$$

are satisfied, where

$$\begin{aligned} \tilde{Q} &= \begin{pmatrix} Q_0 & Q_1 & ... & Q_N \end{pmatrix}, & (7.119) \\ \tilde{R} &= \begin{pmatrix} R_{00} & R_{01} & ... & R_{0N} \\ R_{10} & R_{11} & ... & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & ... & R_{NN} \end{pmatrix}, & (7.120) \\ \tilde{S} &= \mathrm{diag}\begin{pmatrix} \frac{1}{\tilde{h}_0} S_0 & \frac{1}{\tilde{h}_1} S_1 & ... & \frac{1}{\tilde{h}_N} S_N \end{pmatrix}, & (7.121) \\ \tilde{h}_p &= \max\{h_p, h_{p+1}\},\ p = 1, 2, ..., N-1, & (7.122) \\ \tilde{h}_0 &= h_1, & (7.123) \\ \tilde{h}_N &= h_N, & (7.124) \end{aligned}$$

and

$$\Psi(\alpha) = \begin{pmatrix} \Psi_{(0)}(\alpha) \\ \Psi_{(1)}(\alpha) \\ \Psi_{(2)}(\alpha) \\ \vdots \\ \Psi_{(N-1)}(\alpha) \\ \Psi_{(N)}(\alpha) \end{pmatrix} = \begin{pmatrix} \psi_{(1)}(\alpha) \\ \psi_{(2)}(\alpha) + \psi^{(1)}(\alpha) \\ \psi_{(3)}(\alpha) + \psi^{(2)}(\alpha) \\ \vdots \\ \psi_{(N)}(\alpha) + \psi^{(N-1)}(\alpha) \\ \psi^{(N)}(\alpha) \end{pmatrix}, \tag{7.125}$$

$$\begin{aligned} \psi^{(p)}(\alpha) &= h_p \int_0^\alpha \phi^{(p)}(\beta)d\beta, & (7.126) \\ \psi_{(p)}(\alpha) &= h_p \int_\alpha^1 \phi^{(p)}(\beta)d\beta, & (7.127) \\ & \quad p = 1, 2, ..., N. \end{aligned}$$

Proof. The proof for (7.115) is similar to that of (5.133) in Proposition 5.20 for the single delay case. To prove (7.116), we can bound the last term of (7.115) as follows:

$$\begin{aligned} & \sum_{p=1}^N V_{S^p} \triangleq \sum_{p=1}^N \int_0^1 \phi^{(p)T}(\alpha) S^{(p)}(\alpha) \phi^{(p)}(\alpha) h_p d\alpha \\ = & \sum_{p=1}^N \int_0^1 \phi^{(p)T}(\alpha)[(1-\alpha)S_p + \alpha S_{p-1}]\phi^{(p)}(\alpha) h_p d\alpha \\ = & \int_0^1 \alpha h_1 \phi^{(1)T}(\alpha) \frac{1}{h_1} S_0 h_1 \phi^{(1)}(\alpha) d\alpha \\ & + \int_0^1 (1-\alpha) h_N \phi^{(N)T}(\alpha) \frac{1}{h_N} S_N h_N \phi^{(N)}(\alpha) d\alpha \\ & + \sum_{p=1}^{N-1} \int_0^1 [\alpha h_{p+1} \phi^{(p+1)T}(\alpha) \frac{1}{h_{p+1}} S_p h_{p+1} \phi^{(p+1)}(\alpha) \\ & + (1-\alpha) h_p \phi^{(p)T}(\alpha) \frac{1}{h_p} S_p h_p \phi^{(p)}(\alpha)] d\alpha \\ \geq & \int_0^1 \alpha h_1 \phi^{(1)T}(\alpha) \frac{1}{\tilde{h}_0} S_0 h_1 \phi^{(1)}(\alpha) d\alpha \\ & + \int_0^1 (1-\alpha) h_N \phi^{(N)T}(\alpha) \frac{1}{\tilde{h}_N} S_N h_N \phi^{(N)}(\alpha) d\alpha \\ & + \sum_{p=1}^{N-1} \int_0^1 [\alpha h_{p+1} \phi^{(p+1)T}(\alpha) \frac{1}{\tilde{h}_p} S_p h_{p+1} \phi^{(p+1)}(\alpha) \\ & + (1-\alpha) h_p \phi^{(p)T}(\alpha) \frac{1}{\tilde{h}_p} S_p h_p \phi^{(p)}(\alpha)] d\alpha. \end{aligned}$$

In view of (7.117), we can use the quadratic inequalities (Corollary **??** in the Appendix) in each term above to obtain

$$\sum_{p=1}^{N} V_{S^p} \geq \int_0^1 \Psi^T(\alpha)\tilde{S}\Psi(\alpha)d\alpha,$$

from which (7.116) follows. (7.118) is clearly implied by (7.116). ■

The above proposition can also be considered as the special case of Lemma 7.5 and Proposition 7.6.

7.6.4 Lyapunov-Krasovskii derivative condition

Similar to the case of multiple delays, it is useful to define

$$M_p = \min\{i \mid N_i \geq p\}. \tag{7.128}$$

We have

$$A(\xi) = A^{M_p} \text{ independent of } \xi \text{ within } \theta_p \leq \xi < \theta_{p-1}.$$

We will write

$$A_p = A^{M_p} \tag{7.129}$$

for the sake of convenience. Then, after the discretization, dividing the integration interval $[-r, 0]$ into N segments $[\theta_p, \theta_{p-1}]$, $p = 1, 2, ..., N$, the derivative $\dot{V}$ of the Lyapunov-Krasovskii functional V along the trajectory of the system described by (7.98) to (7.102) may be written as

$$\begin{aligned}
\dot{V}(\phi) \;\; = \;\; & -\phi_{0r}^T \Delta \phi_{0r} + 2\phi_{0r}^T \int_0^1 [D^s + (1-2\alpha)D^a]\hat{\phi}(\alpha)d\alpha \\
& - \int_0^1 \hat{\phi}^T(\alpha) S_d \hat{\phi}(\alpha) d\alpha \\
& + 2(\int_0^1 \hat{\phi}(\alpha)d\alpha)^T \int_0^1 [E^s + (1-2\alpha)E^a]\hat{\phi}(\alpha)d\alpha \\
& - (\int_0^1 \hat{\phi}(\alpha)d\alpha)^T R_{ds} (\int_0^1 \hat{\phi}(\alpha)d\alpha) \\
& - \int_0^1 [\int_0^\alpha \begin{pmatrix} \hat{\phi}^T(\alpha) & \hat{\phi}^T(\beta) \end{pmatrix} \\
& \begin{pmatrix} 0 & R_{da} \\ R_{da}^T & 0 \end{pmatrix} \begin{pmatrix} \hat{\phi}(\alpha) \\ \hat{\phi}(\beta) \end{pmatrix} d\beta] d\alpha,
\end{aligned} \tag{7.130}$$

where

$$\Delta = \begin{pmatrix} -PA_0 - A_0^T P - Q_0 - Q_0^T - S_0 & Q_N \\ Q_N^T & S_N \end{pmatrix}, \tag{7.131}$$

$$D^s = \begin{pmatrix} D^s_{01} & D^s_{02} & \dots & D^s_{0N} \\ D^s_{11} & D^s_{12} & \dots & D^s_{1N} \end{pmatrix}, \tag{7.132}$$

$$D^a = \begin{pmatrix} D^a_{01} & D^a_{02} & \dots & D^a_{0N} \\ D^a_{11} & D^a_{12} & \dots & D^a_{1N} \end{pmatrix}, \tag{7.133}$$

$$\begin{aligned} D^s_{ip} &= \frac{h_p}{2}[\Gamma^i(\theta_p + 0) + \Gamma^i(\theta_{p-1} - 0)], \\ D^a_{ip} &= \frac{h_p}{2}[\Gamma^i(\theta_p + 0) - \Gamma^i(\theta_{p-1} - 0)], \end{aligned}$$

or more explicitly

$$\begin{aligned} D^s_{0p} &= \frac{h_p}{2}[A_0^T(Q_p + Q_{p-1}) + 2PA_p + (R_{0p} + R_{0,p-1})] \\ &\quad -(Q_{p-1} - Q_p), && (7.134) \\ D^s_{1p} &= -\frac{h_p}{2}[R_{Np} + R_{N,p-1}], && (7.135) \\ D^a_{0p} &= \frac{h_p}{2}[A_0^T(Q_p - Q_{p-1}) + (R_{0p} - R_{0,p-1})], && (7.136) \\ D^a_{1p} &= \frac{h_p}{2}[R_{N,p-1} - R_{Np}], && (7.137) \end{aligned}$$

$$\begin{aligned} S_d &= \operatorname{diag}\begin{pmatrix} S_{d1} & S_{d2} & \dots & S_{dN} \end{pmatrix}, && (7.138) \\ S_{dp} &= S_{p-1} - S_p, && (7.139) \end{aligned}$$

$$E^s = \begin{pmatrix} E^s_{11} & E^s_{12} & \dots & E^s_{1N} \\ E^s_{21} & E^s_{22} & \dots & E^s_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ E^s_{N1} & E^s_{N2} & \dots & E^s_{NN} \end{pmatrix}, \tag{7.140}$$

$$E^a = \begin{pmatrix} E^a_{11} & E^a_{12} & \dots & E^a_{1N} \\ E^a_{21} & E^a_{22} & \dots & E^a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ E^a_{N1} & E^a_{N2} & \dots & E^a_{NN} \end{pmatrix}, \tag{7.141}$$

$$\begin{aligned} E^s_{pq} &= \frac{h_p h_q}{2} A_p^T(Q_q + Q_{q-1}), && (7.142) \\ E^a_{pq} &= \frac{h_p h_q}{2} A_p^T(Q_q - Q_{q-1}), && (7.143) \end{aligned}$$

$$
\begin{aligned}
R_{ds} &= \begin{pmatrix} R_{ds11} & R_{ds12} & ... & R_{ds1N} \\ R_{ds21} & R_{ds22} & ... & R_{ds2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dsN1} & R_{dsN2} & ... & R_{dsNN} \end{pmatrix}, && (7.144) \\
R_{dspq} &= \frac{1}{2}[(h_p + h_q)(R_{p-1,q-1} - R_{pq}), \\
&\quad +(h_p - h_q)(R_{p,q-1} - R_{p-1,q})], && (7.145)
\end{aligned}
$$

$$
\begin{aligned}
R_{da} &= \begin{pmatrix} R_{da11} & R_{da12} & ... & R_{da1N} \\ R_{da21} & R_{da22} & ... & R_{da2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{daN1} & R_{daN2} & ... & R_{daNN} \end{pmatrix}, && (7.146) \\
R_{dapq} &= \frac{1}{2}(h_p - h_q)(R_{p-1,q-1} - R_{p-1,q} - R_{p,q-1} + R_{pq}), && (7.147)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_{0r}^T &= \begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix}, \\
\hat{\phi}^T(\alpha) &= \begin{pmatrix} \phi^{(1)T}(\alpha) & \phi^{(2)T}(\alpha) & ... & \phi^{(N)T}(\alpha) \end{pmatrix}.
\end{aligned}
$$

From the quadratic expression (7.130), we can obtain the Lyapunov-Krasovskii derivative condition as follows:

Proposition 7.11 *The derivative of the Lyapunov-Krasovskii functional $V(\phi)$ expressed in (7.103) to (7.107), with Q, S and R piecewise linear as in (7.111) to (7.113), along the trajectory of the system expressed by (7.98) to (7.102), satisfies*

$$
\begin{aligned}
&\dot{V}(\phi) \\
\leq\ & -\begin{pmatrix} \phi_{0r}^T & \int_0^1 \hat{\phi}^T(\alpha)d\alpha \end{pmatrix} \\
&\begin{pmatrix} -\frac{1}{3}D^a U D^{aT} + \Delta & -D^s - \frac{1}{3}D^a U E^{aT} \\ -D^{sT} - \frac{1}{3}E^a U D^{aT} & S_d - E^{sa} + R_{ds} - W \end{pmatrix} \begin{pmatrix} \phi_{0r} \\ \int_0^1 \hat{\phi}(\alpha)d\alpha \end{pmatrix} \\
&- \int_0^1 [\int_0^\alpha \begin{pmatrix} \hat{\phi}^T(\alpha) & \hat{\phi}^T(\beta) \end{pmatrix} \begin{pmatrix} W & R_{da} \\ R_{da}^T & W \end{pmatrix} \begin{pmatrix} \hat{\phi}(\alpha) \\ \hat{\phi}(\beta) \end{pmatrix} d\beta, \quad (7.148)
\end{aligned}
$$

provided

$$
\begin{pmatrix} U & -I \\ -I & S_d \end{pmatrix} > 0. \qquad (7.149)
$$

Therefore, the Lyapunov-Krasovskii derivative condition (7.110) is satisfied *for some sufficiently small $\varepsilon > 0$ if*

$$\begin{pmatrix} \Delta & -D^s & -D^a \\ -D^{sT} & S_d + R_{ds} - (E^s + E^{sT}) - W & -E^a \\ -D^{aT} & -E^{aT} & 3S_d \end{pmatrix} > 0, \quad (7.150)$$

$$\begin{pmatrix} W & R_{da} \\ R_{da}^T & W \end{pmatrix} > 0 \quad (7.151)$$

are satisfied.

Proof. The idea is very similar to the single delay and multiple delay case. From (7.130), it can be verified that

$$\begin{aligned}
&\dot{V}(\phi) \\
= \; & -\int_0^1 \begin{pmatrix} \phi_{DE}^T & \hat{\phi}^T(\alpha) \end{pmatrix} \begin{pmatrix} U & -I \\ -I & S_d \end{pmatrix} \begin{pmatrix} \phi_{DE} \\ \hat{\phi}(\alpha) \end{pmatrix} d\alpha \\
& -\phi_{0r}^T \Delta \phi_{0r} + \phi_{0r}^T [D^s U D^{sT} + \frac{1}{3} D^a U D^{aT}] \phi_{0r} \\
& +2\phi_{0r}^T [D^s U E^{sT} + \frac{1}{3} D^a U E^{aT} \int_0^1 \hat{\phi}(\beta) d\beta \\
& + \int_0^1 \hat{\phi}^T(\beta) d\beta [E^s U E^{sT} + \frac{1}{3} E^a U E^{aT}] \int_0^1 \hat{\phi}(\beta) d\beta \\
& -(\int_0^1 \hat{\phi}(\alpha) d\alpha)^T R_{ds} (\int_0^1 \hat{\phi}(\alpha) d\alpha) \\
& - \int_0^1 [\int_0^\alpha \begin{pmatrix} \hat{\phi}^T(\alpha) & \hat{\phi}^T(\beta) \end{pmatrix} \begin{pmatrix} 0 & R_{da} \\ R_{da}^T & 0 \end{pmatrix} \begin{pmatrix} \hat{\phi}(\alpha) \\ \hat{\phi}(\beta) \end{pmatrix} d\beta] d\alpha
\end{aligned}$$

for arbitrary matrix function $U(A_0, A^1, A^2, ..., A^K)$, where

$$\phi_{DE}^T = \phi_{0r}^T [D^s + (1 - 2\alpha) D^a] + \int_0^1 \hat{\phi}^T(\beta) d\beta [E^s + (1 - 2\alpha) E^a].$$

In view of (7.149), we can use the Jensen inequality (Proposition **??** in the Appendix) in the first term above, and collecting terms to obtain

$$\begin{aligned}
&\dot{V}(\phi) \\
\leq \; & -\begin{pmatrix} \phi_{0r}^T & \int_0^1 \hat{\phi}^T(\alpha) d\alpha \end{pmatrix} \begin{pmatrix} -\frac{1}{3} D^a U D^{aT} + \Delta & -D^s - \frac{1}{3} D^a U E^{aT} \\ -D^{sT} - \frac{1}{3} E^a U D^{aT} & S_d - E^{sa} + R_{ds} - W \end{pmatrix} \\
& \begin{pmatrix} \phi_{0r} \\ \int_0^1 \hat{\phi}(\alpha) d\alpha \end{pmatrix} \\
& - \int_0^1 [\int_0^\alpha \begin{pmatrix} \hat{\phi}^T(\alpha) & \hat{\phi}^T(\beta) \end{pmatrix} \begin{pmatrix} W & R_{da} \\ R_{da}^T & W \end{pmatrix} \begin{pmatrix} \hat{\phi}(\alpha) \\ \hat{\phi}(\beta) \end{pmatrix} d\beta
\end{aligned}$$

for arbitrary matrix W, where

$$E^{sa} = E^s + E^{sT} + \frac{1}{3}E^a U E^{aT}.$$

Therefore, the Lyapunov-Krasovskii derivative condition is satisfied if there exist a matrix W and a matrix function U such that (7.149), (7.151) and

$$\begin{pmatrix} -\frac{1}{3}D^a U D^{aT} + \Delta & -D^s - \frac{1}{3}D^a U E^{aT} \\ -D^{sT} - \frac{1}{3}E^a U D^{aT} & S_d - E^{sa} + R_{ds} - W \end{pmatrix} \tag{7.152}$$

are satisfied. Finally, we can use variable elimination discussed in Proposition **??**of the Appendix to eliminate the matrix function U from (7.149) and (7.152) to obtain (7.150). ■

7.6.5 Stability criterion and examples

From the above discussion in this section, we can summarize

Corollary 7.12 *The system described by (7.98) to (7.102) is asymptotically stable if there exist n by n matrices P, Q_p, $S_p = S_p^T$, $R_{pq} = R_{qp}^T$, $p, q = 0, 1, ..., N$; and Nn by Nn real matrix $W = W^T$, such that (7.118), (7.150) and (7.151) are satisfied, with notation defined in (7.119) to (7.124) and (7.131) to (7.147).*

Proof. This follows from Proposition 7.10 and Proposition 7.11. Similar to the single delay and multiple delay case, condition (7.117) is already implied by (7.150). ■

To illustrate the effectiveness of discretized Lyapunov functional method for systems with distributed delay, the following example is considered. The example considers a system resulting from a model transformation from a system with single pointwise delay.

Example 7.4 *Consider the system*

$$\dot{x}(t) = A_0(t)x(t) + \int_{-r/2}^{0} A^1 x(t+\xi)d\xi + \int_{-r}^{-r/2} A^2 x(t+\xi)d\xi, \tag{7.153}$$

where

$$A_0 = \begin{pmatrix} -1.5 & 0 \\ 0.5 & -1 \end{pmatrix}, \tag{7.154}$$

$$A^1 = \begin{pmatrix} 2 & 2.5 \\ 0 & -0.5 \end{pmatrix}, \tag{7.155}$$

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.156}$$

This is the result of model transformation from the system with single delay

$$\dot{x}(t) = \begin{pmatrix} -3 & -2.5 \\ 1 & 0.5 \end{pmatrix} x(t) + \begin{pmatrix} 1.5 & 2.5 \\ -0.5 & -1.5 \end{pmatrix} x(t - r/2), \qquad (7.157)$$

which was discussed in Example 5.5 in Chapter 5. We have scaled the time-delay r to conform to the notations in this section. In Example 5.5 of Chapter 5, it was found that although system (7.157) is asymptotically stable for $r < 4.8368$, *the transformed system (7.153) is asymptotically stable only for* $r < 2$ *due to the presence of additional dynamics. Using Corollary 7.12, the stability limit* $r_{\max}$ *such that system (7.153) is asymptotically stable for* $r \le r_{\max}$ *are estimated. For* $N_{d1} = N_{d2} = 1$ $(N = 2)$*. The calculation results in* $r_{\max} = 1.9725$ *with only about* 1.4% *conservatism. The estimation is improved to* $r_{\max} = 1.9999$ *for* $N_{d1} = N_{d2} = 2$ $(N = 4)$*.*

7.7 Notes

Similar to the single delay case, there are numerous results available for both delay-independent and delay-dependent stability results. See [224] for an overview. For delay-indepent stability using Lyapunov-Krasovskii functional method, see Ge, *et. al.* [79]

For a similar result to Corollary 7.3 using Razumikhin Theorem, see Cao, Sun and Cheng [31] (Similar to the single delay case, scaling factors can be introduced). The derivation used here is original.

For systems with multiple delays, in addition to delay-independent and delay-dependent stability, it is also possible to discuss mixed delay-independent and delay-dependent stability, see Kolmanovskii, *et. al.* [164].

The extension to the general linear time-delay system of the complete quadratic Lyapunov-Krasovskii functional (7.22) was reached by Huang [125]. The modification (7.27) is proposed by Kharitonov.

The discretized Lyapunov functional formulation for systems with multiple delays was first proposed in Gu [94]. The simplification to the current form discussed here is based on Gu [98]. The expression of $\dot{V}(\phi)$ in the proof of Proposition 7.8 can be generalized to

$$
\begin{aligned}
& \dot{V}(\phi) \\
= & -\int_0^1 \begin{pmatrix} \hat{\phi}^T [D^s + (1-2\alpha)D^a] & \tilde{\phi}^T(\alpha) \end{pmatrix} \\
& \begin{pmatrix} U & -I \\ -I & S_d^K - \alpha Z - (1-\alpha)W \end{pmatrix} \begin{pmatrix} [D^s + (1-2\alpha)D^a]^T \hat{\phi} \\ \tilde{\phi}(\alpha) \end{pmatrix} d\alpha \\
& + \hat{\phi}^T (-\Delta + D^s U D^{sT} + \frac{1}{3} D^a U D^{aT}) \hat{\phi} \\
& -[\int_0^1 \tilde{\phi}(\alpha) d\alpha]^T (R_{ds}^{KK} + R_{ds}^{\cdot K})[\int_0^1 \tilde{\phi}(\alpha) d\alpha] \\
& -\int_0^1 [\int_0^\alpha \begin{pmatrix} \tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta) \end{pmatrix} \begin{pmatrix} Z & R_{da}^{KK} \\ R_{da}^{KK} & W \end{pmatrix} \begin{pmatrix} \tilde{\phi}(\alpha) \\ \tilde{\phi}(\beta) \end{pmatrix} d\beta] d\alpha
\end{aligned}
$$

The one used in the proof of Proposition 7.8 is the special case of the above by setting $Z = W$. For nonuniform mesh with large difference of mesh size, a stability criterion based on the above expression can be less conservative but much more complicated.

The discretized Lyapunov functional method for systems with distributed delay was proposed in Gu, *et. al.* [103]. The formulation presented here is based on Gu [99].

8

Stability under dynamic uncertainty

The last section will be deleted under Jie's suggestion.

8.1 Introduction

In this Chapter, we will discuss the stability of time delay systems subject to dynamic uncertainty. We will formulate the problem in the framework of input-output stability. This framework is similar in flavor to what was discussed in Chapter 3. However, we will usually not consider known constant time delay as uncertainty in this chapter. With this framework, we can often use either frequency domain or time domain approaches once the problem is set up, although we will emphasis on time domain approach in this Chapter.

We will begin in Section 8.2 with the basic concept of input-output stability, and the small gain condition for the preservation of input-output stability in feedback configuration. As a simple illustration, we will illustrate the comparison systems approach for stability in Section 8.3. In this approach, the time-delay system is embedded in a system without time-delay with dynamic feedback. The same LMI is obtained by either frequency domain approach using Bounded Real Lemma, and directly from time domain approach. Then we begin a more comprehensive discussion of the methodology of input-output framework, with known constant time delays considered as part of the nominal system rather than uncertainty. In Section 8.4, we set up the scaled small gain problem. In Section 8.5, we explain how the robust stability problem with dynamical feedback uncertainty can be formulated as the scaled small gain problem, and discusses the uncertainty characterization. It will also be shown that some robust stability conditions discussed in Chapter 6 are also sufficient for dynamic uncertainty.

As an application of the above framework, in Section 8.6, we discuss the approximation approach of stability. Specifically, we discuss the approximation of time-varying delays by time-invariant delays, the approximation of arbitrary distributed delay by one with piecewise constant coefficient, and the approximation of arbitrary linear system by on with multiple delays. The errors of such approximations are modeled as dynamic uncertainty.

In this Chapter, for matrices and column vectors, we will use $||\cdot||$ instead of $||\cdot||_2$ to represent the 2-norm. However, we will still keep the subscript for function norms.

8.2 Input-output stability

Let L_{2+}^n represents the set of all the functions $f : \bar{\mathsf{R}}_+ \to \mathsf{R}^n$ which are square integrable, *i.e.*, $\int_0^\infty ||f(t)||^2 dt$ is well defined and finite. In this case, we can define the L_2-norm as

$$||f||_2 = \sqrt{\int_0^\infty ||f(t)||^2 dt}$$

For any $\tau > 0$ and function $f : \bar{\mathsf{R}}_+ \to \mathsf{R}^n$, the truncation operator (or projection operator) $\mathbf{P}_\tau$ is defined as

$$\mathbf{P}_\tau f(t) = \begin{cases} f(t), & 0 \le f \le \tau \\ 0, & t > \tau \end{cases}$$

and the shift operator $\mathbf{S}_\tau$ is defined as

$$\mathbf{S}_\tau f(t) = \begin{cases} f(t-\tau), & f \ge \tau \\ 0, & 0 \le t < \tau \end{cases}$$

Let L_{2e+}^n be defined as

$$L_{2e+}^n = \{f : \bar{\mathsf{R}}_+ \to \mathsf{R}^n \mid \mathbf{P}_\tau f \in L_{2+}^n \text{ for all } \tau > 0\}$$

The superscript n in L_{2+}^n or L_{2e+}^n may be omitted when the dimension n is not important or understood from the context. In input-output approach, a system is a mapping $\mathbf{H} : L_{2e+} \to L_{2e+}$. For a nonsingular matrix X and a system $\mathbf{H}$, we define a new system $\mathbf{H}_X$ by

$$\mathbf{H}_X f = X\mathbf{H}(X^{-1}f)$$

If $\mathbf{F}$ and $\mathbf{G}$ are two systems, then $\mathbf{FG}$ is also a system defined as

$$(\mathbf{FG})f = \mathbf{F}(\mathbf{G}f)$$

A system $\mathbf{H}$ is *linear* if $\mathbf{H}(\alpha f + \beta g) = \alpha\mathbf{H}f + \beta\mathbf{H}g$ for all $\alpha, \beta \in \mathsf{R}$ and $f, g \in L_{2e+}$. It is *causal* if it satisfies $\mathbf{P}_\tau\mathbf{H} = \mathbf{P}_\tau\mathbf{H}\mathbf{P}_\tau$ for all $\tau > 0$. In other words, if $\mathbf{H}$ is a causal system, and $g = \mathbf{H}f$, then, $g(t)$ is independent of $f(\xi)$, $\xi > t$. We will assume all the systems discussed from here on are causal unless specifically mentioned. $\mathbf{H}$ is *time-invariant* if $\mathbf{S}_\tau\mathbf{H} = \mathbf{H}\mathbf{S}_\tau$ for any $\tau > 0$. Clearly, $\mathbf{S}_\tau$ is both causal and time-invariant, and $\mathbf{P}_\tau$ is causal but not time-invariant.

We say that system $\mathbf{H}$ is *L_2-stable* if $\mathbf{H}f \in L_{2+}$ whenever $f \in L_{2+}$. It is *L_2-stable with finite gain* if it is L_2-stable and there exist real scalars $\gamma > 0$ and b such that $||\mathbf{H}f||_2 \leq \gamma ||f||_2 + b$ for all $f \in L_{2+}$. It is *L_2-stable with finite gain and zero bias* if it is L_2-stable and there exists a real scalar $\gamma > 0$ such that $||\mathbf{H}f||_2 \leq \gamma ||f||_2$ for all $f \in L_{2+}$. For the sake of convenience, we will say a system is *input-output stable* if it is L_2-stable with finite gain and zero bias. If $\mathbf{H}$ is input-output stable, we can define its *gain* $\gamma_0(\mathbf{H})$ as

$$\gamma_0(\mathbf{H}) = \inf\{\gamma \mid ||\mathbf{H}f||_2 \leq \gamma ||f||_2 \text{ for all } f \in L_{2+}\}$$

In practice, a system $\mathbf{H}$ is often described using some internal states. We will say the system $\mathbf{H}$ is *internally stable* if the zero state is globally uniformly asymptotically stable. For example, the functional differential equations,

$$\begin{aligned} \dot{x}(t) &= f(t, x_t, u_t) \\ y(t) &= h(t, x_t, u_t) \end{aligned}$$

satisfying

$$\begin{aligned} f(t, 0, 0) &= 0 \\ h(t, 0, 0) &= 0 \end{aligned}$$

describe a system $\mathbf{H}$: The mapping from u to y for zero initial condition

$$x(t) = 0,\ u(t) = 0,\ t \leq 0$$

can be written as $y = \mathbf{H}u$. The system $\mathbf{H}$ is internally stable if the zero state of the functional differential equation

$$\dot{x}(t) = f(t, x_t, 0) \tag{8.1}$$

is asymptotically stable. Notice, we have allowed a slight abuse of notation by using $\mathbf{H}$ to represent both the functional differential equation description (with the possibility of nonzero condition) and the input output mapping corresponding to the zero initial condition. If a system $\mathbf{H}$ is represented by a linear time-invariant functional differential equation, then in view of the solution expression (1.33) in Chapter 1, internal stability implies input-output stability. On the other hand, if certain controllability and observability conditions are satisfied, then input-output stability also implies internal stability. A linear time-invariant system $\mathbf{H}$ can also be represented by its transfer function matrix $H(s)$, and can be studied by frequency domain approach. Furthermore, its gain is equal to its $\mathcal{H}_\infty$ norm

$$\gamma_0(\mathbf{H}) = ||H(s)||_\infty$$

For general nonlinear systems, the relationship between internal stability and input-output stability are more complicated, and will not be pursued here.

Given systems $\mathbf{G}$ and $\mathbf{F}$, connect the two systems in a feedback loop,

$$y = \mathbf{G}u \tag{8.2}$$
$$u = \mathbf{F}y + v \tag{8.3}$$

If the one-to-one relationship between input v and output y is well defined by equations (8.2) and (8.3), then we can write $y = \mathbf{H}v$, and $\mathbf{H}$ can be regarded as a system, and we will write $\mathbf{H} = \text{feedback}(\mathbf{G}, \mathbf{F})$. An important conclusion in input-output stability theory is the *Small Gain Theorem*, which states that the combined system feedback$(\mathbf{G}, \mathbf{F})$ is well defined and input-output stable if both $\mathbf{G}$ and $\mathbf{F}$ are input-output stable and satisfy

$$\gamma_0(\mathbf{G})\gamma_0(\mathbf{F}) < 1 \tag{8.4}$$

Condition (8.4) is known as the *small gain condition*. If the small gain condition is satisfied, then we can easily find an upper bound of $\gamma_0(\text{feedback}(\mathbf{G}, \mathbf{F}))$: from (8.2) and (8.3), we can write

$$\begin{aligned} ||y||_2 &\leq \gamma_0(\mathbf{G})||u||_2 \\ &= \gamma_0(\mathbf{G})||\mathbf{F}y + v||_2 \\ &\leq \gamma_0(\mathbf{G})(||\mathbf{F}y||_2 + ||v||_2) \\ &\leq \gamma_0(\mathbf{G})(\gamma_0(\mathbf{F})||y||_2 + ||v||_2) \end{aligned}$$

Therefore,

$$||y||_2 \leq \frac{\gamma_0(\mathbf{G})}{(1 - \gamma_0(\mathbf{G})\gamma_0(\mathbf{F}))}||v||_2$$

This indicates

$$\gamma_0(\text{feedback}(\mathbf{G}, \mathbf{F})) \leq \frac{\gamma_0(\mathbf{G})}{(1 - \gamma_0(\mathbf{G})\gamma_0(\mathbf{F}))}$$

For real matrix X and feedback system $\mathbf{H} = \text{feedback}(\mathbf{G}, \mathbf{F})$, it is easy to confirm $\mathbf{H}_X = \text{feedback}(\mathbf{G}_X, \mathbf{F}_X)$. This allows us to conclude that feedback$(\mathbf{G}, \mathbf{F})$ is input-output stable if and only if feedback$(\mathbf{G}_X, \mathbf{F}_X)$ is input-output stable.

In practice, the input v in (8.2) and (8.3) is often due to nonzero initial conditions, in which case, we sometimes omit v and write

$$\begin{aligned} y &= \mathbf{G}u \\ u &= \mathbf{F}y \end{aligned}$$

We still say the above system is input-ouput stable to mean that the system described by (8.2) and (8.3) is input-output stable.

8.3 Method of comparison systems

As a simple illustration of input-output approach, we will discuss the method of comparison system. In this method, a time-delay system is embedded in a system without uncertainty subject to dynamical feedback uncertainty known as the *comparison system*. Of course, the stability of the comparison system implies the stability of the original time-delay system. In our development, we will illustrate some features of the input-output approach of robust stability analysis to be developed in later sections.

8.3.1 Frequency domain approach

Consider a system with single delay

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-r),\ t \geq 0 \tag{8.5}$$

where $x(t) \in \mathsf{R}^n$, $A_0, A_1 \in \mathsf{R}^{n\times n}$. As usual, the initial condition is in the form of

$$x_0 = \phi \in \mathcal{C}$$

Then, for arbitrary matrix C, we can write

$$\begin{aligned} \dot{x}(t) &= (A_0 + C)x(t) + (A_1 - C)x(t-r) - C[x(t) - x(t-r)] \\ &= (A_0 + C)x(t) + (A_1 - C)x(t-r) - C\int_{-r}^{0} \dot{x}(t+\theta)d\theta \end{aligned}$$

Using (8.5) in the last term, the above becomes

$$\dot{x}(t) = (A_0+C)x(t)+(A_1-C)x(t-r)-C\int_{-r}^{0}[A_0x(t+\theta)+A_1x(t+\theta-r)]d\theta \tag{8.6}$$

Similar to the model transformation discussed in Chapter 5, the above is valid only for $t \geq r$, and initial condition constraint needs to be imposed for $x(t+\theta)$, $-r \leq \theta \leq r$. The stability analysis disregarding this initial condition constraint will introduce additional conservatism due to additional dynamics. The process of transforming (8.5) to (8.6) without initial condition constraint is known as *parameterized model transformation*. Indeed, if C is chosen as A_1, this becomes the model transformation discussed in Chapter 5. If $C = 0$, then (8.6) reverts back to (8.5).

System (8.6) can be written as

$$\dot{x}(t) = (A_0 + C)x(t) + (A_1 - C)u_1(t) + rCu_2(t) \tag{8.7}$$

$$y_1(t) = x(t) \tag{8.8}$$

$$y_2(t) = A_0 x(t) + A_1 u_1(t) \tag{8.9}$$

with feedback

$$u_1(t) = y_1(t-r) \tag{8.10}$$

$$u_2(t) = -\frac{1}{r}\int_{-r}^{0} y_2(t+\theta)d\theta \tag{8.11}$$

Let $u^T = \begin{pmatrix} u_1^T & u_2^T \end{pmatrix}$ and $y^T = \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}$. Then we can write the forward system described by (8.7) to (8.9) as

$$y = \mathbf{G}u$$

and the feedback system described by (8.10) and (8.11) as

$$u = \mathbf{\Delta}y$$

The forward system $\mathbf{G}$ is a linear time-invariant system without delay, with transfer matrix

$$G(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$$

where

$$\tilde{A} = A_0 + C \tag{8.12}$$

$$\tilde{B} = \begin{pmatrix} A_1 - C & rC \end{pmatrix} \tag{8.13}$$

$$\tilde{C} = \begin{pmatrix} I_n \\ A_0 \end{pmatrix} \tag{8.14}$$

$$\tilde{D} = \begin{pmatrix} 0 & 0 \\ A_1 & 0 \end{pmatrix} \tag{8.15}$$

The feedback system $\mathbf{\Delta}$ has a "block-diagonal structure"

$$u_1 = \mathbf{\Delta}_1 y_1 \tag{8.16}$$

$$u_2 = \mathbf{\Delta}_2 y_2 \tag{8.17}$$

where $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ are described by (8.10) and (8.11), respectively. In fact, both $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ are "linear scalar systems" with $\mathcal{H}_\infty$ norms bounded by one. In other words, the transfer matrices of $\mathbf{\Delta}_k$ are

$$\Delta_k(s) = \delta_k(s)I, \quad k = 1, 2 \tag{8.18}$$

where $\delta_k(s)$, $k = 1, 2$ are stable scalar transfer function satisfying

$$||\delta_k(s)||_\infty = \sup_{\mathrm{Re}(s)\geq 0} |\delta_k(s)| \leq 1, \ k = 1, 2 \tag{8.19}$$

Indeed, $\delta_k(s)$ in (8.18) can be obtained by taking Laplace transform of (8.10) and (8.11)

$$\delta_1(s) = e^{-rs}$$

$$\delta_2(s) = -\frac{1 - e^{-rs}}{rs}$$

which can be shown to satisfy (8.19). Thus, we can embed the system described by (8.5) in the uncertain system feedback$(\mathbf{G}, \mathbf{\Delta})$, consisting of $\mathbf{G}$ described by (8.7) to (8.9) and *dynamical feedback uncertainty* $\mathbf{\Delta}$ described by (8.16) and (8.17), with $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ being *arbitrary* dynamical systems satisfying (8.18) and (8.19).

The transfer matrix of $\mathbf{\Delta}$ can be written as $\Delta(s) = \mathrm{diag}\begin{pmatrix} \delta_1(s)I & \delta_2(s)I \end{pmatrix}$. Using the small gain theorem, since $||\Delta||_\infty \leq 1$, a sufficient condition for the system to be input-output stable is $\gamma_0(\mathbf{G}) = ||G(s)||_\infty < 1$. We can obtain a stronger result by scaling: For any nonsingular block-diagonal matrix

$$X = \mathrm{diag}\begin{pmatrix} X_1 & X_2 \end{pmatrix}, \qquad X_1, X_2 \in \mathsf{R}^{n\times n} \tag{8.20}$$

let $u_X = Xu$, $y_X = Xy$, then the system can be written as

$$\begin{aligned} y_X &= \mathbf{G}_X u_X \\ u_X &= \mathbf{\Delta}_X y_X \end{aligned}$$

Since

$$\gamma_0(\mathbf{\Delta}_X) = ||X\Delta(s)X^{-1}||_\infty = ||\Delta(s)||_\infty \leq 1 \tag{8.21}$$

we can conclude that the system is stable if there exists a nonsingular block-diagonal matrix X such that

$$\gamma_0(\mathbf{G}_X) = ||X^{-1}G(s)X||_\infty < 1 \tag{8.22}$$

A frequency sweeping method can be used to check the satisfaction of (8.22). Indeed, (8.22) for all nonsingular X with block-diagonal structure expressed in (8.20) is equivalent to

$$\begin{aligned} G^T(-j\omega)ZG(j\omega) &< Z, \qquad 0 \leq \omega < \infty \\ \tilde{D}^T Z \tilde{D} &< Z \end{aligned}$$

for any block-diagonal symmetric positive definite matrix

$$Z = Z^T = \mathrm{diag}\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} > 0, \qquad Z_1, Z_2 \in \mathsf{R}^{n\times n}$$

We can use the following bounded real lemma to write (8.22) in a frequency-independent LMI.

Lemma 8.1 (Bounded Real Lemma) *A linear system* $\mathbf{H}$ *described by*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

is internally stable and satisfies

$$\gamma_0(\mathbf{H}) < 1 \tag{8.23}$$

if and only if there exists $P \in \mathsf{R}^{n\times n}$ *such that*

$$P^T = P > 0 \tag{8.24}$$

and

$$\begin{pmatrix} PA + A^TP + C^TC & PB + C^TD \\ (PB + C^TD)^T & -(I - D^TD) \end{pmatrix} < 0 \tag{8.25}$$

are satisfied.

Indeed, using the Bounded Real Lemma, we can conclude that

$$\gamma_0(\mathbf{G}_X) = ||XG(s)X^{-1}|| < 1$$

if and only if there exists a $P \in \mathsf{R}^{n\times n}$ such that (8.24) and

$$\begin{pmatrix} P\tilde{A} + \tilde{A}^TP + \tilde{C}^TX^TX\tilde{C} & (P\tilde{B} + \tilde{C}^TX^TX\tilde{D})X^{-1} \\ [(P\tilde{B} + \tilde{C}^TX^TX\tilde{D})X^{-1}]^T & -I + X^{-T}\tilde{D}^TX^TX\tilde{D}X^{-1} \end{pmatrix} < 0 \tag{8.26}$$

are satisfied. From the above discussion, we can conclude

Proposition 8.2 *The system described by (8.5) is input-output stable if there exist real symmetric matrices* P, Z_1 *and* Z_2 *such that*

$$P > 0$$

$$\begin{pmatrix} M & P(A_1 - C) + A_0^TZ_2A_1 & rPC \\ [P(A_1 - C) + A_0^TZ_2A_1]^T & -Z_1 + A_1^TZ_2A_1 & 0 \\ rC^TP & 0 & -Z_2 \end{pmatrix} < 0 \tag{8.27}$$

where

$$M = P(A_0 + C) + (A_0 + C)^TP + Z_1 + A_0^TZ_2A_0$$

Proof. First, notice that (8.27) implies $Z_1 > 0$ and $Z_2 > 0$ in view of the (3,3) and (2,2) entries of the matrix. From the above discussions, the system is stable if (8.26) is satisfied for some nonsingular X with structure (8.20). Left-multiply the second row of (8.26) by X^T and right multiply the second column by X, define

$$Z = \mathrm{diag}\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = X^TX$$

and use the definitions (8.12) to (8.15). ∎

It is interesting to observe that the above criterion is equivalent to the delay-dependent stability criterion obtained by using implicit model transformation and Lyapunov-Krasovskii functional discussed in Chapter 5. Indeed, by eliminating the variable X in (5.81) and (5.85), changing variables $S = Z_1$, $Y = PC$, $Z = \frac{1}{r}Z_2$, and multiplying -1 in the third row and column, we obtain (8.27). This also implies that the delay-independent stability criterion and the delay-dependent stability criterion using explicit model transformation obtained using Lyapunov-Krasovskii functional discussed in Chapter 5 are equivalent to setting $C = 0$ and $C = A_1$, respectively, in (8.27).

8.3.2 Time domain approach

The above conclusion can also be obtain by using time domain approach. For time domain characterization of uncertainty, first we will show that the operator $\mathbf{\Delta} = \text{diag}\begin{pmatrix} \mathbf{\Delta}_1 & \mathbf{\Delta}_2 \end{pmatrix}$ described by (8.10) to (8.17) satisfies the following *integral quadratic constraint*

$$\int_0^t u^T(\tau) Z u(\tau) d\tau \leq \int_0^t y^T(\tau) Z y(\tau) d\tau \tag{8.28}$$

for any $t > 0$, $y \in L_{2e+}$, $u = \mathbf{\Delta} y$, and $Z = Z^T = \text{diag}\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} > 0$, $Z_1, Z_2 \in \mathsf{R}^{n \times n}$. For X satisfying

$$X^T X = Z$$

(8.28) implies that

$$\gamma_0(\mathbf{\Delta}_X) \leq 1 \tag{8.29}$$

since (8.28) implies

$$\begin{aligned} & \int_0^t ||\mathbf{\Delta}_X y(\tau)||^2 d\tau - \int_0^t ||y(\tau)||^2 d\tau \\ = & \int_0^t [\mathbf{\Delta} \tilde{y}(\tau))]^T Z \mathbf{\Delta} \tilde{y}(\tau)) d\tau - \int_0^t \tilde{y}^T(\tau) Z \tilde{y}(\tau) d\tau \\ \leq & \ 0 \end{aligned}$$

where

$$\tilde{y} = X^{-1} y$$

To show (8.28), let $u = \begin{pmatrix} u_1^T & u_2^T \end{pmatrix}^T = \begin{pmatrix} \mathbf{\Delta}_1 y_1^T & \mathbf{\Delta}_2 y_2^T \end{pmatrix}^T$, and use zero initial condition, we have

$$\begin{aligned} & \int_0^t u_1^T(\tau) Z_1 u_1(\tau) d\tau \\ = & \int_{-r}^{t-r} y_1^T(\tau) Z_1 y_1(\tau) d\tau \\ = & \int_0^{t-r} y_1^T(\tau) Z_1 y_1(\tau) d\tau \\ \leq & \int_0^t y_1^T(\tau) Z_1 y_1(\tau)] d\tau \end{aligned} \tag{8.30}$$

Also, using Jensen's inequality and exchanging the order of integration, we can obtain

$$\begin{aligned}
&\int_0^t u_2^T(\tau) Z_2 u_2(\tau) d\tau \\
= \; & \frac{1}{r^2} \int_0^t [\int_{-r}^0 y_2(\tau+\theta) d\theta]^T Z_2 [\int_{-r}^0 y_2(\tau+\theta) d\theta] d\tau \\
\leq \; & \frac{1}{r} \int_0^t [\int_{-r}^0 y_2^T(\tau+\theta) Z_2 y_2(\tau+\theta) d\theta] d\tau \\
= \; & \frac{1}{r} \int_{-r}^0 [\int_0^t y_2^T(\tau+\theta) Z_2 y_2(\tau+\theta) d\tau] d\theta \\
= \; & \frac{1}{r} \int_{-r}^0 [\int_\theta^{t+\theta} y_2^T(\tau) Z_2 y_2(\tau) d\tau] d\theta \\
\leq \; & \frac{1}{r} \int_{-r}^0 [\int_0^t y_2^T(\tau) Z_2 y_2(\tau) d\tau] d\theta \\
= \; & \int_0^t y_2^T(\tau) Z_2 y_2(\tau) d\tau
\end{aligned} \tag{8.31}$$

Inequalities (8.30) and (8.31) imply (8.28), which is equivalent to (8.29).

With (8.29) established, the input-output stability can be established if there exists a nonsingular X such that

$$\gamma_0(\mathbf{G}_X) < 1 \tag{8.32}$$

To study the condition for (8.32), consider a Lyapunov function

$$V(x) = x^T P x \tag{8.33}$$

The following expression is a quadratic expression of x and u

$$W(x,u) = \dot{V}(x) + y^T Z y - u^T Z u \tag{8.34}$$

where $y = \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}^T$ represents the linear expression (8.8) and (8.9) of x and u. If

$$V(x) \geq \varepsilon ||x||^2 \tag{8.35}$$

$$W(x,u) \leq -\varepsilon(||x||^2 + ||u||^2) \tag{8.36}$$

for some $\varepsilon > 0$, and arbitrary x and u, then (8.32) is satisfied. To show this, integrate (8.36) from 0 to t, using zero initial condition (and therefore $V(x(0)) = 0$), we obtain

$$V(x(t)) + \int_0^t [y^T(\tau) Z y(\tau) - u^T(\tau) Z u(\tau)] d\tau \leq -\varepsilon \int_0^t [||x(\tau)||^2 + ||u(\tau)||^2] d\tau$$

Let

$$\begin{aligned} \delta &= \varepsilon/\lambda_{\max}(Q) \\ X^T X &= Z \end{aligned}$$

Then

$$\begin{aligned} &\int_0^t ||Xy(\tau)||^2 d\tau - (1-\delta)\int_0^t ||Xu(\tau)||^2 d\tau \\ \leq\ & \int_0^t [y^T(\tau)Zy(\tau) - u^T(\tau)(Z-\varepsilon I)u(\tau)]d\tau \\ \leq\ & -\varepsilon\int_0^t ||x(\tau)||^2 d\tau - V(x(t)) \\ \leq\ & 0 \end{aligned}$$

This shows

$$\gamma_0(\mathbf{G}_X) \leq (1-\delta) < 1$$

Condition (8.35) is equivalent to

$$P > 0 \tag{8.37}$$

For condition (8.36), write out the expression of W, we obtain

$$\begin{aligned} W(x,u) &= 2x^T P(\tilde{A}x + \tilde{B}u) + (\tilde{C}x + \tilde{D}u)^T Z(\tilde{C}x + \tilde{D}u) - u^T Z u \\ &= \begin{pmatrix} x^T & u^T \end{pmatrix} \begin{pmatrix} P\tilde{A} + \tilde{A}^T P + \tilde{C}^T Z\tilde{C} & P\tilde{B} + \tilde{C}^T Z\tilde{D} \\ \tilde{B}^T P + \tilde{D}^T Z\tilde{C} & \tilde{D}^T Z\tilde{D} - Z \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \end{aligned}$$

Therefore,

$$\begin{pmatrix} P\tilde{A} + \tilde{A}^T P + \tilde{C}^T Z\tilde{C} & P\tilde{B} + \tilde{C}^T Z\tilde{D} \\ \tilde{B}^T P + \tilde{D}^T Z\tilde{C} & \tilde{D}^T Z\tilde{D} - Z \end{pmatrix} < 0 \tag{8.38}$$

is a sufficient condition for (8.35). This is identical to (8.27).

8.4 Scaled small gain problem

Following the same idea as the last section, we will start developing the general theory under more general setting with the possibility of a time delay system as the nominal forward system. Consider a time-delay system **G** described by the functional differential equation

$$\dot{x}(t) = f(t, x_t, u(t)) \tag{8.39}$$

$$y(t) = h(t, x_t, u(t)) \tag{8.40}$$

where

$$f(t,0,0) = 0 \tag{8.41}$$
$$h(t,0,0) = 0 \tag{8.42}$$

and $x(t) \in \mathsf{R}^n$, $u(t) \in \mathsf{R}^l$, $y(t) \in \mathsf{R}^m$. Notice, we have constrained $u(t)$ and $y(t)$ to have the same dimension. This constraint is imposed for notational convenience, and can be removed. Let $\mathcal{X} \subset \mathsf{R}^{m\times m}$ be a set of nonsingular matrices. The scaled small gain problem is to check whether $\mathbf{G}$ is internally stable and there exists a $X \in \mathcal{X}$ such that

$$\gamma_0(\mathbf{G}_X) < 1 \tag{8.43}$$

The special case of $\mathcal{X} = \{I\}$, *i.e.*, check whether $\mathbf{G}$ is internally stable and

$$\gamma_0(\mathbf{G}) < 1 \tag{8.44}$$

is known as the small gain problem or $\mathcal{H}_\infty$ problem. When $\mathbf{G}$ is linear and time invariant, a frequency domain technique can be used. Indeed, a frequency sweeping method is often possible to check (8.43). In the following, we will discuss the time domain approach.

Let

$$\mathcal{Z} = \{Z \mid Z = X^T X, X \in \mathcal{X}\}$$

Then, the scaled small gain problem is equivalent to finding whether there exists a $Z \in \mathcal{Z}$ such that

$$\int_0^t y^T(\tau)Zy(\tau)d\tau \le \int_0^t u^T(\tau)Zu(\tau)d\tau \tag{8.45}$$

for any t, u and $y = \mathbf{G}u$.

To achieve (8.45), we have

Proposition 8.3 *Let $w_k : \bar{\mathsf{R}}_+ \to \bar{\mathsf{R}}_+$, $k = 1,2,3$, $w_k(0) = 0$, $w_k(s) > 0$, for $s > 0$ for $k = 1,2,3$, and $\lim_{s\to\infty} w_1(s) = \infty$. Let $f, h : \mathsf{R}\times\mathcal{C}\times\mathsf{R}^n \to \mathsf{R}^n$ take $\mathsf{R} \times$ (bounded sets of $\mathcal{C}\times\mathsf{R}^n$) into bounded sets of R^n and satisfy (8.41) and (8.42). Then, the system $\mathbf{G}$ described by (8.39) and (8.40) is internally stable, input-output stable and satisfies (8.43) for some $X \in \mathcal{X}$ if there exists a $Z \in \mathcal{Z}$ and a Lyapunov-Krasovskii functional $V(t, x_t)$ satisfying*

$$w_1(||x(t)||) \le V(t,x_t) \le w_2(||x_t||_c) \tag{8.46}$$

such that the functional

$$W(t,x_t,u(t)) = \dot{V}(t,x_t) + y^T(t)Zy(t) - u^T(t)Zu(t) \tag{8.47}$$

satisfies

$$W(t,x_t,u(t)) \le -w_3(||x(t)||) - \varepsilon||u(t)||^2 \tag{8.48}$$

where $\dot{V}$ denotes the derivative of V along the system trajectory, and $y(t)$ denotes the expression (8.40).

Proof. Setting $u = 0$ in (8.48), we obtain

$$\dot{V}(t, x_t) \le -w_3(||x(t)||) - y^T(t)Zy(t) \le -w_3(||x(t)||)$$

which together with (8.46) proves the internal stability. It remains to be proven that (8.43) is satisfied, or equivalently, (8.45) is satisfied. Integrate (8.48) from 0 to t, we obtain

$$\begin{aligned} & V(t, x_t) - V(0, x_0) + \int_0^t [(y^T(\tau)Zy(\tau) - u^T(\tau)Zu(\tau)]d\tau \\ \le & -\int_0^t w_3(||x(\tau)||)d\tau - \int_0^t \varepsilon u^T(\tau)u(\tau)d\tau \end{aligned}$$

Due to zero initial condition $x_0 = 0$, we have $V(0, x_0) = 0$ in view of (8.46). Therefore,

$$\begin{aligned} & \int_0^t [(y^T(\tau)Zy(\tau) - u^T(\tau)Zu(\tau)]d\tau \\ \le & -\int_0^t w_3(||x(\tau)||)d\tau - \int_0^t \varepsilon u^T(\tau)u(\tau)d\tau - V(t, x_t) \\ \le & -\int_0^t \varepsilon u^T(\tau)u(\tau)d\tau \end{aligned}$$

i.e.,

$$\begin{aligned} \int_0^t y^T(\tau)Zy(\tau)d\tau & \le \int_0^t u^T(\tau)(Z - \varepsilon I)u(\tau)d\tau \\ & \le \gamma^2 \int_0^t u^T(\tau)Zu(\tau)d\tau \end{aligned}$$

where

$$\gamma^2 = 1 - \varepsilon/\lambda_{\max}(Z) < 1$$

■

When the system **G** is linear, and $V(t, x_t)$ is bounded quadratic functional of x_t, then $W(t, x_t, u(t))$ is a quadratic functional of x_t and $u(t)$. The upper bound of $V(t, x_t)$ can always be satisfied for $w_2(||x_t||_c) = K||x_t||_c^2$ for sufficiently large K. We can choose $w_1(||x(t)||) = w_3(||x(t)||) = \varepsilon||x(t)||^2$ for sufficiently small $\varepsilon > 0$, and conditions (8.46) and (8.48) becomes

$$V(t, x_t) \ge \varepsilon||x(t)||^2 \tag{8.49}$$
$$W(t, x_t, u(t)) \le -\varepsilon||x(t)||^2 - \varepsilon||u(t)||^2 \tag{8.50}$$

and it is often possible to reduce the conditions into an LMI problem.

To illustrate this, consider the system with single delay

$$\dot{x}(t) = A_0x(t) + A_1x(t - r) + Eu(t) \tag{8.51}$$
$$y(t) = G_0x(t) + G_1x(t - r) + Du(t) \tag{8.52}$$

Let

$$V(t,\phi) = \phi^T(0)P\phi(0) + \int_{-r}^{0} \phi^T(\tau)S\phi(\tau)d\tau \tag{8.53}$$

where $P^T = P > 0$, $S^T = S > 0$. It can be easily calculated that

$$W(t,\phi,u) = \phi_{xu}^T \Pi \phi_{xu}$$

where

$$\phi_{xu}(t) = \begin{pmatrix} \phi^T(0) & \phi^T(-r) & u^T \end{pmatrix}^T$$

and

$$\Pi = \begin{pmatrix} PA_0 + A_0^T P + S + C_0^T Z G_0 & PA_1 + G_0^T Z G_1 & PE + G_0^T Z D \\ & -S + G_1^T Z G_1 & G_1^T Z D \\ \text{symmetric} & & D^T Z D - Z \end{pmatrix}$$

Therefore, there exists a $X \in \mathcal{X}$ such that $\gamma_0(\mathbf{G}_X) < 1$ if there exists a $Z \in \mathcal{Z}$, $P > 0$ such that $\Pi < 0$, which is an LMI problem.

Of course, similar to the stability problem, the simple Lyapunov-Krasovskii functional (8.53) yields a delay-independent result, and is often too conservative. A more accurate result can be obtained by using a complete quadratic Lyapunov-Krasovskii functional

$$\begin{aligned} V(t,\phi) &= \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\quad + \int_{-r}^{0}[\int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta]d\xi \\ &\quad + \int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \end{aligned} \tag{8.54}$$

where R and S satisfy the symmetry as discussed in Section 5.7 of Chapter

5. Then we can calculate

$$\begin{aligned}
& W(t,\phi,u) \\
= \; & -\phi^T(0)[-PA_0 - A_0^T P - Q(0) - Q^T(0) - S(0)]\phi(0) \qquad (8.55) \\
& -\phi^T(-r)S(-r)\phi(-r) \\
& -\int_{-r}^{0} \phi^T(\xi)\dot{S}(\xi)\phi(\xi)d\xi \\
& -\int_{-r}^{0} d\xi \int_{-r}^{0} \phi^T(\xi)[\frac{\partial}{\partial\xi}R(\xi,\eta) + \frac{\partial}{\partial\eta}R(\xi,\eta)]\phi(\eta)d\eta \\
& +2\phi^T(0)[PA_1 - Q(-r)]\phi(-r) \\
& +2\phi^T(0)\int_{-r}^{0}[A_0^T Q(\xi) - \dot{Q}(\xi) + R(0,\xi)]\phi(\xi)d\xi \\
& +2\phi^T(-r)\int_{-r}^{0}[A_1^T Q(\xi) - R(-r,\xi)]\phi(\xi)d\xi \\
& +2\phi^T(0)PEu + 2u^T E^T \int_{-r}^{0} Q(\xi)\phi(\xi)d\xi - u^T Zu \\
& +(G_0\phi(0) + G_1\phi(-r) + Du)^T Z(G_0\phi(0) + G_1\phi(-r) + Du) \qquad (8.56)
\end{aligned}$$

We can write the conditions in a LMI form by discretized Lyapunov functional method described in Section 5.7 of Chapter 5. Use the same notation as in Section 5.7 of Chapter 5, choose Q, S and R to be piecewise linear, then, it was already shown in Proposition 5.21 in Chapter 5 that a sufficient condition for (8.49) is $S_p > 0$, $p = 0, 1, ..., N$ and

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} + \tilde{S} \end{pmatrix} > 0 \qquad (8.57)$$

For $W(t,\phi,u)$, it is noticed that it is equal to the expression of $\dot{V}$ in (5.122) of Chapter 5 plus the last four terms (three terms involving u and the last term) in (8.56). Therefore, we can easily modify (5.146) in Chapter 5 to obtain

$$\begin{aligned}
W(t,\phi,u) \; = \; & -\phi_{0ru}^T \tilde{\Delta}\phi_{0ru} - \int_0^1 \tilde{\phi}^T(\alpha)S_d\tilde{\phi}(\alpha)d\alpha \\
& - \int_0^1 [\int_0^1 \tilde{\phi}^T(\alpha)R_d\tilde{\phi}(\beta)d\alpha]d\beta \\
& +2\phi_{0ru}^T \int_0^1 [\tilde{D}^s + (1-2\alpha)\tilde{D}^a]\tilde{\phi}(\alpha)hd\alpha \qquad (8.58)
\end{aligned}$$

where

$$\phi_{0ru} = \begin{pmatrix} \phi^T(0) & \phi^T(-r) & u^T \end{pmatrix}^T$$

$$\tilde{\Delta} = \begin{pmatrix} \Delta_{00} & \Delta_{01} & -PE \\ \Delta_{01}^T & \Delta_{11} & 0 \\ -E^T P & 0 & Z \end{pmatrix} - \begin{pmatrix} G_0^T \\ G_1^T \\ D^T \end{pmatrix} Z \begin{pmatrix} G_0 & G_1 & D \end{pmatrix}$$

$$\begin{aligned} \tilde{D}^s &= \begin{pmatrix} D^s \\ D_u^s \end{pmatrix} \\ \tilde{D}^a &= \begin{pmatrix} D^a \\ D_u^a \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} D_u^s &= \frac{h}{2} E^T \begin{pmatrix} Q_1 + Q_0 & Q_2 + Q_1 & \cdots & Q_N + Q_{N-1} \end{pmatrix} \\ D_u^a &= \frac{h}{2} E^T \begin{pmatrix} Q_1 - Q_0 & Q_2 - Q_1 & \cdots & Q_N - Q_{N-1} \end{pmatrix} \end{aligned}$$

Then, the same manipulation as in Proposition 5.21 of Chapter 5 allows us to conclude that (8.50) is satisfied if

$$\begin{pmatrix} \tilde{\Delta} & -\tilde{D}^s & -\tilde{D}^a \\ -\tilde{D}^{sT} & R_d + S_d & 0 \\ -\tilde{D}^{aT} & 0 & 3S_d \end{pmatrix} > 0 \tag{8.59}$$

To conclude, for the system $\mathbf{G}$ described by (8.51) and (8.52), there exists a $X \in \mathcal{X}$ such that $\gamma_0(\mathbf{G}_X) < 1$ if there exist a $Z \in \mathcal{Z}$ and $P = P^T$, Q_p, S_p, $R_{pq} = R_{qp}^T$, $p = 0, 1, ..., N; q = 0, 1, ..., N$ such that (8.57) and (8.59) are satisfied. Notice, $S_p > 0$ is already implied by (8.59).

When $\mathbf{G}$ is a system with multiple pointwise delays or distributed delays with piecewise constant coefficient, we can also make similar modification to the results in Chapter 7 to derive sufficient conditions for $\gamma_0(\mathbf{G}_X) < 1$ in LMI form.

8.5 Robust stability under dynamic uncertainty

8.5.1 Problem setup

Consider a system $\mathbf{H}$ formed by a feedback loop consisting of a forward system $\mathbf{G}$ and a feedback uncertain system $\mathbf{\Delta}$:

$$\begin{aligned} y &= \mathbf{G}u \quad (8.60) \\ u &= \mathbf{\Delta}y \quad (8.61) \end{aligned}$$

where $\mathbf{G}$ is described by the functional differential equations (8.39) and (8.40). The system $\mathbf{\Delta}$ is unknown except that it belongs to a set $\mathcal{D}$ of input-output stable system. The *robust stability* problem is to find conditions such that the feedback system $\mathbf{H}$ so formed is input-output stable for all $\mathbf{\Delta} \in \mathcal{D}$.

Let $\mathcal{X}$ be a set of nonsingular matrices such that

$$\gamma_0(\mathbf{\Delta}_X) \leq 1 \text{ for all } X \in \mathcal{X} \text{ and } \mathbf{\Delta} \in \mathcal{D}$$

For our purpose, the set $\mathcal{D}$ is characterized by the set $\mathcal{X}$. Using Small Gain Theorem, we conclude that the feedback system $\mathbf{H}$ is input-output stable if $\mathbf{G}$ is internally stable and there exists a $X \in \mathcal{X}$ such that

$$\gamma_0(\mathbf{G}_X) < 1.$$

This is the scaled small gain problem discussed in the last Section. Thus, the robust stability problem under dynamical uncertainty is transformed to the scaled small gain problem. Obviously, the larger the set $\mathcal{X}$ is, the less conservative the stability condition is.

As an example, if $\mathbf{G}$ is described by (8.51) and (8.52) and

$$\mathcal{D} = \{\mathbf{\Delta} \mid \mathbf{\Delta} \text{ input-output stable, } \gamma_0(\mathbf{\Delta}) \leq 1\}$$

A special case of this is $u = F(y,t)y$, where $F(y,t)$ is an uncertain matrix satisfying $||F(y,t)|| \leq 1$. Then, we can choose $\mathcal{X} = \{\lambda I \mid \lambda \neq 0\}$. The discussion in the last Section indicates that (8.57) and (8.59) with $Z = \mu I$, $\mu > 0$ guarantees the asymptotic stability of the system. Without loss of generality, we can set $\mu = 1$ since it can be absorbed by the other parameters. The resulting condition is a generalization of Proposition 6.20 in Chapter 6. Indeed, with permutation of rows and columns, it can be see that (8.59) with $Z = I$ and $D = 0$ is equivalent to (6.115) in Chapter 6.

8.5.2 Uncertainty characterization

A very large class of uncertain systems can be modeled by (8.60) and (8.61) with $\mathbf{\Delta}$ having a "block-diagonal" structure through a process known as "pulling out the uncertainty" process. Specifically, u and y are partitioned as

$$\begin{aligned} u^T &= \begin{pmatrix} u_1^T & u_2^T & \cdots & u_\ell^T \end{pmatrix} \\ y^T &= \begin{pmatrix} y_1^T & y_2^T & \cdots & y_\ell^T \end{pmatrix} \end{aligned}$$

with $u_k(t) \in \mathsf{R}^{m_k}$, $y_k(t) \in \mathsf{R}^{m_k}$, $k = 1, 2, ..., \ell$, and $\mathbf{\Delta}$ is such that u_k depends only on y_k:

$$u_k = \mathbf{\Delta}_k y_k, \ \mathbf{\Delta}_k \in \mathcal{D}_k$$

Notice, we have constrained u_k and y_k to have the same dimension. Again, this constraint is imposed for notational convenience, and can be lifted. Symbolically, we can write

$$\mathbf{\Delta} = \text{diag}\begin{pmatrix} \mathbf{\Delta}_1 & \mathbf{\Delta}_2 & \cdots & \mathbf{\Delta}_\ell \end{pmatrix}$$

and

$$\gamma_0(\mathbf{\Delta}_k) \leq 1 \text{ for all } \mathbf{\Delta}_k \in \mathcal{D}_k$$

The scaling matrix set also has block-diagonal structure:

$$\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_\ell$$

Depending on the specific known information about the uncertainty set $\mathcal{D}_k$, we can find the corresponding scaling set $\mathcal{X}_k$ to satisfy

$$\gamma_0(\mathbf{\Delta}_{kX_k}) \leq 1 \text{ for all } X_k \in \mathcal{X}_k \text{ and } \mathbf{\Delta}_k \in \mathcal{D}_k$$

where $\mathbf{\Delta}_{kX_k}$ is defined as

$$\mathbf{\Delta}_{kX_k} f = X_k \mathbf{\Delta}_k (X_k^{-1} f)$$

Correspondingly,

$$\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \times \cdots \times \mathcal{Z}_\ell$$

where

$$\mathcal{Z}_k = \{X_k^T X_k \mid X_k \in \mathcal{X}_k\}$$

If $\mathbf{\Delta}_k$ has no other constraint except $\gamma_0(\mathbf{\Delta}_k) \leq 1$, which is known as a *full block*, then we can let $\mathcal{X}_k = \{\mu_k I_{m_k} \mid \mu_k \neq 0\}$ and $\mathcal{Z}_k = \{\lambda_k I_m \mid \lambda_k > 0\}$. The sets $\mathcal{X}_k$ and $\mathcal{Z}_k$ can often be enlarged when $\mathbf{\Delta}_k$ has more known constraints. An important example is *repeated linear blocks*. In this case, we can further partition

$$\begin{array}{rcl} u_k^T & = & \begin{pmatrix} u_{k1}^T & u_{k2}^T & \cdots & u_{k\ell_k}^T \end{pmatrix} \\ y_k^T & = & \begin{pmatrix} y_{k1}^T & y_{k2}^T & \cdots & y_{k\ell_k}^T \end{pmatrix} \end{array}$$

where $u_{ki}(t) \in \mathsf{R}^{\hat{m}_k}$, $m_{ki}(t) \in \mathsf{R}^{\hat{m}_k}$, $\hat{m}_k = m_k/\ell_k$, and each u_{ki} depends only on y_{ki} with an identical linear mapping

$$u_{ki} = \hat{\mathbf{\Delta}}_k y_{ki},\ i = 1, 2, ..., \ell_k,\ \hat{\mathbf{\Delta}}_k \in \hat{\mathcal{D}}_k$$

where $\hat{\mathbf{\Delta}}_k$ is linear. The most common such linear blocks are: **a.** linear time-invariant system with transfer matrix $\hat{\Delta}_k(s)$: $U_{ki}(s) = \hat{\Delta}_k(s) Y_{ki}(s)$, and **b.** time varying memoryless gain $\hat{\Delta}_k(t)$: $u_{ki}(t) = \hat{\Delta}_k(t) y_{ki}(t)$. Of course, linear time varying systems also fall into this category. Due to linearity, we can write

$$\sum_{i=1}^{k_\ell} \mu_{ki} u_{ki} = \hat{\mathbf{\Delta}}_k \left(\sum_{i=1}^{k_\ell} \mu_{ki} y_{ki} \right)$$

Therefore, we can choose $\mathcal{X}_k = \{M_k \otimes I_{\hat{m}_k} \mid M_k \in \mathsf{R}^{\ell_k \times \ell_k}, M_k \text{ nonsingular}\}$, and $\mathcal{Z}_k = \{\Lambda_k \otimes I_{\hat{m}_k} \mid \Lambda_k \in \mathsf{R}^{\ell_k \times \ell_k}, \Lambda_k > 0\}$, where $\otimes$ represents Kronecker product of matrices. The special case of $\hat{m}_k = 1$ is known as a *scalar block*.

Another important case is the time-invariant algebraic block

$$u_k(t) = f(y_k(t))$$

and the Jacobian matrix $\partial u_k/\partial y_k$ is symmetric and satisfies $||\partial u_k/\partial y_k|| \leq 1$. The symmetry of Jacobian matrix is satisfied by many system components due to physical laws. The conservatism can be further reduced by using *multipliers*. We will not discuss multipliers here.

8.5.3 Robust small gain problem

With the framework discussed so far, it is not too much more difficult to discuss the problem of guaranteed gain under block-diagonal uncertainty. Let

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{G} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $u_k, y_k \in \mathsf{R}^{m_k}$. The system is subject to the feedback uncertainty

$$u_2 = \mathbf{\Delta} y_2, \ \mathbf{\Delta} \in \mathcal{D}$$

The resulting feedback system can be written as

$$y_1 = \mathbf{H} u_1$$

Of course, the system $\mathbf{H}$ depends on $\mathbf{\Delta}$. The robust small gain problem is to check whether the resulting feedback system satisfies

$$\gamma_0(\mathbf{H}) < 1 \text{ for all } \mathbf{\Delta} \in \mathcal{D} \tag{8.62}$$

Let $\mathcal{X}_2$ satisfy

$$\gamma_0(\mathbf{\Delta}_X) \leq 1 \text{ for all } X \in \mathcal{X}_2 \text{ and } \mathbf{\Delta} \in \mathcal{D} \tag{8.63}$$

and let

$$\begin{aligned} \mathcal{X} &= \left\{ \operatorname{diag} \begin{pmatrix} I_m & X \end{pmatrix} \mid X \in \mathcal{X}_2 \right\} \\ \mathcal{Z} &= \left\{ \operatorname{diag} \begin{pmatrix} I_m & X^T X \end{pmatrix} \mid X \in \mathcal{X}_2 \right\} \end{aligned}$$

If the system $\mathbf{G}$ is described by the functional differential equation (8.39) and (8.40) satisfying (8.41) and (8.42), then (8.62) is satisfied if the conditions in Proposition 8.3 is satisfied for the above defined $\mathcal{X}$ and $\mathcal{Z}$. To see this, integrate (8.47) from 0 to t, considering zero initial condition, we obtain

$$
\begin{aligned}
&\int_0^t (||y_1(\tau)||^2 - ||u_1(\tau)||^2)d\tau \\
\leq\ & -\int_0^t w_3(||x(\tau)||) - \varepsilon \int_0^t ||u_1(\tau)||^2 d\tau - \varepsilon \int_0^t ||u_2(\tau)||^2 d\tau \\
& -V(t, x_t) - \int_0^t [y_2^T(\tau) Z y_2(\tau) - u_2^T(\tau) Z u_2(\tau)] d\tau \\
\leq\ & -\varepsilon \int_0^t ||u_1(\tau)||^2 d\tau - \int_0^t [y_2^T(\tau) Z y_2(\tau) - u_2^T(\tau) Z u_2(\tau)] d\tau
\end{aligned}
$$

But (8.63) guarantees

$$\int_0^t [y_2^T(\tau) Z y_2(\tau) - u_2^T(\tau) Z u_2(\tau)] d\tau \geq 0$$

Therefore,

$$\int_0^t ||y_1(\tau)||^2 d\tau \leq (1 - \varepsilon) \int_0^t ||u_1(\tau)||^2 d\tau$$

which implies (8.62).

8.6 Approximation approach

As an application of the above theory, we will consider the application approach of time delay systems. As has been seen from the previous Chapters, the stability problem of time-delay system in general is formidable from numerical point of view, and the class of systems we can treat directly are very limited. To extend the applicability of these methods to more general cases, we can consider appropriate approximations. The errors of such approximation can often be modelled as dynamical feedback uncertainties. The input-output approach is very convenient in analyzing the stability of the resulting feedback system, which guarantees the stability of the original system. The approximation approach can be regarded as an extension to the comparison systems approach where the nominal system is allowed to be a time delay system.

8.6.1 Approximation of time-varying delay

To illustrate the approximation of time-varying delay, consider the following system

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t - r(t)) \tag{8.64}$$

In Chapter 6, we have used the Razumikhin Theorem to study the stability of the above system. Here we assume that additional information about the delay r is available:

$$r_m \leq r(t) \leq r_M, \tag{8.65}$$
$$\dot{r}(t) \leq \rho \tag{8.66}$$

where $r_M > r_m > 0$ and $0 \leq \rho < 1$. It seems reasonable to approximate the time-varying delay by a time-invariant delay r_a, $r_a \in [r_m, r_M]$. Using (8.64), we can write

$$\begin{aligned} x(t - r(t)) &= x(t - r_a) - \int_{t-r(t)}^{t-r_a} \dot{x}(\tau) d\tau \\ &= x(t - r_a) - \int_{t-r(t)}^{t-r_a} [A_0(\tau)x(\tau) + A_1(\tau)x(\tau - r(\tau))] d\tau \end{aligned}$$

Substitute (8.64) by the above, we obtain

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + A_1(t)x(t - r_a) \\ &\quad - A_1(t) \int_{t-r(t)}^{t-r_a} [A_0(\tau)x(\tau)d\tau + A_1(\tau)x(\tau - r(\tau))] d\tau \end{aligned} \tag{8.67}$$

Similar to the case discussed in Chapter 5, (8.67) is equivalent to (8.64) only if a constraint is applied to the initial condition. Without the constraint, the stability of (8.67) implies that of (8.64) but not *vise versa* due to additional dynamics, although when $r_M - r_m$ is sufficiently small, the additional dynamics will not be unstable. The integration term in (8.67) is considered as uncertainty.

System (8.67) can be written as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{G} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{8.68}$$
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{\Delta} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{8.69}$$

where, the forward system $\mathbf{G}$ is described as

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t - r_a) - r_d A_1(t) u_2(t) \tag{8.70}$$
$$y_1(t) = \frac{1}{\sqrt{1-\rho}} x(t) \tag{8.71}$$
$$y_2(t) = A_0(t)x(t) + A_1(t)u_1(t) \tag{8.72}$$

where

$$r_d = \max\{r_M - r_a, r_a - r_m\} \tag{8.73}$$

and the feedback $\mathbf{\Delta}$ is described

$$u_1(t) = \mathbf{\Delta}_1 y_1(t) = \sqrt{1-\rho}\, y_1(t - r(t)) \tag{8.74}$$

$$u_2(t) = \mathbf{\Delta}_2 y_2(t) = \frac{1}{r_d} \int_{t-r(t)}^{t-r_a} y_2(\tau) d\tau \tag{8.75}$$

To obtain the smallest coefficient in the uncertainty term (the last term in (8.70)), we can choose $r_a = (r_M + r_m)/2$. We will show that

$$\gamma_0(\mathbf{\Delta}_{kX_k}) \leq 1, \text{ for all nonsingular } X_k \in \mathsf{R}^{n\times n},\ k = 1, 2 \tag{8.76}$$

Once (8.76) is established, we can conclude

Proposition 8.4 *The system described by (8.64) is input-output stable if the scaled small gain problem*

$$\gamma_0(\mathbf{G}_X) < 1 \textit{ for some } X = \mathrm{diag}\begin{pmatrix} X_1 & X_2 \end{pmatrix},\ X_1, X_2 \in \mathsf{R}^{n\times n} \textit{ nonsingular}$$

has a solution, where $\mathbf{G}$ *is described by (8.70) to (8.73).*

We can use the method discribed in Section 8.4 to obtain an LMI condition. If the coefficient matrices A_0 and A_1 are constant matrices, we can also use frequency domain method.

Proof. From the above discussions, we only need to show (8.76), which is equivalent to

$$\mathcal{I}_k = \int_0^t u_k^T(\tau) Z_k u_k(\tau) d\tau \leq \int_0^t y_k^T(\tau) Z_k y_k(\tau) d\tau$$

for $u_k = \mathbf{\Delta}_k y_k$ and all $Z_k \in \mathsf{R}^{n\times n}$, $Z_k = Z_k^T > 0$. For $k = 1$,

$$\mathcal{I}_1 = \int_0^t u_1^T(\tau) Z_1 u_1(\tau) d\tau = (1-\rho) \int_0^t y_1^T(\tau - r(\tau)) Z_1 y_1(\tau - r(\tau)) d\tau$$

Let $\theta = p(\tau) = \tau - r(\tau)$. Due to the derivative bound, the inverse $\tau = p^{-1}(\theta) = q(\theta)$ is well defined and differentiable, and satisfies

$$\theta + r_m \leq q(\theta) \leq \theta + r_M \tag{8.77}$$

$$0 < q'(\theta) \leq \frac{1}{1-\rho} \tag{8.78}$$

Change integration variable $\theta = p(\tau)$, considering (8.78) and the zero initial condition, we obtain

$$\begin{aligned}
\mathcal{I}_1 &= (1-\rho) \int_{-r(0)}^{t-r(t)} y_1^T(\theta) Z_1 y_1(\theta) q'(\theta) d\theta \\
&= (1-\rho) \int_0^{t-r(t)} y_1^T(\theta) Z_1 y_1(\theta) q'(\theta) d\theta \\
&\leq \int_0^t y_1^T(\theta) Z_1 y(\theta)
\end{aligned}$$

For $k = 2$,

$$\mathcal{I}_2 = \int_0^t u_2^T(\tau) Z_2 u_2(\tau) d\tau = \frac{1}{r_d^2} \int_0^t [\int_{\tau - r(\tau)}^{\tau - r_a} y_2(\theta) d\theta]^T Z_2 [\int_{\tau - r(\tau)}^{\tau - r_a} y_2(\theta) d\theta] d\tau$$

Using Jensen's inequality, we obtain

$$\mathcal{I}_2 \leq \frac{1}{r_d^2} \int_0^t (r(\tau) - r_a) \int_{\tau - r(\tau)}^{\tau - r_a} y_2^T(\theta) Z_2 y_2(\theta) d\theta d\tau$$

Exchange the order of integration, considering the zero initial condition ($y_2(\theta) = 0$ for all $\theta \leq 0$), we can obtain

$$\mathcal{I}_2 \leq \frac{1}{r_d^2} \int_0^{\max\{t - r_a, t - r(t)\}} [\int_{\min\{\theta + r_a, t\}}^{\min\{q(\theta), t\}} (r(\tau) - r_a) d\tau] y_2^T(\theta) Z_2 y_2(\theta) d\theta$$

Since

$$\begin{aligned} |r(\tau) - r_a| &\leq r_d \\ |\min\{q(\theta), t\} - \min\{\theta + r_a, t\}| &\leq r_d \end{aligned}$$

we have

$$0 \leq \int_{\max\{\theta + r_a, t\}}^{\min\{q(\theta), t\}} (r(\tau) - r_a) d\tau \leq r_d^2$$

Therefore,

$$\begin{aligned} \mathcal{I}_2 &\leq \frac{1}{r_d^2} \int_0^{\max\{t - r_a, t - r(t)\}} r_d^2 y_2^T(\theta) Z_2 y_2(\theta) d\theta \\ &\leq \int_0^t y_2^T(\theta) Z_2 y_2(\theta) d\theta \end{aligned}$$

Thus, (8.76) is established for both $k = 1$ and 2. ■

8.6.2 Approximation of distributed delay coefficient matrix

To illustrate the dynamical uncertainty introduced by approximating the coefficient matrix of distributed delays, consider the system

$$\dot{x}(t) = \int_{-r}^0 A(t, \theta) x(t + \theta) d\theta \tag{8.79}$$

We would like to approximate the matrix $A(t, \theta)$ by a piecewise constant matrix in θ. Specifically, let the interval $[-r, 0]$ be divided into N segments, $[\theta_p, \theta_{p-1}]$, $p = 1, 2, ..., N$,

$$-r = \theta_N < \theta_{N-1} < \cdots < \theta_0 = 0$$

We will also write the length of each segment $h_p = \theta_{p-1} - \theta_p$. With this division, we can write the system as

$$\dot{x}(t) = \sum_{p=1}^{N} \int_{\theta_p}^{\theta_{p-1}} A(t,\theta)x(t+\theta)d\theta$$

It seems reasonable to make the following approximation

$$\dot{x}(t) = \sum_{p=1}^{N} \int_{\theta_p}^{\theta_{p-1}} A_p(t)x(t+\theta)d\theta + u(t) \tag{8.80}$$

where

$$A_p(t) = \frac{1}{h_p} \int_{\theta_p}^{\theta_{p-1}} A(t,\theta)d\theta \tag{8.81}$$

and $u(t)$ represents the error of such an approximation, and can be explicitly expressed as

$$u(t) = \sum_{p=1}^{N} \int_{\theta_p}^{\theta_{p-1}} (A_p(t) - A(t,\theta))x(t+\theta)d\theta \tag{8.82}$$

To estimate the error bound, we first give the following result:

Lemma 8.5 *A system $u = \mathbf{F}y$ described by*

$$\begin{aligned} u(t) &= \int_{-r}^{0} U(t,\theta)y(t+\theta)d\theta \\ y(\tau) &= 0,\ \tau \leq 0 \end{aligned}$$

satisfies

$$\gamma_0(\mathbf{F}) \leq \sqrt{r \int_{-r}^{0} (\max_{t\geq 0} ||U(t,\theta)||^2)d\theta}$$

Proof. Using Jensen's inequality, we have

$$\begin{aligned}
&\int_0^t u^T(\tau)u(\tau)d\tau \\
= &\int_0^t [\int_{-r}^0 U(\tau,\theta)y(\tau+\theta)d\theta]^T [\int_{-r}^0 U(\tau,\theta)y(\tau+\theta)d\theta]d\tau \\
\leq &\int_0^t [r\int_{-r}^0 y^T(\tau+\theta)U^T(\tau,\theta)U(\tau,\theta)y(\tau+\theta)d\theta]d\tau \\
= & r\int_{-r}^0 [\int_0^t y^T(\tau+\theta)U^T(\tau,\theta)U(\tau,\theta)y(\tau+\theta)d\tau]d\theta \\
= & r\int_{-r}^0 [\int_\theta^{t+\theta} y^T(\tau)U^T(\tau-\theta,\theta)U(\tau-\theta,\theta)y(\tau)d\tau]d\theta \\
\leq & r\int_{-r}^0 [\int_0^t y^T(\tau)U^T(\tau-\theta,\theta)U(\tau-\theta,\theta)y(\tau)d\tau]d\theta \\
= &\int_0^t y^T(\tau)[r\int_{-r}^0 U^T(\tau-\theta,\theta)U(\tau-\theta,\theta)d\theta]y(\tau)d\tau \\
\leq & [r\int_{-r}^0 (\max_{t\geq 0}||U(t,\theta)||^2)d\theta]\int_0^t y^T(\tau)y(\tau)d\tau
\end{aligned}$$

■

From the above Lemma, we can immediately conclude that:

Proposition 8.6 *The system described by (8.79) is input-output stable if the system* **G** *described by (8.80) and*

$$y(t) = \alpha x(t) \tag{8.83}$$

where α is a constant satisfying

$$\alpha \geq \sqrt{r\sum_{p=1}^N \int_{\theta_p}^{\theta_{p-1}} [\max_{t\geq 0}||A_p(t)-A(t,\theta)||^2]d\theta} \tag{8.84}$$

satisfies

$$\gamma_0(\mathbf{G}) < 1 \tag{8.85}$$

Proof. System (8.80) can be written as

$$\begin{aligned} y &= \mathbf{G}u \\ u &= \mathbf{\Delta} y \end{aligned}$$

where $\mathbf{\Delta}$ is described by

$$u(t) = \frac{1}{\alpha}\sum_{p=1}^N \int_{\theta_p}^{\theta_{p-1}} (A_p(t)-A(t,\theta))y(t+\theta)d\theta \tag{8.86}$$

It is easy to conclude from Lemma 8.5 that $\gamma_0(\mathbf{\Delta}) \leq 1$. Therefore, (8.85) is sufficient for stability. ■

In applying the above proposition, we may choose, for example,

$$\alpha = \sqrt{r \sum_{p=1}^{N} h_p e_p^2} \tag{8.87}$$

where

$$e_p = \max_{\substack{t \geq 0 \\ \theta \in (\theta_p, \theta_{p-1}]}} ||A_p(t) - A(t,\theta)|| \tag{8.88}$$

As pointed out in Section 8.4, a numerical method can be devised to check (8.85).

When h_p, $p = 1, 2, ..., N$ are small, a smaller error bound can often be obtained by carrying out model transformation: for $\theta \in [\theta_p, \theta_{p-1}]$,

$$\begin{aligned} x(t+\theta) &= x(t+\theta_{ap}) - \int_{t+\theta}^{t+\theta_{ap}} \dot{x}(\tau)d\tau \\ &= x(t+\theta_{ap}) - \int_{t+\theta}^{t+\theta_{ap}} [\int_{-r}^{0} A(\tau,\xi)x(\tau+\xi)d\xi]d\tau \end{aligned} \tag{8.89}$$

where

$$\theta_{ap} = (\theta_p + \theta_{p-1})/2 \tag{8.90}$$

Using the above in (8.82), we may conclude

Proposition 8.7 *The system described by (8.79) is input-output stable if the system $\tilde{\mathbf{G}}$ described by (8.80) and*

$$y(t) = \beta x(t) \tag{8.91}$$

where

$$\beta = \frac{r}{2}[\int_{-r}^{0} (\max_{t\geq 0} ||A(t,\theta)||^2)d\theta \sum_{p=1}^{N} h_p^3 e_p^2]^{1/2} \tag{8.92}$$

satisfies

$$\gamma_0(\tilde{\mathbf{G}}) < 1 \tag{8.93}$$

Proof. Use (8.89) in (8.82), taking into account (8.81) and (8.91), we obtain

$$u(t) = -\frac{1}{\beta} \sum_{p=1}^{N} \int_{\theta_p}^{\theta_{p-1}} (A_p(t) - A(t,\theta))[\int_{t+\theta}^{t+\theta_{ap}} \int_{-r}^{0} A(\tau,\xi)y(\tau+\xi)d\xi d\tau]d\theta \tag{8.94}$$

Therefore, system (8.80) can be written as

$$\begin{aligned} y &= \tilde{\mathbf{G}}u \\ u &= \tilde{\mathbf{\Delta}}y \end{aligned}$$

where $\tilde{\mathbf{\Delta}}$ is defined as (8.94). Define

$$z(t) = \mathbf{\Delta}_1 y(t) = \int_{-r}^{0} A(t,\xi)y(t+\xi)d\xi \tag{8.95}$$

Then Lemma 8.5 indicates that

$$\gamma_0(\mathbf{\Delta}_1) \le \sqrt{r\int_{-r}^{0} (\max_{t\ge 0} ||A(t,\theta)||^2)d\theta} \tag{8.96}$$

Use (8.95) in (8.94), we have

$$\begin{aligned} u(t) &= -\frac{1}{\beta}\sum_{p=1}^{N}\int_{\theta_p}^{\theta_{p-1}} (A_p(t) - A(t,\theta))[\int_{t+\theta}^{t+\theta_{ap}} z(\tau)d\tau]d\theta \\ &= -\frac{1}{\beta}\sum_{p=1}^{N}\int_{\theta_p}^{\theta_{p-1}} (A_p(t) - A(t,\theta))[\int_{\theta}^{\theta_{ap}} z(t+\tau)d\tau]d\theta \end{aligned}$$

Exchange the order of integration, we obtain

$$\begin{aligned} u(t) &= \frac{1}{\beta}\sum_{p=1}^{N}[\int_{\theta_{ap}}^{\theta_{p-1}}\int_{\tau}^{\theta_{p-1}} (A_p(t) - A(t,\theta))d\theta z(t+\tau)d\tau \\ &\quad -\int_{\theta_p}^{\theta_{ap}}\int_{\theta_p}^{\tau}(A_p(t) - A(t,\theta))d\theta z(t+\tau)d\tau] \end{aligned}$$

or

$$u(t) = \mathbf{\Delta}_2 z(t) = \frac{1}{\beta}\int_{-r}^{0} H(t,\tau)z(t+\tau)d\tau$$

where

$$H(t,\tau) = \begin{cases} -\int_{\theta_p}^{\tau}(A_p(t) - A(t,\theta))d\theta & \tau \in (\theta_p, \theta_{ap}] \\ \int_{\tau}^{\theta_{p-1}}(A_p(t) - A(t,\theta))d\theta & \tau \in (\theta_{ap}, \theta_{p-1}] \end{cases}$$

Since

$$\max_{t\ge 0} ||H(t,\tau)|| \le \max_{t\ge 0}\frac{h_p}{2}||A_p(t) - A(t,\theta)|| \le h_p e_p/2,\ \tau \in [\theta_p, \theta_{p-1}]$$

Lemma 8.5 indicates that

$$\gamma_0(\mathbf{\Delta}_2) \le \frac{1}{\beta}\sqrt{r\int_{-r}^{0} (\max_{t\ge 0} ||H(t,\theta)||^2)d\theta} \le \frac{1}{\beta}\sqrt{r\sum_{p=1}^{N} h_p^3 e_p^2/4}$$

we have

$$\gamma_0(\tilde{\mathbf{\Delta}}) \le \gamma_0(\mathbf{\Delta}_2)\gamma_0(\mathbf{\Delta}_1) \le 1$$

Therefore, (8.93) is sufficient for stability. ∎

To compare the two criteria, assume uniform division $h_p = h = r/N$, and use (8.87) to calculate α, then we have

$$\beta/\alpha = \frac{h}{2}[r \int_{-r}^{0} (\max_{t\ge 0} ||A(t,\theta)||^2) d\theta]^{1/2}$$

Therefore, when h is sufficiently small, $\beta < \alpha$, and therefore, $\gamma_0(\tilde{\mathbf{G}}) < \gamma_0(\mathbf{G})$, Proposition 8.7 is less conservative.

8.6.3 *Approximation by multiple delays*

Consider

$$\dot{x}(t) = \int_{-r}^{0} dF(\theta)x(t+\theta) \tag{8.97}$$

where F is of bounded variation, continuous from the left on $(-r, 0)$, and satisfies $F(\theta) = 0$, $\theta \ge 0$; $F(\theta) = F(-r)$, $\theta < -r$. We would like to approximate it by a system with multiple pointwise delays. Let

$$\begin{aligned} h &= r/N \\ r_k &= kh \\ r_{mk} &= (k-1/2)h \\ r_{Mk} &= (k+1/2)h \end{aligned}$$

We can write

$$\dot{x}(t) = \sum_{k=0}^{N} \int_{-r_{Mk}}^{-r_{mk}} dF(\theta)x(t+\theta) \tag{8.98}$$

Using

$$\begin{aligned} x(t+\theta) &= x(t-r_k) - \int_{\theta}^{-r_k} \dot{x}(t+\tau)d\tau \\ &= x(t-r_k) - \int_{\theta}^{-r_k} [\int_{-r}^{0} dF(\theta)x(t+\tau+\theta)]d\tau \end{aligned}$$

in (8.98), we obtain

$$\dot{x}(t) = \sum_{k=0}^{N} A_k x(t-r_k) + u(t) \tag{8.99}$$

where

$$A_k = \int_{-r_{Mk}}^{-r_{mk}} dF(\theta)$$

and

$$u(t) = -\sum_{k=0}^{N} \int_{-r_{Mk}}^{-r_{mk}} dF(\theta) \int_{\theta}^{-r_k} z(t+\tau)d\tau \tag{8.100}$$

where

$$z(t) = \int_{-r}^{0} dF(\theta)x(t+\theta)$$

Let

$$y(t) = \alpha x(t) \tag{8.101}$$

where

$$\alpha = \mu^2\sqrt{rh}$$

and μ is the total variation of F in $[-r, 0]$:

$$\mu = \sup_{\substack{K>0 \\ -r=\theta_0<\theta_1<\cdots<\theta_K=0}} \sum_{i=1}^{K} ||F(\theta_i) - F(\theta_{i-1})|| = \operatorname*{Var}_{[-r,0]} F(\theta)$$

Then we can write the system (8.97) as

$$\begin{aligned} y &= \mathbf{G}u \\ u &= \mathbf{\Delta}y \end{aligned}$$

where $\mathbf{G}$ is described by (8.99) and (8.101), and

$$\mathbf{\Delta} = \mathbf{\Delta}_1\mathbf{\Delta}_2$$

where $u = \mathbf{\Delta}_1 z$ is described by (8.100), and $z = \mathbf{\Delta}_2 y$ is described by

$$z(t) = \frac{1}{\alpha}\int_{-r}^{0} dF(\theta)y(t+\theta)$$

With this framework, we will show in the following Proposition that $\gamma_0(\mathbf{\Delta}) \leq 1$, and therefore, the stability is assured if $\gamma_0(\mathbf{G}) < 1$. An important observation is $||u||_2 \leq \alpha||x||_2$, and $\alpha \to 0$ as $h \to 0$, representing asymptotically close approximation, which is intuitively clear.

Proposition 8.8 *The system described by (8.97) is input-output stable if the system* $\mathbf{G}$ *described by (8.99) and (8.101) satisfies* $\gamma_0(\mathbf{G}) < 1$.

Proof. We only need to show $\gamma_0(\mathbf{\Delta}) \leq 1$. For arbitrary interval $[a, b]$ and scalar function $\beta(\theta) \geq 0, \theta \in [a, b]$, define

$$\int_a^b \beta(\theta)|dF(\theta)| = \sup_{\substack{K>0 \\ a=\theta_0<\theta_1<\cdots<\theta_K=b}} \sum_{i=1}^{K} \beta(\theta_i)||F(\theta_i) - F(\theta_{i-1})||$$

The special case of $\beta(\theta) = 1$ is the total variation of $F(\theta)$ in $[a, b]$:

$$\int_a^b |dF(\theta)| = \operatorname*{Var}_{[a,b]} F(\theta)$$

especially,

$$\int_{-r}^0 |dF(\theta)| = \mu$$

With this definition, we have another form of Jensen's inequality (see [256]):

$$[\int_a^b dF(\theta)\eta(\theta)]^T[\int_a^b dF(\theta)\eta(\theta)] \le [\int_a^b |dF(\theta)|][\int_a^b |dF(\theta)|\eta^T(\theta)\eta(\theta)]$$

We can write

$$\begin{aligned}
\int_0^t z^T(\tau)z(\tau)d\tau &= \int_0^t [\frac{1}{\alpha}\int_{-r}^0 dF(\theta)y(\tau+\theta)]^T[\frac{1}{\alpha}\int_{-r}^0 dF(\theta)y(\tau+\theta)]d\tau \\
&\le \frac{1}{\alpha^2}\int_0^t \mu \int_{-r}^0 |dF(\theta)|\ ||y(\tau+\theta)||^2 d\tau \\
&= \frac{\mu}{\alpha^2}\int_{-r}^0 |dF(\theta)| \int_0^t ||y(\tau+\theta)||^2 d\tau \\
&= \frac{\mu}{\alpha^2}\int_{-r}^0 |dF(\theta)| \int_\theta^{t+\theta} ||y(\tau)||^2 d\tau \\
&\le \frac{\mu}{\alpha^2}\int_{-r}^0 |dF(\theta)| \int_0^t ||y(\tau)||^2 d\tau \\
&= \frac{\mu^2}{\alpha^2}\int_0^t y^T(\tau)y(\tau)d\tau
\end{aligned}$$

Therefore

$$\gamma_0(\mathbf{\Delta}_2) \le \mu/\alpha$$

Also, exchange the order of integration in (8.100), we obtain

$$\begin{aligned}
u(t) &= \sum_{k=0}^N \int_{-r_{Mk}}^{-r_{mk}} [\int_\tau^{-r_{mk}} dF(\theta) - \int_{-r_{Mk}}^\tau dF(\theta)]z(t+\tau)d\tau \\
&= \sum_{k=0}^N \int_{-r_{Mk}}^{-r_{mk}} [F(-r_{mk}) - F(\tau) - F(\tau) + F(-r_{Mk})]z(t+\tau)d\tau \\
&= \int_{-r}^0 T(\tau)z(t+\tau)d\tau
\end{aligned}$$

where

$$T(\tau) = [F(-r_{mk}) - F(\tau)] - [F(\tau) - F(-r_{Mk})],\ \tau \in [-r_{Mk}, -r_{mk}]$$

Using Lemma 8.5, we can conclude that

$$\begin{aligned}
\frac{\gamma_0^2(\mathbf{\Delta}_1)}{r} &\leq \int_{-r}^{0} ||T(\tau)||^2 d\tau \\
&= \sum_{k=0}^{N} \int_{-r_{Mk}}^{-r_{mk}} ||T(\tau)||^2 d\tau \\
&= \sum_{k=0}^{N} \int_{-h/2}^{h/2} ||T(\tau - kh)||^2 d\tau \\
&= \int_{-h/2}^{h/2} \sum_{k=0}^{N} ||T(\tau - kh)||^2 d\tau \\
&\leq \int_{-h/2}^{h/2} \left(\sum_{k=0}^{N} ||T(\tau - kh)|| \right)^2 d\tau \\
&\leq \int_{-h/2}^{h/2} \mu^2 d\tau \\
&= h\mu^2
\end{aligned}$$

Therefore

$$\gamma_0(\mathbf{\Delta}_1) \leq \sqrt{r\mu^2 h}$$

We obtain

$$\begin{aligned}
\gamma_0(\mathbf{\Delta}) &\leq \gamma_0(\mathbf{\Delta}_1)\gamma_0(\mathbf{\Delta}_2) \\
&\leq \sqrt{r\mu^2 h}\mu/\alpha \\
&= 1
\end{aligned}$$

■

8.7 Passivity and generalization

A system $\mathbf{G} : L_{2e+}^n \to L_{2e+}^n$ is *passive* if for any $u \in L_{2e+}^n$, and $y = \mathbf{G}u$, we have

$$\int_0^t y^T(\tau)u(\tau) \geq 0 \text{ for all } t > 0 \tag{8.102}$$

It is *strictly passive* if there exists an $\varepsilon > 0$ such that (8.102) can be replaced by

$$\int_0^t y^T(\tau)u(\tau) \geq \varepsilon \int_0^t ||u(\tau)||^2 d\tau \text{ for all } t > 0 \tag{8.103}$$

Passivity theory plays an important role in stability theory, mainly due to the fact that many natural systems without external power supply are often passive. Also, roughly speaking, the passivity is preserved under negative feedback. It also has important applications in adaptive control.

For finite dimensional linear time-invariant systems, passivity can be checked by the celebrated Kalman-Yacubovitch-Popov Lemma, which can also be written as an LMI. For a time delay system described by (8.39) and (8.40) satisfying (8.41) and (8.42), then it can be easily shown that the system is internally stable and strictly passive if the the conditions in Proposition 8.3 are satisfied after (8.47) is replaced by

$$W(t, x_t, u(t)) = \dot{V}(t, x_t) + y^T(t)u(t) \tag{8.104}$$

It is passive if the conditions for passivity is satisfied after (8.48) is replaced by

$$W(t, x_t, u(t)) \leq \varepsilon ||x(t)||^2$$

More generally, it is sometimes desirable to consider systems satisfying *integral quadratic constraint (IQC)*

$$\int_0^t \begin{pmatrix} y^T(\tau) & u^T(\tau) \end{pmatrix} Z \begin{pmatrix} y(\tau) \\ u(\tau) \end{pmatrix} d\tau \geq \varepsilon \int_0^t ||u(\tau)||^2 d\tau \text{ for all } t > 0$$

Consider time delay system described by (8.39) and (8.40) satisfying (8.41) and (8.42), we can also show that the system is internally stable and satisfies the above IQC if the the conditions in Proposition 8.3 are satisfied after (8.47) is replaced by

$$W(t, x_t, u(t)) = \dot{V}(t, x_t) + \begin{pmatrix} y^T(\tau) & u^T(\tau) \end{pmatrix} Z \begin{pmatrix} y(\tau) \\ u(\tau) \end{pmatrix}$$

8.8 Notes

Input-output approach to stability and Small Gain Theorem are discussed in more details in Desoer and Vidyasagar [60] and Vidyasagar [301], where more general settings are used.

The comparison system approach were explored by, for example, Halanay [107], Lakshmikhantam and Leela [171], Driver [57] and Huang and Zhou [127]. The material presented here is a generalization of Zhang, *et. al.* [316]. In Zhang, *et. al.* [316], C is constrained to $A_1 M$, and frequency domain method was used, and equivalence was established with the stability criterion proposed in Park [240] (see also the Notes in Chapter 5).

The scaled small gain problem and related passivity problem for finite dimensional linear time invariant systems have been discussed in details in, for example, Boyd and Yang [25], Packard and Doyle [229], with emphasis on robust stability. For linear time-delay systems, see Kokame, *et. al.* [156], De Souza and Li [255], Gu [93] and Han and Gu [116] for time domain approach and Huang and Zhou [128] for frequency domain approach. The process of "Pulling out uncertainty" was discussed in Doyle, *et. al.* [64]. For multiplier method used for nonlinear time-invariant uncertainty, see Popov [235] and Bliman [15] for single variable case, and Desoer and Wu [59] for multiple variable case with symmetric Jacobian matrix (known as reciprocal relations, see Karnopp, *et. al.* [169]).

The time-varying system described by (8.64) to (8.66) was discussed in Gu and Han [101], where a more primitive discretized Lyapunov functional method was used and r_a was set at r_m. The special case of $r_m = 0$ were discussed in, for example, Kim [152] for delay-dependent case, and Cao and Sun [32], Phoojaruenchanachai and Furuta [234] for delay-independent case. The framework presented as an approximation problem is new. It should also be noted that it is also possible for time-varying delays to have stabilizing effects, see Louisell [184].

The importance of studying the stability problem under approximation is illustrated by a number of examples in the literature: Louisell [187] gave a system which is stable independent of delays when delays are constrained to be commensurate, and becomes unstable with a small deviation from this constraint. Another example was given by Engelborghs, *et. al.* [66] where a stable system with distributed delays obtained by some infinite dimensional control design (such as spectrum assignment) may become unstable when numerical quadrature is used to approximate the distributed delays.

The passivity problem is discussed in more details in Desoer and Vidyasagar [60] and Vidyasagar [301], and Popov [236]. For some extension of bounded real lemma and Kalman-Yacubovitch-Popov Lemma, see Shaked, *et. al.* [252], Bliman [15], Halanay [107], Popov and Halanay [237]. For the significance of integral quadratic constraint, see Scherer, *et. al.* [254].

Appendix A

Matrix facts

In this appendix, we will discuss some useful facts about matrices. We assume the readers are familiar with the basic matrix algebra. The materials presented here is not intended to be self-contained. Rather, it is intended to fix the notations and point out some facts which may not be familiar to some readers. For a systematic presentation of these materials, the readers are referred to the references discussed in the *Notes* section at the end of this Appendix.

We use R^n to represent the set of real column vectors with n components, and C^n of complex column vectors. Similarly, $\mathsf{R}^{m\times n}$ and $\mathsf{C}^{m\times n}$ represent the sets of real and complex m by n matrices, respectively. We will use x_i to denote the ith component of column vector x, and a_{ij} to represent the element of matrix A at the ith row and jth column. A^T is the transpose of A. For complex matrices, the Hermitian transpose defined as $A^H \triangleq \bar{A}^T$ is often used, where $\bar{A}$ is the complex conjugate of A. In the following, we will mainly discuss complex matrices, but with obvious specialization to real matrices. I denotes an identity matrix. If we need to emphasize the dimention, we may write, for example, I_n for n-dimensional identity matrix. The rank of A, the number of linearly independent columns, is denoted as $\mathrm{rank}(A)$. If $\mathrm{rank}(A) = \min\{m, n\}$, then we say A is full rank. We say A is of full row rank if $\mathrm{rank}(A) = m$, and full column rank if $\mathrm{rank}(A) = n$. For a square matrix $A \in \mathsf{C}^{nxn}$, we also use $\mathrm{tr}(A)$ to represent the trace of A, *i.e.*,

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

A block-diagonal matrix with $A_1, A_2, \cdots, A_p$ in the diagonal entries is denoted as $\mathrm{diag}\begin{pmatrix} A_1 & A_2 & \cdots & A_p \end{pmatrix}$.

A square matrix A is *Hermitian* if $A^H = A$. We use $A > 0$ to denote the fact that A is Hermitian positive definite. Similarly, "$\geq$", "$<$" and "$\leq$" denote positive semi-definiteness, negative definiteness and negative semi-definiteness in the matrix context. We also use $A > B$ to mean $A - B > 0$, with obvious extension to other inequality signs.

A.1 Determinant

For a square matrix $A \in \mathsf{C}^{n\times n}$, its determinant is denoted as $\det(A)$. If A and B are both square matrices, then it can be shown that

$$\det(AB) = \det(A)\det(B) \tag{A.1}$$

Another well known fact is that

$$\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(C), \tag{A.2}$$

whenever A and D are both square. If A is square and nonsingular, then we can use (A.1), (A.2) and the fact

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

to obtain

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B), \tag{A.3}$$

which is known as *Schur (determinant) complement*. Similarly, if D is nonsingular, we can show

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det(A - BD^{-1}C). \tag{A.4}$$

Let $A = I$ and $D = I$ in the Schur complements (A.3) and (A.4), we arrive at another very useful identity: if CB is square

$$\det(I - CB) = \det(I - BC).$$

Especially, if k and g are column vectors, we have

$$\det(I - kg^T) = 1 - g^T k.$$

A.2 Eigenvalue problems

For a matrix $A \in \mathsf{C}^{n\times n}$, any solution λ of the following nth order polynomial equation

$$\det(\lambda I - A) = 0 \tag{A.5}$$

is known as an *eigenvalue*, and is often denoted as $\lambda(A)$. Obviously, there are always n eigenvalues for an n by n matrix counting multiplicity. The eigenvalues are often ordered according to some scheme, in which case, we often use $\lambda_i(A)$ to denote the ith eigenvalue of A. The set of all the eigenvalues of matrix A is known as the spectrum of A, and is denoted as $\sigma(A)$. The spectrum radius is defined as

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i(A)|.$$

We also write, similarly,

$$\underline{\rho}(A) = \min_{1 \le i \le n} |\lambda_i(A)|.$$

The eigenvalues of a matrix satisfy the following simple relations

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i(A) &= \operatorname{tr}(A), \\ \prod_{i=1}^{n} \lambda_i(A) &= \det(A). \end{aligned}$$

Also, if $A \in \mathsf{C}^{m\times n}$ and $B \in \mathsf{C}^{n\times m}$, then AB and BA share the same nonzero eigenvalues.

Corresponding to any eigenvalue λ, an *eigenvector* is any $\xi \in \mathsf{C}^n$, $\xi \neq 0$ satisfying

$$A\xi = \lambda\xi. \tag{A.6}$$

For an eigenvalue λ of multiplicity m, if there do not exist m linearly independent eigenvectors, then we can always find one or more *generalized eigenvectors* η satisfying

$$(A - \lambda I)\eta = \xi, \tag{A.7}$$

where ξ is either an eigenvector or a generalized eigenvector. We can always find a set of m linearly independent vectors consisting of the eigenvectors and generalized eigenvectors corresponding to an eigenvalue of multiplicity m.

If A is Hermitian, then all the eigenvalues are real, and there are always m independent eigenvectors corresponding to an eigenvalues of multiplicity m. In which case, we use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ to denote the minimum and maximum eigenvalues.

Given two square matrices A and B, a *generalized eigenvalue* λ is a solution of the equation

$$\det(\lambda B - A) = 0,$$

which is often denoted as $\lambda(A, B)$. The spectrum $\sigma(A, B)$ is the set of all the generalized eigenvalues. Similarly, an eigenvector in the generalized eigenvalue problem is any $\xi \neq 0$ satisfying

$$A\xi = \lambda B\xi.$$

Notice the difference between the generalized eigenvector and the eigenvector in the generalized eigenvalue problem.

A.3 Singular value decomposition

For a given $A \in \mathsf{C}^{m\times n}$, let $p = \min\{m, n\}$, then we can always write

$$A = U\Sigma V^H, \tag{A.8}$$

where Σ is a real diagonal matrix

$$\begin{aligned} \Sigma &= \operatorname{diag}\begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_p \end{pmatrix}, \\ \sigma_1 &\geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0, \end{aligned}$$

and $U \in \mathsf{C}^{m\times p}$, $V \in \mathsf{C}^{n\times p}$ satisfy

$$U^H U = V^H V = I_p.$$

Expression (A.8) is knonwn as the *singular value decomposition* of A, and σ_i, $i = 1, 2, \ldots, n$ are its *singular values.* We write

$$\begin{aligned} \sigma_{\max}(A) &= \max_{1\leq i\leq p} \sigma_i, \\ \sigma_{\min}(A) &= \min_{1\leq i\leq p} \sigma_i. \end{aligned}$$

We sometimes write $\sigma_{\max}$ and $\sigma_{\min}$ as $\bar{\sigma}$ and $\underline{\sigma}$, respectively. Singular values have the following relations with the eigenvalues

$$\sigma_{\min}(A) \leq \min_{1\leq i\leq n} |\lambda(A)| \leq \max_{1\leq i\leq n} |\lambda(A)| \leq \sigma_{\max}(A).$$

A.4 Norms

A norm $||\cdot||$ is a function defined on C^n satisfying the following three conditions

1. $||x|| = 0$ if and only if $x = 0$,
2. $||\alpha x|| = |\alpha| \cdot ||x||$ for any scalar α,
3. $||x + y|| \leq ||x|| + ||y||$.

For $x \in \mathsf{C}^n$, let

$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T.$$

Then the three most common norms are:

1. 1-norm:

$$||x||_1 = \sum_{i=1}^{n} |x_i|,$$

2. 2-norm:

$$||x||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2},$$

3. ∞-norm

$$||x||_\infty = \max_{1 \leq i \leq \infty} |x_i|.$$

The 2-norm is geometrically the most appealing, but 1-norm and ∞-norms are sometimes more convenient to use. Obviously, an m by n matrix can be regarded as a vector with $m \times n$ components, and the norms can be defined accordingly. For example, corresponding to the 2-norm, we can define the following norm of a matrix $A \in \mathsf{C}^{m \times n}$

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.$$

However, the most useful norms for matrices are the *induced norms* defined in terms of the corresponding vector norm

$$||A||_k = \max_{||x||_k \leq 1} ||Ax||_k = \max_{||x||_k \neq 0} \frac{||Ax||_k}{||x||_k}.$$

It can be shown that the three most common induced norms are

1. 1-norm

$$||A||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|,$$

2. 2-norm

$$||A||_2 = \sqrt{\lambda_{\max}(A^H A)},$$

3. ∞-norm

$$||A||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |a_{ij}|.$$

Clearly, $||A||_2 = \sigma_{\max}(A)$. An important advantage of induced norms is the fact that it is compatible with matrix multiplication operation in the following sense:

$$||AB|| \leq ||A|| \cdot ||B||.$$

A.5 Kronecker product and sum

For $A \in \mathsf{C}^{m\times n}$, $B \in \mathsf{C}^{p\times q}$, the Kronecker product of A and B, $A \otimes B$, is a mp by nq matrix defined as

$$A \otimes B \triangleq \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

For square matrices $A \in \mathsf{C}^{m\times m}$, $B \in \mathsf{C}^{n\times n}$, we can also define their Kronecker sum as

$$A \oplus B \triangleq A \otimes I_n + I_m \otimes B.$$

If λ_i, $i = 1, 2, \ldots, m$ are the eigenvalues of A, and μ_j, $j = 1, 2, \ldots, n$ are the eigenvalues of B, then, the mn eigenvalues of $A \otimes B$ and $A \oplus B$ are $\lambda_i \mu_j$, and $\lambda_i + \mu_j$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, respectively.

A.6 Notes

Most of the materials presented here can be found in Golub and Van Loan [81] and Horn and Johnson [122]. The Kronecker product and sum are covered in Graham [88].

Appendix B

Linear Matrix Inequalities and Quadratic Integral Inequalities

B.1 Basic LMI problem

A strict linear matrix inequality (LMI) has the general form of

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{B.1}$$

where

$$x = \begin{pmatrix} x_1 & x_2 & ... & x_m \end{pmatrix}^T \in R^m$$

is a vector consisting of m variables, and $F_i = F_i^T \in \mathsf{R}^{n\times n}$, $i = 0, 1, 2, ..., m$ are $m + 1$ given constant matrices. An LMI may also be non-strict where "$>$" is replaced by "$\geq$". We will only discuss strict LMI here. Notice, the variables appear linearly on the left hand side of the inequality. The basic LMI problem is to find whether or not there exists an $x \in \mathsf{R}^m$ such that (B.1) is satisfied. The importance of LMI lies in two facts: i) There exist efficient computational methods to solve LMI; ii) Many practical problems may be formulated in the standard LMI form. In this appendix, we will illustrate some simple cases where it is possible to transform some common problems to the standard LMI form (B.1).

Obviously,

$$F(x) < 0$$

may be written as

$$-F(x) > 0,$$

which is in the standard form (B.1). Also, a set of LMIs may be viewed as one LMI. For example,

$$F^{(i)}(x) > 0,\ i = 1, 2$$

is equivalent to one LMI

$$\begin{pmatrix} F^{(1)}(x) & 0 \\ 0 & F^{(2)}(x) \end{pmatrix} > 0.$$

For the sake of convenience, the variables often appear in matrix form. For example,

$$\begin{aligned} (PA + A^T P) &< 0, && \text{(B.2)} \\ P &> 0, && \text{(B.3)} \end{aligned}$$

the Lyapunov inequality, is an LMI with symmetric matrix $P = P^T \in \mathsf{R}^{n\times n}$ as its variable, which is equivalent to $\frac{1}{2}n(n+1)$ independent scalar variables and can be written in the standard form of (B.1). With this understanding, it is common practice to leave an LMI with matrix variables.

Another important fact worth mentioning is that the positive definiteness of a matrix implies the positive definiteness of all its principle minors, where a principle minor is a matrix (say, $m \times m$) consisting of the elements in the $i_1, i_2, ..., i_m$th rows and columns of the original matrix ($i_1, i_2, ..., i_m$ disctinct). For example,

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} > 0,$$

implies

$$a > 0,\ d > 0,\ f > 0,$$

and

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} > 0,\ \begin{pmatrix} a & c \\ c & f \end{pmatrix} > 0,\ \begin{pmatrix} d & e \\ e & f \end{pmatrix} > 0.$$

B.2 Generalized Eigenvalue Problem (GEVP)

The generalized eigenvalue problem (GEVP) is to find

$$\inf \lambda, \tag{B.4}$$

subject to the constraint

$$\begin{aligned} \lambda B(x) - A(x) &> 0, && \text{(B.5)} \\ B(x) &> 0, && \text{(B.6)} \\ C(x) &> 0, && \text{(B.7)} \end{aligned}$$

where $A(x)$, $B(x)$ and $C(x)$ are matrices depending on the variable $x \in \mathsf{R}^m$ linearly. As a special case, the constraint (B.7) may be absent. The terminology is motivated by the generalized eigenvalue problem in the classical eigenvalue theory of matrices, which is also very useful in this book. To see the connection, consider the special case of the above problem: when A and B are constant matrices, and the additional constraint (B.7) is absent, then the solution of the problem is the greatest generalized eigenvalue $\lambda_{\max}$ to satisfy the following generalized eigenvalue problem

$$A\xi = \lambda B\xi,\ \ \xi \neq 0.$$

We will refer to the problem described by (B.4) to (B.7) by the abbreviation GEVP in order to distinguish it from the classical generalized eigenvalue problem.

A number of problems encountered in this book may be conveniently transformed into the standard GEVP. For example, consider the problem

$$\bar{\alpha} = \sup \alpha, \tag{B.8}$$

subject to

$$\begin{aligned} P(x) + \alpha Q(x) &> 0, && \text{(B.9)} \\ P(x) &> 0, && \text{(B.10)} \\ R(x) &< 0. && \text{(B.11)} \end{aligned}$$

This problem can be transformed to the standard GEVP

$$\frac{1}{\bar{\alpha}} = \inf \beta,$$

subject to

$$\begin{aligned} \beta P(x) + Q(x) &> 0, \\ P(x) &> 0, \\ -R(x) &> 0. \end{aligned}$$

Another problem that can be easily transformed to the standard GEVP is

$$\bar{\alpha} = \sup \alpha,$$

subject to

$$\begin{aligned} P(x) + \alpha Q(x) &< 0, \\ Q(x) &> 0. \end{aligned}$$

This is equivalent to the standard GEVP

$$\frac{1}{\bar{\alpha}} = \inf \beta,$$

subject to

$$\begin{aligned} -\beta P(x) - Q(x) &> 0, \\ -P(x) &> 0, \\ Q(x) &> 0. \end{aligned}$$

We will simply refer to these problems also as GEVP without further comments.

B.3 Transforming nonlinear matrix inequalities to LMI form

In many practical problems, the parameters may appear nonlinearly in its most natural form. The fact that many of them can be transformed into an LMI form is an important observation. Here, we only mention some important techniques:

Proposition B.1 *For any nonsingular matrix A of compatible dimension, a matrix inequality*

$$F > 0 \tag{B.12}$$

is satisfied if and only if

$$AFA^T > 0. \tag{B.13}$$

This obvious fact has some important consequences. Especially, we may choose A to be block version of "elementary matrices", in which case, left-multiplying F by A is equivalent to one of the following operations on matrix F:

1. Multiply some row block by a nonsingular matrix;
2. Left multiply a row block by a matrix and add to another row block;
3. Interchange two row blocks.

Proposition B.1 implies that we may perform a series of above mentioned operations on (B.12) and the symmetric operations on the columns to arrive at an equivalent matrix inequality. A very important special case of the above is the following *Schur complement*.

Corollary B.2 *For matrices A, B, C, the inequality*

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} > 0 \tag{B.14}$$

is equivalent to the following two inequalities

$$\begin{aligned} A &> 0, & \text{(B.15)} \\ C - B^T A^{-1} B &> 0. & \text{(B.16)} \end{aligned}$$

Proof. Using Proposition B.1: multiply the first row of (B.14) by $-B^T A^{-1}$ and add to the second row, and perform the symmetric operation on the columns. ■

In formulating practical problems to LMI, one often needs to transform inequalities of the form (B.15) and (B.16) to the form (B.14).

B.4 $\mathcal{S}$-procedure

$\mathcal{S}$-procedure plays important role in robust stability theory. It can be stated as follows:

Proposition B.3 *Let $F_i \in \mathsf{R}^{n\times n}$, $i = 0, 1, 2, ..., p$. Then the following statement is true*

$$\xi^T F_0 \xi > 0 \text{ for any } \xi \in \mathsf{R}^n \text{ satisfying } \xi^T F_i \xi \geq 0, \tag{B.17}$$

if there exist real scalars $\tau_i \geq 0$, $i = 1, 2, ..., p$ such that

$$F_0 - \sum_{i=1}^{p} \tau_i F_i > 0. \tag{B.18}$$

For $p = 1$, these two statements are equivalent.

There are a number of variations. For example, the proposition is still valid if "$>$" in both (B.17) and (B.18) are replaced by "$\geq$".

B.5 Elimination of matrix variables

A technique especially useful in time-delay systems is the elimination of matrix variables in LMIs. The following Proposition states that a matrix variable appearing in two off-diagonal symmetric entries in a linear matrix inequality can be eliminated.

Proposition B.4 *There exists a matrix X such that*

$$\begin{pmatrix} P & Q & X \\ Q^T & R & V \\ X^T & V^T & S \end{pmatrix} > 0, \tag{B.19}$$

if and only if

$$\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} > 0, \tag{B.20}$$

$$\begin{pmatrix} R & V \\ V^T & S \end{pmatrix} > 0. \tag{B.21}$$

Notice that (B.20) and (B.21) are obtained by deleting the row and column containing X in two different ways.

Proof. Necessity is obvious since the left hand side of both (B.20) and (B.21) are principal minors of the left hand side of (B.19). For sufficiency, left-multiply second row of (B.19) by $-V^T R^{-1}$ and add to the third row, with symmetric operation for the columns. Equivalently, left-multiply (B.19) by

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -V^T R^{-1} & I \end{pmatrix},$$

and right-multiply by its transpose, it is seen that (B.19) is equivalent to

$$\begin{pmatrix} P & Q & X - QR^{-1}V \\ Q^T & R & 0 \\ X^T - V^T R^{-1} Q^T & 0 & S - V^T R^{-1} V \end{pmatrix} > 0.$$

But the above is clearly satisfied for

$$X = QR^{-1}V \tag{B.22}$$

if (B.20) and (B.21) are satisfied in view of Schur complement. ■

In some cases, matrix variables appearing in multiple entries can also be eliminated. For example:

Corollary B.5 *There exists a matrix X such that*

$$\begin{pmatrix} P + XE_0 + E_0^T X^T & Q + XE & X \\ (Q + XE)^T & R & V \\ X^T & V^T & S \end{pmatrix} > 0, \tag{B.23}$$

if and only if

$$\begin{pmatrix} P + E_0^T S E_0 & M \\ M^T & N \end{pmatrix} > 0, \tag{B.24}$$

$$\begin{pmatrix} R & V \\ V^T & S \end{pmatrix} > 0, \tag{B.25}$$

where

$$\begin{aligned} M &= Q - E_0^T V^T + E_0^T S E_1, \\ N &= R - VE - E^T V^T + E^T S E. \end{aligned}$$

Proof. Left-multiply the third row of the left hand side of (B.23) by $-\begin{pmatrix} E_0 & E \end{pmatrix}^T$ and add to the second row, with symmetric operation on the columns, then use Proposition B.4, one realizes that (B.23) is equivalent to (B.24) and

$$\begin{pmatrix} R - VE - E^T V^T + E^T S E & V - E^T S \\ (V - E^T S)^T & S \end{pmatrix} > 0. \tag{B.26}$$

Since (B.25) can be obtained from (B.26) by left-multiplying the second row by E^T and add to the first row with symmetric operation on the columns, (B.26) is equivalent to (B.25). ■

It is also possible to eliminate a common matrix variables in a diagonal entry of two matrix inequalities, as is shown in the following proposition.

Proposition B.6 *There exists a symmetric matrix X such that*

$$\begin{pmatrix} P_1 - LXL^T & Q_1 \\ Q_1^T & R_1 \end{pmatrix} > 0, \tag{B.27}$$

$$\begin{pmatrix} P_2 + X & Q_2 \\ Q_2^T & R_2 \end{pmatrix} > 0 \tag{B.28}$$

if and only if

$$\begin{pmatrix} P_1 + LP_2L^T & Q_1 & LQ_2 \\ Q_1^T & R_1 & 0 \\ Q_2^T L^T & 0 & R_2 \end{pmatrix} > 0. \tag{B.29}$$

Proof. Using the Schur complement, it is easy to see that (B.29) is equivalent to

$$R_1 > 0, \tag{B.30}$$

$$R_2 > 0, \tag{B.31}$$

and

$$\Delta = P_1 + LP_2L^T - Q_1R_1^{-1}Q_1^T - LQ_2R_2^{-1}Q_2^TL^T > 0. \tag{B.32}$$

Also, (B.27) and (B.28) are equivalent to (B.30), (B.31) and

$$\Delta_1 = P_1 - LXL^T - Q_1R_1^{-1}Q_1^T > 0, \tag{B.33}$$

$$\Delta_2 = P_2 + X - Q_2R_2^{-1}Q_2^T > 0. \tag{B.34}$$

Therefore, we only need to show that, given (B.30) and (B.31), there exists an X satisfying (B.33) and (B.34) if and only if (B.32) is satisfied.

The existence of X to satisfy (B.33) and (B.34) implies (B.32) since $\Delta = \Delta_1 + L\Delta_2L^T$. On the other hand, if (B.32) is satisfied, let

$$X = Q_2R_2^{-1}Q_2^T - P_2 + \varepsilon I$$

for some sufficiently small $\varepsilon > 0$ results in

$$\Delta_2 = \varepsilon I > 0,$$

and

$$\Delta_1 = \Delta - L\Delta_2L^T = \Delta - \varepsilon LL^T > 0,$$

thus (B.33) and (B.34) are satisfied. ∎

The above results (Proposition B.4, Corollary B.5 and Proposition B.6) can be extended to the case where the constant matrices are replaced by continuous matrix functions depending on variables within a compact set. For example, Proposition B.4 can be extended to the following.

Corollary B.7 *For given continuous matrix functions $P(\theta)$, $Q(\theta)$, $R(\theta)$, $V(\theta)$ and $S(\theta)$, with variable θ varying with a compact set $\Omega \subset \mathsf{R}^k$, there exists a continuous matrix function $X(\theta)$ such that*

$$\begin{pmatrix} P(\theta) & Q(\theta) & X(\theta) \\ Q^T(\theta) & R(\theta) & V(\theta) \\ X^T(\theta) & V^T(\theta) & S(\theta) \end{pmatrix} > 0 \text{ for all } \theta \in \Omega, \tag{B.35}$$

if and only if

$$\begin{pmatrix} P(\theta) & Q(\theta) \\ Q^T(\theta) & R(\theta) \end{pmatrix} > 0, \tag{B.36}$$

$$\begin{pmatrix} R(\theta) & V(\theta) \\ V^T(\theta) & S(\theta) \end{pmatrix} > 0, \tag{B.37}$$

for all $\theta \in \Omega$.

Proof. Based on Proposition B.4 and its proof, it is sufficient to prove that X as expressed in (B.22) is continuous. From (B.36), $R > 0$ for all $\theta \in \Omega$. This implies that $R > \varepsilon I$ for some constant $\varepsilon > 0$ since R is continuous and Ω is compact. Therefore, R^{-1} exists, is continuous and bounded. Since Q and V are also continuous, X is continuous. ∎

B.6 Quadratic Integral Inequalities

The following integral inequality is known as the Jensen Inequality, which plays an important role in the stability problem of time-delay systems.

Proposition B.8 (Jensen Inequality) *For any constant matrix $M \in \mathsf{R}^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathsf{R}^m$ such that the integrations concerned are well defined, then*

$$\gamma \int_0^\gamma \omega^T(\beta) M \omega(\beta) d\beta \geq \left(\int_0^\gamma \omega(\beta) d\beta \right)^T M \left(\int_0^\gamma \omega(\beta) d\beta \right). \tag{B.38}$$

Proof. It is easy to see, using Schur complement, that

$$\begin{pmatrix} \omega^T(\beta) M \omega(\beta) & \omega^T(\beta) \\ \omega(\beta) & M^{-1} \end{pmatrix} \geq 0$$

for any $0 \leq \beta \leq \gamma$. Integration of the above inequality from 0 to γ yields

$$\begin{pmatrix} \int_0^\gamma \omega^T(\beta) M \omega(\beta) d\beta & \int_0^\gamma \omega^T(\beta) d\beta \\ \int_0^\gamma \omega(\beta) d\beta & \gamma M^{-1} \end{pmatrix} \geq 0.$$

Use Shur complement in above to reach (B.38). ∎

Corollary B.9 *For any constant matrix $M \in \mathsf{R}^{m\times m}$, $M = M^T > 0$ and vector functions $\omega, \omega_1, \omega_2 : [0,1] \to \mathsf{R}^m$, such that the integrations in the following are well defined, then*

$$\begin{aligned}
&\int_0^1 [(1-\alpha)\omega_1^T(\alpha)M\omega_1(\alpha) + \alpha\omega_2^T(\alpha)M\omega_2(\alpha)]d\alpha \\
\geq\ & \int_0^1 \left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right)^T \\
&\qquad M\left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right) d\alpha, \qquad \text{(B.39)}
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 (1-\alpha)\omega^T(\alpha)M\omega(\alpha)d\alpha \\
\geq\ & \int_0^1 \left(\int_0^\alpha \omega(\beta)d\beta\right)^T \frac{1}{\alpha}M\left(\int_0^\alpha \omega(\beta)d\beta\right) d\alpha \qquad \text{(B.40)}\\
\geq\ & \int_0^1 \left(\int_0^\alpha \omega(\beta)d\beta\right)^T M\left(\int_0^\alpha \omega(\beta)d\beta\right) d\alpha, \qquad \text{(B.41)}
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \alpha\omega^T(\alpha)M\omega(\alpha)d\alpha \\
\geq\ & \int_0^1 \left(\int_\alpha^1 \omega(\beta)d\beta\right)^T \frac{1}{1-\alpha}M\left(\int_\alpha^1 \omega(\beta)d\beta\right) d\alpha \qquad \text{(B.42)}\\
\geq\ & \int_0^1 \left(\int_\alpha^1 \omega(\beta)d\beta\right)^T M\left(\int_\alpha^1 \omega(\beta)d\beta\right) d\alpha. \qquad \text{(B.43)}
\end{aligned}$$

Proof. Use Proposition B.8 for the case $\gamma = 1$, and

$$\omega(\beta) = \begin{cases} \omega_1(\beta), & 0 \leq \beta \leq \alpha, \\ \omega_2(\beta), & \alpha < \beta \leq 1, \end{cases}$$

to obtain

$$\begin{aligned}
&\int_0^\alpha \omega_1^T(\beta)M\omega_1(\beta)d\beta + \int_\alpha^1 \omega_2^T(\beta)M\omega_2(\beta)d\beta \\
\geq\ & \left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right)^T M\left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right) d\alpha.
\end{aligned}$$

Integrate the above from $\alpha = 0$ to $\alpha = 1$, and exchange the order of integration between α and β in both terms of the left hand side to arrive at inequality (B.39). To see (B.40),

$$
\begin{aligned}
& \int_0^1 (1-\alpha)\omega^T(\alpha) M\omega(\alpha) d\alpha \\
= & \int_0^1 \int_\alpha^1 d\beta \omega^T(\alpha) M\omega(\alpha) d\alpha \\
= & \int_0^1 \int_0^\beta \omega^T(\alpha) M\omega(\alpha) d\alpha d\beta \\
\geq & \int_0^1 \left(\int_0^\beta \omega(\alpha) d\alpha \right)^T \frac{1}{\beta} M \left(\int_0^\beta \omega(\alpha) d\alpha \right) d\beta
\end{aligned}
$$

which is (B.40). In the above, the equality in the second step is the result of exchanging the order of integration, and the inequality in the last step results from the Jensen Inequality (B.38). Considering the fact that $1/\alpha \geq 1$ in the integration interval, (B.41) is obvious. The proof of (B.42) and (B.43) are similar. ■

B.7 Notes

The importance of matrix inequalities, especially Linear Matrix Inequalities (LMI), in systems theory and control has long been recognized since Lyapunov theory was published around 1890 [192]. Indeed, solutions of some matrix inequalities have appeared as early as early 1960's by Yakubovich [311] [312] [313], see also the work by Horisberger and Bélanger [123]. Largely due to unavailability of efficient numerical algorithms for the general form of LMI, most of the earlier works on problems related to LMI's were reformulated in such forms as Lyapunov equations and algebraic Riccati Equations (ARE). The realization that LMI is a convex optimization problem [23], and the development of efficient interior point method [221] have spurred tremendous interest among researchers in control systems theory to formulate many control problems in LMI form. See the book by Boyd, et. al. [24] for more concepts on LMI, GEVP and applications in system and control theory. For more recent progress, and generalization to semidefinite programming, see [278] and [80]. Commercial software of LMI solver in MATLAB is available [77].

For a more comprehensive and systematic coverage of the materials in Sections B.1 to B.3, see the book by Boyd, et. al. [24].

The elimination of variables in LMI in Section B.5 was discussed in [95] and [97]. Some other cases of variable elimination are discussed in [24].

The quadratic integral inequalities discussed in Section B.6 was discussed in [97] in the time-delay stability context. The Jensen Inequality is covered in many books on probability theory. See, for example, [256].

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The bibliography is neither complete nor necessary. A lot of new entries will be added, and some entries will be deleted.

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