HW 3 Lotanna Ohazuruike

1. Energy of a pendulum

$$E_{tot} = \frac{mL^2}{2} \omega^2(t) + \frac{mgL}{2} \theta^2(t) - mgL$$

Analytically show that E monotonically increases with time when the Euler method is used to compute the motion.

Sol:

For the Euler method,

$$\theta_{n+1} = \theta_n + \Delta t\, \omega_n \Rightarrow \frac{d\theta}{dt} = \frac{\theta_{n+1} - \theta_n}{\Delta t} \approx \omega_n$$

$$\omega_{n+1} = \omega_n + \Delta t \frac{F}{m} \Rightarrow \frac{d\omega}{dt} = \frac{\omega_{n+1} - \omega_n}{\Delta t} \approx F/m$$

$$\frac{dE}{dt} = 2\omega \cdot \frac{mL^2}{\cancel{2}} \frac{d\omega}{dt} + 2\theta \cdot \frac{mgL}{\cancel{2}} \frac{d\theta}{dt} - 0$$

$$= \omega_n \cancel{m} L^2 \cdot F/\cancel{m} + \theta_n mgL\omega_n$$

$$= \omega_n L^2 F + \theta_n \omega_n mgL > 0$$

This expression is greater than 0 for a pendulum motion, implying that E would be increasing with time.

2 Harmonic oscillator has the eqns of motions

$$\ddot{x} = -\omega^2 x$$

with solution $\quad x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$

a) Show that the eqn can be non dimensionalized using the natural times $\tau = \omega t$.
What is the initial velocity in these units?

Soln

If $\quad \tau = \omega t$

$$d\tau = \omega dt$$

$$dt = \frac{1}{\omega} d\tau$$

The eqn of motion becomes

$$\frac{d^2x}{dt^2} = \cancel{\omega^2} \frac{d^2x}{d\tau^2} = -\cancel{\omega^2} x$$

$$\boxed{\frac{d^2x}{d\tau^2} = -x}$$ Ans.

which has non-dimensional coefficients.

Velocity, $\quad v = dx/dt = -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t)$

$$v(t) = -\omega x_0 \sin(\omega t) + \underbrace{v_0}_{\text{init vel.}} \cos(\omega t)$$

If $\quad \tau = \omega t$

$$v(\tau) = \frac{dx}{dt} \cdot dt/d\tau = \left[-\omega x_0 \sin(\omega t) + v_0 \cos(\omega t)\right] \cdot \frac{1}{\omega}$$

$$v(\tau) = -x_0 \sin \tau + \underbrace{\frac{v_0}{\omega}}_{\text{init vel.}} \cos \tau$$

$\boxed{\text{Initial vel. becomes } v_0/\omega}$ Ans.

which is understandable because if we scaled time by τ, velocity should be scaled by $1/\omega$ since it is inversely proportional to time.

3. Korteweg-de Vries (KdV) eqn

$$\frac{\partial u}{\partial t} + 6u(x,t)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

a) Show that the KdV eqn permits the solutions $u(x,t) = v(x-ct)$, and reduces to the ODE $-cv'(y) + 6v(y)v'(y) + v'''(y) = 0$ with this substitution.

Soln

If $u(x,t) = v(x-ct)$

$$\partial u/\partial x = \partial v/\partial x$$

$$\partial u/\partial t = -c\,\partial v/\partial t$$

Substituting into the KdV eqn:

$$-c\,\partial v/\partial t + 6v(x-ct)\,\partial v/\partial x + \frac{\partial^3 v}{\partial x^3} = 0$$

If $y = x-ct$, the ODE becomes

$$-cv'(y) + 6v(y)v'(y) + v'''(y) = 0$$

Ans.

b) Show that the steady-state KdV eqn can be made parameter-free (no c) by substituting $z = y\sqrt{c}$ and $v(y) = c\,w(z)$

Soln

$$v(y) = c\,w(z)$$

$$\partial v/\partial y = c\frac{\partial w}{\partial z}\cdot \partial z/\partial y$$

From

$$z = y\sqrt{c}\,, \quad \partial z/\partial y = \sqrt{c}$$

which leads to

$$\partial v/\partial y = c\sqrt{c}\,\partial w/\partial z\,, \quad \partial^2 v/\partial y^2 = c^2\frac{\partial^2 w}{\partial z^2}\,, \quad \frac{\partial^3 v}{\partial y^3} = c^2\sqrt{c}\,\frac{\partial^3 w}{\partial z^3}$$

Similarly,

$$\partial v/\partial t = c\,\partial w/\partial z \cdot \partial z/\partial t = c\,\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot \partial y/\partial t$$

Since $y = x-ct$, $\partial y/\partial t = -c$

$$\partial v/\partial t = -c^2\sqrt{c}\,\partial w/\partial z$$

Substituting $\partial v/\partial t$ and $\partial v/\partial y$ in the previous ODE gives (and $\partial^3 v/\partial y^3$)

$$-c^3\sqrt{c}\,\partial w/\partial z + 6\left[c\,w(z)\right]\left[c\sqrt{c}\,\partial w/\partial z\right] + c^2\sqrt{c}\,\partial^3 w/\partial z^3 = 0$$

$$\underbrace{-c^3\sqrt{c}\,\partial w/\partial z}_{\text{from time derivative which is neglected for steady state}} + 6c^2\sqrt{c}\,w\,\partial w/\partial z + c^2\sqrt{c}\,\partial^3 w/\partial z^3 = 0$$

Divide by $c^2\sqrt{c}$ $\Rightarrow$ $\boxed{6\,w(z)\,\partial w/\partial z + \partial^3 w/\partial z^3 = 0}$

Ans.