Grammars

class 5

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Grammars

Grammars express languages

Example: the English language

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

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$$\langle article \rangle \rightarrow a$$
$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow cat$$
$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$
$$\langle verb \rangle \rightarrow walks$$

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A derivation of "the dog walks":

$$\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ walks
\end{aligned}$$

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A derivation of "a cat runs":

$$\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ runs
\end{aligned}$$

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Language of the grammar:

L = { "a cat runs",
"a cat walks",
"the cat runs",
"the cat walks",
"a dog runs",
"a dog walks",
"the dog runs",
"the dog walks" }

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Notation

Production Rules

$$\underbrace{\begin{array}{l}\langle noun\rangle \to cat \\ \langle noun\rangle \to dog\end{array}}$$

Variable

Terminal

Another Example

Grammar: $S \to aSb$

$S \to \lambda$

Derivation of sentence ab:

$$S \Rightarrow aSb \Rightarrow ab$$

$S \to aSb$ $\quad$ $S \to \lambda$

Grammar: $S \to aSb$

$S \to \lambda$

Derivation of sentence $aabb$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$S \to aSb$ $\quad$ $S \to \lambda$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Language of the grammar

$S \to aSb$

$S \to \lambda$

$$L = \{a^n b^n : n \geq 0\}$$

More Notation

Grammar $\quad G = (V, T, S, P)$

V : Set of variables

T : Set of terminal symbols

S : Start variable

P : Set of Production rules

Example

Grammar G : $S \to aSb$

$S \to \lambda$

$G = (V, T, S, P)$

$V = \{S\}$ $T = \{a, b\}$

$P = \{S \to aSb,\ S \to \lambda\}$

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More Notation

Sentential Form:

A sentence that contains variables and terminals

Example:

$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$

Sentential Forms sentence

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We write: $S \overset{*}{\Rightarrow} aaabbb$

Instead of:

$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$

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In general we write: $w_1 \overset{*}{\Rightarrow} w_n$

If: $w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$

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By default: $w \overset{*}{\Rightarrow} w$

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Example

Grammar

$S \to aSb$

$S \to \lambda$

Derivations

$S \overset{*}{\Rightarrow} \lambda$

$S \overset{*}{\Rightarrow} ab$

$S \overset{*}{\Rightarrow} aabb$

$S \overset{*}{\Rightarrow} aaabbb$

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Example

Grammar

$S \to aSb$

$S \to \lambda$

Derivations

$S \overset{*}{\Rightarrow} aaSbb$

$aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$

Another Grammar Example

Grammar G:

$S \to Ab$

$A \to aAb$

$A \to \lambda$

Derivations:

$S \Rightarrow Ab \Rightarrow b$

$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow abb$

$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aabbb$

More Derivations

$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aaaAbbbb$
$\Rightarrow aaaaAbbbbb \Rightarrow aaaabbbbb$

$S \overset{*}{\Rightarrow} aaaabbbbb$

$S \overset{*}{\Rightarrow} aaaaaabbbbbbb$

$S \overset{*}{\Rightarrow} a^n b^n b$

Language of a Grammar

For a grammar G
with start variable S :

$$L(G) = \{w : \quad S \overset{*}{\Rightarrow} w\}$$

String of terminals

Example

For grammar G: $S \to Ab$

$A \to aAb$

$A \to \lambda$

$$L(G) = \{a^n b^n b : \quad n \geq 0\}$$

Since: $S \overset{*}{\Rightarrow} a^n b^n b$

A Convenient Notation

$A \to aAb$
$A \to \lambda$

$A \to aAb \mid \lambda$

$\langle article \rangle \to a$
$\langle article \rangle \to the$

$\langle article \rangle \to a \mid the$

Linear Grammars

Linear Grammars

Grammars with
at most one variable at the right side
of a production

Examples:

$S \to aSb$

$S \to \lambda$

$S \to Ab$

$A \to aAb$

$A \to \lambda$

A Non-Linear Grammar

Grammar G :

$S \to SS$

$S \to \lambda$

$S \to aSb$

$S \to bSa$

$$L(G) = \{w: \; n_a(w) = n_b(w)\}$$

Number of a in string w

Another Linear Grammar

Grammar G :

$S \to A$

$A \to aB \mid \lambda$

$B \to Ab$

$$L(G) = \{a^n b^n : n \ge 0\}$$

Right-Linear Grammars

All productions have form: $A \to xB$

or

$A \to x$

string of terminals

Example:

$S \to abS$

$S \to a$

Left-Linear Grammars

All productions have form: $A \to Bx$

or

$A \to x$

string of terminals

Example:

$S \to Aab$

$A \to Aab \mid B$

$B \to a$

Regular Grammars

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Regular Grammars

A regular grammar is any right-linear or left-linear grammar

Examples:

G_1

$$S \to abS$$
$$S \to a$$

G_2

$$S \to Aab$$
$$A \to Aab \mid B$$
$$B \to a$$

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Observation

Regular grammars generate regular languages

Examples:

G_1

$$S \to abS$$
$$S \to a$$

$$L(G_1) = (ab)^* a$$

G_2

$$S \to Aab$$
$$A \to Aab \mid B$$
$$B \to a$$

$$L(G_2) = aab(ab)^*$$

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Regular Grammars Generate Regular Languages

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Theorem

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

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Theorem - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular grammar generates a regular language

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Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Any regular language is generated by a regular grammar

Proof - Part 1

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

The language $L(G)$ generated by any regular grammar G is regular

The case of Right-Linear Grammars

Let G be a right-linear grammar

We will prove: $L(G)$ is regular

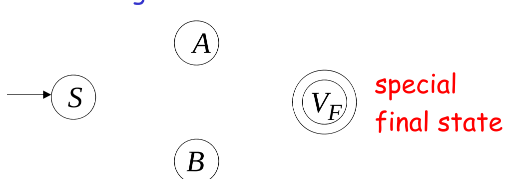
Proof idea: We will construct NFA M with $L(M) = L(G)$

Grammar G is right-linear

Example:

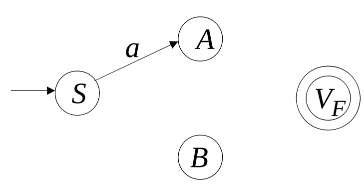
$$S \to aA \mid B$$
$$A \to aa\,B$$
$$B \to b\,B \mid a$$

Construct NFA M such that every state is a grammar variable:

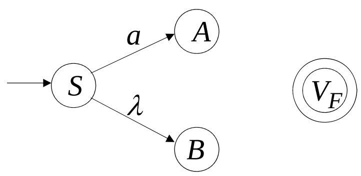


$$S \to aA \mid B$$
$$A \to aa\,B$$
$$B \to b\,B \mid a$$

Add edges for each production:



$$S \to aA$$



$S \to aA \mid B$

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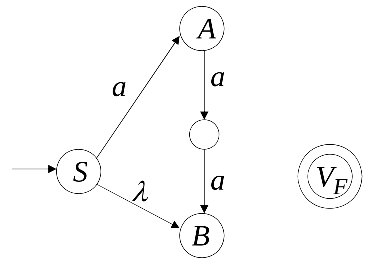

$S \to aA \mid B$

$A \to aa\ B$

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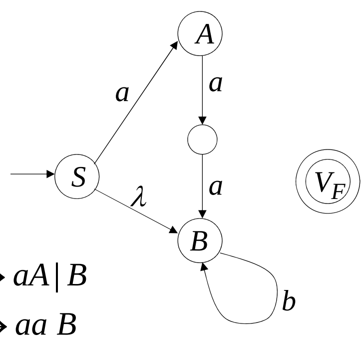

$S \to aA \mid B$

$A \to aa\ B$

$B \to bB$

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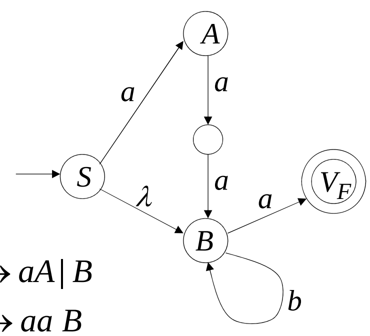

$S \to aA \mid B$

$A \to aa\ B$

$B \to bB \mid a$

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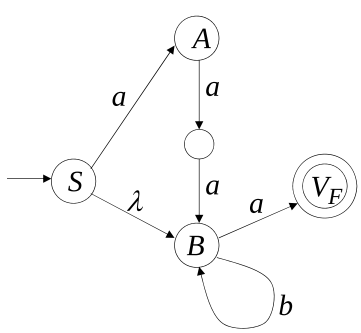

$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$

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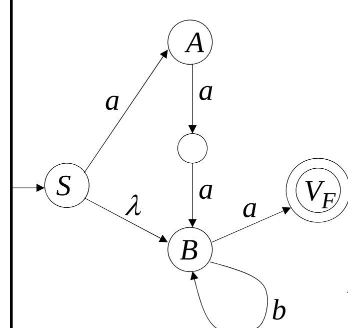
NFA M



Grammar G

$S \to aA \mid B$

$A \to aa\ B$

$B \to bB \mid a$

$L(M) = L(G) =$

$aaab^*a + b^*a$

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In General

A right-linear grammar G

has variables: $V_0, V_1, V_2, \ldots$

and productions: $V_i \rightarrow a_1 a_2 \cdots a_m V_j$

or

$V_i \rightarrow a_1 a_2 \cdots a_m$

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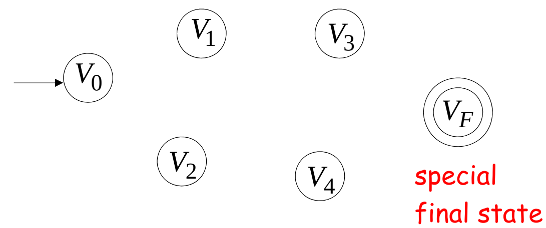
We construct the NFA M such that:

each variable V_i corresponds to a node:



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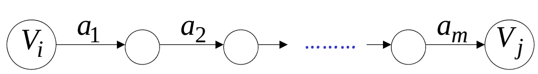
For each production: $V_i \rightarrow a_1 a_2 \cdots a_m V_j$

we add transitions and intermediate nodes



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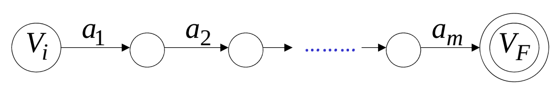
For each production: $V_i \rightarrow a_1 a_2 \cdots a_m$

we add transitions and intermediate nodes



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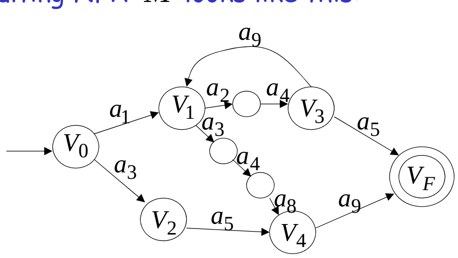
Resulting NFA M looks like this:



It holds that: $L(G) = L(M)$

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The case of Left-Linear Grammars

Let G be a left-linear grammar

We will prove: $L(G)$ is regular

Proof idea:

We will construct a right-linear grammar G' with $L(G) = L(G')^R$

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Since G is left-linear grammar
the productions look like:

$$A \to B a_1 a_2 \cdots a_k$$

$$A \to a_1 a_2 \cdots a_k$$

Construct right-linear grammar G'

Left linear G

$$A \to B a_1 a_2 \cdots a_k$$

$$A \to Bv$$

$\Downarrow$

Right linear G'

$$A \to a_k \cdots a_2 a_1 B$$

$$A \to v^R B$$

Construct right-linear grammar G'

Left linear G

$$A \to a_1 a_2 \cdots a_k$$

$$A \to v$$

$\Downarrow$

Right linear G'

$$A \to a_k \cdots a_2 a_1$$

$$A \to v^R$$

It is easy to see that: $L(G) = L(G')^R$

Since G' is right-linear, we have:

$L(G')$	$\Rightarrow$	$L(G')^R$	$\Rightarrow$	$L(G)$
Regular Language		Regular Language		Regular Language

Proof - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language L is generated
by some regular grammar G

Any regular language L is generated
by some regular grammar G

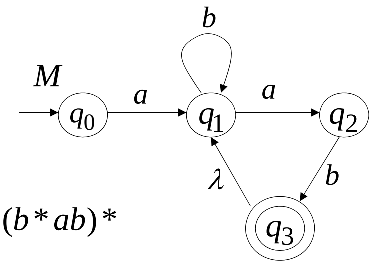
Proof idea:

Let M be the NFA with $L = L(M)$.

Construct from M a regular grammar G
such that $L(M) = L(G)$

Since L is regular
there is an NFA M such that $L = L(M)$

Example:



$L = ab^*ab(b^*ab)^*$

$L = L(M)$

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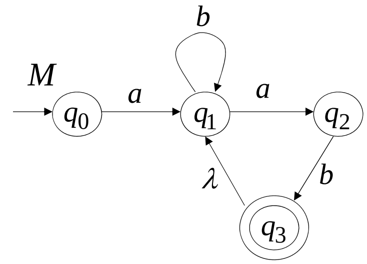
Convert M to a right-linear grammar



$q_0 \to aq_1$

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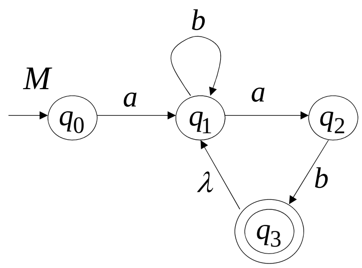

$q_0 \to aq_1$

$q_1 \to bq_1$

$q_1 \to aq_2$

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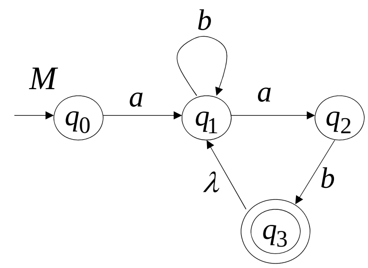
$q_0 \to aq_1$

$q_1 \to bq_1$

$q_1 \to aq_2$

$q_2 \to bq_3$



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$L(G) = L(M) = L$

G

$q_0 \to aq_1$

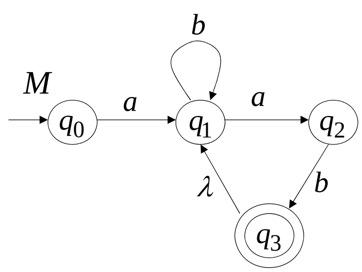
$q_1 \to bq_1$

$q_1 \to aq_2$

$q_2 \to bq_3$

$q_3 \to q_1$

$q_3 \to \lambda$



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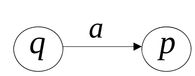
In General

For any transition:



Add production: $q \to ap$

variable terminal variable

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For any final state: q_f

Add production: $q_f \rightarrow \lambda$

Since G is right-linear grammar

G is also a regular grammar

with $L(G) = L(M) = L$