HW 4 10/6/2020

1. The Hopf model

$$\frac{dx}{dt} = ax + y - x(x^2+y^2) \qquad \frac{dy}{dt} = -x + ay - y(x^2+y^2)$$

a) Rewrite these eqns in polar coordinates. Analytically show that they fall into the origin if $a<0$, and limit to a circle of radius $\sqrt{a}$ for $a>0$.

Ans.

In polar coordinates,

$$x = r\cos\theta \qquad \partial x/\partial\theta = -r\sin\theta \qquad \frac{\partial x}{\partial r} = \cos\theta \quad (1)$$

$$y = r\sin\theta \qquad \partial y/\partial\theta = r\cos\theta \qquad \partial y/\partial r = \sin\theta \quad (2)$$

$$x^2+y^2 = r^2$$

From total derivative,

$$\frac{dx}{dt} = \frac{\partial x}{\partial\theta}\cdot\frac{d\theta}{dt} + \frac{\partial x}{\partial r}\cdot\frac{dr}{dt} = -r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt} \quad (3)$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial\theta}\cdot\frac{d\theta}{dt} + \frac{\partial y}{\partial r}\cdot\frac{dr}{dt} = r\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dr}{dt} \quad (4)$$

Substituting (3) and (4) in the Hopf model,

$$\frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt} = ar\cos\theta + r\sin\theta - r\cos\theta(r^2)$$

$$-r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt} = r\sin\theta + (ar-r^3)\cos\theta \quad (5)$$

[Shorter: Comparing b/s; $-r\,d\theta/dt = r \Rightarrow d\theta/dt = -1$; $dr/dt = ar - r^3$]

$$\frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dr}{dt} = -r\cos\theta + ar\sin\theta - r\sin\theta(r^2)$$

$$r\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dr}{dt} = -r\cos\theta + (ar-r^3)\sin\theta \quad (6)$$

Eqn (5) × $\cos\theta$ + Eqn (6) × $\sin\theta$

$$\cos^2\theta\frac{dr}{dt} + \sin^2\theta\frac{dr}{dt} = \cancel{r\cos\theta\sin\theta} + (ar-r^3)\cos^2\theta - \cancel{r\cos\theta\sin\theta} + (ar-r^3)\sin^2\theta$$

$$\frac{dr}{dt} = ar - r^3$$

Eqn (6) × $\cos\theta$ − Eqn (5) × $\sin\theta$

$$r\cos^2\theta\frac{d\theta}{dt} + r\sin^2\theta\frac{d\theta}{dt} = -r\cos^2\theta + \cancel{(ar-r^3)\sin\theta\cos\theta} - r\sin^2\theta - \cancel{(ar-r^3)\cos\theta\sin\theta}$$

$$r\,d\theta/dt = -r\cos^2\theta - r\sin^2\theta$$

$$d\theta/dt = -1 \qquad \boxed{\frac{dr}{dt} = ar - r^3 \;;\; \frac{d\theta}{dt} = -1}$$

Since $\frac{dr}{dt} = ar - r^3 = r(a - r^2)$

If $a < 0$, then $a - r^2 < 0$

$$\frac{dr}{dt} < 0 \qquad [\because r > 0]$$

Hence, the radius decreases w.r.t time towards to origin

For $a > 0$, the circle increases until some inflexion point at which $dr/dt = 0$

$$r(a - r^2) = 0$$

$$a = r^2$$

$$r = \sqrt{a}$$

Physically, we know this is the limit of r because r cannot be -ve, and we have shown the case of $a < 0$.

b) Compute the dynamics for the Hopf model using an adaptive RK technique.

2. Suppose an $N \times N$ matrix M has n orthogonal eig. vectors v_i, such that $M v_i = \lambda_i v_i$ with all λ_i distinct.

a) Show analytically that $M = VLV^{-1}$, where L is a diagonal matrix of eig. values and V is the matrix formed of columns of eig. vectors (that is $V_{ij} = (v_j)_i$).

Soln

if $i = 1$, $\quad M v_1 = \lambda_1 v_1 \quad — (1)$

$M v_2 = \lambda_2 v_2 \quad —$

$\vdots$

$M v_N = \lambda_N v_N \quad — (n)$

Adding all eqns from (1) to (n)

$$M(v_1 + v_2 + \cdots + v_N) = (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_N v_N)$$

In matrix form, both sides can be expressed as

$$M \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_N \end{bmatrix}$$

$$MV = VL$$

where $V = [v_1 \; v_2 \cdots v_N]$ and $L = \text{diag}\{\lambda_1 \; \lambda_2 \cdots \lambda_N\}$

Multiplying b/s by V^{-1} from the right [or right-inverse of V]

$$MVV^{-1} = VLV^{-1}$$

$$\boxed{M = VLV^{-1}}$$

Ans.

b) Use (a) to analytically show that M^n has eig. values λ_i^n, with the same eig. vectors. Use that to show that $M^n b = c_1 \lambda_1^n v_1 + c_2 \lambda_2 v_2^n + \cdots$ for some c_i you should determine. Argue that this means $\lim_{n \to \infty} M^n b \propto v_{max}$ (for a const. of proportionality you should determine), with v_{max} the eig. vector associated with the largest eig. value of M.

Soln

From (a), $\quad M = VLV^{-1}$

$$M^2 = (VLV^{-1})(VLV^{-1}) = VLV^{-1}VLV^{-1} = VL^2V^{-1}$$

Similarly, $\quad M^n = (VLV^{-1})(VLV^{-1})(VLV^{-1})(VLV^{-1}) \cdots$

all of which reduces to only the 1st matrix V, and last V^{-1}, while L is multiplied n times

$$M^n = V L^n V^{-1}$$

which shows that the eig. value matrix for M^n is L^n

But $L = \text{diag}\{\lambda_1\ \lambda_2 \cdots \lambda_N\}$

$\therefore L^n = \text{diag}\{\lambda_1^n\ \lambda_2^n \cdots \lambda_N^n\}$ Ans.

Hence, the eig. values of M^n are λ_i^n, while the eig. vectors are same.

$$M^n b = V L^n V^{-1} b = [V L^n][V^{-1} b]$$

Let $V^{-1} b = c$ where $\in \mathbb{R}^N$

Also, $V L^n = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_N^n \end{bmatrix}$

Then, $V L^n [V^{-1} b] = V L^n c = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_N^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$

$$M^n b = V L^n V^{-1} b = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1 \lambda_1^n \\ c_2 \lambda_2^n \\ \vdots \\ c_N \lambda_N^n \end{bmatrix}$$

$$M^n b = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 + \cdots c_N \lambda_N^n v_N$$ Ans.

where c_i are the elements of the vector from the product $V^{-1} b$

If we factor out λ_{max}^n

$$M^n b = c_m \lambda_{max}^n \left[\frac{c_1}{c_m} \left(\frac{\lambda_1}{\lambda_{max}} \right)^n v_1 + \frac{c_2}{c_m} \left(\frac{\lambda_2}{\lambda_{max}} \right)^n v_2 + \frac{c_m}{c_m} \left(\frac{\lambda_{max}}{\lambda_{max}} \right)^n v_m + \cdots \frac{c_N}{c_m} \left(\frac{\lambda_N}{\lambda_{max}} \right)^n v_N \right]$$

$$M^n b = c_m \lambda_{max}^n \left[\frac{c_1}{c_m} \left(\frac{\lambda_1}{\lambda_{max}} \right)^n v_1 + \frac{c_2}{c_m} \left(\frac{\lambda_2}{\lambda_{max}} \right)^n v_2 + \cdots + v_{max} + \cdots + \frac{c_N}{c_m} \left(\frac{\lambda_N}{\lambda_{max}} \right)^n v_N \right]$$

Since $\frac{\lambda_i}{\lambda_{max}} < 0$, $\lim_{n \to \infty} \left(\frac{\lambda_i}{\lambda_{max}} \right)^n \approx 0$

$\therefore \lim_{n \to \infty} M^n b \approx c_m \lambda_{max}^n v_{max}$ Ans.

$\lim_{n \to \infty} M^n b \propto v_{max}$ with const. of prop $= c_m \lambda_{max}^n$