PHYS 6350 HW7. Due Monday Nov 23 at 9am.

1. **7 points**. The dimensionless Navier-Stokes (NS) equations are $\nabla \cdot \mathbf{u} = 0$ and $\dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + Re^{-1}\nabla^2 \mathbf{u}$. This problem will consider this equation *in two dimensions only*, with $\mathbf{u} = (u_x, u_y, 0)$. The fluid is trapped between two walls (with the boundary conditions $\mathbf{u}(y = 1) = \mathbf{u}(y = -1) = 0$, and periodic boundary conditions on $\mathbf{u}$ in the x direction).

   (a) Show analytically that *a* solution to the NS equations are $p = p_0 - \alpha x$, $u_x = \alpha Re(1 - y^2)/2$, and $u_y = 0$, for any α. This parabolic flow profile is referred to as Poiseuille flow.

   (b) The vorticity of a flow is defined as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, with $\boldsymbol{\omega} = (0, 0, \omega)$ for a 2D system. For an arbitrary 2D flow (*not* the Poiseuille flow in (a)), show that $\dot{\omega} + (\mathbf{u} \cdot \nabla)\omega = Re^{-1}\nabla^2\omega$. What are the boundary conditions on ω?

   (c) Can the PDE in (b) be solved using the methods for solving PDE's we described in class? If so, explain how. If not, describe a predictor-corrector method (like the SIMPLE algorithm described in class) that could be used?

2. **3 points**. Find $\delta A[n(r)]/\delta n(r)$ for

   (a) $A[n(r)] = \int d\boldsymbol{r}' d\boldsymbol{r}'' \frac{n(\boldsymbol{r}')n(\boldsymbol{r}'')}{|\boldsymbol{r}' - \boldsymbol{r}''|}$

   (b) $A[n(r)] = \int d\boldsymbol{r}' |\nabla n(\boldsymbol{r}')|^2$

3. **10 points**. In class we discussed an algorithm for solving eigenvalue problems. In this question, you'll implement that algorithm. You may use any language you choose, you do not need to use all three languages. Your code should solve the problem $x''(t) = E * f(x)$ with the boundary conditions $x(0) = x(\pi) = 0$ by:

   (a) Starting with with $E = 0.01$

   (b) Use rk4 to solve the initial value problem $x''(t) = Ef(x)$ with $x(0) = 0$ and $x'(0) = 1$. Set $b(E) = x(\pi)$ from this numerical calculation.

   (c) Use rk4 to solve the initial value problem $x''(t) = (E + dE)f(x)$ with $x(0) = 0$ and $x'(0) = 1$. Set $b(E + dE) = x(\pi)$ from this numerical calculation, and define $b'(E) \approx (b(E + dE) - b(E))/dE$.

   (d) Use Newton's method to update the eigenvalue, $E = E - b(E)/b'(E)$.

   (e) Return to step (b), unless $b(E)$ is less than some tolerance.

   You are free to choose the tolerance, dE, and dt (the latter for the timestep in rk4). *Hint*: Make sure you integrate all the way to $t = \pi$ in your implementation of rk4.

   Use your algorithm to find one eigenvalue for the following two boundary value problems:

- $x''(t) = -Ex(t)$, with $x(0) = x(\pi) = 0$. You should find an eigenvalue satisfying $E \approx n^2$, with n an integer ≥ 1.
- $x''(t) = -Ex^{1/3}(t)$, with $x(0) = x(\pi) = 0$. *Note:* $E = 1$ is not an eigenvalue for this boundary value problem.