# Regional Stabilization of Time-Delay Systems with Actuator Saturation and Delay

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**Abstract**

to be added

**Keywords:** to be added

## 1 Introduction

## 2 Problem Formulation

In this section, we first consider the following linear time-delay system with actuator saturation:

$$\left\{\begin{array}{l}\dot{x}(t)=Ax(t)+A_{\mathrm{r}}x(t-r)+B\mathrm{sat}(u(t))\\ x(t)=\phi(t),u(t)=\omega(t),\forall t\in[-r,0],\end{array}\right.\tag{1}$$

where $r$ is nonnegative scalar, and $A, A_{\mathrm{r}}$ and $B$ are known matrices. The function $\mathrm{sat}(u)$: $\mathbf{R}^m\rightarrow\mathbf{R}^m$ is the standard saturation defined as follows:

$$\mathrm{sat}(u(t))=\left[\begin{array}{ccc}\mathrm{sat}(u_1) & \cdots & \mathrm{sat}(u_2)\end{array}\right]^{\mathrm{T}},$$

with $\mathrm{sat}(u_i)=\mathrm{sign}(u_i)\min\{1,|u_i|\}$. Let $\mathcal{C}_{n,r}=\mathcal{C}([-r,0],\mathbf{R}^n)$ denote the Banach space of continuous vector functions mapping the interval $[-r,0]$ into $\mathbf{R}^n$ with the topology of uniform convergence, and $x_t\in\mathcal{C}_{n,r}$ denote the restriction of $x(t)$ to the interval $[t-r,t]$ translated to $[-r,0]$. When a state feedback $u(t)=F^{\mathrm{T}}x(t)$ is applied on to the system (1), the resulting closed-loop system reads

$$\left\{\begin{array}{l}\dot{x}(t)=Ax(t)+A_{\mathrm{r}}x(t-r)+B\mathrm{sat}\left(F^{\mathrm{T}}x(t)\right)\\ x(t)=\phi(t),\forall t\in[-r,0].\end{array}\right.\tag{2}$$

For an initial condition $x_0\in\mathcal{C}_{n+m,r}$, denote the solution of system (2) as $x(t,x_0)$. Assume that the trivial solution $x(t,x_0)\equiv 0$ is asymptotically stable, then the domain of attraction of the origin is defined as

$$\mathcal{G}=\left\{x_0\in\mathcal{C}_{n,r}:\lim_{t\rightarrow\infty}\|x(t,x_0)\|=0\right\}.$$

A set $\mathcal{S}\subset\mathcal{C}_{n,r}$ is said to be invariant for system (2) if $x_0\in\mathcal{S}\Rightarrow x(t,x_0)\in\mathcal{S},\forall t\geq 0$. Moreover, a set $\mathcal{S}$ is called a contractively invariant set if it is an invariant set and is in the domain of attraction.

**Problem 1** *Given a state gain $F$, how to determine whether the closed-loop system y (**??**) is locally stable, and then to determine the local stability of the system x (2)?*

**Problem 2** *For a given $F$, if system (**??**) is locally stable, how to find a set $\mathcal{L}_y\subset\bar{\mathcal{C}}_{n,r}$ such that $\mathcal{L}_y$ is an estimate of the domian of attraciton for the system (**??**), and is as large as possible? As a result, to get a corresponding estimate of the domian of attraciton $\mathcal{L}_x\subset\mathcal{C}_{n+m,r}$ for the system (2).*

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**Problem 3** *How to disign an $F$ such that an estimate of the domain of attraction $\mathcal{L}_y$ for $y$ system is maximized? And it follows that the corresponding estimate of the domian of attraciton $\mathcal{L}_x$ for the system (2) is also maximized.*

Let $\mathcal{V}$ be the set of all $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0, then there are $2^m$ elements in $\mathcal{V}$. Assume that each element in $\mathcal{V}$ is labeled as $D_i, i \in \mathcal{V}_m \triangleq \{1, 2, \ldots, 2^m\}$ and let $D_i^- = I - D_i, i \in \mathcal{V}_m$. Then we give the following lemma for later use.

**Lemma 1** *For two vectors $u, v \in \mathbf{R}^m$, if $\|v\|_\infty \leq 1$, then*

$$\operatorname{sat}(u) \in \operatorname{co}\left\{D_i u + D_i^- v, i \in \mathcal{V}_m\right\},$$

*where $\operatorname{co}\{\cdot\}$ denotes the convex hull of a set.*

**Lemma 2** *For a given positive-define matrice $R \in \mathbf{R}^{n\times n}$, any differentiable function $\omega$ in $[a, b] \longrightarrow \mathbf{R}^n$, the following inequality holds:*

$$\int_a^b \dot{\omega}^{\mathrm{T}}(s) R \dot{\omega}^{\mathrm{T}}(s) \mathrm{d}s \geq \frac{1}{b-a} (\omega(b) - \omega(a))^{\mathrm{T}} R (\omega(b) - \omega(a)). \tag{3}$$

**Lemma 3** *Let $x_i \in \mathbf{R}^n, i \in \mathbf{I}[1, m], m \geq 1$, be a series of vectors and $P > 0$ be given. Then*

$$\left(\sum_{i1}^{m} x_i\right)^{\mathrm{T}} P \left(\sum_{i1}^{m} x_i\right) \leq m \left(\sum_{i1}^{m} x_i^{\mathrm{T}} P x_i\right). \tag{4}$$

**Lemma 4** *Let $\mathcal{A}, \mathcal{B}, \mathcal{H}, \mathcal{P}$ and $\mathcal{F}$ be real matrices of appropriate dimensions such that $\mathcal{P} > 0$ and $\mathcal{F}^{\mathrm{T}}\mathcal{F} \leq I$. Then for any $\varepsilon > 0$ such that $\mathcal{P}^{-1} - \varepsilon^{-1}\mathcal{B}\mathcal{B}^{\mathrm{T}} > 0$,*

$$(\mathcal{A} + \mathcal{B}\mathcal{F}\mathcal{H})^{\mathrm{T}} \mathcal{P} (\mathcal{A} + \mathcal{B}\mathcal{F}\mathcal{H}) \leq \mathcal{A}^{\mathrm{T}} \left(\mathcal{P}^{-1} - \varepsilon^{-1}\mathcal{B}\mathcal{B}^{\mathrm{T}}\right)^{-1} \mathcal{A} + \varepsilon \mathcal{H}^{\mathrm{T}}\mathcal{H}. \tag{5}$$

# 3 Main Results

For $x_t \in \mathcal{C}_{n,r}$ and given positive-define matrices $P \in \mathbf{R}^{n\times n}, Q \in \mathbf{R}^{Nn\times Nn}, R \in \mathbf{R}^{n\times n}$, and an integer $N \geq 1$, we choose a Laypunov-Krasovskii functional candidate as

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \tag{6}$$

where

$$\begin{aligned}
V_1(x_t) &= x^{\mathrm{T}}(t) P x(t), \\
V_2(x_t) &= \int_{t-\frac{r}{N}}^{t} \pi^{\mathrm{T}}(s) Q \pi(s) \mathrm{d}s, \\
V_3(x_t) &= \int_{-r}^{0} \int_{t+s}^{t} \dot{x}^{\mathrm{T}}(\theta) R \dot{x}(\theta) \mathrm{d}\theta \mathrm{d}s,
\end{aligned}$$

and

$$\pi(s) = \begin{bmatrix} x(s) \\ x\left(s - \frac{1}{N}r\right) \\ \vdots \\ x\left(s - \frac{N-1}{N}\right) \end{bmatrix}.$$

Then we can present the following theorem.

**Theorem 1** *Consider the linear time-delay system (??). Given $F \in \mathbf{R}^{n\times m}, r \neq 0$ and $\rho > 0$. If there exist positive-define matrices $P \in \mathbf{R}^{n\times n}, Q \in \mathbf{R}^{Nn\times Nn}, R \in \mathbf{R}^{n\times n}$, and $X_{\mathrm{r}} = \begin{bmatrix} X_{\mathrm{r}1}^{\mathrm{T}} & X_{\mathrm{r}2}^{\mathrm{T}} & \cdots & X_{\mathrm{r}(N+2)}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbf{R}^{(N+2)n\times n}, H \in \mathbf{R}^{n\times m}$, non-singular $Y \in \mathbf{R}^{n\times n}$ such that the following hold*

$$\begin{bmatrix} \varpi(i) & \mathcal{X}_{\mathrm{r}} \\ * & -\frac{1}{r}\mathcal{R} \end{bmatrix} < 0, i \in \mathcal{V}_m, \tag{7}$$

*and the relations*

$$\begin{array}{c} \left|H_i^{\mathrm{T}} y(t)\right| \leq 1, i \in \{1, m\}, \\ H = \begin{bmatrix} H_1 & H_2 & \cdots & H_m \end{bmatrix}, \end{array}$$

*are satisfied for all $x_t \in \mathcal{L}_V$, where*

$$\mathcal{L}_V = \left\{\psi \in \bar{\mathcal{C}}_{n,r} : V(\psi) \leq \rho\right\},$$

*and*

$$\begin{aligned} \varpi(i) = {} & \Gamma_{\mathrm{P}i}^{\mathrm{T}} \begin{bmatrix} 0 & \mathcal{P} \\ \mathcal{P} & 0 \end{bmatrix} \Gamma_{\mathrm{P}i} + \Gamma_{\mathrm{R}i}^{\mathrm{T}} \begin{bmatrix} 0 & \mathcal{R} \\ \mathcal{R} & 0 \end{bmatrix} \Gamma_{\mathrm{R}i} + \Gamma_{\mathrm{Q}i}^{\mathrm{T}} \begin{bmatrix} 0 & \mathcal{Q} \\ \mathcal{Q} & 0 \end{bmatrix} \Gamma_{\mathrm{Q}i} \\ & + \mathrm{He}\left( \begin{bmatrix} \mathcal{X}_{\mathrm{r}1} \\ \mathcal{X}_{\mathrm{r}2} \\ \vdots \\ \mathcal{X}_{\mathrm{r}(N+2)} \end{bmatrix} \begin{bmatrix} I_n & 0_{n,(N-1)n} & I_n & 0_{n,n} \end{bmatrix} \right) \\ & + \mathrm{He}\left( \begin{bmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{bmatrix} \begin{bmatrix} A & 0_{n,(N-1)n} & A_{\mathrm{r}} & -I_n \end{bmatrix} \mathcal{Y}^{\mathrm{T}} \right) \\ & + \mathrm{He}\left( \begin{bmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{bmatrix} \begin{bmatrix} BD_i F^{\mathrm{T}} \mathfrak{Y}^{\mathrm{T}} + BD_i^{-} \mathcal{H} & 0_{n,(N+1)n} \end{bmatrix} \right), \end{aligned}$$

*with*

$$\begin{aligned} & \Gamma_{\mathrm{P}i} = \begin{bmatrix} 0_{n,n} & 0_{n,Nn} & I_n \\ I_n & 0_{n,Nn} & 0_{n,n} \end{bmatrix}, \Gamma_{\mathrm{R}i} = \begin{bmatrix} 0_{n,n} & 0_{n,Nn} & 0_{n,n} \\ 0_{n,n} & 0_{n,Nn} & \sqrt{r} I_n \end{bmatrix}, \Gamma_{\mathrm{Q}i} = \begin{bmatrix} I_{Nn} & 0_{Nn,n} & 0_{Nn,n} \\ 0_{Nn,n} & I_{Nn} & 0_{Nn,n} \end{bmatrix}, \\ & \mathfrak{Y} = Y^{-1}, \mathcal{Y} = \mathrm{diag}\{\mathfrak{Y}, \mathfrak{Y}, \cdots, \mathfrak{Y}\}_{(N+2)n}, \mathcal{Y}_1 = \mathrm{diag}\{\mathfrak{Y}, \mathfrak{Y}, \cdots, \mathfrak{Y}\}_{Nn}, \\ & \mathcal{P} = \mathfrak{Y} P \mathfrak{Y}^{\mathrm{T}}, \mathcal{R} = \mathfrak{Y} R \mathfrak{Y}^{\mathrm{T}}, \mathcal{Q} = \mathcal{Y}_1 Q \mathcal{Y}_1^{\mathrm{T}}, \mathcal{H} = \mathfrak{Y} H, \\ & \mathcal{X}_{\mathrm{r}} = \begin{bmatrix} \mathcal{X}_{\mathrm{r}1}^{\mathrm{T}} & \mathcal{X}_{\mathrm{r}2}^{\mathrm{T}} & \cdots & \mathcal{X}_{\mathrm{r}(N+2)}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \mathcal{X}_{\mathrm{r}j} = \mathfrak{Y} X_{\mathrm{r}j} \mathfrak{Y}^{\mathrm{T}}, j \in \mathbf{I}[1, N+2], \end{aligned} \tag{8}$$

*then the solution $x(t) \equiv 0$ is asymptotically stable for the closed-loop system (??) with the set $\mathcal{L}_V$ contained in the domain of attraction.*

**Proof.** We choose the Lyapunov-Krasovskii functional in the form of (6). Then the derivatives of $V_i(x_t), i = 1, 2, 3$, are given by

$$\begin{array}{l} \dot{V}_1(x_t) = \dot{x}^{\mathrm{T}}(t) P x(t) + x^{\mathrm{T}}(t) P \dot{x}(t), \\ \dot{V}_2(x_t) = \pi^{\mathrm{T}}(t) Q \pi(t) - \pi^{\mathrm{T}}\left(t - \frac{r}{N}\right) Q \pi\left(t - \frac{r}{N}\right), \\ \dot{V}_3(x_t) = r \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - \int_{-r}^{0} \dot{x}^{\mathrm{T}}(t+s) R \dot{x}(t+s) \mathrm{d}s. \end{array} \tag{9}$$

By denoting $\eta^{\mathrm{T}}(t) = \begin{bmatrix} \pi^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-r) & \dot{x}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, we have

$$\begin{aligned} \dot{V}(x_t) &= \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t) \\ &= \eta^{\mathrm{T}}(t) \Gamma_1 \eta(t) - \int_{t-r}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) \mathrm{d}s, \end{aligned} \tag{10}$$

where $\Gamma_1$ is given by

$$\Gamma_1 = \Gamma_{\mathrm{P}i}^{\mathrm{T}} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \Gamma_{\mathrm{P}i} + \Gamma_{\mathrm{R}i}^{\mathrm{T}} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \Gamma_{\mathrm{R}i} + \Gamma_{\mathrm{Q}i}^{\mathrm{T}} \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} \Gamma_{\mathrm{Q}i}, \tag{11}$$

where $\Gamma_{\mathrm{P}i}, \Gamma_{\mathrm{R}i}, \Gamma_{\mathrm{Q}i}$ are in the form of (??). By using the Newton-Leibniz formula and the system (2), we have the following two identities:

$$\begin{array}{l} 0 = 2\eta^{\mathrm{T}}(t) X_{\mathrm{r}} \left( x(t) - x(t-r) - \int_{t-r}^{t} \dot{x}(s)\,\mathrm{d}s \right) \\ 0 = 2\eta^{\mathrm{T}}(t) Y_\eta \left( Ax(t) + A_{\mathrm{r}} x(t-r) + B\mathrm{sat}\left(F^{\mathrm{T}} x(t)\right) - \dot{x}(t) \right), \end{array} \tag{12}$$

where $Y_\eta \in \mathbf{R}^{(N+2)n\times n}$ is defined as

$$Y_\eta = \begin{bmatrix} Y & Y & \cdots & Y \end{bmatrix}^{\mathrm{T}}. \tag{13}$$

Inserting (12) into (10) gives

$$\begin{aligned} \dot{V}(x_t) =& \eta^{\mathrm{T}}(t) \Gamma_3 \eta(t) + 2\eta^{\mathrm{T}}(t) Y_\eta B\mathrm{sat}\left(F^{\mathrm{T}} x(t)\right) \\ & - \int_{t-r}^{t} z^{\mathrm{T}}(s) R^{-1} z(s)\,\mathrm{d}s \\ \leq & \eta^{\mathrm{T}}(t) \Gamma_3 \eta(t) + 2\eta^{\mathrm{T}}(t) Y_\eta B\mathrm{sat}\left(F^{\mathrm{T}} x(t)\right), \end{aligned} \tag{14}$$

where

$$z(s) = X_{\mathrm{r}}^{\mathrm{T}} \eta(t) + R\dot{x}(s),$$

and

$$\Gamma_3 = \Gamma_1 + \Gamma_2 + rX_{\mathrm{r}} R^{-1} X_{\mathrm{r}}^{\mathrm{T}} \tag{15}$$

with $\Gamma_2$ defined as

$$\Gamma_2 = \mathrm{He}\left(X_{\mathrm{r}} \begin{bmatrix} I_n & 0_{n,(N-1)n} & I_n & 0_{n,n} \end{bmatrix}\right) + \mathrm{He}\left(Y_\eta \begin{bmatrix} A & 0_{n,(N-1)n} & A_{\mathrm{r}} & -I_n \end{bmatrix}\right) \tag{16}$$

From Lemma 1, there exist a series of nonnegative number $\lambda_i, i \in \mathcal{V}_m$ such that

$$\mathrm{sat}\left(F^{\mathrm{T}} y(t)\right) = \sum_{i\in\mathcal{V}_m} \lambda_i \left(D_i F^{\mathrm{T}} + D_i^{-} H^{\mathrm{T}}\right) y(t), \quad \sum_{i\in\mathcal{V}_m} \lambda_i = 1. \tag{17}$$

We conclude from (17) that,

$$\begin{aligned} 2\eta^{\mathrm{T}}(t) Y_\eta B\mathrm{sat}\left(F^{\mathrm{T}} x(t)\right) &= 2\eta^{\mathrm{T}}(t) Y_\eta B \sum_{i\in\mathcal{V}_m} \lambda_i \left(D_i F^{\mathrm{T}} + D_i^{-} H^{\mathrm{T}}\right) x(t) \\ &= 2\eta^{\mathrm{T}}(t) \sum_{i\in\mathcal{V}_m} \lambda_i \begin{bmatrix} Y_\eta B\left(D_i F^{\mathrm{T}} + D_i^{-} H^{\mathrm{T}}\right) & 0_{n,(N+1)n} \end{bmatrix} \eta(t) \\ &\leq \eta^{\mathrm{T}}(t) \max_{i\in\mathcal{V}_m} \left\{\Gamma_4(i)\right\} \eta(t), \end{aligned} \tag{18}$$

where

$$\Gamma_4(i) = \mathrm{He}\left(\begin{bmatrix} Y_\eta B\left(D_i F^{\mathrm{T}} + D_i^{-} H^{\mathrm{T}}\right) & 0_{n,(N+1)n} \end{bmatrix}\right). \tag{19}$$

Then substituting (18) into (14) gives

$$\dot{V}(x_t) \leq \max_{i\in\mathcal{V}_m} \eta^{\mathrm{T}}(t) \varsigma(i) \eta(t),$$

where $\varsigma(i) = \Gamma_1 + \Gamma_2 + \Gamma_4(i) + rX_{\mathrm{r}} R^{-1} X_{\mathrm{r}}^{\mathrm{T}}$.

By the Schur complement, $\varsigma(i) < 0, i \in \mathcal{V}_m$ is equivalent to

$$\begin{bmatrix} \Gamma_1 + \Gamma_2 + \Gamma_4(i) & X_{\mathrm{r}} \\ * & -\frac{1}{r} R \end{bmatrix} < 0, \tag{20}$$

which is equivalent to (7) by setting $\mathfrak{Y}=Y^{-1}, \mathcal{Y}=\mathrm{diag}\left\{\mathfrak{Y}, \mathfrak{Y}, \cdots, \mathfrak{Y}\right\}_{(N+2)n}, \mathcal{Y}_1=\mathrm{diag}\left\{\mathfrak{Y}, \mathfrak{Y}, \cdots, \mathfrak{Y}\right\}_{Nn}$, $\mathcal{P}=\mathfrak{Y}P\mathfrak{Y}^{\mathrm{T}}, \mathcal{R}=\mathfrak{Y}R\mathfrak{Y}^{\mathrm{T}}, \mathcal{Q}=\mathcal{Y}_1 Q\mathcal{Y}_1^{\mathrm{T}}, \mathcal{H}=\mathfrak{Y}H, \mathcal{X}_{\mathrm{r}j}=\mathfrak{Y}X_{\mathrm{r}j}\mathfrak{Y}^{\mathrm{T}}, j \in \mathbf{I}[1, N+2]$ and performing a congruence transformation by $\mathcal{Y}$ to (20). That is to say that (20) holds if (7) is satisfied. Then we have $\dot{V}(x_t) < 0$. ■