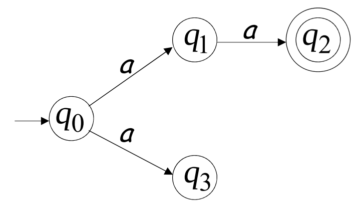
Non Deterministic Automata

Class 3

Nondeterministic Finite Accepter (NFA)

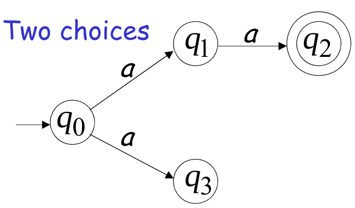
Alphabet = $\{a\}$



Nondeterministic Finite Accepter (NFA)

Alphabet = $\{a\}$

Two choices



Nondeterministic Finite Accepter (NFA)

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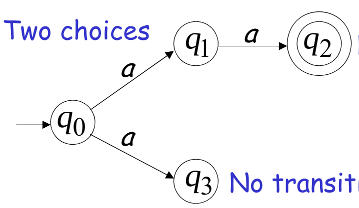
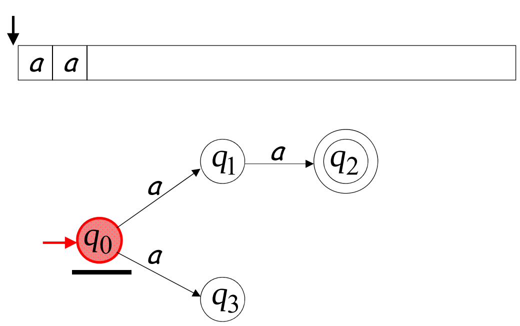
Two choices



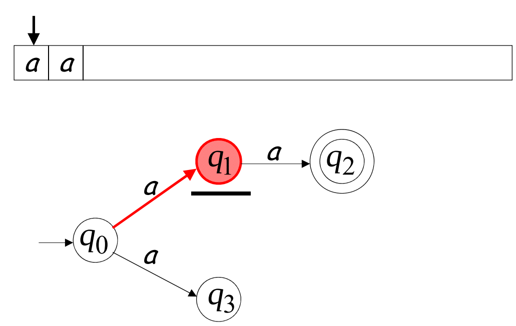
No transition

No transition

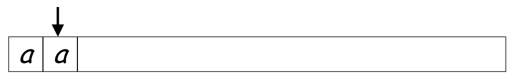
First Choice

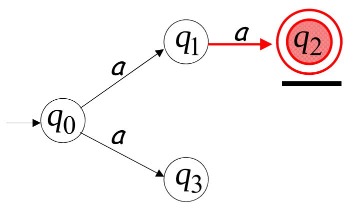


First Choice

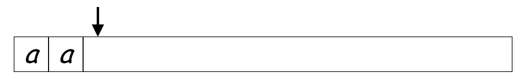


First Choice

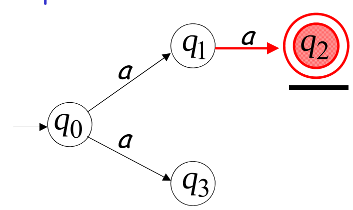




First Choice

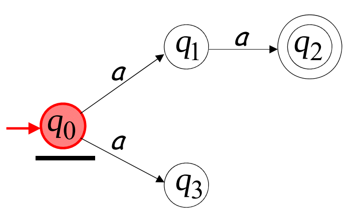


All input is consumed

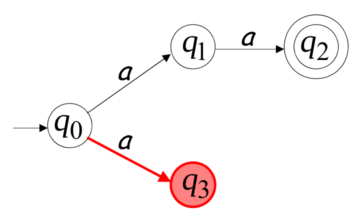


"accept"

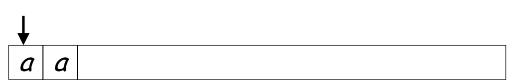
Second Choice

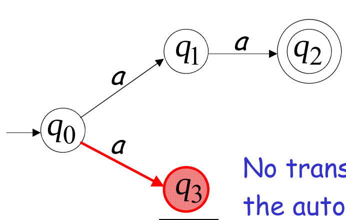


Second Choice



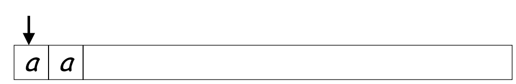
Second Choice



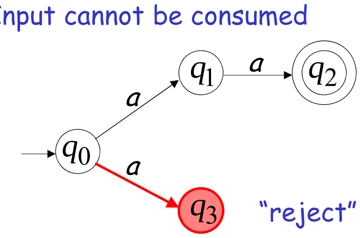


No transition:
the automaton hangs

Second Choice



Input cannot be consumed



"reject"

An NFA accepts a string:
when there is a computation of the NFA
that accepts the string

AND

all the input is consumed and the automaton
is in a final state

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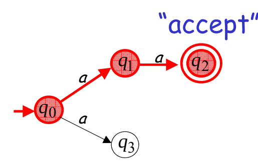
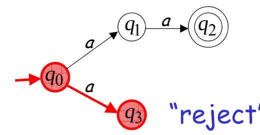
Example

aa is accepted by the NFA:





because this
computation
accepts aa

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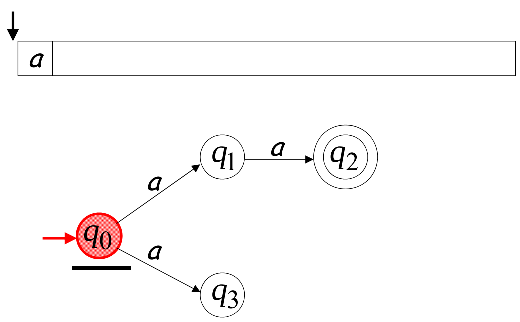
Rejection example



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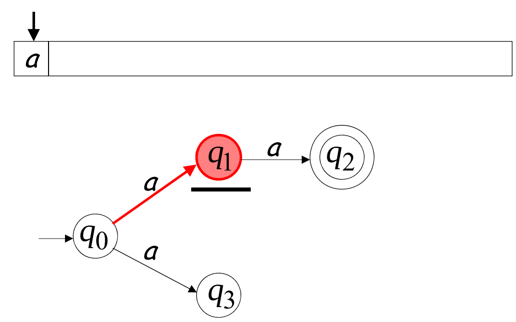
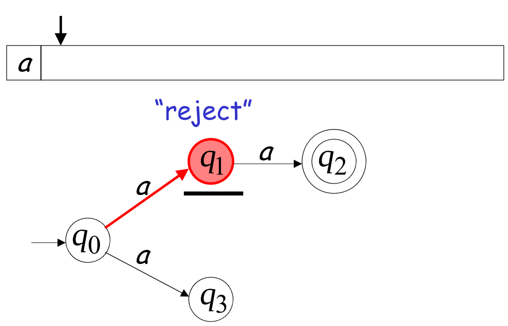
First Choice



16

First Choice



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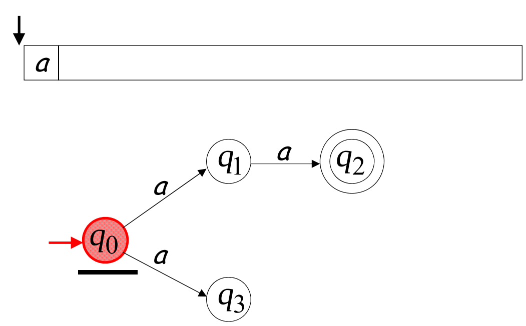
Second Choice



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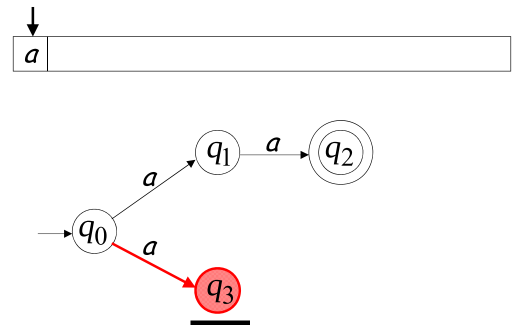
Second Choice



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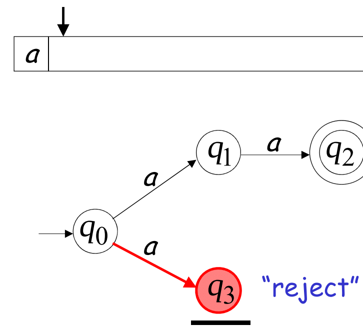
Second Choice



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An NFA rejects a string:
when there is no computation of the NFA that accepts the string:

- All the input is consumed and the automaton is in a non final state

OR

- The input cannot be consumed

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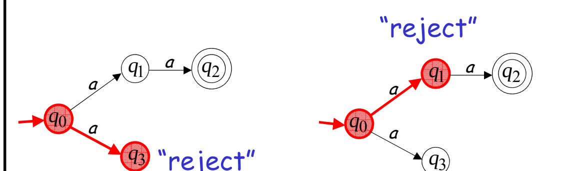
Example

a is rejected by the NFA:



All possible computations lead to rejection

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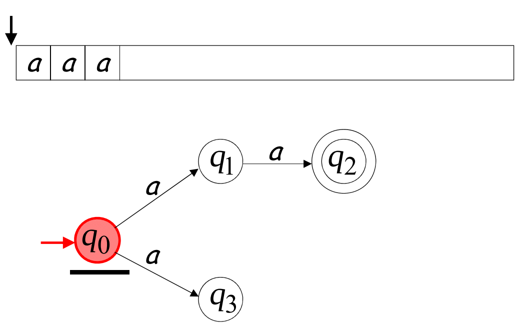
Rejection example



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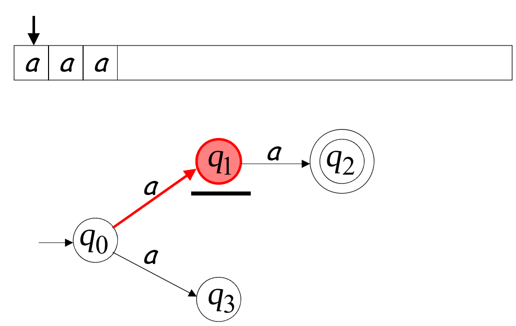
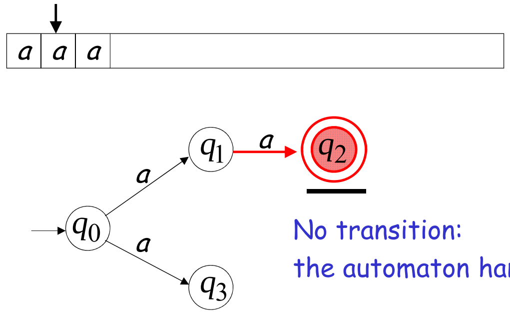
First Choice



24

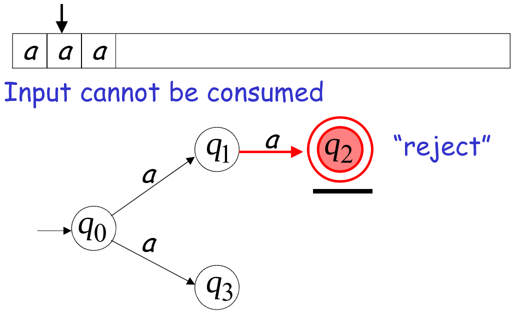
First Choice



No transition:
the automaton hangs

25

First Choice

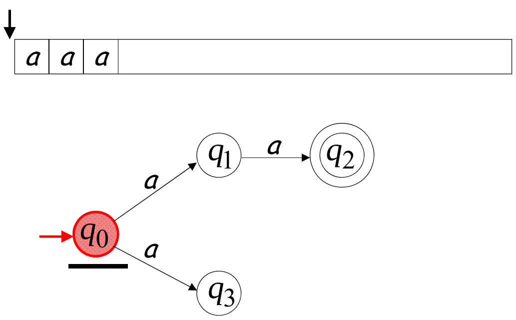


Input cannot be consumed

"reject"

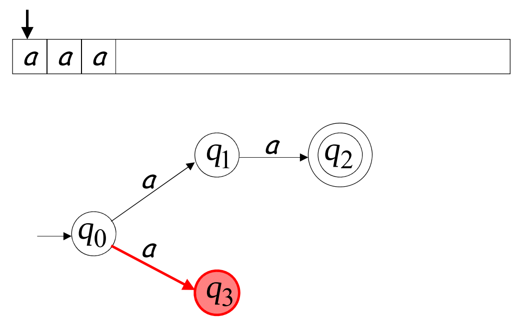
26

Second Choice



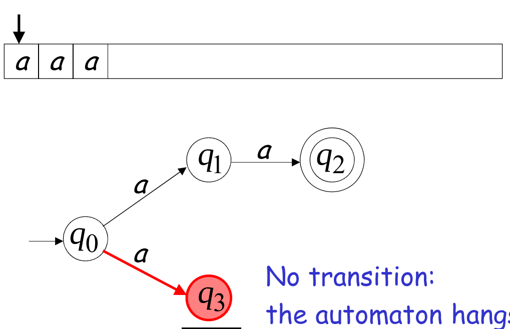
27

Second Choice



28

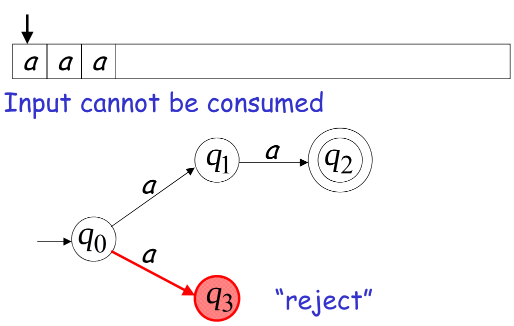
Second Choice



No transition:
the automaton hangs

29

Second Choice

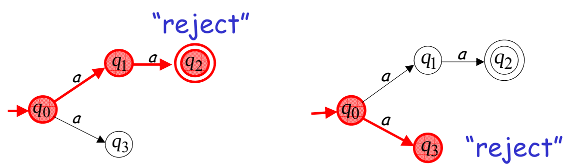


Input cannot be consumed

"reject"

30

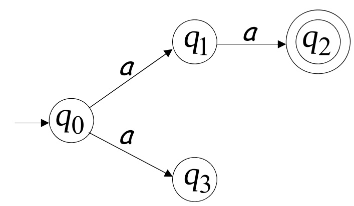
aaa is rejected by the NFA:



All possible computations lead to rejection

31

Language accepted: $L = \{aa\}$



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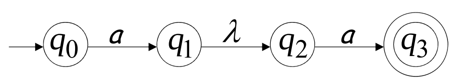
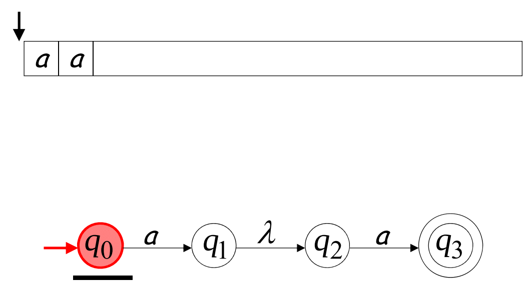
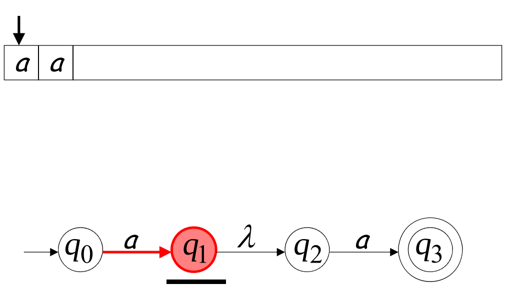
Lambda Transitions



33

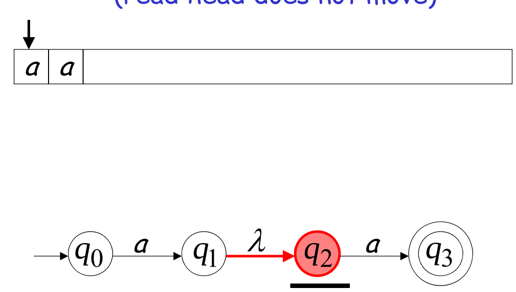


34

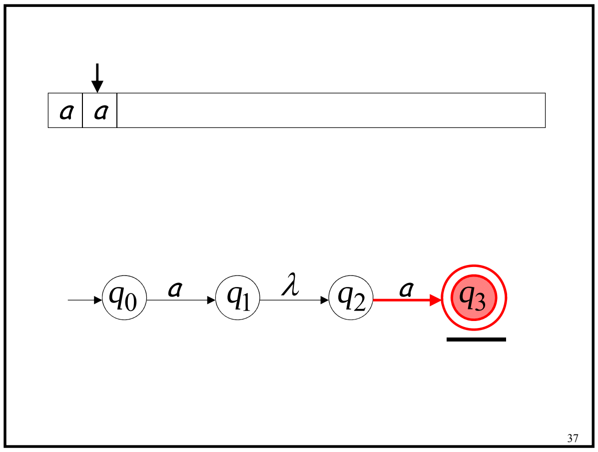


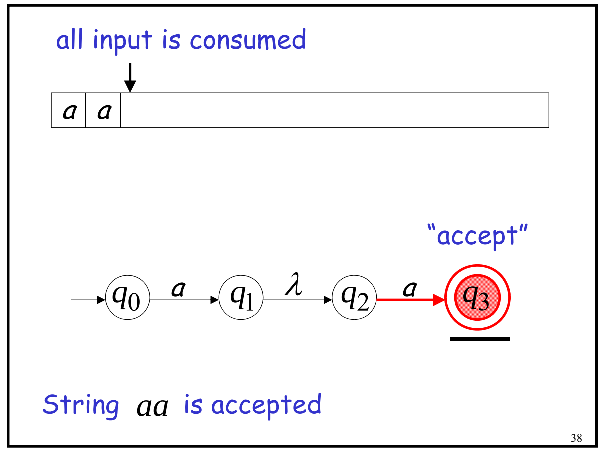
35

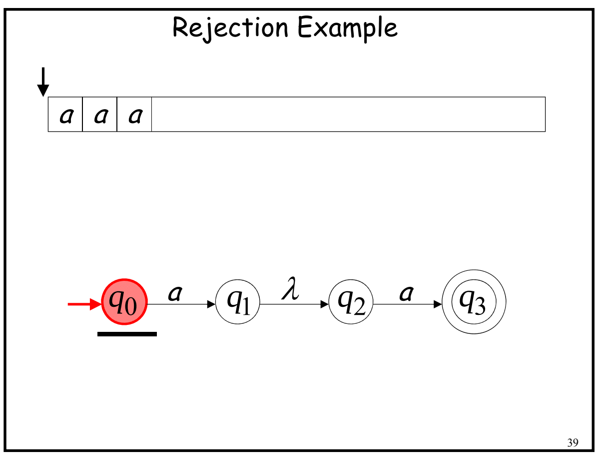
(read head does not move)

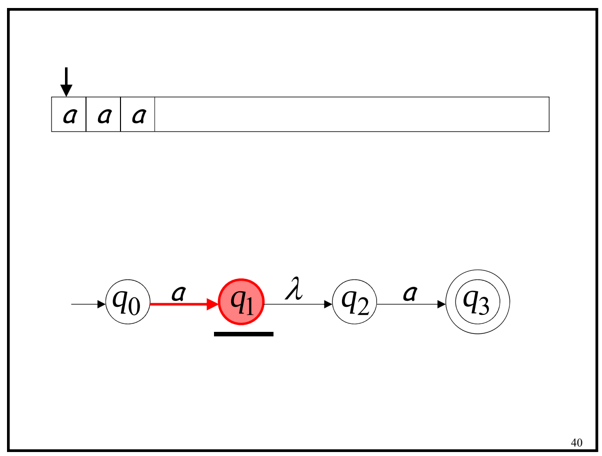


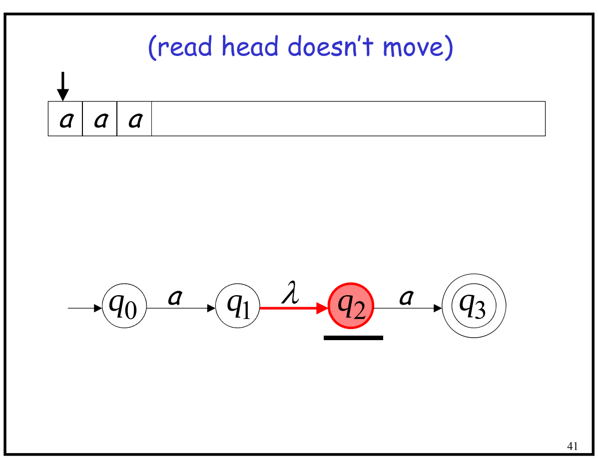
36

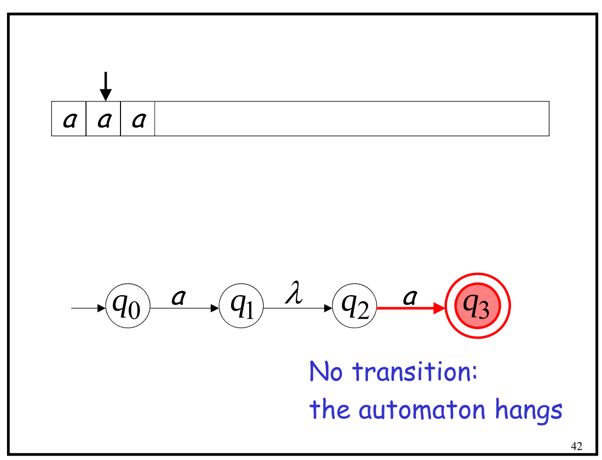

a a
q0 a q1 λ q2 a q3
37


all input is consumed
a a
"accept"
q0 a q1 λ q2 a q3
String aa is accepted
38

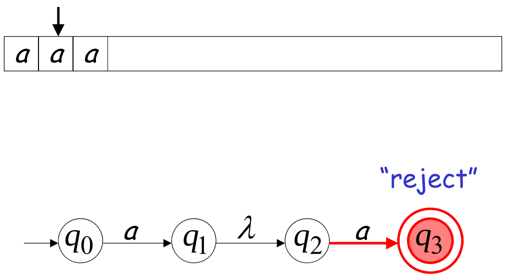

Rejection Example
a a a
q0 a q1 λ q2 a q3
39


a a a
q0 a q1 λ q2 a q3
40


(read head doesn't move)
a a a
q0 a q1 λ q2 a q3
41


a a a
q0 a q1 λ q2 a q3
No transition:
the automaton hangs
42

Input cannot be consumed



String aaa is rejected

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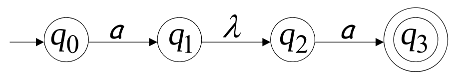
Language accepted: $L = \{aa\}$



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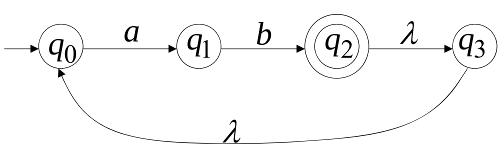
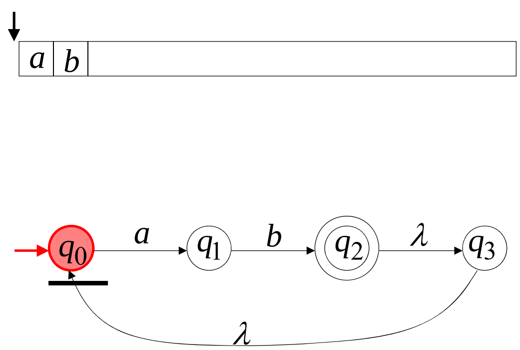
Another NFA Example



45



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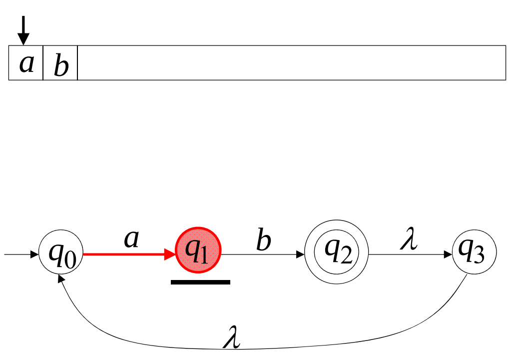

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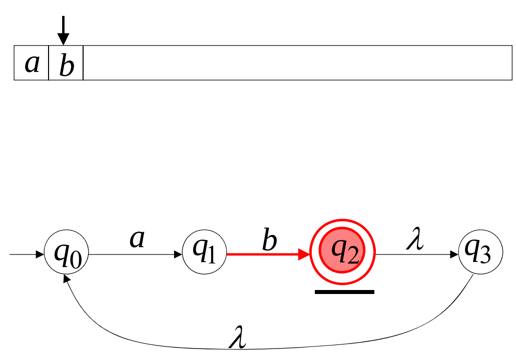
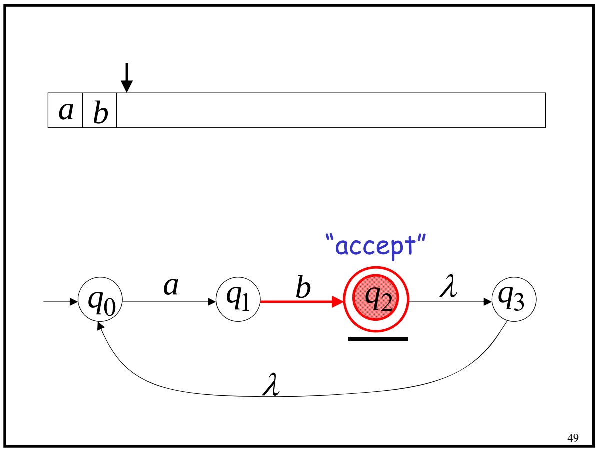
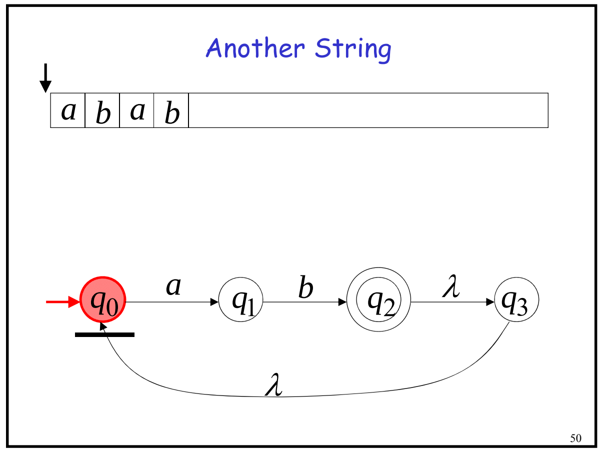


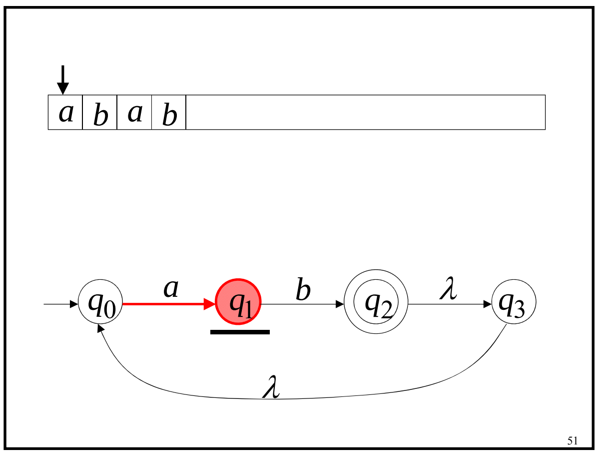
48



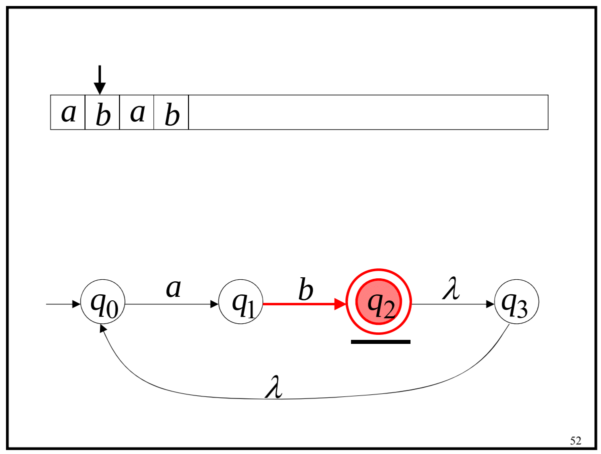
a b
"accept"
q_0 a q_1 b q_2 λ q_3
λ
49



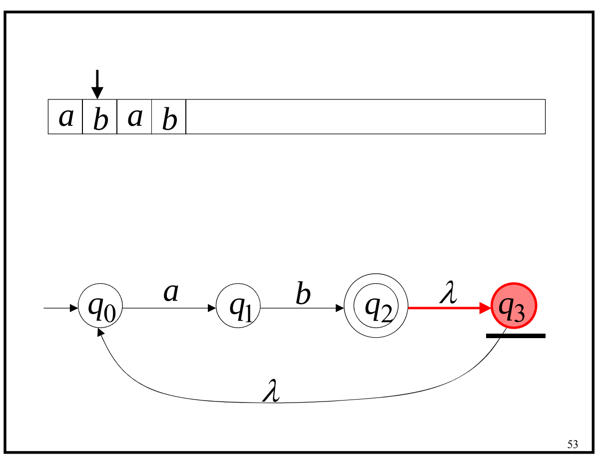
Another String
a b a b
q_0 a q_1 b q_2 λ q_3
λ
50



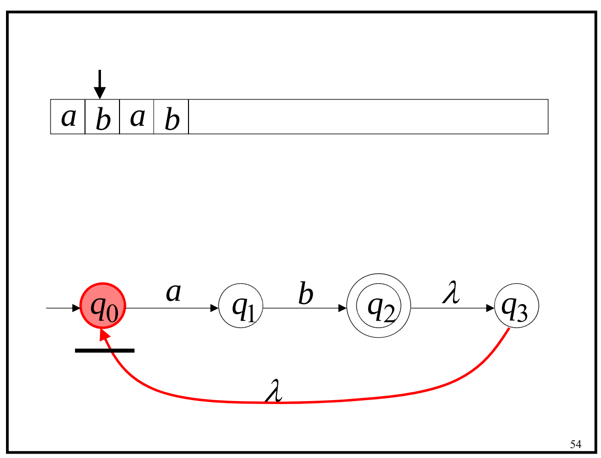
a b a b
q_0 a q_1 b q_2 λ q_3
λ
51



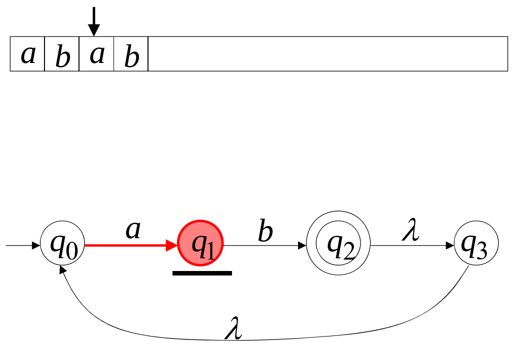
a b a b
q_0 a q_1 b q_2 λ q_3
λ
52

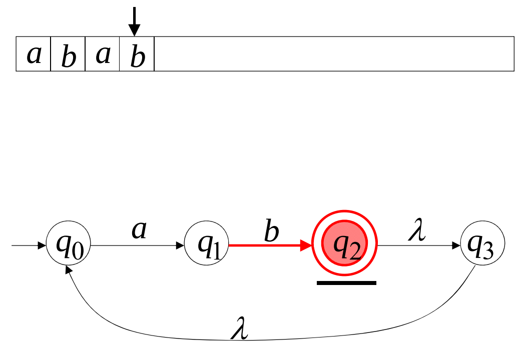


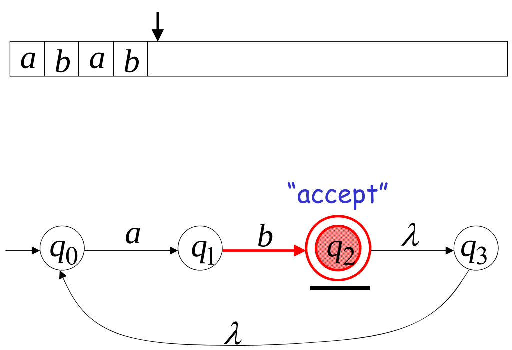
a b a b
q_0 a q_1 b q_2 λ q_3
λ
53



a b a b
q_0 a q_1 b q_2 λ q_3
λ
54

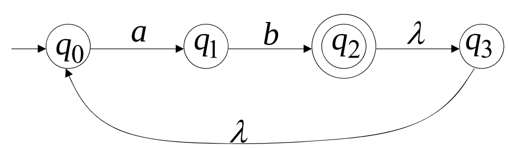




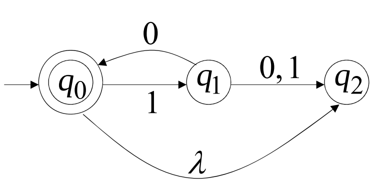


Language accepted

$$L = \{ab,\ abab,\ ababab,\ \ldots\}$$
$$= \{ab\}^{+}$$

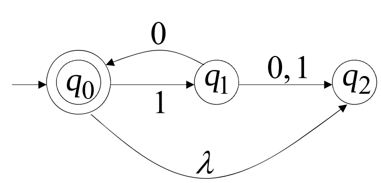


Another NFA Example



Language accepted

$$L(M) = \{\lambda,\ 10,\ 1010,\ 101010,\ \ldots\}$$
$$= \{10\}^*$$



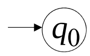
(redundant state)

Remarks:

- The λ symbol never appears on the input tape

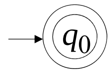
- Simple automata:

M_1



$L(M_1) = \{\}$

M_2



$L(M_2) = \{\lambda\}$

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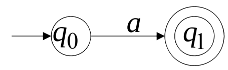
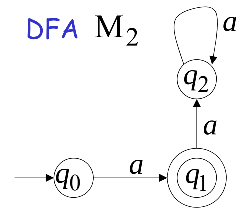
- NFAs are interesting because we can express languages easier than DFAs

NFA M_1



DFA M_2



$L(M_1) = \{a\}$

$L(M_2) = \{a\}$

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Formal Definition of NFAs

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

Q : Set of states, i.e. $\{q_0, q_1, q_2\}$

Σ : Input aplhabet, i.e. $\{a, b\}$

δ : Transition function

q_0 : Initial state

F : Final states

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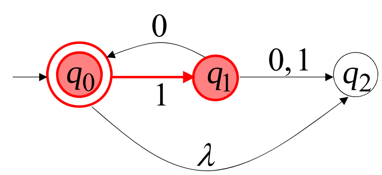
Transition Function δ

$$\delta(q_0, 1) = \{q_1\}$$



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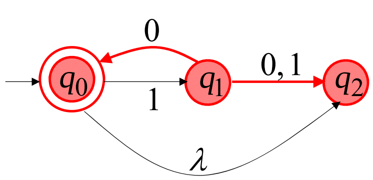
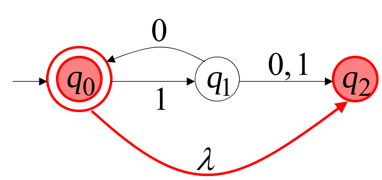
$$\delta(q_1, 0) = \{q_0, q_2\}$$



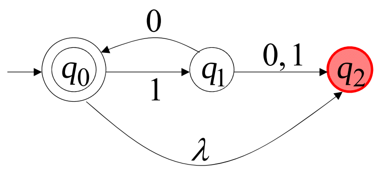
65

$$\delta(q_0, \lambda) = \{q_0, q_2\}$$



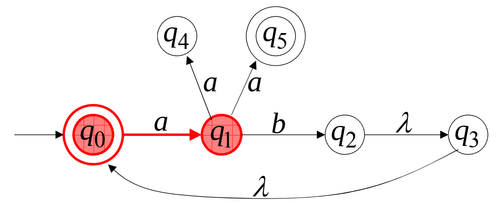
66

$$\delta(q_2, 1) = \varnothing$$

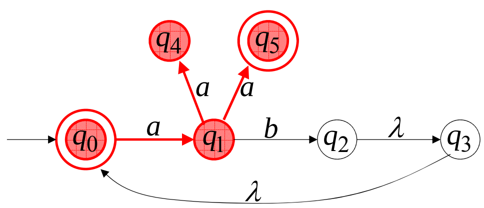


Extended Transition Function δ^*

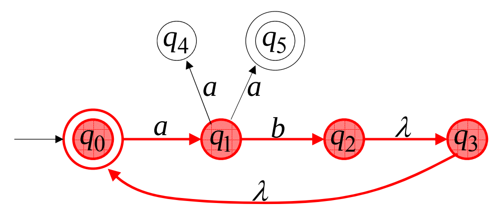
$$\delta^*(q_0, a) = \{q_1\}$$



$$\delta^*(q_0, aa) = \{q_4, q_5\}$$

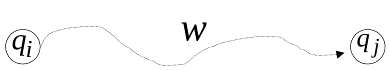


$$\delta^*(q_0, ab) = \{q_2, q_3, q_0\}$$

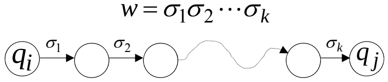


Formally

$q_j \in \delta^*(q_i, w)$: there is a walk from q_i to q_j with label w

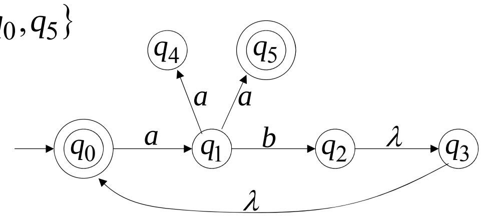


$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



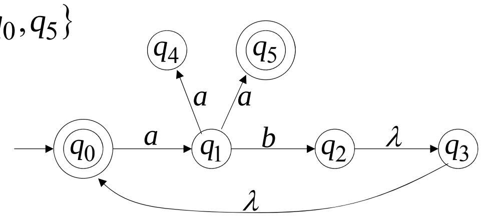
The Language of an NFA M

$F = \{q_0, q_5\}$



$\delta^*(q_0, aa) = \{q_4, \underbrace{q_5}_{\in F}\}$ $\qquad aa \in L(M)$

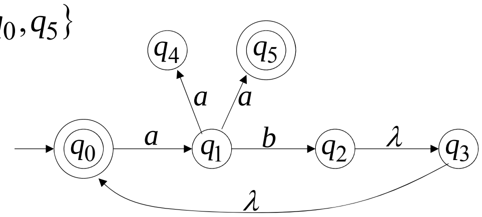
$F = \{q_0, q_5\}$



$\delta^*(q_0, ab) = \{q_2, q_3, \underline{q_0}\}$ $\quad ab \in L(M)$

$\searrow \in F$

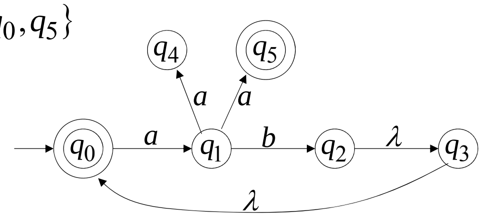
$F = \{q_0, q_5\}$



$\delta^*(q_0, abaa) = \{q_4, \underline{q_5}\}$ $\quad abaa \in L(M)$

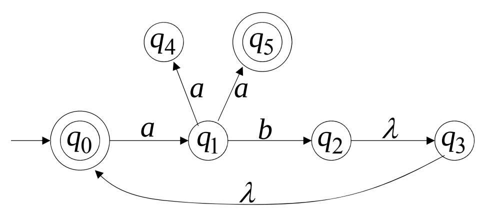
$\searrow \in F$

$F = \{q_0, q_5\}$



$\delta^*(q_0, aba) = \{q_1\}$ $\quad aba \notin L(M)$

$\searrow \notin F$



$L(M) = \{\lambda\} \cup \{ab\}^* \{aa\}$

Formally

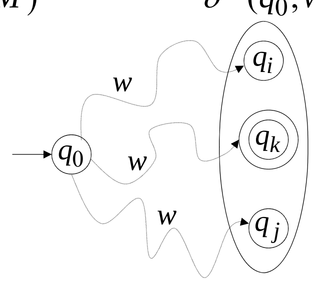
The language accepted by NFA M is:

$$L(M) = \{w_1, w_2, w_3, \ldots\}$$

where $\quad \delta^*(q_0, w_m) = \{q_i, q_j, \ldots, q_k, \ldots\}$

and there is some $\quad q_k \in F \quad$ (final state)

$w \in L(M)$ $\qquad \delta^*(q_0, w)$



$q_k \in F$

NFAs accept the Regular Languages

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Equivalence of Machines

Definition for Automata:

Machine M_1 is equivalent to machine M_2

if $L(M_1) = L(M_2)$

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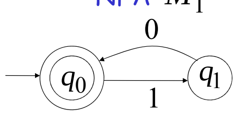
Example of equivalent machines

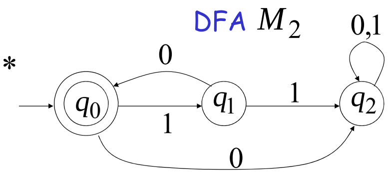
NFA M_1

$L(M_1) = \{10\}^*$



DFA M_2

$L(M_2) = \{10\}^*$



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We will prove:

$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$

Languages accepted by DFAs

NFAs and DFAs have the same computation power

82

Step 1

$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$

Proof: Every DFA is trivially an NFA

⇓

Any language L accepted by a DFA is also accepted by an NFA

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Step 2

$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$

Proof: Any NFA can be converted to an equivalent DFA

⇓

Any language L accepted by an NFA is also accepted by a DFA

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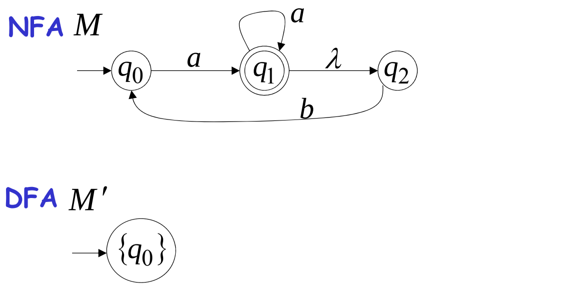
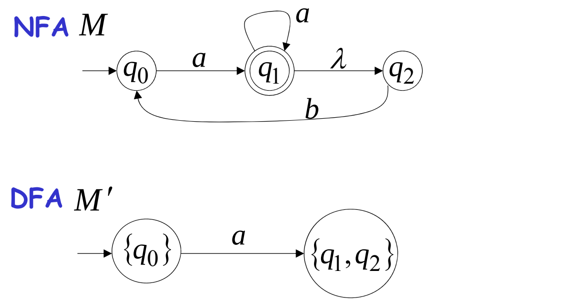
Convert NFA to DFA



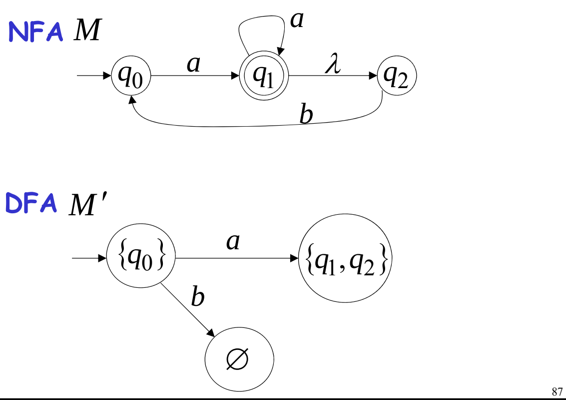
85

Convert NFA to DFA



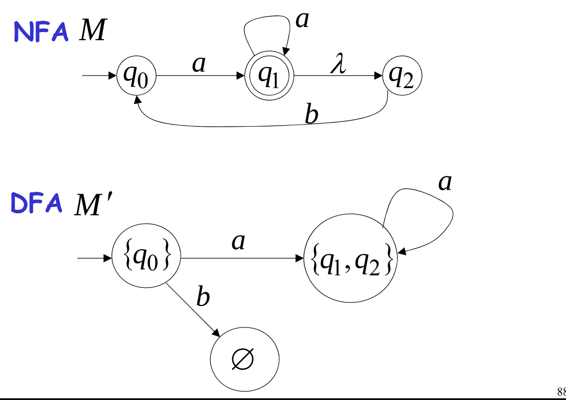
86

Convert NFA to DFA



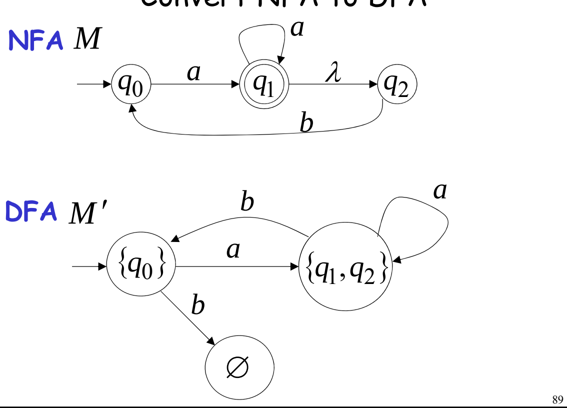
87

Convert NFA to DFA



88

Convert NFA to DFA



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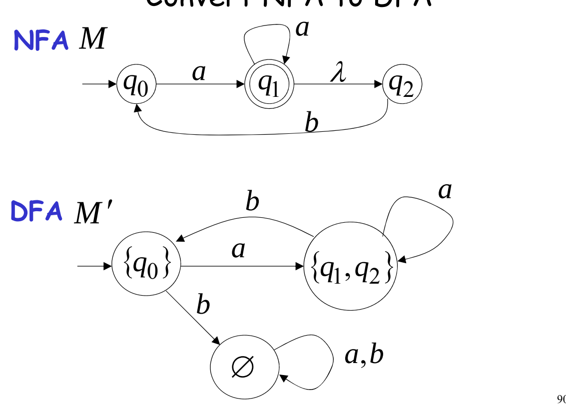
Convert NFA to DFA



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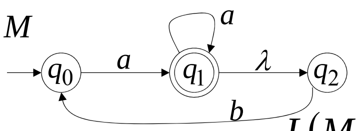
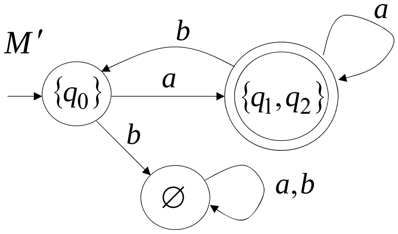
Convert NFA to DFA

NFA M



$L(M) = L(M')$

DFA M'



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NFA to DFA: Remarks

We are given an NFA M

We want to convert it
to an equivalent DFA M'

With $L(M) = L(M')$

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If the NFA has states

$$q_0, q_1, q_2, \dots$$

the DFA has states in the powerset

$$\emptyset, \{q_0\}, \{q_1\}, \{q_1, q_2\}, \{q_3, q_4, q_7\}, \dots$$

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Procedure NFA to DFA

1. Initial state of NFA: q_0

$\Downarrow$

Initial state of DFA: $\{q_0\}$

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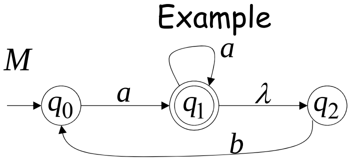
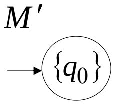
Example

NFA M



DFA M'



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Procedure NFA to DFA

2. For every DFA's state $\{q_i, q_j, \dots, q_m\}$

Compute in the NFA

$$\left.\begin{array}{l}\delta^*(q_i, a),\\ \delta^*(q_j, a),\\ \dots\end{array}\right\} = \{q'_i, q'_j, \dots, q'_m\}$$

Add transition to DFA

$$\delta(\{q_i, q_j, \dots, q_m\},\ a) = \{q'_i, q'_j, \dots, q'_m\}$$

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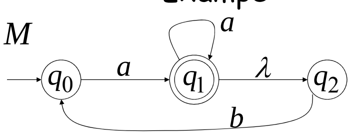
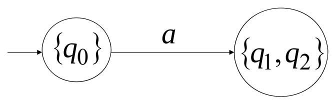
Exampe

NFA M



$$\delta^*(q_0, a) = \{q_1, q_2\}$$

DFA M'



$$\delta(\{q_0\}, a) = \{q_1, q_2\}$$

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Procedure NFA to DFA

Repeat Step 2 for all letters in alphabet,
until
no more transitions can be added.

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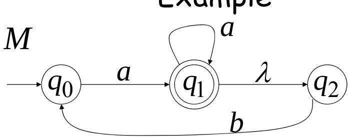
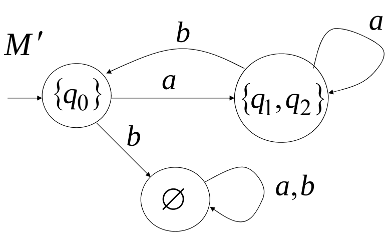
Example

NFA M



DFA M'



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Procedure NFA to DFA

3. For any DFA state $\{q_i, q_j, \ldots, q_m\}$

If some q_j is a final state in the NFA

Then, $\{q_i, q_j, \ldots, q_m\}$
is a final state in the DFA

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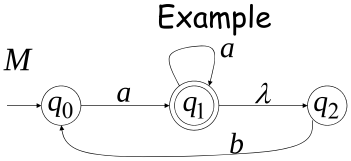
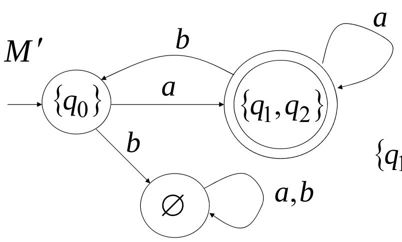
Example

NFA M



$q_1 \in F$

DFA M'



$\{q_1, q_2\} \in F'$

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Theorem

Take NFA M

Apply procedure to obtain DFA M'

Then M and M' are equivalent :

$$L(M) = L(M')$$

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Proof

$$L(M) = L(M')$$

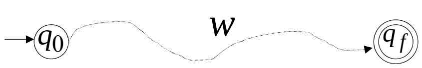
$$L(M) \subseteq L(M') \quad \text{AND} \quad L(M) \supseteq L(M')$$

First we show: $L(M) \subseteq L(M')$

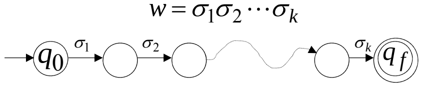
Take arbitrary: $w \in L(M)$

We will prove: $w \in L(M')$

$$w \in L(M)$$

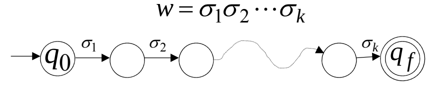
M : 

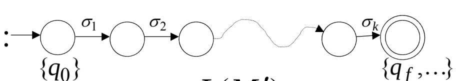
$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

M : 

We will show that if $w \in L(M)$

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

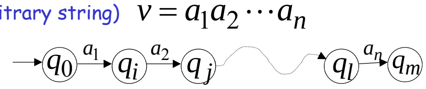
M : 

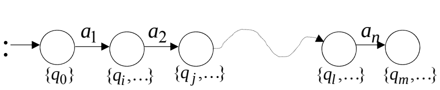
M' : 

$$w \in L(M')$$

More generally, we will show that if in M:

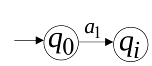
(arbitrary string) $v = a_1 a_2 \cdots a_n$

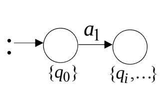
M : 

M' : 

Proof by induction on $|v|$

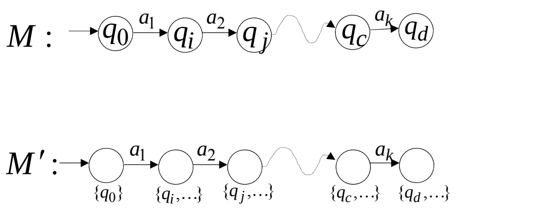
Induction Basis: $v = a_1$

M : 

M' : 

Induction hypothesis: $1 \le |v| \le k$

$$v = a_1 a_2 \cdots a_k$$



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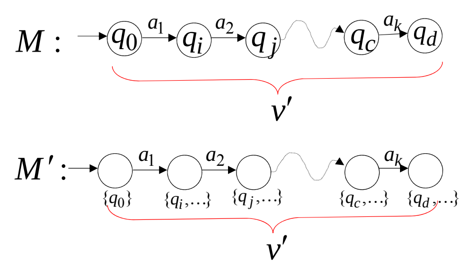
Induction Step: $|v| = k+1$

$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$



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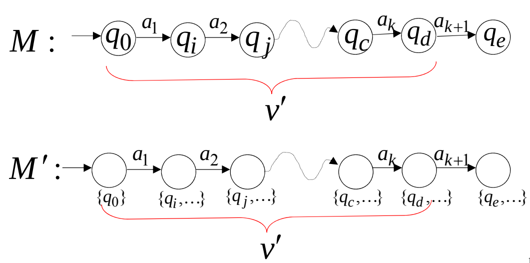
Induction Step: $|v| = k+1$

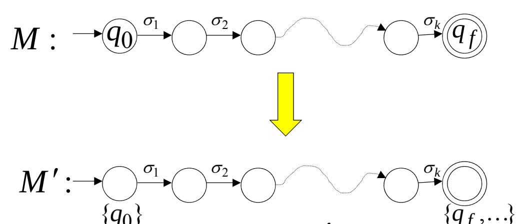
$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$



111

Therefore if $w \in L(M)$

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



$$w \in L(M')$$

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We have shown: $L(M) \subseteq L(M')$

We also need to show: $L(M) \supseteq L(M')$

(proof is similar)

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