A Review of Stabilization of Linear Systems with Input Saturation

Bin Zhou* Guang-Ren Duan*

Abstract

We give a brief over review of the stabilization of linear systems with input saturation.

Keywords: Input saturation; global stabilization; semi-global stabilization; local stabilization.

1 Introduction and Notation

Notation:

1. $\sigma(\cdot) : \mathbf{R}^m \to \mathbf{R}^m$ denotes the unit saturation function

$$\sigma(u) = \begin{bmatrix} \sigma(u_1) \\ \sigma(u_2) \\ \vdots \\ \sigma(u_m) \end{bmatrix}, \ \sigma(u_i) = \operatorname{sign}(u_i) \min\{|u_i|, 1\}.$$

2. $|\cdot|$ and $|\cdot|_\infty$ refers to respectively the 2 and ∞-norm of a matrix.
3. $\mathscr{E}(P, \rho) = \{x : x^{\mathrm{T}} P x \leq \rho\}$ denotes the ellipsoid, where $P > 0$. If $\rho = 1$, we use $\mathscr{E}(P)$ for short.
4. $\mathscr{L}(H) = \{x : |Hx|_\infty \leq 1\}$ denotes the region in which $\sigma(Hx) = Hx$.
5. $\mathbf{I}[p, q] = \{p, p+1, \cdots, q\}$, where $p \leq q$ are two integers.

2 Problem Formulation and Preliminaries

Definition 1 *(Domain of Attraction) Consider the nonlinear system*

$$\dot{x} = f(x), \ x \in \mathbf{R}^n. \tag{1}$$

The set $\mathcal{D} \subset \mathbf{R}^n$ is called the domain of attraction for this system if

$$x_0 \in \mathcal{D} \Longrightarrow \lim_{t \to \infty} |x(t)| = 0.$$

Definition 2 *(Invariant Set) The set $\mathcal{S} \subset \mathbf{R}^n$ is called an invariant set for the nonlinear system (1) if*

$$x_0 \in \mathcal{S} \Longrightarrow x(t) \in \mathcal{S}, \forall t \geq 0.$$

If, moreover,

$$x_0 \in \mathcal{S} \Longrightarrow \begin{cases} x(t) \in \mathcal{S}, \forall t \geq 0, \\ \lim_{t\to\infty} |x(t)| = 0, \end{cases}$$

then $\mathcal{S}$ is called a contractively invariant set.

*Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin, 150001, China.

Definition 3 *(Null Controllable Region) Consider the linear system with input saturation*

$$\dot{x}(t) = Ax + B\sigma(u), \tag{2}$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^m$. The set $\mathcal{C} \subseteq \mathbf{R}^n$ is called the null controllable region for system (2) if there exists a control $u \in \mathbf{R}^m$ and a $T \in [0, \infty]$ such that

$$x_0 \in \mathcal{C} \Longrightarrow x(T) = 0, \forall t \geq T.$$

Proposition 1 *Let $\mathcal{C}$ be the null controllable region for system (2).*

1. *$\mathcal{C}$ is convex.*
2. *If all the eigenvalues of A are on the closed-left half plane, namely, $\lambda(A) \subset \mathbf{C}^{\mathrm{Re}(s)\leq 0}$, and (A, B) is stabilizable, then $\mathcal{C} = \mathbf{R}^n$. In this case, we call that (A, B) is asymptotically null controllable by bounded controls (ANCBC).*
3. *If all the eigenvalues of A are on the open-right half plane, namely, $\lambda(A) \subset \mathbf{C}^{\mathrm{Re}(s)>0}$, then $\mathcal{C}$ is bounded.*
4. *Let (A, B) be partitioned as*
$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad B = \begin{bmatrix} B_s \\ B_u \end{bmatrix},$$
in which $A_s \in \mathbf{R}^{n_s}, \lambda(A_s) \subset \mathbf{C}^{\mathrm{Re}(s)\leq 0}$ and $\lambda(A_u) \subset \mathbf{C}^{\mathrm{Re}(s)>0}$. Then $\mathcal{C} = \mathbf{R}^{n_s} \oplus \mathcal{C}_u$, where $\mathcal{C}_u$ is the null controllable region for the subsystem
$$\dot{x}_u = A_u x_u + \sigma(B_u v).$$

Proof. We only prove Item 3. This proof is initially given in [21].

Since A is anti-stable, there exists a scalar $\delta > 0$ such that $\delta I_n - A$ is Hurwitz. Hence there exists a $P > 0$ such that

$$(\delta I_n - A)^{\mathrm{T}} P + P(\delta I_n - A) < 0,$$

or, equivalently,

$$A^{\mathrm{T}} P + PA > 2\delta P.$$

Consider $V = x^{\mathrm{T}} P x$. Then

$$\begin{aligned}
\dot{V}(x) &= x^{\mathrm{T}}\left(A^{\mathrm{T}}P + PA\right)x + 2x^{\mathrm{T}}PB\sigma(u) \\
&\geq 2\delta V(x) - 2\left|x^{\mathrm{T}}P^{\frac{1}{2}}\right|\left|P^{\frac{1}{2}}B\right||\sigma(u)| \\
&\geq 2\delta V(x) - 2\sqrt{m}\left|x^{\mathrm{T}}P^{\frac{1}{2}}\right|\left|P^{\frac{1}{2}}B\right| \\
&= 2\delta V(x) - 2\sqrt{m}\sqrt{V(x)}\left|P^{\frac{1}{2}}B\right| \\
&= 2\delta\sqrt{V(x)}\left(\sqrt{V(x)} - \frac{\sqrt{m}}{\delta}\left|P^{\frac{1}{2}}B\right|\right) \\
&= \frac{2\delta\sqrt{V(x)}}{\sqrt{V(x)} + \frac{\sqrt{m}}{\delta}\left|P^{\frac{1}{2}}B\right|}\left(V(x) - \frac{m}{\delta^2}\left|P^{\frac{1}{2}}B\right|^2\right).
\end{aligned}$$

Hence if

$$V(x_0) \geq \frac{m}{\delta^2}\left|P^{\frac{1}{2}}B\right|^2,$$

we have $\dot{V}(x) \geq 0$ and thus

$$V(x(t)) \geq V(x_0) \geq \frac{m}{\delta^2}\left|P^{\frac{1}{2}}B\right|^2.$$

This indicates that

$$\mathcal{C} \subset \mathscr{E}\left(P, \frac{m}{\delta^2}\left|P^{\frac{1}{2}}B\right|^2\right),$$

which is bounded. ■

For the linear system (2) with input saturation, the problems that are of interest can be collected in the following.

Problem 1 *For the linear system (2) with input saturation.*

(i) Compute the null controllable region $\mathcal{C}$ and design the controller $u = u(\mathcal{C})$ such that the closed-loop system is asymptotically stable with the domain of attraction being $\mathcal{C}$.

(ii) Design a linear state feedback $u = Fx$ such that the domain of attraction for the closed-loop system is maximized.

(iii) Design a linear state feedback $u = Fx$ such that a contractively invariant set for the closed-loop system is maximized (in this case, such a contractively invariant set is referred to as the estimation of domain of attraction).

Remark 1 *For the above problem, we have the following remark.*

1. *If we use $\iota \succ \iota\iota$ to denote that ι is more difficult than $\iota\iota$, then*

$$(i) \succ (ii) \succ (iii).$$

2. *Problem (i) has been solved completely only when $\lambda(A) \subset \mathbf{C}^{\mathrm{Re}(s)\leq 0}$. Since in this case $\mathcal{C} = \mathbf{R}^n$, this problem is referred to as global stabilization. In [2] it is shown that a triple integrator system cannot be globally stabilized by saturated linear feedback. A general result that multiple integrators system with length larger than 2 cannot be stabilized globally by saturated linear feedback was proven in [12]. Hence u must be a nonlinear function of x. For the same system (a chain of integrators), a nonlinear feedback law consisting of nested saturation elements was initially established in [13]. The idea was then extended to solve the global stabilization problem for general ANCBC linear system in [11].*

3. *If the controller u is restricted to be a linear function of the state x, namely, $u = F(\gamma)x$, then there exist an $F(\gamma) : (0,1) \to \mathbf{R}^{m\times n}$ such that the estimation of the domain of attraction of the closed-loop system (denoted by $\mathcal{S}(\gamma)$) satisfies*

$$\lim_{\gamma\to 0^+} \mathcal{S}(\gamma) = \mathbf{R}^n.$$

   This problem is referred to as semi-global stabilization problem (by linear feedback). This problem was originally studied in [7] where a constructive solution by eigenstructure assignment was established.

4. *For general system, only Problem (iii) is tractable at present. This problem is generally referred to as local stabilization.*

3 Global Stabilization

3.1 Teel Canonical Form and a Fundamental Lemma

Definition 4 *(Teel Canonical Form) [21] The matrix pair $(A_{\mathrm{T}}, b_{\mathrm{T}}) \in (\mathbf{R}^{n\times n}, \mathbf{R}^{n\times 1})$ is in Teel's canonical form if it is controllable and*

$$A_{\mathrm{T}} = \begin{bmatrix} A_1 & b_1 f_2 & \cdots & b_1 f_{k-1} & b_1 f_k \\ & A_2 & \cdots & b_2 f_{k-1} & b_2 f_k \\ & & \ddots & \vdots & \vdots \\ & & & A_{k-1} & b_{k-1} f_k \\ & & & & A_k \end{bmatrix}, \quad b_{\mathrm{T}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \end{bmatrix}, \tag{3}$$

where $b_i \in \mathbf{R}^{n_i \times 1}, A_i \in \mathbf{R}^{n_i \times n_i}, f_i \in \mathbf{R}^{1 \times n_i}$ are arbitrary matrices and $n_1 + n_2 + \cdots + n_k = n$.

The following lemma implies a method for transforming a general linear system with a single input into its Teel's canonical form.

Lemma 1 *[19] Let (A, b) and $(\mathfrak{A}, \mathfrak{b})$ be two given matrix pairs and (A, b) is controllable. Then there exists a nonsingular matrix T such that*

$$\mathfrak{A} = TAT^{-1}, \quad \mathfrak{b} = Tb, \tag{4}$$

if and only if $(\mathfrak{A}, \mathfrak{b})$ is also controllable and $\lambda(A) = \lambda(\mathfrak{A})$. In this case, the unique transformation matrix T is given by

$$T = Q_{\mathrm{c}}(\mathfrak{A}, \mathfrak{b}) Q_{\mathrm{c}}^{-1}(A, b). \tag{5}$$

3.2 A Chain of Integrators

In this section, we consider the following n-th order multiple integrators system

$$\dot{x}_i = x_{i+1}, \quad i \in \mathbf{I}[1, n-1], \quad \dot{x}_n = u, \tag{6}$$

where u satisfies

$$|u| \le u_{\max}.$$

Here $u_{\max}$ can be any positive number. This system is clearly ANCBC.

3.2.1 Nested and Cascade Saturation Functions

Lemma 2 *[20] Let $\lambda_i, i \in \mathbf{I}[1, n]$ be a series of priori given positive numbers. Then there exists a nonsingular matrix T such that the linear transformation $y = Tx$ puts the linear system (6) into*

$$\dot{y} = Ay + bu, \tag{7}$$

where A and b are given by (Teel canonical form)

$$A = \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \lambda_{n-1} & \lambda_n \\ 0 & 0 & \cdots & 0 & \lambda_n \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{8}$$

Lemma 3 *[20] Consider the following scalar nonlinear system*

$$\dot{x} = u, \ u = -\varepsilon\sigma\left(\frac{\lambda x}{\varepsilon}\right) + e(t), \quad t \ge t_0, \tag{9}$$

where $e(t) : [t_0, \infty) \to \mathbf{R}$ is uniformly bounded. If

$$|e(t)| < \varepsilon, \quad \forall t \ge t_0,$$

then there exists a $T > t_0$ such that for $\forall t > T$, there holds $|x| \le \frac{\varepsilon}{\lambda}$. Furthermore, the input u in system (9) can be simplified as $u = -\lambda x + e(t)$.

Applying this lemma recursively on system (7) gives the following theorem.

Theorem 1 *[20] Let $\lambda_i, i \in \mathbf{I}[1,n]$ be a series of priori given positive numbers and $\varepsilon_i, i \in \mathbf{I}[2,n]$ satisfy the following series of inequalities*

$$\varepsilon_j > \sum_{i=1}^{j-1} \varepsilon_i, \ j \in \mathbf{I}[2,n], \ \varepsilon_1 > 0, \ \sum_{i=1}^{n} \varepsilon_i \leq u_{\max}. \tag{10}$$

Then the nonlinear control law

$$u = -\sum_{i=1}^{n} \varepsilon_i \sigma \left(\frac{\lambda_i y_i}{\varepsilon_i} \right),$$

with y given in (7) globally stabilizes system (6). Furthermore, the closed-loop system will operate in linear region at finite time with eigenvalues $-\lambda_i, i \in \mathbf{I}[1,n]$.

Lemma 4 *[20] Consider the following scalar system*

$$\dot{x} = u, \quad u = -\varepsilon\sigma \left(\frac{\lambda x - e(t)}{\varepsilon} \right), \quad t \geq t_0, \tag{11}$$

where $e(t) : [t_0, \infty) \to \mathbf{R}$ is uniformly bounded. If

$$|e(t)| < \frac{1}{2}\varepsilon, \ \forall t \geq t_0,$$

then there exists a number $T > t_0$ such that for $\forall t > T$, there holds $|x| \leq \frac{\varepsilon}{2\lambda}$. Moreover, the function u in (11) can be simplified as $u = -\lambda x + e(t)$.

Applying the above lemma on system (7) recursively gives the following theorem.

Theorem 2 *[20] Let λ_i, $i \in \mathbf{I}[1,n]$ be a series of priori given positive numbers and ε_i, $i \in \mathbf{I}[1,n]$ be some positive numbers satisfying*

$$\varepsilon_n \leq u_{\max}, \ \varepsilon_i > 2\varepsilon_{i-1}, \ i \in \mathbf{I}[2,n], \ \varepsilon_1 > 0. \tag{12}$$

Then the nonlinear control law $u = u_n$ with

$$\begin{aligned} u_i &= -\varepsilon_i \sigma \left(\frac{\lambda_i y_i}{\varepsilon_i} - \frac{1}{\varepsilon_i} u_{i-1} \right), \ i \in \mathbf{I}[2,n], \\ u_1 &= -\varepsilon_1 \sigma \left(\frac{\lambda_1 y_1}{\varepsilon_1} \right), \end{aligned} \tag{13}$$

and y given in (7) globally stabilizes system (6). Moreover, the closed-loop system will operate in linear region at finite time with eigenvalues $-\lambda_i$, $i \in \mathbf{I}[1,n]$.

In both theorems, the saturation functional can be made state dependent to improve the transient performances [20].

3.2.2 Control Laws with Less Nested Saturation Functions

Lemma 5 *[17] Let γ_i and $\rho_i \geq \frac{1}{2}, i \in \mathbf{I}[2,p]$ be a series of given positive constants, where $p = \left[\frac{n+1}{2}\right]$ with $[a]$ being the integer part of a. Then there exits a transformation $y = Tz$ such that system (6) is transformed into*

$$\dot{y} = A_{\mathrm{o}} y + b_p u, \tag{14}$$

where A_{o} and b_p are respectively given by (Teel canonical form)

$$A_{\mathrm{o}} = \begin{bmatrix} eA_{\mathrm{d}}e^{\mathrm{T}} & eA_2 & \cdots & eA_{p-1} & eA_p \\ 0 & A_{\mathrm{d}} & \cdots & A_{p-1} & A_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{\mathrm{d}} & A_p \\ 0 & 0 & \cdots & 0 & A_{\mathrm{d}} \end{bmatrix}, \ b_p = \begin{bmatrix} eb \\ b \\ \vdots \\ b \\ b \end{bmatrix},$$

in which $b = [0,1]^{\mathrm{T}}$ *and*

$$A_{\mathrm{d}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ A_i = \begin{bmatrix} 0 & 0 \\ \rho_i \gamma_i^2 & 2\rho_i\gamma_i \end{bmatrix}, \quad i \in \mathbf{I}[1,p],$$
$$e = \begin{cases} I_2, & n \text{ is even}, \\ \begin{bmatrix} 0 & 1 \end{bmatrix}, & n \text{ is odd}. \end{cases}$$

Let P_{d} be the unique positive definite solution to the parametric Lyapunov equation (PLE)

$$A_{\mathrm{d}}^{\mathrm{T}} P_{\mathrm{d}} + P_{\mathrm{d}} A_{\mathrm{d}} - P_{\mathrm{d}} b b^{\mathrm{T}} P_{\mathrm{d}} = -\gamma P_{\mathrm{d}}, \tag{15}$$

where γ is a positive constant, namely,

$$P_{\mathrm{d}} = \begin{bmatrix} \gamma^3 & \gamma^2 \\ \gamma^2 & 2\gamma \end{bmatrix}. \tag{16}$$

Moreover, for any constant $\rho \geq \frac{1}{2}$, we define

$$A_{\mathrm{c}} = A_{\mathrm{d}} - \rho b b^{\mathrm{T}} P_{\mathrm{d}} = \begin{bmatrix} 0 & 1 \\ -\rho\gamma^2 & -2\rho\gamma \end{bmatrix}. \tag{17}$$

Then we have the following lemma.

Lemma 6 *[17] Consider the following second order nonlinear system*

$$\dot{x} = A_{\mathrm{d}} x + bu, \quad x \in \mathbf{R}^2, \tag{18}$$

in which u *is defined as*

$$u = -\varepsilon_2 \sigma \left(\frac{\rho b^{\mathrm{T}} P_{\mathrm{d}}}{\varepsilon_2} x + \frac{\varepsilon_1}{\varepsilon_2} \sigma(z) \right), \tag{19}$$

where $\omega, \rho \geq \frac{1}{2}, \varepsilon_1$ *and* ε_2 *are given positive scalars, and* z *is an external signal. If*

$$\varepsilon_2 \geq \left(\frac{8\rho - 1}{4\rho - 1} \right) \varepsilon_1. \tag{20}$$

then there exists a finite time T *such that the control and the system can be simplified as*

$$\begin{cases} u = -\rho b^{\mathrm{T}} P x - \varepsilon_1 \sigma(z), \\ \dot{x} = A_{\mathrm{c}} x - b\varepsilon_1 \sigma(z), \ \forall t \geq T. \end{cases}$$

Now denote

$$y^{\mathrm{T}} = \begin{bmatrix} y_1^{\mathrm{T}} & y_2^{\mathrm{T}} & \cdots & y_p^{\mathrm{T}} \end{bmatrix}, \ y_i \in \mathbf{R}^2, \ i \in \mathbf{I}[2,p],$$
$$y_1 \in \begin{cases} \mathbf{R}^2, & n \text{ is even}, \\ \mathbf{R}, & n \text{ is odd}. \end{cases}$$

Then applying Lemma 6 on system (14) recursively gives the following theorem.

Theorem 3 *[17] Let* γ_i *and* $\rho_i \geq \frac{1}{2}, i \in \mathbf{I}[1,p]$ *be a series of given positive constants. Let the positive constants* $\varepsilon_i, i \in \mathbf{I}[1,p]$ *be such that*

$$\varepsilon_i \geq \left(\frac{8\rho_i - 1}{4\rho_i - 1} \right) \varepsilon_{i-1}, \ i \in \mathbf{I}[2,p], \ \varepsilon_p \leq u_{\max}, \tag{21}$$

then the following nonlinear control law $u = u_p$ *stabilizes system (6) globally*

$$u_i = -\varepsilon_i \sigma \left(\frac{\rho_i b^{\mathrm{T}} P_i}{\varepsilon_i} y_i - \frac{u_{i-1}}{\varepsilon_i} \right), \ i \in \mathbf{I}[2,p], \tag{22}$$

where

$$u_1 = \begin{cases} -\varepsilon_1 \sigma \left(\frac{\rho_1 b^{\mathrm{T}} P_1}{\varepsilon_1} y_1 \right), & n \text{ is even}, \\ -\varepsilon_1 \sigma \left(\frac{\rho_1 \gamma_1^2}{\varepsilon_1} y_1 \right), & n \text{ is odd}, \end{cases}$$

in which $P_i, i \in \mathbf{I}[1,p]$ *is in the form of (16) with* γ *being replaced by* γ_i. *Moreover, the closed-loop system becomes a linear system after finite time and has the characteristic equation*

$$\alpha(s) = \begin{cases} \prod_{i=1}^{p} \left(s^2 + 2\rho_i \gamma_i s + \rho_i \gamma_i^2 \right), & n \text{ is even}, \\ \left(s + \rho_1 \gamma_1^2 \right) \prod_{i=2}^{p} \left(s^2 + 2\rho_i \gamma_i s + \rho_i \gamma_i^2 \right), & n \text{ is odd}. \end{cases} \tag{23}$$

3.3 A Chain of Oscillator

In this section, we consider the following linear system

$$\dot{x} = Ax + Bu, \quad x \in \mathbf{R}^n, \tag{24}$$

with bounded input

$$|u| \leq u_{\max},$$

where $u_{\max}$ is some known scalar representing the amplitude limitation of the control. Assume that (A, B) is controllable and all the eigenvalues of A have zero real parts and nonzero imaginary parts, namely,

$$\lambda(A) = \{\pm \omega_i \mathrm{j}\}_{i=1}^{p}, \ p = \frac{n}{2} \text{ and } \omega_i \neq 0, \ i \in \mathbf{I}[1,p]. \tag{25}$$

Linear systems that satisfy these assumptions contain multiple oscillators.

Lemma 7 *[23] Let* γ_i *and* $\rho_i \geq \frac{1}{2}, i \in \mathbf{I}[2,p]$, *be some given positive numbers. Then, system (24) is algebraically equivalent to*

$$\dot{y} = A_{\mathrm{o}} y + b_p u, \tag{26}$$

where A_{o}, b_p *are, respectively, given by*

$$A_{\mathrm{op}} = \begin{bmatrix} A_{\omega_1} & A_2 & \cdots & A_{p-1} & A_p \\ 0 & A_{\omega_2} & \cdots & A_{p-1} & A_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{\omega_{p-1}} & A_p \\ 0 & 0 & \cdots & 0 & A_{\omega_p} \end{bmatrix}, \ b_p = \begin{bmatrix} b \\ b \\ \vdots \\ b \\ b \end{bmatrix},$$

with $b = [0, 1]^{\mathrm{T}}$ *and*

$$A_{\omega_i} = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 \\ \frac{\rho_i \gamma_i^2}{\omega_i} & 2\rho_i \gamma_i \end{bmatrix}, \quad i \in \mathbf{I}[1,p]. \tag{27}$$

Let P be the unique positive definite solution to the following PLE

$$A_\omega^{\mathrm{T}} P + P A_\omega - P b b^{\mathrm{T}} P = -\gamma P, \tag{28}$$

where A_ω is in the form of (27) with $\omega_i = \omega$ and γ is a positive scalar. Solving (28), we obtain

$$P = \begin{bmatrix} \frac{\gamma^3}{\omega^2} + 2\gamma & \frac{\gamma^2}{\omega} \\ \frac{\gamma^2}{\omega} & 2\gamma \end{bmatrix}. \tag{29}$$

Moreover, for some scalar $\rho \geq \frac{1}{2}$, we define

$$A_{\mathrm{c}\omega} = A_\omega - \rho b b^{\mathrm{T}} P = \begin{bmatrix} 0 & \omega \\ -\omega - \frac{\rho \gamma^2}{\omega} & -2\rho\gamma \end{bmatrix}. \tag{30}$$

Lemma 8 *[23] Consider the following second-order nonlinear system*

$$\dot{x} = A_\omega x + bu, \quad \boldsymbol{x} \in \mathbf{R}^2, \tag{31}$$

where

$$u = -\varepsilon_2 \sigma \left(\frac{\rho b^{\mathrm{T}} P \boldsymbol{x}}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \sigma(z) \right), \tag{32}$$

with $\omega, \rho \geq \frac{1}{2}, \varepsilon_1, \varepsilon_2$ *being some positive scalars and* $z = z(t)$ *an arbitrary external signal. If*

$$\varepsilon_2 \geq \left(\frac{8\rho - 1}{4\rho - 1} \right) \varepsilon_1, \tag{33}$$

then, after some finite time T*, the following holds for all* $t \geq T$*,*

$$\begin{cases} \dot{x} = A_{\mathrm{c}\omega} x - b\varepsilon_1 \sigma(z(t)), \\ u = -\rho b^{\mathrm{T}} P x - \varepsilon_1 \sigma(z). \end{cases}$$

Let the state vector y be partitioned as

$$y = \begin{bmatrix} y_1^{\mathrm{T}} & y_2^{\mathrm{T}} & \cdots & y_p^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \ y_i \in \mathbf{R}^2, \ i \in \mathbf{I}[1, p]. \tag{34}$$

Then applying Lemma 8 on system (26) recursively gives the following theorem.

Theorem 4 *[23] Let* γ_i *and* $\rho_i \geq \frac{1}{2}, i \in \mathbf{I}[1, p]$ *be some given positive numbers. If the positive scalars* $\varepsilon_i, i \in \mathbf{I}[1, p]$*, are chosen such that*

$$\varepsilon_i \geq \left(\frac{8\rho_i - 1}{4\rho_i - 1} \right) \varepsilon_{i-1}, \ i \in \mathbf{I}[2, p], \ \varepsilon_p \leq u_{\max}, \tag{35}$$

then the nonlinear function $u = u_p$ *with*

$$\begin{aligned} u_i &= -\varepsilon_i \sigma \left(\frac{\rho_i b^{\mathrm{T}} P_i y_i}{\varepsilon_i} - \frac{u_{i-1}}{\varepsilon_i} \right), \ i \in \mathbf{I}[2, p], \\ u_1 &= -\varepsilon_1 \sigma \left(\frac{\rho_1 b^{\mathrm{T}} P_1 y_1}{\varepsilon_1} \right), \end{aligned} \tag{36}$$

where $P_i, i \in \mathbf{I}[1, p]$*, are in the form of (29) with* $\gamma = \gamma_i$ *and* $\omega = \omega_i$*, stabilizes system (24) globally. Moreover the closed-loop system will operate linearly after a finite time with the characteristic polynomial*

$$\alpha(s) = \prod_{i=1}^{p} \left(s^2 + 2\rho_i \gamma_i s + \omega_i^2 + \rho_i \gamma_i^2 \right). \tag{37}$$

3.4 General Linear System with Multiple Inputs

For general linear system with multiple inputs, we need to transform it into a series of linear systems with a single input. This can be accomplished by the Wonham canonical form decomposition.

Lemma 9 *(Wonham Canonical Form) [16] Assume that* $(A, B) \in (\mathbf{R}^{n \times n}, \mathbf{R}^{n \times m})$ *is controllable. Then there exists a nonsingular matrix* T *such that*

$$T^{-1} A T = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1l} \\ & A_2 & \ddots & A_{2l} \\ & & \ddots & \vdots \\ & & & A_l \end{bmatrix}, \ T^{-1} B = \begin{bmatrix} b_1 & & & * \\ & b_2 & & * \\ & & \ddots & \vdots \\ & & b_l & * \end{bmatrix}, \tag{38}$$

where $(A_i, b_i) \in \left(\mathbf{R}^{n_i \times n_i}, \mathbf{R}^{n_i \times 1} \right), i \in \mathbf{I}[1, l]$ *are controllable and* $l \leq m$ *is an integer.*

For the purpose of recursive design, we introduce the following simple lemma.

Lemma 10 *[21] Consider the following linear system*

$$\left\{ \begin{array}{l} \dot{x}_1 = A_1 x_1 + A_{12} x_2 + b_1 \sigma(u_1), \\ \dot{x}_2 = A_2 x_2 \end{array} \right. \tag{39}$$

Assume that $\lambda(A_2) \cap \lambda(A_1) = \varnothing$. Let T_{12} be the unique solution to the following Sylvester matrix equation

$$A_1 T_{12} - T_{12} A_2 = A_{12}.$$

Then, via the following state of transformation

$$\left[\begin{array}{c} y_1 \\ y_2 \end{array} \right] = \left[\begin{array}{cc} I & T_{12} \\ 0 & I \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right],$$

system (39) is algebraically equivalent to

$$\left\{ \begin{array}{ll} \dot{y}_1 & = A_1 y_1 + b_1 \sigma(u_1), \\ \dot{y}_l & = A_2 y_2. \end{array} \right.$$

We first consider the l-th subsystem of (38), namely,

$$\dot{x}_l = A_l x_l + b_l \sigma(u_l).$$

According to the development in the above two subsections, there exists a controller $u_l = u_l(x_l)$ and a $T_l > 0$ such that

$$\begin{aligned} \dot{x}_l &= (A_l + b_l k_l) x_l \\ &\triangleq A_{\mathrm{c}l} x_l, \ \forall t \geq T_l. \end{aligned}$$

Now consider the $l-1$-th subsystem of (38), namely,

$$\left\{ \begin{array}{ll} \dot{x}_{l-1} & = A_{l-1} x_{l-1} + A_{l-1,l} x_l + b_{l-1} \sigma(u_{l-1}), \\ \dot{x}_l & = A_{\mathrm{c}l} x_l. \end{array} \right.$$

Consider the following change of the state variable

$$\left[\begin{array}{c} y_{l-1} \\ y_l \end{array} \right] = \left[\begin{array}{cc} I_{n_{l-1}} & T_{l-1,l} \\ 0 & I_{n_l} \end{array} \right],$$

where $T_{l-1,l}$ is the unique solution to the following Sylvester matrix equation

$$A_{l-1} T_{l-1,l} - T_{l-1,l} A_{\mathrm{c}l} = A_{l-1,l}.$$

Notice that $\lambda(A_{l-1}) \cap \lambda(A_{cl})$ is satisfied. Then, by Lemma 10, we have

$$\left\{ \begin{array}{ll} \dot{y}_{l-1} & = A_{l-1} y_{l-1} + b_{l-1} \sigma(u_{l-1}), \\ \dot{y}_l & = A_{\mathrm{c}l} y_l. \end{array} \right.$$

Again, by the developments in the above two subsections, there exists a controller $u_{l-1} = u_{l-1}(y_{l-1})$ and a $T_{l-1} > T_l$ such that the closed-loop system is linear for all $t \geq T_{l-1}$, namely,

$$\begin{aligned} \left[\begin{array}{c} \dot{x}_{l-1} \\ \dot{x}_l \end{array} \right] &= \left(\left[\begin{array}{cc} A_{l-1} & A_{l-1,l} \\ 0 & A_l \end{array} \right] + \left[\begin{array}{cc} b_{l-1} & 0 \\ 0 & b_l \end{array} \right] \left[\begin{array}{cc} k_{l-1} & k_{l-1,l} \\ 0 & k_l \end{array} \right] \right) \left[\begin{array}{c} x_{l-1} \\ x_l \end{array} \right] \\ &= \left[\begin{array}{cc} A_{l-1} + b_{l-1} k_{l-1} & A_{l-1,l} + b_{l-1} k_{l-1,l} \\ 0 & A_l + b_l k_l \end{array} \right] \left[\begin{array}{c} x_{l-1} \\ x_l \end{array} \right] \\ &\triangleq A_{\mathrm{c}l-1} \left[\begin{array}{c} x_{l-1} \\ x_l \end{array} \right], \ t \geq T_{l-1}. \end{aligned}$$

Repeating the above process produces the controller $u_i, i = l, l-1, \cdots, 1$.

4 Semi-global Stabilization

Consider a continuous-time time-invariant linear system subject to actuator saturation

$$\dot{x} = Ax + B\sigma(u), \tag{40}$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$ are, respectively, the state and input vectors. The problem to be solved in this section is stated as follows.

Problem 2 *(L_∞ Semi-global Stabilization) Let $\Omega \subset \mathbf{R}^n$ be a prescribed bounded set. Design a linear state feedback controller $u = Fx$ such that the closed-loop system is asymptotically stable with Ω contained in the domain of attraction, namely, $x(0) \in \Omega \Rightarrow x(t) \in \Omega, \forall t \geq 0,$ and $\lim_{t\to\infty} |x(t)| = 0$.*

4.1 L_∞ Semi-global Stabilization by L_∞ Low Gain Feedback

We consider a parameterized linear state feedback

$$u(t) = F(\gamma)\,x(t),\ F(\gamma) : (0, D_\infty] \to \mathbf{R}^{m\times n}, \tag{41}$$

where D_∞ is a positive constant and $F(\gamma)$ is a matrix function to be determined and is such that $A+BF(\gamma)$ is Hurwitz. Then, if the actuator does not saturated for all time and $x_0 \in \Omega$, namely,

$$|u|_{L_\infty} \triangleq \sup_{t\geq 0} |u(t)|_\infty \leq 1,\ \forall x_0 \in \Omega, \tag{42}$$

the closed-loop system can be expressed as

$$\dot{x}(t) = (A + BF(\gamma))\,x(t),\ x(0) = x_0, \tag{43}$$

which is linear, and is thus asymptotically stable. Since

$$u(t) = F(\gamma)\,\mathrm{e}^{(A+BF(\gamma))t}x_0, \tag{44}$$

and Ω is bounded and can be arbitrarily large, (42) is satisfied if and only if

$$\lim_{\gamma\to 0^+} \sup_{t\geq 0} \left|F(\gamma)\,\mathrm{e}^{(A+BF(\gamma))t}\right| = 0. \tag{45}$$

Motivated by this observation, we give the following definition.

Definition 5 *(L_∞ Low Gain Feedback) Assume that (A, B) is ANCBC. Then a bounded matrix gain $F(\gamma)$: $(0, D_\infty] \to \mathbf{R}^{m\times n}$ is called an L_∞ low gain for the matrix pair (A, B) if (45) is satisfied.*

The following proposition is a consequence of the Definition 5.

Proposition 2 *[21] Assume that (A, B) is ANCBC and $F(\gamma) : (0, D_\infty] \to \mathbf{R}^{m\times n}$ is an L_∞ low gain for (A, B). Then,*

1. *For any $|x_0| < \infty$ and any integer $l \geq 1$, the control $u(t)$ for the closed-loop system (43) satisfies*

$$\lim_{\gamma\to 0^+} \left|u^{(l)}\right|_{L_\infty} = 0, \tag{46}$$

   namely, the L_∞ norm of any order time-derivative of $u(t)$ can also be made arbitrarily small by decreasing γ.

2. *$F(\gamma) : (0, 1] \to \mathbf{R}^{m\times n}$ is an L_∞ low gain for (A, B) only if*

$$\lim_{\gamma\to 0^+} F(\gamma) = 0. \tag{47}$$

   If, moreover, A is critically stable, then $F(\gamma)$ is an L_∞ low gain for (A, B) if and only if (47) is satisfied.

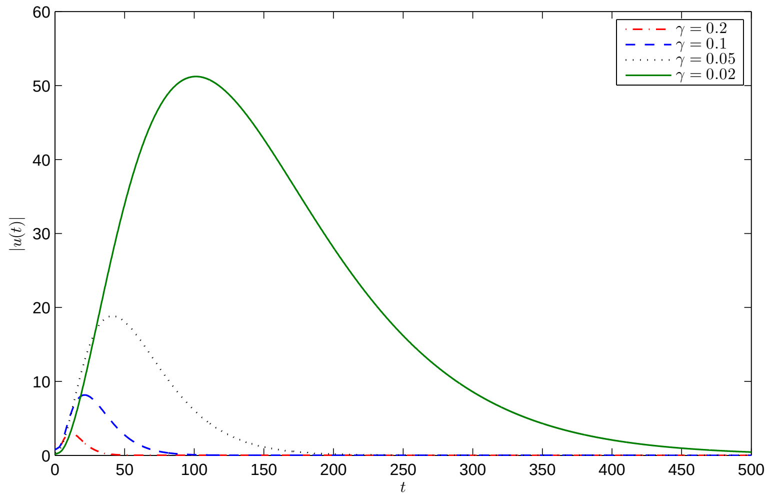

Figure 1: L_∞ slow peaking phenominon for system (48) with low gain (49)

4.2 Characterization of L_∞ Low Gain Feedback

From this proposition we can see that (47) is only a necessary condition for guaranteeing that $F(\gamma)$ is an L_∞ low gain for (A, B). We have example which verifies that (47) is not sufficient.

Example 1 *Consider a linear system in the form of (40) with*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{48}$$

The following feedback gain

$$F(\gamma) = -\begin{bmatrix} \gamma^3 + 3\gamma^2 & 3\gamma^2 + \gamma & 3\gamma \\ -\gamma & 0 & 0 \end{bmatrix}, \ \gamma > 0, \tag{49}$$

is such that $\lambda(A + BF(\gamma)) = \{-\gamma, -\gamma, -\gamma\}$ and (47) is satisfied. However,

$$\begin{aligned} \lim_{\gamma \to 0^+} \sup_{t \geq 0} \left| F(\gamma) \, \mathrm{e}^{(A+BF(\gamma))t} \right| &\geq \lim_{\gamma \to 0^+} \left| F(\gamma) \, \mathrm{e}^{(A+BF(\gamma))\frac{1}{\gamma}} \right|^2 \\ &\geq \lim_{\gamma \to 0^+} \frac{1}{2} \left| F(\gamma) \, \mathrm{e}^{(A+BF(\gamma))\frac{1}{\gamma}} \right|_{\mathrm{F}}^2 \\ &= \lim_{\gamma \to 0^+} \left(h(\gamma) + \frac{1}{8\gamma^2} \right) \\ &= \infty, \end{aligned} \tag{50}$$

which violates (45), where $h(\gamma) = \frac{1}{8}\gamma^6 + \frac{1}{4}\gamma^4 + \frac{31}{8}\gamma^2 + \frac{9}{4} - \gamma^3$ is a polynomial of γ. Hence there exists bounded initial condition x_0 such that the peak value of $|u(t)|$ approaches to infinity as γ approaches to zero, namely, the actuator saturation can not be "avoided" no matter how small γ is. For illustration such a "L_∞ slow peaking phenomenon", we select an initial condition $x_0 = [-4, 5, -4]^\mathsf{T}$ and different parameter γ, and plot the control signals in Fig. 1 which clearly shows such a phenomenon.

We next present a sufficient condition for guaranteeing that a given $F(\gamma)$ is an L_∞ low gain.

Theorem 5 *[21] Assume that (A, B) is ANCBC and $F(\gamma) : (0, D_\infty] \to \mathbf{R}^{m\times n}$ be stated in Definition 5. Then $F(\gamma)$ is an L_∞ low gain for (A, B) if there exists a scalar $\gamma_\infty^* \in (0, D_\infty]$ and a matrix $P_\infty = P_\infty(\gamma) : (0, \gamma_\infty^*] \to \mathbf{R}^{n\times n}$, which is continuous in γ, positive definite for all $\gamma \in (0, \gamma_\infty^*]$, such that, for all $\gamma \in (0, \gamma_\infty^*]$,*

$$(A + BF(\gamma))^\mathsf{T} P_\infty + P_\infty (A + BF(\gamma)) < 0, \tag{51}$$

$$F^\mathsf{T}(\gamma) F(\gamma) \le P_\infty, \tag{52}$$

$$\lim_{\gamma\to 0^+} P_\infty(\gamma) = 0. \tag{53}$$

4.3 Design of L_∞ Low Gain Feedback

We next introduce several methods for the L_∞ low gain design. We first recall the ARE based low gain design.

Lemma 11 *[8] Assume that (A, B) is ANCBC. Let $Q(\gamma) : (0, \infty) \to \mathbf{R}^{n\times n}$ be continuously differentiable positive definite for $\forall\gamma \in (0, \infty)$ and such that*

$$\lim_{\gamma\to 0^+} Q(\gamma) = 0, \quad \frac{\mathrm{d}Q(\gamma)}{\mathrm{d}\gamma} > 0. \tag{54}$$

Then the following ARE

$$A^\mathsf{T} P(\gamma) + P(\gamma) A - P(\gamma) BB^\mathsf{T} P(\gamma) = -Q(\gamma), \ \gamma \in (0, \infty), \tag{55}$$

has a unique positive definite solution $P(\gamma)$ such that $A - BB^\mathsf{T} P(\gamma)$ is Hurwitz and

$$\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma} > 0, \quad \lim_{\gamma\to 0^+} P(\gamma) = 0. \tag{56}$$

We have the following corollary of this lemma.

Corollary 1 *[21] (ARE Based L_∞ Low Gain Feedback) Let $P(\gamma)$ be determined in Lemma 11. Then the gain*

$$F_{\mathrm{ARE}}(\gamma) = -B^\mathsf{T} P(\gamma), \tag{57}$$

is an L_∞ low gain for (A, B) in the sense of Definition 5. Moreover, all the three conditions of Theorem 5 are satisfied with

$$P_\infty = P_\infty(\gamma) = \mathrm{tr}\left(B^\mathsf{T} P(\gamma) B\right) P(\gamma). \tag{58}$$

We next introduce the parametric Lyapunov equation based L_∞ low gain design

Lemma 12 *[22] Assume that (A, B) is ANCBC and, moreover, all the eigenvalues of A are on the imaginary axis, namely, $\lambda(A) \subset \mathbf{C}^{\mathrm{Re}\, s=0}$. Consider the following parametric ARE*

$$A^\mathsf{T} P + PA - PBB^\mathsf{T} P = -\gamma P. \tag{59}$$

1. *The ARE (59) has a unique positive definite solution $P = P(\gamma) = W^{-1}(\gamma)$ for all $\gamma > 0$, where $W(\gamma)$ is the unique positive definite solution to the following PLE*

$$W\left(A + \frac{\gamma}{2} I_n\right)^\mathsf{T} + \left(A + \frac{\gamma}{2} I_n\right) W = BB^\mathsf{T}. \tag{60}$$

2. *If we define the associated feedback gain*

$$F(\gamma) = F_{\mathrm{PLE}}(\gamma) = -B^\mathsf{T} P(\gamma), \tag{61}$$

then $A + BF(\gamma)$ is Hurwitz for all $\gamma > 0$ and, moreover

$$\lambda(A + BF(\gamma)) = \{-\gamma + s : s \in \lambda(A)\}. \tag{62}$$

3. $P(\gamma)$ *is a polynomial matrix of* γ *and such that*

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P(\gamma) > 0, \ \lim_{\gamma \to 0^+} P(\gamma) = 0, \ \operatorname{tr}\left(B^\intercal P(\gamma) B\right) = n\gamma. \tag{63}$$

The following corollary is then a consequence of the above lemma.

Corollary 2 *[21] (PLE Based L_∞ Low Gain Feedback) Let $P(\gamma)$ be determined in Lemma 12. Then the gain is an L_∞ low gain for (A, B) in the sense of Definition 5. Moreover, all the three conditions of Theorem 5 are satisfied for any $\gamma_\infty^* > 0$ with*

$$P_\infty = P_\infty(\gamma) = n\gamma P(\gamma). \tag{64}$$

The following remark concerns a comparison between the ARE based low gain feedback and PLE based low gain feedback.

Remark 2 *The PLE based L_∞ low gain design approach has at least the following advantages over the ARE based L_∞ low gain design approach:*

1. *The PLE based L_∞ low gain design approach needs only to solve the linear matrix equation (60), while the ARE based L_∞ low gain design approach needs to solve the nonlinear ARE (55), which may become numerically ill-conditioned as γ goes to zero (see Section 2.4 in [5]);*
2. *Explicit expression of P to the PLE (60) can be obtained, and thus explicit expression of $F_{\mathrm{PLE}}(\gamma)$ can be obtained, which is not the case for the ARE based L_∞ low gain design approach;*
3. *It follows from (62) that the PLE based L_∞ low gain design approach guarantees that the real part of the eigenvalues of the closed-loop system matrix $A - BB^\intercal P(\gamma)$ is $-\gamma$. Hence in this approach the low gain parameter γ has an obvious system meaning, namely, it represents the convergence rate of the closed-loop system. This is not the case for the ARE based L_∞ low gain design approach.*
4. *For the PLE based L_∞ low gain design approach, if we consider the Lyapunov function $V(x) = x^\intercal P x$ for the closed-loop system $\dot{x} = (A - BB^\intercal P(\gamma))\, x$, we have $\dot{V}(x) \leq -\gamma V(x)$, namely, $\dot{V}(x)$ is proportional to $V(x)$. This property is helpful for the related Lyapunov analysis for more complicated problem, for example, the simultaneous internal and external global stabilization of linear systems with actuator saturation [15], in which it is commented that the PLE based L_∞ low gain design approach "greatly simplifies the expression for (our) controllers and the subsequent analysis".*

5 Local Stabilization

5.1 Different Treatments of the Saturation Nonlinearity

The saturation nonlinearity can be naturally treated as a sector nonlinearity.

Lemma 13 *For any $u \in \mathbf{R}^m$ and any diagonal matrix $T \geq 0$, there holds*

$$\sigma^{\mathrm{T}}(u)\, T\, (\sigma(u) - u) \leq 0.$$

By this lemma, the stability analysis and stabilization problems can be respectively recast into the absolute stability and stabilization problems.

Lemma 14 *[10] Let $\alpha_i \in (0,1], i \in \mathbf{I}[1,m]$ be some given scalars. Assume that $|u_i| \leq \frac{1}{\alpha_i}, i \in \mathbf{I}[1,m]$. Then, for any $u \in \mathbf{R}^m$, there holds*

$$\sigma(u) \in \operatorname{co}\left\{\mathit{\Gamma}_i u : i \in \mathbf{I}\left[1, 2^m\right]\right\},$$

where

$$\Gamma_i = \begin{bmatrix} \gamma_{i1} & & & \\ & \gamma_{i2} & & \\ & & \ddots & \\ & & & \gamma_{im} \end{bmatrix},$$

in which $\gamma_{ij} = 1$ *or* $\alpha_j, j \in \mathbf{I}[1, m]$.

Example 2 *Consider* $m = 2$. *Then by this lemma, for any* $\alpha_1, \alpha_2 \in (0, 1]$, *we have*

$$\sigma(u) \in \mathrm{co}\left\{ \begin{bmatrix} u_1 \\ \alpha_2 u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} \alpha_1 u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} \alpha_1 u_1 \\ \alpha_2 u_2 \end{bmatrix} \right\}.$$

Lemma 15 *[4] Let* $v \in R^m$ *be a vector and such that* $|v|_\infty \leq 1$. *Then, for any* $u \in \mathbf{R}^m$, *there holds*

$$\sigma(u) \in \mathrm{co}\{D_i u + (I_m - D_i) v : i \in \mathbf{I}[1, 2^m]\},$$

where $D_i \in \mathscr{D}_m$ *with*

$$\mathscr{D}_m = \left\{ \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{bmatrix} : d_i = 1 \text{ or } 0, i \in \mathbf{I}[1, m] \right\}.$$

Example 3 *Consider* $m = 2$. *Then by this lemma, for any* $|v_1| \leq 1, |v_2| \leq 1$, *we have*

$$\sigma(u) \in \mathrm{co}\left\{ \begin{bmatrix} u_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\}.$$

Associated with the set $\mathscr{D}_m$ whose elements are labeled as $D_i, i \in \mathbf{I}[1, 2^m]$, we define a function $f_m : \mathbf{I}[1, 2^m] \to \mathbf{I}\left[1, 2^{m-1}\right]$ as follows:

$$f_m(i) = \begin{cases} f_m(i-1) + 1, & D_i + D_j \neq I_m, \ \forall j \in \mathbf{I}[1, i], \\ f_m(j), & D_i + D_j = I_m, \ \exists j \in \mathbf{I}[1, i]. \end{cases} \tag{65}$$

Lemma 16 *[18] Let* $\overleftrightarrow{m} = m2^{m-1}$ *and* $v \in \mathbf{R}^{\overleftrightarrow{m}}$ *be such that* $\|v\|_\infty \leq 1$. *Then, for any* $u \in \mathbf{R}^m$, *there holds*

$$\sigma(u) \in \mathrm{co}\left\{D_i u + \mathcal{D}_i^- v : i \in \mathbf{I}[1, 2^m]\right\}, \tag{66}$$

where $\mathcal{D}_i^- \in \mathbf{R}^{m \times \overleftrightarrow{m}}, i \in \mathbf{I}[1, 2^m]$ *are defined as*

$$\mathcal{D}_i^- = e_{f_m(i)} \otimes (I_m - D_i), \ \forall i \in \mathbf{I}[1, 2^m], \tag{67}$$

where $e_i \in \mathbf{R}^{2^m - 1}$ *is a row vector whose i-th column is 1 and the others are zero.*

Example 4 *Consider* $m = 2$. *Then by the above lemma, for any* $|v_1| \leq 1, |v_2| \leq 1, |v_3| \leq 1$, *and* $|v_4| \leq 1$, *we have*

$$\sigma(u) \in \mathrm{co}\left\{ \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_2 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} \right\}.$$

It follows that if we choose $v_i = v$ in Lemma 16, then this lemma reduces to Lemma 15. If we choose $v = \mathrm{diag}\{\alpha_1, \cdots, \alpha_m\} u$ in Lemma 15, then this lemma reduces to Lemma 14.

5.2 Local Stabilization

Consider the following linear system subject to input saturation

$$\dot{x} = Ax + B\sigma(u), \tag{68}$$

where $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are some given matrices. Consider the linear state feedback

$$u = Fx.$$

The closed-loop system is given by

$$\dot{x} = Ax + B\sigma(Fx).$$

Proposition 3 *[21] If there exists a matrix $P > 0 \in \mathbf{R}^{n\times n}$ and a matrix $H \in \mathbf{R}^{\overleftrightarrow{m}\times n}$, such that*

$$\mathscr{E}(P) \subseteq \mathscr{L}(H), \tag{69}$$

and the following inequalities

$$\left(A + B\left(D_i F + \mathcal{D}_i^- H\right)\right)^{\mathrm{T}} P + P\left(A + B\left(D_i F + \mathcal{D}_i^- H\right)\right) < 0,\ i \in \mathbf{I}[1, 2^m], \tag{70}$$

are satisfied, then system (68) is asymptotically stable with $\mathscr{E}(P)$ contained in the domain of attraction.

By denoting $Q = P^{-1}$, $Z = HP^{-1}, W = FP^{-1}$ and the k-th row of Z as Z_k, then (69) is equivalent to [4]

$$\begin{bmatrix} 1 & Z_k \\ \star & Q \end{bmatrix} \geq 0,\ k \in \mathbf{I}[1, \overleftrightarrow{m}], \tag{71}$$

and the LMIs in (70) can be equivalently rewritten as

$$AQ + BD_iW + B\mathcal{D}_i^- Z + \left(AQ + BD_iW + B\mathcal{D}_i^- Z\right)^{\mathrm{T}} < 0,\ i \in \mathbf{I}[1, 2^m]. \tag{72}$$

Similar to the argument given in [4], the largeness of the ellipsoid $\mathscr{E}(P)$ with respect to a shape reference set $\mathscr{X}_{\mathrm{R}} \subset \mathbf{R}^n$ can be measured by the scalar α which is the maximal number such that $\alpha\mathscr{X}_{\mathrm{R}} \subseteq \mathscr{E}(P)$ is satisfied, where $\alpha\mathscr{X}_{\mathrm{R}} \triangleq \{\alpha x : x \in \mathscr{X}_{\mathrm{R}}\}$. If the reference set is chosen as the polyhedron, namely,

$$\mathscr{X}_{\mathrm{R}} = \mathrm{co}\{x_1, x_2, \cdots, x_l\},\ l \geq 1, \tag{73}$$

where $x_i \in \mathbf{R}^n, i \in \mathbf{I}[1, l]$ are some given vectors, then $\alpha\mathscr{X}_{\mathrm{R}} \subseteq \mathscr{E}(P)$ is equivalent to

$$\begin{bmatrix} \gamma & \star \\ x_j & Q \end{bmatrix} \geq 0,\ j \in \mathbf{I}[1, l], \tag{74}$$

where $\gamma = 1/\alpha^2$.

With these transformations, the estimation of the domain of attraction for the discrete-time nonlinear system (68) can then be formulated as

$$\inf_{Q>0, Z} \gamma \quad \text{s.t.} \quad (71),\ (72) \text{ and } (74), \tag{75}$$

which is an LMIs-based optimization problem. The resulting maximal estimation of the domain of attraction can be recovered from $\mathscr{E}(P) = \mathscr{E}\left(Q^{-1}\right)$.

6 Performance Issue

6.1 Gain Scheduling

This idea is as follows.

1. When the state is far from the origin, the control gain is small so as to avoiding over-saturation;
2. When the state is near the origin, the control gain can be made larger so as to utilize the actuator capacities.

There are three kinds of gain scheduling controllers. Consider the linear system with input saturation

$$\dot{x} = Ax + B\sigma(u). \tag{76}$$

1. Discrete Gain Scheduling: Let $\mathscr{E}(P_0) \supset \mathscr{E}(P_1) \supset \cdots \supset \mathscr{E}(P_N)$ be a series of ellipsoids and such that $\mathscr{E}(P_i)$ is a contractively invariant set for the closed-loop system

$$\dot{x} = Ax + B\sigma(F_i x). \tag{77}$$

Then the gain scheduling controller [14]

$$u = \begin{cases} F_0 x, & x \in \mathscr{E}(P_0) \backslash \mathscr{E}(P_1), \\ F_1 x, & x \in \mathscr{E}(P_1) \backslash \mathscr{E}(P_2), \\ \vdots & \\ F_{N-1} x, & x \in \mathscr{E}(P_{N-1}) \backslash \mathscr{E}(P_N), \\ F_N x, & x \in \mathscr{E}(P_N). \end{cases}$$

2. Continuous static gain scheduling: Let $F(\gamma) : (\gamma_0, \gamma_{\max}) \to R^{m\times n}$ be a parameterized stabilizing gain. The continuous static gain scheduling controller is of the form

$$u = F(\gamma)x, \ \gamma = \gamma(x).$$

A typical function $\gamma(x)$ is given by [9]

$$\gamma(x) = \inf\{\gamma : \mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right) x^{\mathrm{T}} P(\gamma) x = 1\},$$

where $F(\gamma) = B^{\mathrm{T}} P(\gamma)$ and $A^{\mathrm{T}} P(\gamma) + P(\gamma) P - P(\gamma) BB^{\mathrm{T}} P(\gamma) = -Q(\gamma) < 0$.

3. Dynamic Gain Scheduling: The controller is of the form [24]

$$\begin{cases} u = F(\gamma) x \\ \dot{\gamma} = \gamma(\gamma, x, u) \geq 0, \ \gamma(0) = \gamma_0. \end{cases}$$

The design of $\gamma(x, u)$ is constructive and is not an easy task.

6.2 Gutman and Hagander Re-design

Consider the linear system with input saturation

$$\dot{x} = Ax + B\sigma(u). \tag{78}$$

Let F be a matrix such that $A + BF$ is Hurwitz and $P > 0$ be such that

$$(A + BF)^{\mathrm{T}} P + P(A + BF) < 0.$$

Let $\mathscr{E}(P, \rho) \subseteq \mathscr{L}(F)$. Then $E(P, \rho)$ is a contractively invariant set for the closed-loop system

$$\dot{x} = Ax + B\sigma(Fx), \tag{79}$$

since, for any $x \in \mathscr{E}(P, \rho) \subseteq \mathscr{L}(F)$, the closed-loop system is linear and is asymptotically stable.

Proposition 4 *[3] Let F and P be stated above and $\mu \in \mathbf{R}^m$ be any positive semi-definite diagonal matrix. Let*

$$\begin{aligned} u &= Fx - \mu B^{\mathrm{T}} Px \\ &= \left(F - \mu B^{\mathrm{T}} P\right) x. \end{aligned} \tag{80}$$

Then $\mathscr{E}(P, \rho)$ is also a contractively invariant set for the closed-loop system consisting of (78) and (80), namely

$$\dot{x} = Ax + B\sigma\left(Fx - \mu B^{\mathrm{T}} Px\right).$$

Proof. (A simple proof) Rewrite the closed-loop system as

$$\dot{x} = (A + BF)\,x + B\left[\sigma\left(Fx - \mu B^{\mathrm{T}} Px\right) - Fx\right].$$

Consider the Lyapunov function $V(x) = x^{\mathrm{T}} Px$. Then

$$\begin{aligned} \dot{V}(x) &= x^{\mathrm{T}}\left((A + BF)^{\mathrm{T}} P + P(A + BF)\right)x + 2x^{\mathrm{T}} PB\left[\sigma\left(Fx - \mu B^{\mathrm{T}} Px\right) - Fx\right] \\ &= x^{\mathrm{T}}\left((A + BF)^{\mathrm{T}} P + P(A + BF)\right)x + \sum_{i=1}^{m} 2x^{\mathrm{T}} PB_i\left[\sigma\left(F_i x - \mu_i B_i^{\mathrm{T}} Px\right) - F_i x\right]. \end{aligned}$$

For any $x \in \mathscr{E}(P, \rho)$, we have $|Fx|_\infty \leq 1$. Then, for any $i \in \mathbf{I}[1, m]$,

$$x^{\mathrm{T}} PB_i\left(\sigma\left(F_i x - \mu_i B_i^{\mathrm{T}} Px\right) - F_i x\right)\begin{cases} \leq 0, & F_i x \geq 0 \text{ and } B_i^{\mathrm{T}} Px \geq 0, \\ \leq 0, & F_i x \geq 0 \text{ and } B_i^{\mathrm{T}} Px < 0, \\ \leq 0, & F_i x < 0 \text{ and } B_i^{\mathrm{T}} Px \geq 0, \\ \leq 0, & F_i x < 0 \text{ and } B_i^{\mathrm{T}} Px < 0. \end{cases}$$

As a result, we can derive

$$\begin{aligned} \dot{V}(x) &\leq x^{\mathrm{T}}\left((A + BF)^{\mathrm{T}} P + P(A + BF)\right)x \\ &< 0, x \neq 0. \end{aligned}$$

Therefore, $\mathscr{E}(P, \rho)$ is a contractively invariant set. ■

Remark 3 *Some explanations on Gutman and Hagander re-design.*

1. *The merit of Gutman and Hagander re-design is that, the gain F is designed such that the closed-loop system (79) has a large domain of attraction and then the feedback gain is redesigned as (80) to improve the transient performances.*
2. *If the gain F is an L_∞ low gain, then this design is referred to as low-and-high gain feedback [5].*
3. *Notice that the gain factor μ can be dependent on time and the state x. Particularly, μ can be a nonlinear function of x,*

   $$\mu = \mu(x, t).$$

   In this case, this control law is referred to as composite nonlinear feedback (CNF) [1], [6].
4. *The discrete-time version is less elegant [3].*

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