1. When numerically solving a differential equation, it is always better to choose a smaller Δt. Explain why or why not.

Ans. False

This is because while using a smaller Δt will reduce the numerical discretization error, it leads to an increased number of computations to reach a desired total duration. This increase in computations results in higher round-off errors due to limits of machine accuracy.

2.

Ans.

Adaptive methods are methods in which the time step size is dynamic. Δt is computed at each timestep in order to optimize computational efficiency.

Non-adaptive methods use a constant Δt.

Adaptive methods are most useful in system with non-uniform dynamics. That is, systems that have a fairly simple behaviour during run times, and significant dynamics at a specific section. We essentially adapt our integrator to focus on regions of key dynamism.

A major disadvantage is that we cannot preallocate memory because we do not know how long it will take to run as Δt changes in each timestep.

4)

a)

The mth element in the expansion is

$$-\frac{(I-A)^m}{m} \qquad m \in K$$

The following element in the expansion will be

$$-\frac{(I-A)^{m+1}}{m+1}$$

If we expand up to m,

$$\log(A) = -\left\{\frac{(I-A)}{1} + \frac{(I-A)^2}{2} + \dots + \frac{(I-A)^m}{m}\right\}$$

If we expand up to m+1,

$$\log(A) = -\left\{\frac{(I-A)}{1} + \frac{(I-A)^2}{2} + \dots + \frac{(I-A)^m}{m} + \frac{(I-A)^{m+1}}{m+1}\right\}$$

This converges if each new term added gets smaller as m increases

$$\lim_{m\to\infty}\left(-\frac{(I-A)^{m+1}}{m+1}\right) \to 0$$

$$-\frac{(I-A)^{m+1}}{m+1} = -\frac{(I-A)^m}{m}\cdot\left[\frac{m}{m+1}(I-A)\right]$$

For this to approach, the term in square brackets should $\to 0$

$$\left\|\frac{m}{m+1}(I-A)\right\| < 1$$

$$\boxed{\|(I-A)\| < \frac{m+1}{m} \qquad m \in K}$$

4 b)

$$\log(A) = -\left[\frac{I-A}{1} + \frac{(I-A)^2}{2} + \frac{(I-A)^3}{3} + \frac{(I-A)^4}{4} + \frac{(I-A)^5}{5} + \cdots\right]$$

Each matrix addition / subtraction is $O(n^2)$ operations

" " multiplication is $O(n^3)$ "

$$\left.\begin{array}{l} \left.\begin{array}{l} \left.\begin{array}{l} I-A \Rightarrow O(n^2) \text{ ops} \\ (I-A)^2 \Rightarrow O(2n^2+n^3) \end{array}\right\} \text{Total of } 3n^2+n^3 \\ (I-A)^3 \Rightarrow O(3n^2+2n^3) \end{array}\right\} \text{Total of } 6n^2+3n^3 \\ (I-A)^4 \Rightarrow O(4n^2+3n^3) \\ (I-A)^5 \Rightarrow O(5n^2+4n^3) \end{array}\right\} \text{Total of } 15n^2+10n^3$$

Hence, for each new term added, we get additional $O(kn^2 + (k-1)n^3)$ operations.

While the total operations is a series sum amounting to

$$= O\left[\frac{k(k+1)}{2}n^2 + \frac{k(k-1)}{2}n^3\right]$$

5a)

The LHS of the dimensioned form is equivalent to $\frac{1}{GM}\frac{\partial^2 r}{\partial t^2}$

The LHS of the dimensionless form is equivalent to $\frac{\partial^2 r'}{\partial t^2}$ r' is di

Comparing both eqns,

$$\frac{1}{GM} \cdot r = r'$$

$$\underset{\text{dim-less}}{r'} = \underbrace{\frac{1}{GM}}_{\substack{\text{scale factor} \\ \text{units of } L^{-1}}} \cdot \underset{\text{dimension } L}{r}$$

1 unit length of r' is equivalent to $(GM) \cdot m$

Velocity $= r/t$

If $r/t = 1$ km/s

$$\frac{r' \cdot GM}{t} = 10^3 \text{ m/s}$$

Arbitrary velocity units

$$\boxed{r'/t = \frac{10^3}{GM} \text{ length/sec}}$$

5b) Analytically show that if the first star is initially stationary at the origin, and if the second and third stars satisfy $r_2 = -r_3$ and with velocities $v_2 = -v_3$, that the first star will remain stationary for all time. Is this solution stable if r_1 is perturbed?

Soln

$$\ddot{r}_1 = -\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} \qquad (1)$$

If first star is initially at origin then $\vec{r}_1 = (0,0)$

$$\vec{r}_1 - \vec{r}_2 = -\vec{r}_2 \qquad (2)$$

$$\vec{r}_1 - \vec{r}_3 = -\vec{r}_3 \qquad (3)$$

$$\ddot{r}_1 = -\frac{(-\vec{r}_2)}{|(-\vec{r}_2)|^3} - \frac{(-\vec{r}_3)}{|(-\vec{r}_3)|^3} \qquad (4)$$

$$= \frac{\vec{r}_2}{|\vec{r}_2|^3} + \frac{\vec{r}_3}{|\vec{r}_3|^3}$$

If $\vec{r}_2 = -\vec{r}_3$

$$\ddot{r}_1 = -\frac{\vec{r}_3}{|\vec{r}_3|^3} + \frac{\vec{r}_3}{|\vec{r}_3|^3}$$

$$\ddot{r}_1 = 0$$

Hence, the first star remains at rest and never accelerates

If $\vec{r}_1$ is slightly perturbed, eqns (2) and (3) no longer hold and star 1 begins to move.