A Universal Turing Machine

class 17

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A limitation of Turing Machines:

Turing Machines are "hardwired"

they execute only one program

Real Computers are re-programmable

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Solution: Universal Turing Machine

Attributes:

- Reprogrammable machine
- Simulates any other Turing Machine

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Universal Turing Machine simulates any other Turing Machine M

Input of Universal Turing Machine:

Description of transitions of M

Initial tape contents of M

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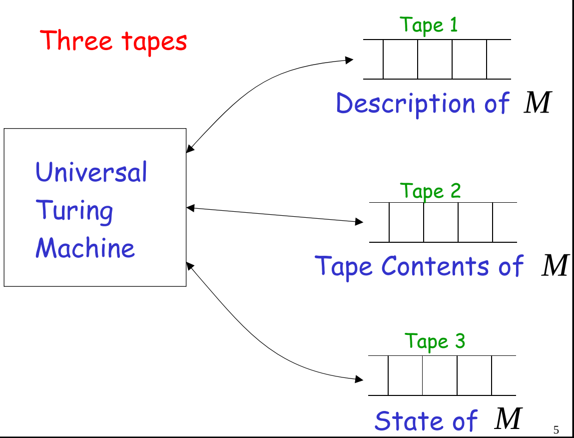
Three tapes



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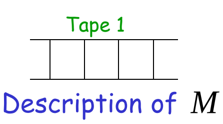

We describe Turing machine M as a string of symbols:

We encode M as a string of symbols

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Alphabet Encoding

Symbols:	a	b	c	d	$\cdots$
Encoding:	1	11	111	1111	

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State Encoding

States:	q_1	q_2	q_3	q_4	$\cdots$
Encoding:	1	11	111	1111	

Head Move Encoding

Move:	L	R
Encoding:	1	11

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Transition Encoding

Transition: $\delta(q_1, a) = (q_2, b, L)$

Encoding: $1\,0\,1\,0\,11\,0\,1\,0\,1$

separator

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Machine Encoding

Transitions:

$\delta(q_1, a) = (q_2, b, L)$ $\quad$ $\delta(q_2, b) = (q_3, c, R)$

Encoding:

$1\,0\,1\,0\,11\,0\,11\,0\,1 \;\; 00 \;\; 11\,0\,11\,0\,111\,0\,111\,0\,11$

separator

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Tape 1 contents of Universal Turing Machine:

encoding of the simulated machine M
as a binary string of 0's and 1's

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A Turing Machine is described
with a binary string of 0's and 1's

Therefore:

The set of Turing machines forms a language:

each string of the language is
the binary encoding of a Turing Machine

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Language of Turing Machines

L = { 010100101, (Turing Machine 1)

00100100101111, (Turing Machine 2)

111010011110010101, ...

...... }

Countable Sets

Infinite sets are either: Countable

or

Uncountable

Countable set:

Any finite set

or

Any *Countably infinite* set:

There is a one to one correspondence
between
elements of the set
and
Natural numbers

Example: The set of even integers is countable

Even integers: $0, 2, 4, 6, \ldots$

Correspondence: $\downarrow \; \downarrow \; \downarrow \; \downarrow$

Positive integers: $1, 2, 3, 4, \ldots$

$2n$ corresponds to $n+1$

Example: The set of rational numbers is countable

Rational numbers: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$

Naïve Proof

Rational numbers: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Correspondence: $\downarrow \quad \downarrow \quad \downarrow$

Positive integers: $1, 2, 3, \ldots$

Doesn't work:

we will never count numbers with nominator 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$

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Better Approach

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \ldots & \\
\frac{3}{1} & \frac{3}{2} & \ldots & & \\
\frac{4}{1} & \ldots & & &
\end{array}
$$

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$$
\begin{array}{ccccc}
\frac{1}{1} & \rightarrow \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \ldots & \\
\frac{3}{1} & \frac{3}{2} & \ldots & & \\
\frac{4}{1} & \ldots & & &
\end{array}
$$

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$$
\begin{array}{ccccc}
\frac{1}{1} & \rightarrow \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \ldots & \\
\frac{3}{1} & \frac{3}{2} & \ldots & & \\
\frac{4}{1} & \ldots & & &
\end{array}
$$

22

$$
\begin{array}{ccccc}
\frac{1}{1} & \rightarrow \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \ldots & \\
\frac{3}{1} & \frac{3}{2} & \ldots & & \\
\frac{4}{1} & \ldots & & &
\end{array}
$$

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$$
\begin{array}{ccccc}
\frac{1}{1} & \rightarrow \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \ldots & \\
\frac{3}{1} & \frac{3}{2} & \ldots & & \\
\frac{4}{1} & \ldots & & &
\end{array}
$$

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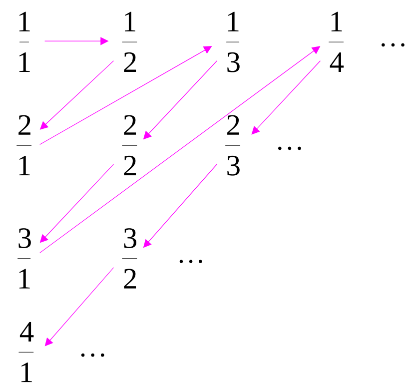

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Rational Numbers: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \ldots$

Correspondence:

Positive Integers: $1, \; 2, \; 3, \; 4, \; 5, \; \ldots$

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We proved:

the set of rational numbers is countable
by describing an **enumeration procedure**

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Definition

Let S be a set of strings

An **enumeration procedure** for S is a
Turing Machine that generates
all strings of S one by one

and

Each string is generated in finite time

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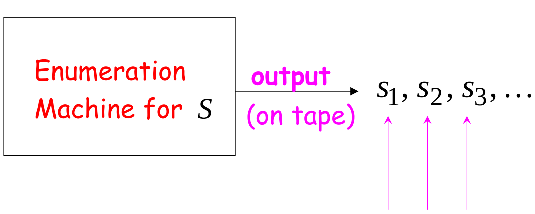
strings $s_1, s_2, s_3, \ldots \in S$



Finite time: $t_1, t_2, t_3, \ldots$

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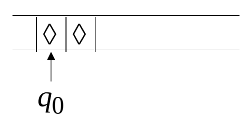
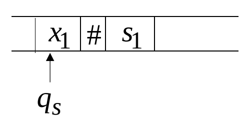
Enumeration Machine

Configuration

Time 0



Time t_1



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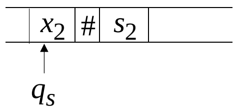
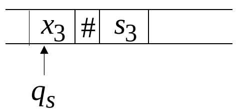
Time t_2



Time t_3



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Observation:

If for a set there is an enumeration procedure, then the set is countable

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Example:

The set of all strings $\{a,b,c\}^+$ is countable

Proof:

We will describe an enumeration procedure

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Naive procedure:

Produce the strings in lexicographic order:

a
aa
aaa
$aaaa$
......

Doesn't work:

strings starting with b will never be produced

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Better procedure: **Proper Order**

1. Produce all strings of length 1
2. Produce all strings of length 2
3. Produce all strings of length 3
4. Produce all strings of length 4

....

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Produce strings in **Proper Order**:

a, b, c — length 1

aa, ab, ac, ba, bb, bc, ca, cb, cc — length 2

aaa, aab, aac, ... — length 3

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Theorem: The set of all Turing Machines is countable

Proof: Any Turing Machine can be encoded with a binary string of 0's and 1's

Find an enumeration procedure for the set of Turing Machine strings

Enumeration Procedure:

Repeat

1. Generate the next binary string of 0's and 1's in proper order

2. Check if the string describes a Turing Machine
   - if **YES:** print string on output tape
   - if **NO:** ignore string

Uncountable Sets

Definition: A set is uncountable if it is not countable

Theorem:

Let S be an infinite countable set

The powerset 2^S of S is uncountable

Proof:

Since S is countable, we can write

$$S = \{s_1, s_2, s_3, \ldots\}$$

Elements of S

Elements of the powerset have the form:

$\{s_1, s_3\}$

$\{s_5, s_7, s_9, s_{10}\}$

……

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We encode each element of the power set with a binary string of 0's and 1's

Powerset element	Encoding s_1	s_2	s_3	s_4	$\cdots$
$\{s_1\}$	1	0	0	0	$\cdots$
$\{s_2, s_3\}$	0	1	1	0	$\cdots$
$\{s_1, s_3, s_4\}$	1	0	1	1	$\cdots$

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Let's assume (for contradiction) that the powerset is countable.

Then: we can enumerate the elements of the powerset

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Powerset element	Encoding				
t_1	1	0	0	0	0 $\cdots$
t_2	1	1	0	0	0 $\cdots$
t_3	1	1	0	1	0 $\cdots$
t_4	1	1	0	0	1 $\cdots$
$\cdots$					

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Take the powerset element whose bits are the complements in the diagonal

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t_1	1	0	0	0	0 $\cdots$
t_2	1	1	0	0	0 $\cdots$
t_3	1	1	0	1	0 $\cdots$
t_4	1	1	0	0	1 $\cdots$

New element: $0011\ldots$

(birary complement of diagonal)

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The new element must be some t_i of the powerset

However, that's impossible:

from definition of t_i

the i-th bit of t_i must be the complement of itself

Contradiction!!!

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Since we have a contradiction:

The powerset 2^S of S is uncountable

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An Application: Languages

Example Alphabet : $\{a,b\}$

The set of all Strings:

$$S = \{a,b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

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Example Alphabet : $\{a,b\}$

The set of all Strings:

$$S = \{a,b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

A language is a subset of S :

$$L = \{aa, ab, aab\}$$

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Example Alphabet : $\{a,b\}$

The set of all Strings:

$$S = \{a,b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

infinite and countable

The powerset of S contains all languages:

$$2^S = \{\underbrace{\{\lambda\}}_{L_1}, \underbrace{\{a\}}_{L_2}, \underbrace{\{a,b\}}_{L_3} \underbrace{\{aa, ab, aab\}}_{L_4}, \underbrace{\ldots}_{\cdots}\}$$

uncountable

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Languages: uncountable

$$\begin{array}{cccccc} L_1 & L_2 & L_3 & \cdots & L_k & \cdots \\ \downarrow & \downarrow & \downarrow & & \downarrow & \\ M_1 & M_2 & M_3 & & ? & \end{array}$$

Turing machines: countable

There are more languages than Turing Machines

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Conclusion:

There are some languages not accepted by Turing Machines

(These languages cannot be described by algorithms)

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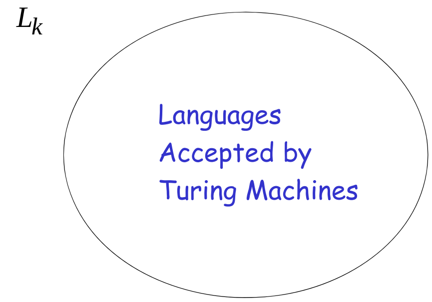
Languages not accepted by Turing Machines



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