HW3, due Sept 25 (Friday) at 9am

In this assignment, you are free to use any language you wish to answer all computational questions. You *do not* need to use all three languages. Please be aware that the homework solutions will be written in only one language as well.

1. **4 points** Garcia 2.15, modified: In the small angle approximation, the energy of a pendulum is

$$E_{tot} = \frac{mL^2}{2}\omega^2(t) + \frac{mgL}{2}\theta^2(t) - mgL \tag{1}$$

   Analytically show that E monotonically increases with time when the Euler method is used to compute the motion.

2. **5 points**A harmonic oscillator has the equations of motion $\ddot{x} = -\omega^2 x$, with solution $x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$.

   (a) Show that the equation can be nondimensionalized by using the natural times $\tau = \omega t$. What is the initial velocity in these units?

   (b) Write a program (in your preferred language) that numerically solves the nondimensionalized harmonic oscillator, $\ddot{x} = -x$, with $x_0 = 0$ and $v_0 = 1$, using both the Verlet and Runge-Kutta 4^{th} order methods. Compute $x^{(verlet)}(T)$ and $x^{(rk4)}(T)$ (the estimated x at time T using the verlet and rk4 methods, respectively), where $T = 5$ and with $\Delta t \in \{0.5, 0.1, 0.05, 0.01, 0.005, 0.001\}$. Plot $g_1(\Delta t) = |\sin(T) - x^{Verlet}(T)|$ and $g_2 = |\sin(T) - x_N^{rk4}(T)|$ on log-log axes and comment on any differences between them.

3. **11 points**. The Korteweg-de Vries (KdV) equation describes wave motion in shallow water, with the normalized height $u(x,t)$ satisfying the PDE

$$\frac{\partial u}{\partial t} + 6u(x,t)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{2}$$

   (a) Show that the KdV equation permits the solutions $u(x,t) = v(x - ct)$, and reduces to the ordinary differential equation $-cv'(y) + 6v(y)v'(y) + v'''(y) = 0$ with this substitution.

   (b) Show that the steady-state KdV equation can be made parameter-free (no c) by substituting $y = z\sqrt{c}$ and $v(y) = cw(z)$

   (c) Use rk4 to solve the parameter-free KdV equation, $w''' = (1 - 6w)w'$, with $w(0) = 1/2$, $w'(0) = 0$, and $w''(0) = -1/4$, for $z \in (0, 5)$. Plot your numerical result as points in the same figure as the exact solution to this nonlinear equation, $w_{exact} = \frac{1}{2}\text{sech}^2(z/2)$.

   (d) Repeat your numerical integration over the domain $z \in (0, 50)$, and plot your computed $w(z)$. You should observe a sequence of "solitons": localized traveling peaks in the fluid that are well separated. The KdV was the first model to successfully describe experimentally observed solitons in water.