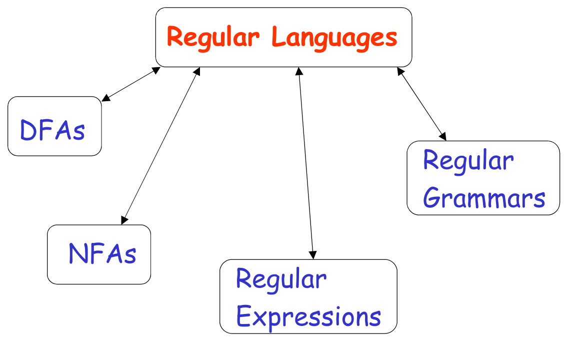
Standard Representations of Regular Languages



When we say: We are given a Regular Language L

We mean: Language L is in a standard representation

Elementary Questions

about

Regular Languages

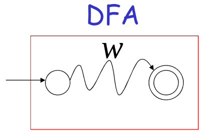
class 6

Membership Question

Question: Given regular language L and string w how can we check if $w \in L$ **?**

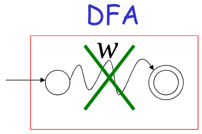
Answer: Take the DFA that accepts L and check if w is accepted

DFA



$w \in L$

DFA



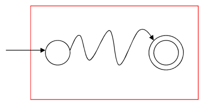
$w \notin L$

Question: Given regular language L how can we check if L is empty: $(L = \emptyset)$?

Answer: Take the DFA that accepts L

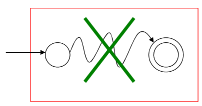
Check if there is any path from the initial state to a final state

DFA



$L \neq \varnothing$

DFA



$L = \varnothing$

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Question: Given regular language L
how can we check
if L is finite?

Answer: Take the DFA that accepts L

Check if there is a walk with cycle
from the initial state to a final state

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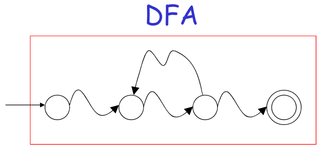
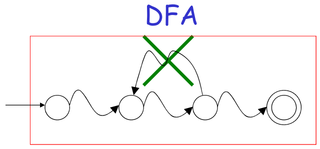
DFA



L is infinite

DFA



L is finite

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Question: Given regular languages L_1 and L_2
how can we check if $L_1 = L_2$?

Answer: Find if $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \varnothing$

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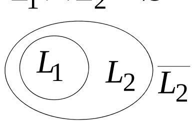
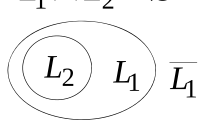
$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \varnothing$$

$L_1 \cap \overline{L_2} = \varnothing$ and $\overline{L_1} \cap L_2 = \varnothing$





$L_1 \subseteq L_2$ $\qquad$ $L_2 \subseteq L_1$

$$L_1 = L_2$$

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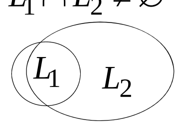
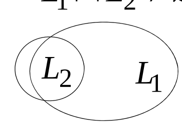
$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \varnothing$$

$L_1 \cap \overline{L_2} \neq \varnothing$ or $\overline{L_1} \cap L_2 \neq \varnothing$





$L_1 \not\subset L_2$ $\qquad$ $L_2 \not\subset L_1$

$$L_1 \neq L_2$$

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Non-regular languages

Non-regular languages $\{a^n b^n : \; n \geq 0\}$

$\{vv^R : \; v \in \{a,b\}^*\}$

Regular languages

a^*b $\quad b^*c+a$

$b+c(a+b)^*$

$etc...$

How can we prove that a language L is not regular?

Prove that there is no DFA that accepts L

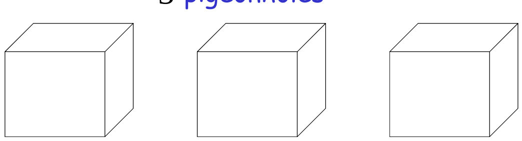
Problem: this is not easy to prove

Solution: the Pumping Lemma !!!

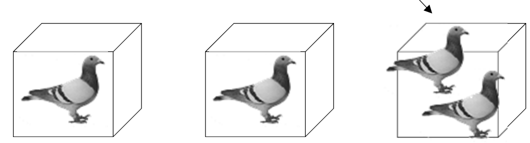
The Pigeonhole Principle

4 pigeons

3 pigeonholes

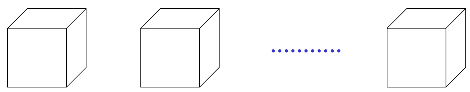


A pigeonhole must contain at least two pigeons



n pigeons

m pigeonholes $n > m$



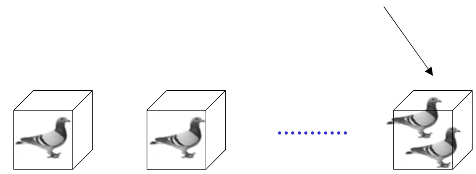
The Pigeonhole Principle

n pigeons

m pigeonholes

$n > m$

There is a pigeonhole with at least 2 pigeons

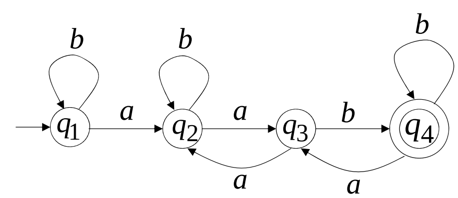


The Pigeonhole Principle

and

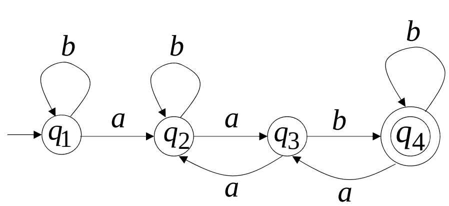
DFAs

DFA with 4 states



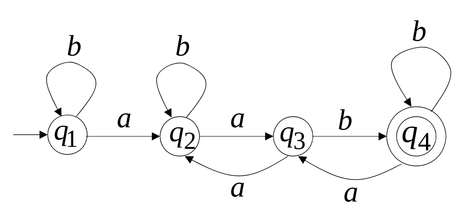
In walks of strings: a, aa, aab

no state is repeated



In walks of strings: $aabb$, $bbaa$, $abbabb$, $abbbabbabb...$

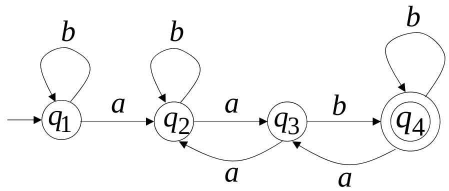
a state is repeated



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

Thus, a state must be repeated

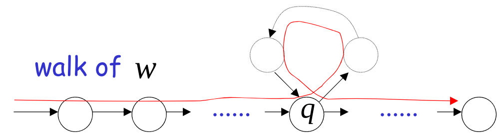


In general, for any DFA:

String w has length $\geq$ number of states

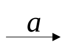
$\Downarrow$

A state q must be repeated in the walk of w

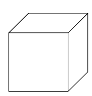


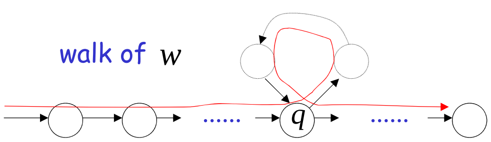
Repeated state

In other words for a string w:



transitions are pigeons 

states are pigeonholes 

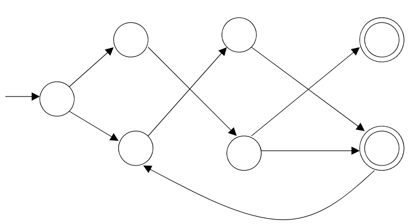


Repeated state

The Pumping Lemma

Take an infinite regular language L

There exists a DFA that accepts L



m states

Take string w with $w \in L$

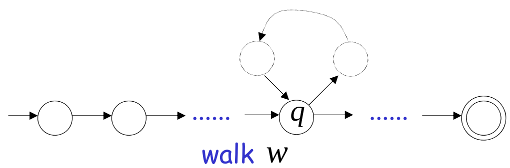
There is a walk with label w:

walk w

If string w has length $|w| \geq m$ (number of states of DFA)

then, from the pigeonhole principle:

a state is repeated in the walk w



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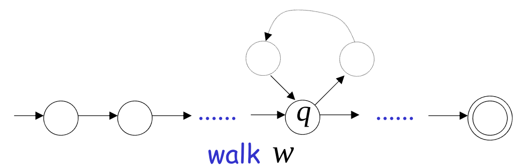
Let q be the first state repeated in the walk of w



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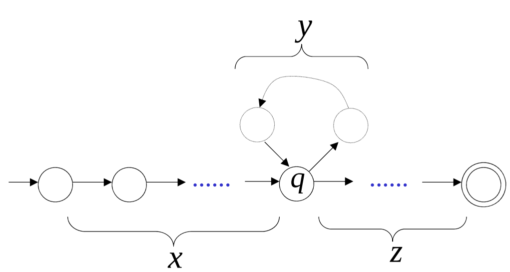
Write $w = x\,y\,z$



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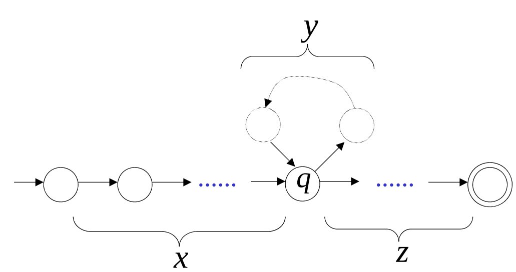
Observations: length $|x\,y| \leq m$ number of states of DFA

length $|y| \geq 1$



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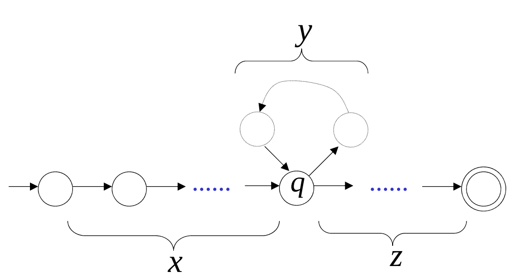
Observation: The string $x\,z$ is accepted



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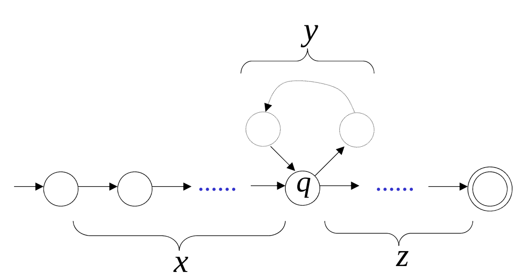
Observation: The string $x\,y\,y\,z$ is accepted



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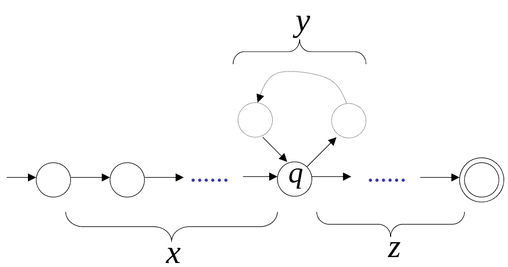
Observation: The string $x\ y\ y\ y\ z$ is accepted



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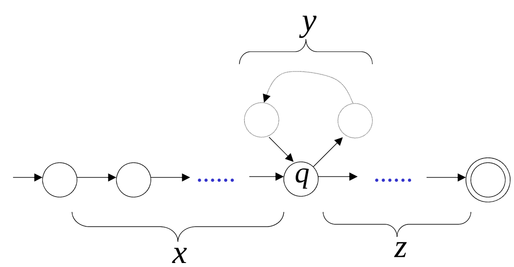
In General: The string $x\ y^i\ z$ is accepted $i = 0, 1, 2, \ldots$



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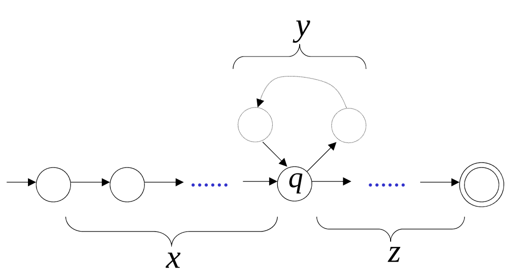
In General: $x\ y^i\ z \in L$ $\qquad i = 0, 1, 2, \ldots$

Language accepted by the DFA



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In other words, we described:

The Pumping Lemma !!!

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The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\ y\ z$
- with $|x\ y| \leq m$ and $|y| \geq 1$
- such that: $x\ y^i\ z \in L$ $\qquad i = 0, 1, 2, \ldots$

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Applications

of

the Pumping Lemma

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Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

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$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

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$$L = \{a^n b^n : n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^m b^m$

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Write: $a^m b^m = x\,y\,z$

From the Pumping Lemma
it must be that length $|x\,y| \leq m, \;\; |y| \geq 1$

$$xyz = a^m b^m \;=\; \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}$$

$$x \quad y \quad z$$

Thus: $y = a^k, \;\; k \geq 1$

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$$x\,y\,z = a^m b^m \qquad y = a^k, \;\; k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \;\in L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^2\,z \;\in L$

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$$x\,y\,z = a^m b^m \qquad y = a^k, \;\; k \geq 1$$

From the Pumping Lemma: $x\,y^2\,z \;\in L$

$$xy^2z = \overbrace{a...aa...aa...aa...a}^{m+k}\overbrace{b...b}^{m} \in L$$

$$x \quad y \quad y \quad z$$

Thus: $a^{m+k} b^m \in L$

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$a^{m+k}b^m \in L \qquad k \geq 1$

BUT: $L = \{a^n b^n : n \geq 0\}$

$\Downarrow$

$a^{m+k}b^m \notin L$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages $\{a^n b^n : \ n \geq 0\}$