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Delay-dependent methods and the first delay interval

Kun Liu [a,*], Emilia Fridman [b]

[a] ACCESS Linnaeus Centre and School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden

[b] School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel

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ABSTRACT

This paper deals with the solution bounds for time-delay systems via delay-dependent Lyapunov–Krasovskii methods. Solution bounds are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation. We show that an additional bound for solutions is needed for the first time-interval, where $t < \tau(t)$, both in the continuous and in the discrete time. This first time-interval does not influence on the stability and the exponential decay rate analysis. The analysis of the first time-interval is important for nonlinear systems, e.g., for finding the domain of attraction. Regional stabilization of a linear (probably, uncertain) system with unknown and bounded input delay under actuator saturation is revisited, where the saturation avoidance approach is used.

1. Introduction

Consider the following continuous-time system with input delay

$$\dot{x}(t) = Ax(t) + Bu(t - \tau(t)), \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $u(t) = 0,\ t < 0$ and $\tau(t)$ is the time-varying delay $\tau(t) \in [0, h]$. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n_u}$ are system matrices. These matrices can be uncertain with polytopic type uncertainty. We seek a stabilizing state-feedback $u(t) = Kx(t)$ that leads to the exponentially stable closed-loop system

$$\dot{x}(t) = Ax(t) + A_1x(t - \tau(t)), \quad A_1 = BK \tag{2}$$

with (the discontinuous for $x(0) \neq 0$) initial condition

$$x(0) = x_0, \qquad x(\theta) = 0, \quad \theta \in [-h, 0). \tag{3}$$

There may be a problem with the bounds on the solutions when the delay-dependent analysis is performed via a Lyapunov–Krasovskii Functional (LKF) V. This is because for $t < \tau(t)$ (2) coincides with $\dot{x}(t) = Ax(t)$ and it may happen that $\dot{V} < 0,\ x \neq 0$ does not hold (e.g., if A is not Hurwitz). Therefore, an additional bound for solutions is needed for the first time-interval with $t < \tau(t)$. The length of this interval may be smaller than h. Clearly, this first time-interval (where the solution $x(t)$ is bounded) is not important for the stability and for the exponential decay rate analysis.

In the present paper, we show that the first time-interval of the delay length needs a special analysis when we deal with the solution bounds of time-delay systems via the Lyapunov–Krasovskii method, both in the continuous and in the discrete time. Local stabilization of a linear continuous-time plant with delayed saturated input is revisited. The conditions are given in terms of Linear Matrix Inequalities (LMIs). Finally, the results are applied to the stabilization of discrete-time time-delay systems with actuator saturation. Polytopic uncertainties in the system model can be easily included in our analysis. Some preliminary results have been presented in [1].

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. For any matrix $A \in \mathbb{R}^{n\times n}$ and vector $x \in \mathbb{R}^n$, the notations A_j and x_j denote, respectively, the jth line of matrix A and the jth component of vector x. $\mathbb{Z}$ denotes the set of non-negative integers. Given $\bar{u} = [\bar{u}_1, \ldots, \bar{u}_{n_u}]^T,\ 0 < \bar{u}_i,\ i = 1, \ldots, n_u$, for any $u = [u_1, \ldots, u_{n_u}]^T$ we denote by $\mathrm{sat}(u)$ the vector with coordinates $\mathrm{sign}(u_i)\min(|u_i|,\ \bar{u}_i)$.

2. Solution bounds via delay-dependent Lyapunov–Krasovskii methods: continuous-time

Solution bounds are important for nonlinear systems, where we are interested in the domain of attraction. They are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation.

* Corresponding author.
E-mail addresses: liukun@eng.tau.ac.il, kunliu@kth.se (K. Liu), emilia@eng.tau.ac.il (E. Fridman).

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Consider the initial value problem (2), (3). We assume the following:

A1. There exists a unique t^* such that $t-\tau(t)<0,\ t<t^*$ and $t-\tau(t)\geq 0,\ t\geq t^*$.

It is clear that $t^*\leq h$. We suppose that t^* is either known or unknown but upper-bounded by the known $h_1\leq h$. Assumption A1 always holds for the slowly-varying delays, where $\dot{\tau}<1$, since the function $t-\tau(t)$ is monotonically increasing with $\frac{d}{dt}(t-\tau(t))>0$. A1 also holds for piecewise-continuous delays with $\dot{\tau}\leq 1$, if the delays do not grow in the jumps (e.g. in Networked Control Systems (NCSs)). Under A1, (2), (3) for $t\geq 0$ is equivalent to

$$\begin{aligned}&\dot{x}(t)=Ax(t),\quad t\in[0,t^*),\\&x(0)=x_0\end{aligned}\tag{4}$$

and (2), where $t\geq t^*$.

Consider e.g., the standard LKF for the exponential stability of systems with $\tau(t)\in[0,h]$:

$$\begin{aligned}V(x_t,\dot{x}_t)&=\bar{V}(t)\\&=x^T(t)Px(t)+\int_{t-h}^{t}e^{2\alpha(s-t)}x^T(s)Sx(s)ds\\&\quad+h\int_{-h}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta,\\&\quad P>0,\ S>0,\ R>0,\ \alpha>0.\end{aligned}\tag{5}$$

Assume that along (2)

$$\dot{\bar{V}}+2\alpha\bar{V}\leq 0,\quad \alpha\geq 0,\ t\geq t^*.\tag{6}$$

Then

$$V(x_t,\dot{x}_t)\leq e^{-2\alpha(t-t^*)}V(x_{t^*},\dot{x}_{t^*}).$$

Remark 1. In many cases, e.g. in NCSs, t^* may be smaller than h. In order to derive less conservative exponential bounds, it is important to guarantee $\dot{\bar{V}}+2\alpha\bar{V}\leq 0$ for $t\geq t^*$ and not only for $t\geq h$.

Note that for $t-\tau(t)<0$ the system (2), (3) has the form (4) and, for the unstable A, (6) is clearly not feasible on $t\in[0,t^*)$ since otherwise it would yield that

$$x^T(t)Px(t)\leq V(x_t,\dot{x}_t)\leq e^{-2\alpha t}x_0^TPx_0,\quad t\in[0,t^*),$$

which is not true. Formally for $t\in[0,t^*)$ we have the same system (2) on $[0,t^*)$. Why it may happen that (6) does not hold for $t\in[0,t^*)$? This is for two reasons.

(1) The stabilizing A_1-term does not appear in the dynamics for $t\in[0,t^*)$.
(2) The expression $\dot{\bar{V}}+2\alpha\bar{V}\leq 0$ along (4) for $t\in[0,h)$ is different from the one along (2) for $t\geq h$ (as compared in (7) and (8) below).

For $t\in[0,h)$ and the zero initial condition (3) for $t<0$ we have

$$\begin{aligned}\bar{V}(t)&=x^T(t)Px(t)+\int_{0}^{t}e^{2\alpha(s-t)}x^T(s)Sx(s)ds\\&\quad+h\int_{-t}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta\\&\quad+h\int_{-h}^{-t}\int_{0}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta,\quad t\in[0,h).\end{aligned}$$

Then

$$\begin{aligned}\dot{\bar{V}}(t)+2\alpha\bar{V}(t)&=2x^T(t)P\dot{x}(t)\\&\quad+x^T(t)[S+2\alpha P]x(t)+h^2\dot{x}^T(t)R\dot{x}(t)\\&\quad-h\int_{0}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)ds,\quad t\in[0,h)\end{aligned}\tag{7}$$

to be compared with

$$\begin{aligned}\dot{\bar{V}}(t)+2\alpha\bar{V}(t)&=2x^T(t)P\dot{x}(t)+x^T(t)[S+2\alpha P]x(t)\\&\quad-x^T(t-h)Sx(t-h)+h^2\dot{x}^T(t)R\dot{x}(t)\\&\quad-h\int_{t-h}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)ds,\quad t\geq h.\end{aligned}\tag{8}$$

The feasibility of $\dot{\bar{V}}(t)+2\alpha\bar{V}(t)\leq 0$ along (2) for $t\geq h$ cannot guarantee $\dot{\bar{V}}(t)+2\alpha\bar{V}(t)\leq 0$ for $t^*\leq t<h$, where e.g., the term with S is useless.

Our objectives now are as follows:

(a) to guarantee that (8) holds for $t\geq t^*$ and not only for $t\geq h$,
(b) to derive simple bound on $V(x_{t^*},\dot{x}_{t^*})$ in terms of x_0.

Since the solution to (2), (4) does not depend on the values of $x(t)$ for $t<0$, we redefine the initial condition to be constant:

$$x(t)=x_0,\quad t\leq 0.\tag{9}$$

Then $V(x_t,\dot{x}_t)$ will have the form

$$\begin{aligned}V(x_t,\dot{x}_t)&=x^T(t)Px(t)+\int_{t-h}^{t}e^{2\alpha(s-t)}x^T(s)Sx(s)ds\\&\quad+h\int_{-t}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta\\&\quad+h\int_{-h}^{-t}\int_{0}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta,\quad t\in[0,h]\end{aligned}\tag{10}$$

leading to (8) for all $t\geq t^*$.

Our next objective is to derive a simple bound on $V(x_{t^*},\dot{x}_{t^*})$ in terms of x_0. If A is constant and known, one could substitute into $V(x_t,\dot{x}_t)$ of (10), where $t=t^*$, the following expressions:

$$\begin{aligned}&x(t)=e^{At}x_0,\quad t\in[0,t^*];\qquad x(t)=x_0,\quad t<0;\\&\dot{x}(t)=Ae^{At}x_0,\quad t\in[0,t^*]\end{aligned}$$

and then use upper-bounding. However, this may be complicated and conservative, especially if A is uncertain. Instead we develop below the direct Lyapunov approach for finding the bound on $V(x_{t^*},\dot{x}_{t^*})$.

As mentioned above, $\dot{\bar{V}}(t)+2\alpha\bar{V}(t)\leq 0$ along (4) is not guaranteed for $t\in[0,t^*)$ if A is not Hurwitz. Therefore, we consider $V_0(t)=x^T(t)Px(t),\ P>0$, and add the following conditions to (6): let there exist $\delta>0$ such that along (4)

$$\dot{V}_0(t)-2\delta V_0(t)\leq 0,\quad t\in[0,t^*),\tag{11a}$$

$$\dot{\bar{V}}(t)+2\alpha\bar{V}(t)-2\delta V_0(t)\leq 0,\quad t\in[0,t^*),\tag{11b}$$

then from (11a), $V_0(t)\leq e^{2\delta t}V_0(0)$ for $t\in[0,t^*)$.

Under the constant initial function, where $\dot{x}(t)=0,\ t<0$ and $\bar{V}(t)=V(x_t,\dot{x}_t)$ of (5), we have

$$\bar{V}(0)=x_0^TPx_0+\int_{-h}^{0}e^{2\alpha s}x_0^TSx_0ds.$$

Hence, $\bar{V}(0)\leq x_0^T(P+hS)x_0$.

Then (11b) implies

$$\begin{aligned}V(x_t,\dot{x}_t)&\leq e^{-2\alpha t}\bar{V}(0)+(e^{2\delta t}-1)x_0^TPx_0\\&\leq e^{-2\alpha t}x_0^T(P+hS)x_0+(e^{2\delta t}-1)x_0^TPx_0,\quad t\in[0,t^*).\end{aligned}$$

The latter yields

$$V(x_{t^*},\dot{x}_{t^*})\leq e^{-2\alpha t^*}x_0^T(P+hS)x_0+(e^{2\delta t^*}-1)x_0^TPx_0.$$

Therefore, (6) and (11) guarantee

$$\begin{aligned}V(x_t,\dot{x}_t)&\leq e^{-2\alpha(t-t^*)}[e^{-2\alpha t^*}x_0^T(P+hS)x_0\\&\quad+(e^{2\delta t^*}-1)x_0^TPx_0],\quad t\geq t^*.\end{aligned}\tag{12}$$

We have proved the following:

Lemma 1. *Under* A1 *and* (9), *let LKF given by* (5) *satisfy* (6) *along* (2) *and* (11) *along* (4). *Then the solution of the initial value problem* (2), (4) *satisfies* (12).

3. State-feedback control with input saturation: continuous-time

In this section, the result of Lemma 1 is applied to the stabilization of continuous-time time-delay systems with actuator saturation. Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t - \tau(t)), \qquad u(t) = Kx(t), \tag{13}$$

with the control law which is subject to the following amplitude constraints

$$|u_i(t)| \le \bar{u}_i, \quad 0 < \bar{u}_i,\ i = 1, \dots, n_u. \tag{14}$$

The time-varying delay $\tau(t)$ belongs to $[0, h]$ and satisfies the assumption A1. We will consider two cases:

(1) t^* is known,
(2) t^* is unknown but upper-bounded by the known $h_1 \le h$.

The state-feedback can be presented as $u(t) = \mathrm{sat}(Kx(t))$ leading to the following closed-loop system:

$$\dot{x}(t) = Ax(t) + B\mathrm{sat}(Kx(t - \tau(t))), \quad t \ge t^*. \tag{15}$$

Suppose for simplicity that $u(t - \tau(t)) = 0$ for $t - \tau(t) < 0$. The initial condition is then given by (4).

Denote by $x(t, x_0)$ the state trajectory of (4), (15) with the initial condition $x_0 \in \mathbb{R}^n$. Then the domain of attraction of the closed-loop nonlinear system (4), (15) is the set $\mathcal{A} = \{x_0 \in \mathbb{R}^n : \lim_{t\to\infty} x(t, x_0) = 0\}$. We seek conditions for the existence of a gain matrix K which lead to the exponentially stable closed-loop system. Having met these conditions, a simple procedure for finding the gain K should be presented. Moreover, we obtain an estimate $\mathcal{X}_\beta \subset \mathcal{A}$ (as large as we can get) on the domain of attraction, where

$$\mathcal{X}_\beta = \{x_0 \in \mathbb{R}^n : x_0^T P x_0 \le \beta^{-1}\}, \tag{16}$$

and where $\beta > 0$ is a scalar, $P > 0$ is an $n \times n$-matrix.

We define the polyhedron

$$\mathcal{L}(K, \bar{u}) = \{x(t) \in \mathbb{R}^n : |K_i x(t)| \le \bar{u}_i,\ i = 1, \dots, n_u\}.$$

If the control is such that $x(t) \in \mathcal{L}(K, \bar{u})$, then the system (15) admits the linear representation

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)), \quad \tau(t) \in [0, h]. \tag{17}$$

The objective is to compute a controller gain K and an associated set of initial conditions that make the system (17) exponentially stable.

Theorem 1. *Assume t^* is known. Given $\epsilon \in \mathbb{R}$ and positive scalars $\alpha, \beta, \delta, \sigma, h$, let there exist $n \times n$ matrices $\bar{P} > 0, \bar{P}_2, \bar{S}_{12}, \bar{R} > 0, \bar{S} > 0$, $n_u \times n$-matrix Y such that $\bar{S} \le \sigma \bar{P}$ and the following LMIs hold:*

$$\begin{bmatrix} \bar{R} & \bar{S}_{12} \\ * & \bar{R} \end{bmatrix} \ge 0, \tag{18}$$

$$\begin{bmatrix} A\bar{P}_2 + \bar{P}_2^T A^T - 2\delta\bar{P} & \bar{P} - \bar{P}_2 + \epsilon\bar{P}_2^T A^T \\ * & -\epsilon\bar{P}_2 - \epsilon\bar{P}_2^T \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} \bar{\Sigma}_{11} & \bar{\Sigma}_{12} & \bar{S}_{12}e^{-2\alpha h} & BY + (\bar{R} - \bar{S}_{12})e^{-2\alpha h} \\ * & \bar{\Sigma}_{22} & 0 & \epsilon BY \\ * & * & -(\bar{S} + \bar{R})e^{-2\alpha h} & (\bar{R} - \bar{S}_{12}^T)e^{-2\alpha h} \\ * & * & * & (-2\bar{R} + \bar{S}_{12} + \bar{S}_{12}^T)e^{-2\alpha h} \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} \bar{P}\bar{\rho}^{-1} & Y_j^T \\ * & \beta\bar{u}_j^2 \end{bmatrix} \ge 0, \quad j = 1, \dots, n_u, \tag{21}$$

$$\begin{bmatrix} \bar{\Sigma}_{11} - 2\delta\bar{P} & \bar{\Sigma}_{12} & \bar{S}_{12}e^{-2\alpha h} & (\bar{R} - \bar{S}_{12})e^{-2\alpha h} \\ * & \bar{\Sigma}_{22} & 0 & 0 \\ * & * & -(\bar{S} + \bar{R})e^{-2\alpha h} & (\bar{R} - \bar{S}_{12}^T)e^{-2\alpha h} \\ * & * & * & (-2\bar{R} + \bar{S}_{12} + \bar{S}_{12}^T)e^{-2\alpha h} \end{bmatrix} < 0, \tag{22}$$

where

$$\begin{aligned}
\bar{\Sigma}_{11} &= A\bar{P}_2 + \bar{P}_2^T A^T + \bar{S} - \bar{R}e^{-2\alpha h} + 2\alpha\bar{P}, \\
\bar{\Sigma}_{12} &= \bar{P} - \bar{P}_2 + \epsilon\bar{P}_2^T A^T, \\
\bar{\Sigma}_{22} &= -\epsilon\bar{P}_2 - \epsilon\bar{P}_2^T + h^2\bar{R}, \\
\bar{\rho} &= e^{-2\alpha t^*}(1 + h\sigma) + (e^{2\delta t^*} - 1).
\end{aligned}$$

Then, for all initial conditions x_0 belonging to $\mathcal{X}_\beta$, where $P = \bar{P}_2^{-T}\bar{P}\bar{P}_2^{-1}$, the closed-loop system (17) *is exponentially stable for all delays $\tau(t) \in [0, h]$, where $K = Y\bar{P}_2^{-1}$.*

Moreover, if t^* is unknown but $t^* \le h_1$ with $h_1 \le h$, where h_1 is a known bound, the term $\bar{P}\bar{\rho}^{-1}$ in (21) *is replaced by $\bar{P}(h\sigma + e^{2\delta h_1})^{-1}$.*

Proof. Suppose that $x(t) \in \mathcal{L}(K, \bar{u})$. Consider the LKF of (5). We analyze first the case when $t \ge t^*$. Differentiating $\bar{V}(t)$ along (17), we have

$$\begin{aligned}
\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \le\ & 2x^T(t)P\dot{x}(t) + x^T(t)[S + 2\alpha P]x(t) \\
& + h^2\dot{x}^T(t)R\dot{x}(t) - x^T(t-h)Se^{-2\alpha h}x(t-h) \\
& - he^{-2\alpha h}\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds.
\end{aligned} \tag{23}$$

Then, by Jensen's inequality and Theorem 1 of [2] we arrive at

$$\begin{aligned}
& -h\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds \\
&= -h\int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds - h\int_{t-h}^{t-\tau(t)} \dot{x}^T(s)R\dot{x}(s)ds \\
&\le -\frac{h}{\tau(t)}f_1(t) - \frac{h}{h-\tau(t)}f_2(t) \\
&\le -f_1(t) - f_2(t) - 2g_{1,2}(t) \\
&= -\lambda^T(t)\Omega\lambda(t),
\end{aligned}$$

where

$$\Omega = \begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \ge 0, \tag{24}$$

and

$$\begin{aligned}
f_1(t) &= [x(t) - x(t-\tau(t))]^T R[x(t) - x(t-\tau(t))], \\
f_2(t) &= [x(t-\tau(t)) - x(t-h)]^T R[x(t-\tau(t)) - x(t-h)], \\
g_{1,2}(t) &= [x(t) - x(t-\tau(t))]^T S_{12}[x(t-\tau(t)) - x(t-h)], \\
\lambda(t) &= \mathrm{col}\{x(t) - x(t-\tau(t)), x(t-\tau(t)) - x(t-h)\}.
\end{aligned}$$

We use the descriptor method [3], where the right-hand side of the expression

$$2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][Ax(t) + BKx(t-\tau(t)) - \dot{x}(t)] = 0,$$

with some $n \times n$-matrices P_2, P_3 is added to $\dot{\bar{V}}(t)$.

Hence, setting $\xi(t) = \mathrm{col}\{x(t), \dot{x}(t), x(t-h), x(t-\tau(t))\}$, we conclude that $\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \le \xi^T(t)\Psi\xi(t) \le 0,\ t \ge t^*$, if LMIs (24) and

$$\Psi = \begin{bmatrix} \psi_{11} & P - P_2^T + A^T P_3 & S_{12}e^{-2\alpha h} & \psi_{14} \\ * & -P_3 - P_3^T + h^2 R & 0 & P_3^T BK \\ * & * & -(S + R)e^{-2\alpha h} & \psi_{34} \\ * & * & * & \psi_{44} \end{bmatrix} < 0, \tag{25}$$

are feasible, where

$$\psi_{11} = A^T P_2 + P_2^T A + S - Re^{-2\alpha h} + 2\alpha P,$$
$$\psi_{14} = P_2^T BK + (R - S_{12})e^{-2\alpha h},$$
$$\psi_{34} = (R - S_{12}^T)e^{-2\alpha h},$$
$$\psi_{44} = (-2R + S_{12} + S_{12}^T)e^{-2\alpha h}.$$

Following [4], choose $P_3 = \varepsilon P_2$ and denote $P_2^{-1} = \bar{P}_2, \bar{P}_2^T P \bar{P}_2 = \bar{P}, K\bar{P}_2 = Y, \bar{P}_2^T S \bar{P}_2 = \bar{S}, \bar{P}_2^T R \bar{P}_2 = \bar{R}, \bar{P}_2^T S_{12} \bar{P}_2 = \bar{S}_{12}$. Multiplying (24) by diag$\{\bar{P}_2, \bar{P}_2\}$ and its transpose, (25) by diag$\{\bar{P}_2, \bar{P}_2, \bar{P}_2, \bar{P}_2\}$ and its transpose, from the right and the left, we conclude that (18) and (20) guarantee $\dot{\bar{V}}(t) + 2\alpha \bar{V}(t) \leq 0,\ t \geq t^*$.

Consider further the case where $0 \leq t < t^*$ and, thus the system is given by (4). For $0 \leq t < t^*$, LKF (5) under the constant initial condition (9) has the form

$$\begin{aligned}\bar{V}(t) &= x^T(t)Px(t) + \int_{t-h}^{t} e^{2\alpha(s-t)} x^T(s)Sx(s)ds \\ &\quad + h\int_{-t}^{0}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta \\ &\quad + h\int_{-h}^{-t}\int_{0}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta.\end{aligned}$$

Along (4), this leads to (23) since $x(t-h) \equiv x_0,\ t \leq h$. Similar to the case when $t \geq t^*$, we can prove that the LMIs (18) and (22) guarantee (11b) along (4) for $0 \leq t < t^*$.

Then differentiating $V_0(t)$ along (4) and applying the descriptor method, we have

$$\dot{V}_0(t) - 2\delta V_0(t) = \xi_{\text{sat}}^T(t)\Pi_{\text{sat}}\xi_{\text{sat}}(t) \leq 0,$$

where $\xi_{\text{sat}}(t) = \text{col}\{x(t), \dot{x}(t)\}$, if

$$\Pi_{\text{sat}} = \begin{bmatrix} A^T P_2 + P_2^T A - 2\delta P & P - P_2^T + A^T P_3 \\ * & -P_3 - P_3^T \end{bmatrix} < 0. \tag{26}$$

Choose $P_3 = \varepsilon P_2$ and denote $P_2^{-1} = \bar{P}_2$. Multiplying (26) by diag$\{\bar{P}_2, \bar{P}_2\}$ and its transpose, from the right and the left, we conclude that the LMI (19) yields $\dot{V}_0(t) - 2\delta V_0(t) \leq 0,\ 0 \leq t < t^*$.

Noting that $\bar{S} \leq \sigma \bar{P}$ implies $S \leq \sigma P$, from (12) and $x_0 \in \mathcal{X}_\beta$, we have for all $x(t)$:

$$\begin{aligned}x^T(t)Px(t) &\leq \bar{V}(t) \\ &\leq e^{-2\alpha(t-t^*)}[e^{-2\alpha t^*}x_0^T(P + hS)x_0 + (e^{2\delta t^*} - 1)x_0^T P x_0] \\ &\leq e^{-2\alpha(t-t^*)}[e^{-2\alpha t^*}x_0^T(P + h\sigma P)x_0 + (e^{2\delta t^*} - 1)x_0^T P x_0] \\ &\leq e^{-2\alpha(t-t^*)}\bar{\rho}x_0^T P x_0 \\ &\leq \bar{\rho}\beta^{-1}, \quad t \geq t^*.\end{aligned}$$

So for all

$$x(t) : x^T(t)Px(t) \leq \bar{\rho}\beta^{-1} \Rightarrow x^T(t)K_i^T K_i x(t) \leq \bar{u}_i^2,$$

if $x^T(t)K_i^T K_i x(t) \leq \beta\bar{\rho}^{-1}x^T(t)Px(t)\bar{u}_i^2$. The latter inequality is guaranteed if $\beta\bar{\rho}^{-1}P\bar{u}_i^2 - K_i^T K_i \geq 0$, and, thus, by Schur complements if

$$\begin{bmatrix} P\bar{\rho}^{-1} & K_i^T \\ * & \beta\bar{u}_i^2 \end{bmatrix} \geq 0$$

or if (21) is feasible, where $Y_i = K_i P_2^{-1} = K_i \bar{P}_2$ and $\bar{P} = P_2^{-T} P P_2^{-1} = \bar{P}_2^T P \bar{P}_2$. Hence LMI conditions in Theorem 1 ensure that the trajectories of the system (17) converge to the origin exponentially, provided that $x_0 \in \mathcal{X}_\beta$.

Remark 2. Consider the following continuous-time system controlled through a network:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{27}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input. We suppose that the control input is subject to amplitude constraints (14). We assume that the state vector is sampled at s_k, satisfying

$$0 = s_0 < s_1 < \cdots < s_k < \cdots, \quad k \in \mathbb{Z}, \quad \lim_{k\to\infty} s_k = \infty.$$

The sampled state vector experiences an uncertain, time varying delay η_k as it is transmitted through the network. The delay η_k is bounded, i.e., $0 \leq \eta_k \leq \eta_M$. The actuator is updated with new control signals at the instants $t_k = s_k + \eta_k,\ k \in \mathbb{Z}$. An event driven zero-order hold keeps the control signal constant through the interval $[t_k, t_{k+1})$, i.e., until the arrival of new data at t_{k+1}. As in [5], we assume that $t_{k+1} - t_k + \eta_k \leq \tau_M,\ k \in \mathbb{Z}$. Note that the first updating time t_0 corresponds to the first data received by the actuator. Then for $t \in [0, t_0)$, (27) is given by

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \quad t \in [0, t_0).$$

The effective control signal to be applied to the system (27) is given by $u(t) = \text{sat}(Kx(t_k - \eta_k)),\ t_k \leq t < t_{k+1}$. Defining $\tau(t) = t - t_k + \eta_k,\ t_k \leq t < t_{k+1}$, we obtain the following closed-loop system:

$$\dot{x}(t) = Ax(t) + B\text{sat}(Kx(t - \tau(t))), \tag{28}$$

with $0 \leq \tau(t) < t_{k+1} - t_k + \eta_k \leq \tau_M$ and $\dot{\tau}(t) = 1$ for $t \neq t_k$. Then Theorem 1 holds for (28) with $t^* = t_0, h_1 = \eta_M, h = \tau_M$.

4. Solution bounds via delay-dependent Lyapunov–Krasovskii methods: discrete-time

In this section, we present the discrete-time counterpart of the results obtained in the previous one. Consider the discrete-time system with input delay

$$\begin{aligned}&x(k+1) = Ax(k) + Bu(k - \tau(k)), \\ &x(0) = x_0, \quad k \in \mathbb{Z},\end{aligned} \tag{29}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $u(k) = 0,\ k < 0$ and $\tau(k)$ is the time-varying delay $\tau(k) \in [0, h]$, where h is a known positive integer. A and B are system matrices with appropriate dimensions. These matrices can be uncertain with polytopic type uncertainty. Similar to Section 1, we seek a stabilizing state-feedback $u(k) = Kx(k)$ that leads to the exponentially stable closed-loop system

$$x(k+1) = Ax(k) + A_1 x(k - \tau(k)), \quad A_1 = BK \tag{30}$$

with the initial condition

$$x(0) = x_0, \qquad x(k) = 0, \quad k = -h, -h+1, \ldots, -1. \tag{31}$$

The problem of the first time-interval may arise when the delay-dependent analysis is performed via a LKF V to deal with the bounds on the solutions. This is because for $k < \tau(k)$ (30) coincides with $x(k+1) = Ax(k)$ and it may happen that $\Delta V(k) = V(k+1) - V(k) < 0$ does not hold (e.g., if A is not Schur stable). Therefore, an additional bound for solutions is also needed for the first time sequence with $k < \tau(k)$.

Consider the initial value problem (30), (31). Similar to A1, we assume the following:

A2. There exists a unique $k^* \in \mathbb{Z}$ such that $k - \tau(k) < 0,\ k < k^*$ and $k - \tau(k) \geq 0,\ k \geq k^*$.

It is clear that $k^* \leq h$. We suppose that k^* is either known or unknown but upper-bounded by the known $h_1 \leq h$. Under A2, the initial value problem (30), (31) for $k \geq 0$ is equivalent to

$$x(k+1) = Ax(k), \quad k = 0, 1, \ldots, k^* - 1, \qquad x(0) = x_0 \tag{32}$$

and (30), where $k = k^*, k^* + 1, \ldots$.

Consider now the standard LKF for the exponential stability of discrete-time systems with $\tau(k) \in [0, h]$ (see e.g., [6]):

$$\begin{aligned} V(k) &= x^T(k)Px(k) + \sum_{s=k-h}^{k-1} \lambda^{k-s-1} x^T(s)Sx(s) \\ &\quad + h \sum_{j=-h}^{-1} \sum_{s=k+j}^{k-1} \lambda^{k-s-1} \eta^T(s)R\eta(s), \\ &\quad P > 0,\ S > 0,\ R > 0,\ 0 < \lambda < 1, \\ \eta(k) &= x(k+1) - x(k). \end{aligned} \tag{33}$$

Assume that along (30)

$$V(k+1) - \lambda V(k) \leq 0, \quad 0 < \lambda < 1,\ k = k^*, k^* + 1, \ldots. \tag{34}$$

Then

$$V(k) \leq \lambda^{k-k^*} V(k^*), \quad k = k^*, k^* + 1, \ldots.$$

Note that for $k - \tau(k) < 0$ the system (30), (31) has the form (32) and, for the non-Schur A, (34) is clearly not feasible on $k = 0, 1, \ldots, k^* - 1$ since otherwise it would follow that

$$x^T(k)Px(k) \leq V(k) \leq \lambda^k x_0^T P x_0, \quad k = 0, 1, \ldots, k^*,$$

which is not true.

For $k = 0, 1, \ldots, h-1$ and the zero initial condition (31) (substituted for $x(k)$) we have

$$V(k) = x^T(k)Px(k) + V_S(k) + V_{1R}(k) + V_{2R}(k),$$

where

$$\begin{aligned} V_S(k) &= \sum_{s=-1}^{k-1} \lambda^{k-s-1} x^T(s)Sx(s), \\ V_{1R}(k) &= h \sum_{j=-k}^{-1} \sum_{s=k+j}^{k-1} \lambda^{k-s-1} \eta^T(s)R\eta(s), \\ V_{2R}(k) &= h \sum_{j=-h}^{-k-1} \sum_{s=-1}^{k-1} \lambda^{k-s-1} \eta^T(s)R\eta(s). \end{aligned} \tag{35}$$

Taking into account that

$$\begin{aligned} & x^T(k+1)Px(k+1) - \lambda x^T(k)Px(k) \\ &\quad = [x^T(k) + \eta^T(k)]P[x(k) + \eta(k)] - \lambda x^T(k)Px(k) \\ &\quad = 2x^T(k)P\eta(k) + \eta^T(k)P\eta(k) + (1-\lambda)x^T(k)Px(k), \\ & V_S(k+1) = \lambda V_S(k) + x^T(k)Sx(k), \\ & V_{1R}(k+1) = \lambda V_{1R}(k) + (k+1)h\eta^T(k)R\eta(k), \\ & V_{2R}(k+1) = \lambda V_{2R}(k) + [h-(k+1)]h\eta^T(k)R\eta(k) \\ &\qquad - h \sum_{s=-1}^{k-1} \lambda^{k-s} \eta^T(s)R\eta(s), \end{aligned}$$

we have

$$\begin{aligned} V(k+1) - \lambda V(k) &= \eta^T(k)(h^2R + P)\eta(k) + 2x^T(k)P\eta(k) \\ &\quad + x^T(k)[S + (1-\lambda)P]x(k) \\ &\quad - h \sum_{s=-1}^{k-1} \lambda^{k-s} \eta^T(s)R\eta(s), \\ &\quad k = 0, 1, \ldots, h-1 \end{aligned}$$

to be compared with

$$\begin{aligned} V(k+1) - \lambda V(k) &= \eta^T(k)(h^2R + P)\eta(k) + 2x^T(k)P\eta(k) \\ &\quad + x^T(k)[S + (1-\lambda)P]x(k) \\ &\quad - x^T(k-h)S\lambda^h x(k-h) \\ &\quad - h \sum_{s=k-h}^{k-1} \lambda^{k-s} \eta^T(s)R\eta(s) \end{aligned} \tag{36}$$

for $k \geq h$. The feasibility of $V(k+1) - \lambda V(k) \leq 0$ along (30) for $k \geq h$ cannot guarantee $V(k+1) - \lambda V(k) \leq 0$ for $k = k^*, \ldots, h-1$, where e.g., the term with S is useless.

Our objectives now are as follows: (a) to guarantee (34) for $k = k^*, k^* + 1, \ldots$ and not only for $k = h, h+1, \ldots$, (b) to derive a simple bound on $V(k^*)$ in terms of x_0. Since the solution to (30), (32) does not depend on the values of $x(k)$ for $k < 0$, we redefine the initial condition to be constant for $k \leq 0$:

$$x(k) = x_0, \quad k = -h, -h+1, \ldots, 0. \tag{37}$$

Then $V(k)$ will have the form

$$\begin{aligned} V(k) &= x^T(k)Px(k) + \sum_{s=k-h}^{k-1} \lambda^{k-s-1} x^T(s)Sx(s) \\ &\quad + V_{1R}(k) + V_{2R}(k), \quad k = 0, 1, \ldots, h-1 \end{aligned} \tag{38}$$

leading to (36) for all $k \geq k^*$, where $V_{iR}(k),\ i = 1, 2$, are given by (35).

If A is constant and known, one could substitute into $V(k)$ of (38), where $k = k^*$, the following expressions:

$$\begin{aligned} & x(k) = A^k x_0, \quad 0 \leq k \leq k^*; \qquad x(k) = x_0, \quad k < 0; \\ & \eta(k) = A^k(A - I)x_0, \quad 0 \leq k \leq k^* \end{aligned}$$

and then use upper-bounding. However, this may be complicated and conservative, especially if A is uncertain. Instead we develop below the direct Lyapunov approach for finding the bound on $V(k^*)$.

As mentioned above, $V(k+1) - \lambda V(k) \leq 0$ along (32) is not guaranteed for $0, 1, \ldots, k^* - 1$ if A is not Schur. Therefore, we consider $V_0(k) = x^T(k)Px(k), P > 0$, and add the following conditions to (34): let there exist $\mu > 1$ such that along (32)

$$V_0(k+1) - \mu V_0(k) \leq 0, \quad k = 0, 1, \ldots, k^* - 1, \tag{39a}$$

$$V(k+1) - \lambda V(k) - (\mu - 1)V_0(k) \leq 0, \quad k = 0, 1, \ldots, k^* - 1. \tag{39b}$$

Then from (39a), $V_0(k) \leq \mu^k V_0(0)$ for $k = 0, 1, \ldots, k^*$.

Under the constant initial condition, where $\eta(k) = 0,\ k < 0$ and $V(k)$ of (33), we have for $k = 0$

$$V(0) = x_0^T P x_0 + \sum_{s=-h}^{-1} \lambda^{-s-1} x_0^T S x_0.$$

Hence, $V(0) \leq x_0^T(P + hS)x_0$.

Then (39b) implies

$$\begin{aligned} V(k) &\leq \lambda^k V(0) + (\mu^k - 1)x_0^T P x_0 \\ &\leq \lambda^k x_0^T(P + hS)x_0 + (\mu^k - 1)x_0^T P x_0, \quad k = 0, 1, \ldots, k^*. \end{aligned}$$

The latter yields

$$V(k^*) \leq \lambda^{k^*} x_0^T(P + hS)x_0 + (\mu^{k^*} - 1)x_0^T P x_0.$$

Therefore, (34) and (39) guarantee

$$\begin{aligned} V(k) &\leq \lambda^{k-k^*}[\lambda^{k^*} x_0^T(P + hS)x_0 + (\mu^{k^*} - 1)x_0^T P x_0], \\ &\quad k = k^*, k^* + 1, \ldots. \end{aligned} \tag{40}$$

We have proved the following:

Lemma 2. *Under A2, let LKF given by (33) satisfy (34) along (30) and (39) along (32). Then the solution of the initial value problem (30), (32) satisfies (40).*

Remark 3. If the system (1) or (29) has only state delay (and no input delay), the first delay interval should also be analyzed separately similar to Lemma 1 or 2, respectively. Indeed, in the continuous-time case, consider

$$\dot{x}(t) = Ax(t) + A_1x(t-h),$$

where $h > 0$ is a constant delay and A is not Hurwitz. Choose the initial condition to be zero for $t \in [-h, -\varepsilon]$ with $\varepsilon \to 0^+$. Then for $t \in [0, h-\varepsilon]$, the system has a form $\dot{x}(t) = Ax(t)$, i.e. $\dot{\bar{V}} + 2\alpha\bar{V} \le 0$ for $\bar{V}$ given by (5) cannot be feasible for $t \in [0, h-\varepsilon]$.

5. State-feedback control with input saturation: discrete-time

Consider the system

$$\begin{aligned} &x(k+1) = Ax(k) + Bu(k-\tau(k)), \\ &u(k) = Kx(k), \quad k \in \mathbb{Z} \end{aligned} \tag{41}$$

with the control law which is subject to the following amplitude constraints

$$|u_i(k)| \le \bar{u}_i, \quad 0 < \bar{u}_i,\ i = 1, \ldots, n_u,\ k \in \mathbb{Z}. \tag{42}$$

The time-varying delay $\tau(k)$ belongs to $[0, h]$ and satisfies the assumption A2, where h is a positive integer. We will consider two cases: (1) k^* is known, (2) k^* is unknown but bounded by a known positive integer $h_1 \le h$. Then the state-feedback has the following form $u(k) = \mathrm{sat}(Kx(k))$. Applying the latter control law, the closed-loop system obtained is

$$\begin{aligned} &x(k+1) = Ax(k) + B\mathrm{sat}(Kx(k-\tau(k))), \\ &\quad k = k^*, k^*+1, \ldots. \end{aligned} \tag{43}$$

Suppose for simplicity that $u(k-\tau(k)) = 0$ for $k - \tau(k) < 0$. The initial condition is then given by (32).

If the control is such that $x(k) \in \mathcal{L}(K, \bar{u})$ then the system (43) admits the linear representation

$$x(k+1) = Ax(k) + BKx(k-\tau(k)), \quad \tau(k) \in [0, h]. \tag{44}$$

Our objective is to compute a controller gain K and an associated set of initial conditions that make the solutions of system (44) exponentially stable. We apply LKF (33) to system (44) with time-varying delay from the maximum delay interval $[0, h]$. By using arguments similar to Theorem 1, we arrive at

Theorem 2. *Assume k^* is known. Given $\epsilon \in \mathbb{R}$, positive scalars $\lambda < 1, \beta, \mu > 1, \sigma$ and positive integer h. Let there exist $n \times n$ matrices $\bar{P} > 0$, $\bar{P}_2, \bar{S}_{12}, \bar{R} > 0, \bar{S} > 0$, $n_u \times n$-matrix Y such that $\bar{S} \le \sigma\bar{P}$, (18) and the following LMIs hold:*

$$\begin{bmatrix} (A-I)\bar{P}_2 + \bar{P}_2^T(A-I)^T + (1-\mu)\bar{P} & \bar{P} - \bar{P}_2 + \epsilon\bar{P}_2^T(A-I)^T \\ * & -\epsilon\bar{P}_2 - \epsilon\bar{P}_2^T + \bar{P} \end{bmatrix} < 0, \tag{45}$$

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \bar{S}_{12}\lambda^h & BY + (\bar{R} - \bar{S}_{12})\lambda^h \\ * & \Sigma_{22} & 0 & \epsilon BY \\ * & * & -(\bar{S} + \bar{R})\lambda^h & (\bar{R} - \bar{S}_{12}^T)\lambda^h \\ * & * & * & \Sigma_{44} \end{bmatrix} < 0, \tag{46}$$

$$\begin{bmatrix} \bar{P}\rho^{-1} & Y_j^T \\ * & \beta\bar{u}_j^2 \end{bmatrix} \ge 0, \quad j = 1, \ldots, n_u, \tag{47}$$

$$\begin{bmatrix} \Sigma_{11} + (1-\mu)\bar{P} & \Sigma_{12} & \bar{S}_{12}\lambda^h & (\bar{R} - \bar{S}_{12})\lambda^h \\ * & \Sigma_{22} & 0 & 0 \\ * & * & -(\bar{S} + \bar{R})\lambda^h & (\bar{R} - \bar{S}_{12}^T)\lambda^h \\ * & * & * & \Sigma_{44} \end{bmatrix} < 0, \tag{48}$$

where

$$\begin{aligned} &\Sigma_{11} = (A-I)\bar{P}_2 + \bar{P}_2^T(A-I)^T + \bar{S} - \bar{R}\lambda^h + (1-\lambda)\bar{P}, \\ &\Sigma_{12} = \bar{P} - \bar{P}_2 + \epsilon\bar{P}_2^T(A-I)^T, \\ &\Sigma_{22} = -\epsilon\bar{P}_2 - \epsilon\bar{P}_2^T + h^2\bar{R} + \bar{P}, \\ &\Sigma_{44} = (-2\bar{R} + \bar{S}_{12} + \bar{S}_{12}^T)\lambda^h, \\ &\rho = \lambda^{k^*}(1 + h\sigma) + \mu^{k^*} - 1. \end{aligned}$$

Then, for all initial conditions x_0 belonging to $\mathcal{X}_\beta$, where $P = \bar{P}_2^{-T}\bar{P}\bar{P}_2^{-1}$, the closed-loop system (44) is exponentially stable for all delays $0 \le \tau(k) \le h$, where $K = Y\bar{P}_2^{-1}$.

Moreover, if k^* is unknown but $k^* \le h_1$ with $h_1 \le h$, where h_1 is a known bound, the term $\bar{P}\rho^{-1}$ in (47) is replaced by $\bar{P}(h\sigma + \mu^{k^*})^{-1}$.

Remark 4. Note that

$$x_0^T Px_0 \le \lambda_{\max}(P)|x_0|^2 \le \beta^{-1},$$

where $\lambda_{\max}(P)$ denotes the largest eigenvalue of P. Hence the following initial region $|x_0|^2 \le \beta^{-1}/\lambda_{\max}(P)$ is inside of $\mathcal{X}_\beta$. Similar to [7], in order to have a bigger initial ball, i.e., to minimize $\lambda_{\max}(P)$ we add the constraint

$$\begin{bmatrix} -\varrho I & I \\ * & -\bar{P}_2 - \bar{P}_2^T + \bar{P} \end{bmatrix} < 0, \tag{49}$$

to Theorems 1 and 2. Since $P > 0$ and $P = \bar{P}_2^{-T}\bar{P}\bar{P}_2^{-1}$, i.e., $\bar{P} = \bar{P}_2^T P\bar{P}_2$, we have

$$(P^{-1} - \bar{P}_2^T)P(P^{-1} - \bar{P}_2^T)^T \ge 0,$$

which implies that $P^{-1} \ge \bar{P}_2 + \bar{P}_2^T - \bar{P}$ or $P \le (\bar{P}_2 + \bar{P}_2^T - \bar{P})^{-1}$. Hence, by Schur complements, if (49) holds, it follows that

$$\varrho I > (\bar{P}_2 + \bar{P}_2^T - \bar{P})^{-1} \ge P,$$

which implies that $P < \varrho I$. So we need to minimize ϱ in order to minimize $\lambda_{\max}(P)$.

Remark 5. It should be pointed out that the results presented in Theorems 1 and 2 can be improved by the use of a generalized sector condition [8] or a polytopic modeling [9].

Remark 6. LMIs of Theorems 1 and 2 are affine in the system matrices. Therefore, in the case of system matrices from the uncertain time-varying polytope

$$\Theta = \sum_{j=1}^{M} g_j(t)\Theta_j, \quad 0 \le g_j(t) \le 1,$$

$$\sum_{j=1}^{M} g_j(t) = 1, \qquad \Theta_j = \begin{bmatrix} A^{(j)} & B^{(j)} \end{bmatrix},$$

one have to solve these LMIs simultaneously for all the M vertices Θ_j, applying the same decision matrices.

6. Examples

Example 1. Consider the system from [10]:

$$\dot{x}(t) = \begin{bmatrix} 1.1 & -0.6 \\ 0.5 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-\tau(t)), \tag{50}$$

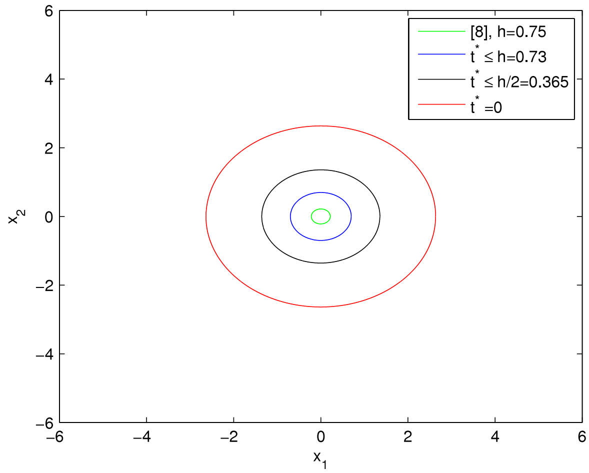

Fig. 1. Example 1: largest ball of admissible initial conditions for different t^*.

where $\bar{u} = 5$. Choose $\varepsilon = 0.97, \sigma = 1.0 \times 10^{-3}, \beta = 1$. First we assume that t^* is unknown and bounded by $h_1 = h$. Application of Theorem 1 with $\alpha = 0$ and Remark 4 leads to the asymptotic stability of the closed-loop system for all delays $\tau(t) \le 0.73$ with $\delta \in [1.81, 13.01]$. When $\delta = 1.82$, we achieve the largest ball of admissible initial conditions $|x_0| \le 0.6985$. The resulting controller gain is $K = [-1.8116\ 0.5586]$. Then for different t^*, applying Theorem 1 with $\alpha = 0$ and Remark 4 we give the corresponding largest ball of admissible initial conditions (see Fig. 1).

Example 2 *([10])*. We consider (13) with the following matrices:

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1(t) & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 + g_2(t) \\ -1 \end{bmatrix},$$

where $|g_1(t)| \le 0.1$, $|g_2(t)| \le 0.3$. Suppose that $\bar{u} = 10$. Choose $\varepsilon = 0.8, \sigma = 1.0 \times 10^{-3}, \beta = 1$. First we assume that t^* is unknown and bounded by $h_1 = h$. Application of Theorem 1 with $\alpha = 0$ and Remarks 4, 6 leads to the asymptotic stability of the closed-loop system for all delays $\tau(t) \le 0.36$ with $\delta \in [1.37, 32.53]$. When $\delta = 1.42$, the obtained set of admissible initial conditions in this case is given by

$$\mathcal{X} = \left\{ x_0 \in \mathbb{R}^2 : x_0^T \begin{bmatrix} 0.1772 & 0.0435 \\ 0.0435 & 0.011 \end{bmatrix} x_0 \le 1 \right\}$$

and the corresponding largest ball of admissible initial conditions is $|x_0| \le 2.3067$ (see Fig. 2). The resulting controller gain is $K = -[2.5215\ 0.6251]$. Then for different t^*, applying Theorem 1 with $\alpha = 0$ and Remarks 4, 6, we give the corresponding largest ball of admissible initial conditions (see Table 1).

Example 3. Discretize the system (50) with a sampling time $T_s = 0.01$:

$$x(k+1) = \begin{bmatrix} 1.0110 & -0.006 \\ 0.005 & 0.99 \end{bmatrix} x(k) + \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} u(k - \tau(k)), \quad (51)$$

and where $\bar{u} = 5$. Choose $\varepsilon = 0.97, \sigma = 1.0 \times 10^{-3}, \beta = 1$. First we assume that k^* is unknown and bounded by $h_1 = h$. Application of Theorem 2 with $\lambda = 1$ and Remark 4 leads to the asymptotic stability of the closed-loop system for all delays $\tau(k) \le 5$ with $\mu \in [1.66, 66.12]$. When $\delta = 1.77$ we achieve the largest ball of admissible initial conditions $|x_0| \le 0.6179$. The resulting controller gain is $K = [-1.8550\ 0.5720]$.

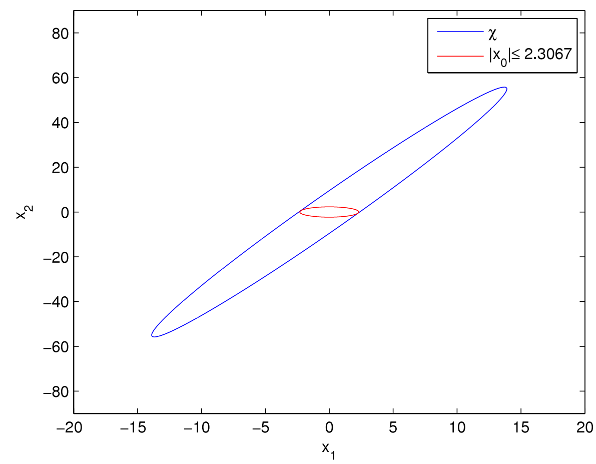

Fig. 2. Example 2: set of admissible initial conditions.

Table 1
Example 2: largest ball of admissible initial conditions for different t^*.

$t^*(h = 0.36)$	$t^* \le h$	$t^* \le h/2$	0
$\|x_0\|$	2.3067	2.9784	3.8456

7. Conclusions

In this paper, we show that the first time interval of the delay length needs a special analysis when we deal with the solution bounds of time-delay system via the Lyapunov–Krasovskii method, both in the continuous and in the discrete time. Regional stabilization of linear continuous/discrete-time plant with input saturation is revisited. The conditions are given in terms of LMIs. Numerical examples illustrate the efficiency of the method.

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