HW3, due Wed Oct 7 at 9am

In this assignment, you are free to use any language you wish to answer all computational questions. You *do not* need to use all three languages. Please be aware that the homework solutions will be written in only one language as well.

1. **6 points** Garcia 3.23: The Hopf model is given by

$$\frac{dx}{dt} = ax + y - x(x^2 + y^2) \qquad \frac{dy}{dt} = -x + ay - y(x^2 + y^2) \tag{1}$$

   (a) Rewrite these equations in polar coordinates. Analytically show that they fall into the origin if $a < 0$, and limit to a circle of radius $\sqrt{a}$ for $a > 0$.

   (b) Compute the dynamics for the Hopf model using an adaptive runge kutta technique (you are free to use rk45 in matlab or python, or use Garcia's code).

2. **10 points** Suppose an $n \times n$ matrix $\mathbf{M}$ has n orthogonal eigenvectors $\mathbf{v}_i$, such that $\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ with all λ_i distinct.

   (a) Show analytically that $\mathbf{M} = \mathbf{V}\mathbf{L}\mathbf{V}^{-1}$, where $\mathbf{L}$ is a diagonal matrix of eigenvalues and $\mathbf{V}$ is the matrix formed of columns of eigenvectors (that is, $\mathbf{V}_{ij} = (\mathbf{v}_j)_i$).

   (b) Use (a) to analytically show that $\mathbf{M}^n$ has eigenvalues λ_i^n, with the same eigenvectors. Use that to show that $\mathbf{M}^n\mathbf{b} = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots$ for some c_i you should determine. Argue that this means $\lim_{n\to\infty} \mathbf{M}^n\mathbf{b} \propto \mathbf{v}_{max}$ (for a constant of proportionality you should determine), with $\mathbf{v}_{max}$ the eigenvector associated with the largest eigenvalue of $\mathbf{M}$.

   (c) Define the 50×50 matrix $\mathbf{M}_{ij} = 1$ if $|i-j| < 3$ and 0 otherwise, and the vector $(\mathbf{b}_0)_i = 1$. Compute the largest eigenvalue, λ_{max}, and eigenvector, $\mathbf{v}_{max}$, of this matrix (you may use any numerical method).

   (d) Iteratively compute $\mathbf{b}_k = \mathbf{M}\cdot\mathbf{b}_{k-1}/|\mathbf{b}_{k-1}|$, iterating until $|\mathbf{b}_k - \mathbf{b}_{k-1}| < 10^{-6}|\mathbf{b}_k|$ (with the total number of iterations k_{max}). This is called a power iteration of $\mathbf{M}$. Show that $\mathbf{b}_{k_{max}} \propto \mathbf{v}_{max}$, and that $\mathbf{M}\mathbf{b}_{k_{max}} \approx \lambda_{max}\mathbf{b}_{k_{max}}$.