1. When numerically solving a differential equation, it is always better to choose a smaller $\Delta t$. Explain why or why not.

Ans. False

This is because while using a smaller $\Delta t$ will reduce the numerical discretization error, it leads to an increased number of computations to reach a desired total duration. This increase in computations results in higher round-off errors due to limits of machine accuracy.

2.

Ans.

Adaptive methods are methods in which the time step size is dynamic. $\Delta t$ is computed at each timestep in order to optimize computational efficiency.

Non-adaptive methods use a constant $\Delta t$.

Adaptive methods are most useful in system with non-uniform dynamics. That is, systems that have a fairly simple behaviour during run times, and significant dynemics at a specific section. We essentially adapt our integrator to focus on regions of key dynamism.

A major disadvantage is that we cannot preallocate memory because we do not know how long it will take to run as $\Delta t$ changes in each timestep.

4) a

a)

The mth element in the expansion is

$$-\frac{(I-A)^m}{m} \qquad m \in K$$

The following element in the expansion will be

$$-\frac{(I-A)^{m+1}}{m+1}$$

If we expand up to m,

$$\log(A) = -\left\{\frac{(I-A)}{1} + \frac{(I-A)^2}{2} + \dots + \frac{(I-A)^m}{m}\right\}$$

If we expand up to m+1,

$$\log(A) = -\left\{\frac{(I-A)}{1} + \frac{(I-A)^2}{2} + \dots + \frac{(I-A)^m}{m} + \frac{(I-A)^{m+1}}{m+1}\right\}$$

This converges if each new term added gets smaller as m increases

$$\lim_{m\to\infty}\left(-\frac{(I-A)^{m+1}}{m+1}\right) \to 0$$

$$-\frac{(I-A)^{m+1}}{m+1} = -\frac{(I-A)^m}{m}\cdot\left[\frac{m}{m+1}(I-A)\right]$$

For this to approach, the term in square brackets should $\to 0$

$$\left\|\frac{m}{m+1}(I-A)\right\| < 1$$

$$\boxed{\|(I-A)\| < \frac{m+1}{m} \qquad m \in K}$$

4 b)

$$\log(A) = -\left[\frac{I-A}{1} + \frac{(I-A)^2}{2} + \frac{(I-A)^3}{3} + \frac{(I-A)^4}{4} + \frac{(I-A)^5}{5} + \cdots\right]$$

Each matrix addition/subtraction is $O(n^2)$ operations

" " multiplication is $O(n^3)$ "

$$\left.\begin{array}{l} \left.\begin{array}{l} \left.\begin{array}{l} I-A \Rightarrow O(n^2) \text{ ops} \\ (I-A)^2 \Rightarrow O(2n^2+n^3) \end{array}\right\} \text{Total of } 3n^2+n^3 \\ (I-A)^3 \Rightarrow O(3n^2+2n^3) \end{array}\right\} \text{Total of } 6n^2+3n^3 \\ (I-A)^4 \Rightarrow O(4n^2+3n^3) \\ (I-A)^5 \Rightarrow O(5n^2+4n^3) \end{array}\right\} \text{Total of } 15n^2+10n^3$$

Hence, for each new term added, we get additional $O(kn^2 + (k-1)n^3)$ operations.

While the total operations is a series sum amounting to

$$= O\left[\frac{k(k+1)}{2}n^2 + \frac{k(k-1)}{2}n^3\right]$$

5a)

The LHS of the dimensioned form is equivalent to $\frac{1}{GM}\frac{\partial^2 r}{\partial t^2}$

The LHS of the dimensionless form is equivalent to $\frac{\partial^2 r'}{\partial t^2}$ r' is di

Comparing both eqns,

$$\frac{1}{GM} \cdot r = r'$$

$$\underbrace{r'}_{\text{dim-less}} = \underbrace{\frac{1}{GM}}_{\substack{\text{scale factor}\\ \text{units of } L^{-1}}} \cdot r$$

dimension L

1 unit length of $r'$ is equivalent to $(GM).m$

Velocity $= r/t$

If $r/t = 1$ Km/s

$$\frac{r' \cdot GM}{t} = 10^3 \, m/s$$

Arbitrary velocity units

$$\boxed{r'/t = 10^3/GM \quad \text{length / sec}}$$

5b) Analytically show that if the first star is initially stationary at the origin, and if the second and third stars satisfy $r_2 = -r_3$ and with velocities $v_2 = -v_3$, that the first star will remain stationary for all time. Is this solution stable if $r_1$ is perturbed?

Soln

$$\ddot{\vec{r}}_1 = -\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} \quad (1)$$

If first star is initially at origin then $\vec{r}_1 = (0,0)$

$$\vec{r}_1 - \vec{r}_2 = -\vec{r}_2 \quad (2)$$

$$\vec{r}_1 - \vec{r}_3 = -\vec{r}_3 \quad (3)$$

$$\ddot{\vec{r}}_1 = -\frac{(-\vec{r}_2)}{|(-\vec{r}_2)|^3} - \frac{(-\vec{r}_3)}{|(-\vec{r}_3)|^3} \quad (4)$$

$$= \frac{\vec{r}_2}{|\vec{r}_2|^3} + \frac{\vec{r}_3}{|\vec{r}_3|^3}$$

If $\vec{r}_2 = -\vec{r}_3$

$$\ddot{\vec{r}}_1 = -\frac{\vec{r}_3}{|\vec{r}_3|^3} + \frac{\vec{r}_3}{|\vec{r}_3|^3}$$

$$\ddot{\vec{r}}_1 = 0$$

Hence, the first star remains at rest and never accelerates

If $\vec{r}_1$ is slightly perturbed, eqns (2) and (3) no longer hold and star 1 begins to move.