Turing Machines

class 15

1

The Language Hierarchy

$a^n b^n c^n$ **?** ww **?**

Context-Free Languages

$a^n b^n$ ww^R

Regular Languages

$a*$ $a*b*$

2

Languages accepted by

Turing Machines

$a^n b^n c^n$ ww

Context-Free Languages

$a^n b^n$ ww^R

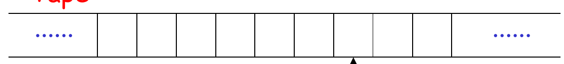
Regular Languages

$a*$ $a*b*$

3

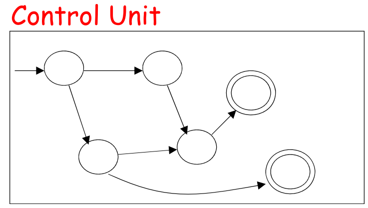
A Turing Machine

Tape



Read-Write head

Control Unit



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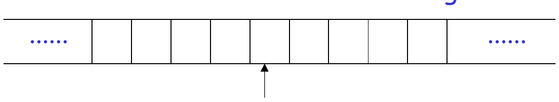
The Tape

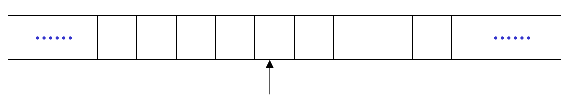
No boundaries -- infinite length



Read-Write head

The head moves Left or Right

5



Read-Write head

The head at each time step:

1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

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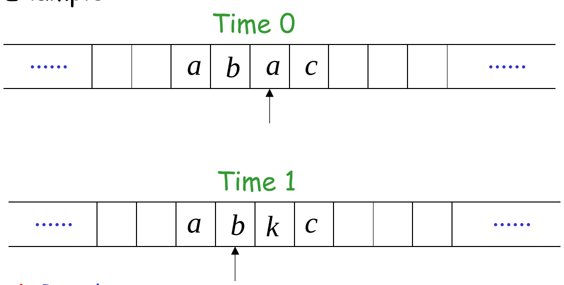
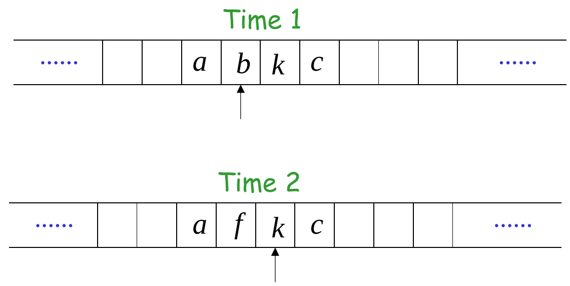
Example:



1. Reads a
2. Writes k
3. Moves Left

7



1. Reads b
2. Writes f
3. Moves Right

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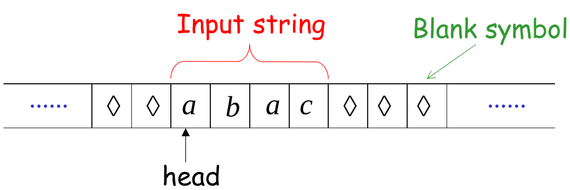
The Input String



Head starts at the leftmost position
of the input string

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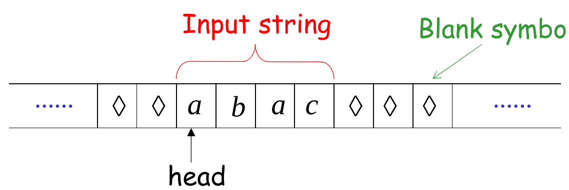

Remark: the input string is never empty

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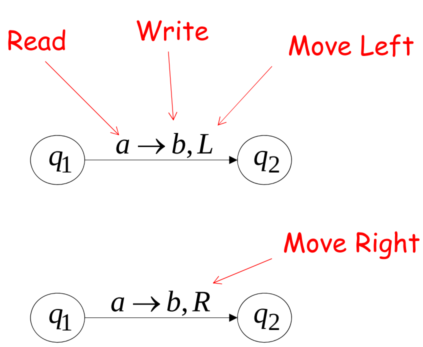
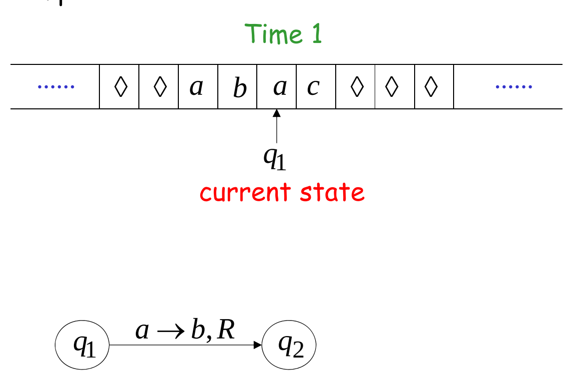
States & Transitions

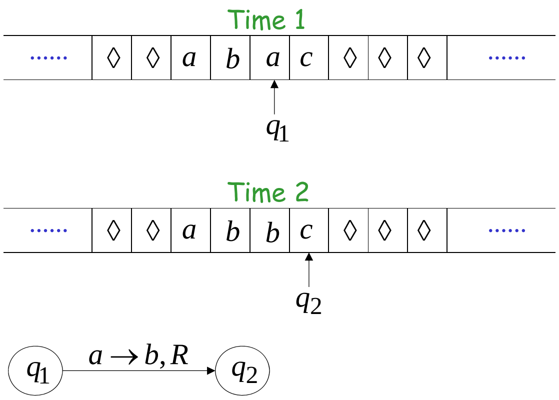


11

Example:



12



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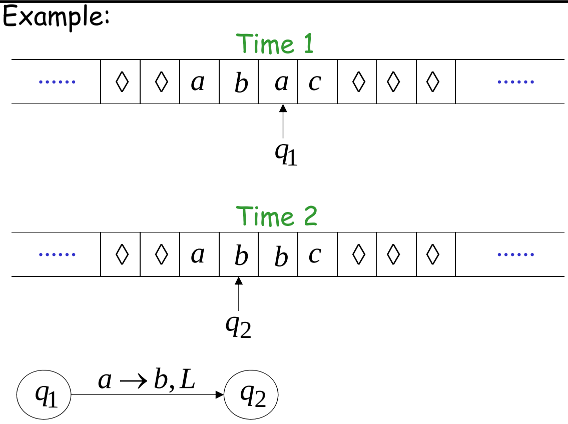
Example:



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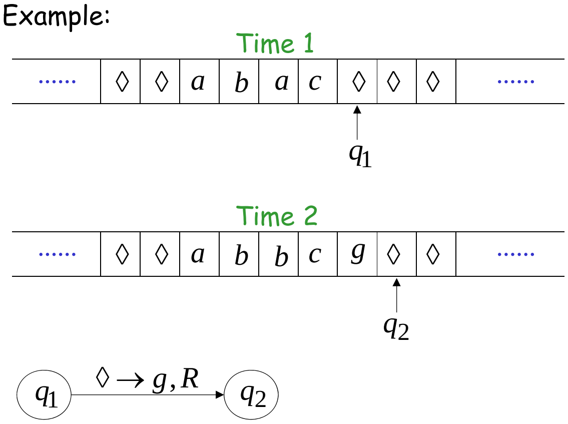
Example:



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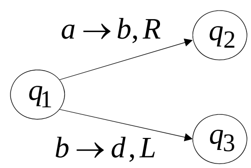
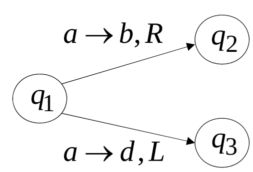
Determinism

Turing Machines are deterministic

Allowed



Not Allowed



No lambda transitions allowed

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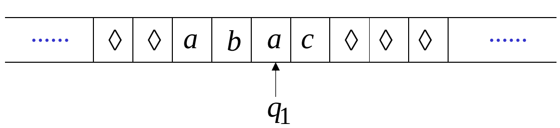
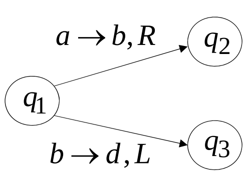
Partial Transition Function

Example:





Allowed:

No transition
for input symbol c

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Halting

The machine ***halts*** if there are
no possible transitions to follow

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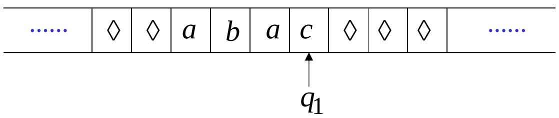
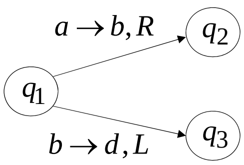
Example:





No possible transition

HALT!!!

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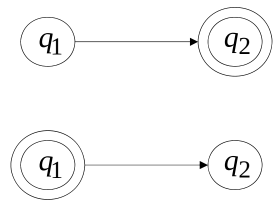
Final States



Allowed

Not Allowed

- Final states have no outgoing transitions
- In a final state the machine halts

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Acceptance

Accept Input → If machine halts in a final state

Reject Input → If machine halts in a non-final state
or
If machine enters an *infinite loop*

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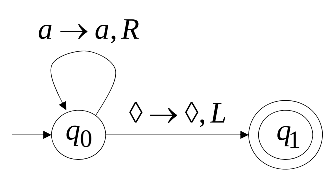
Turing Machine Example

A Turing machine that accepts the language:

$aa*$



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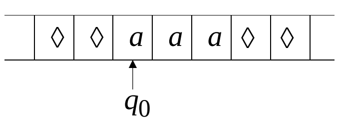
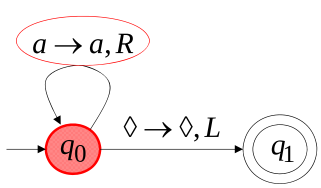
Time 0





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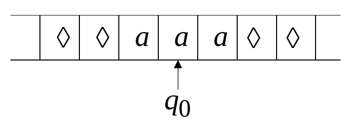
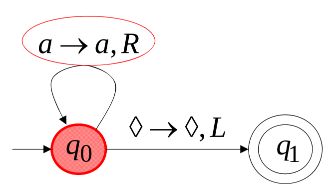
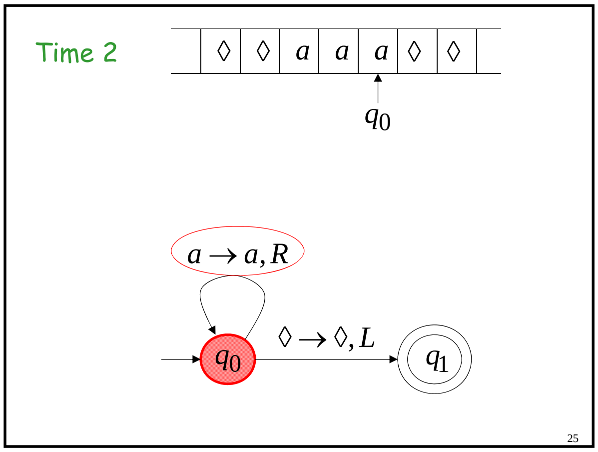
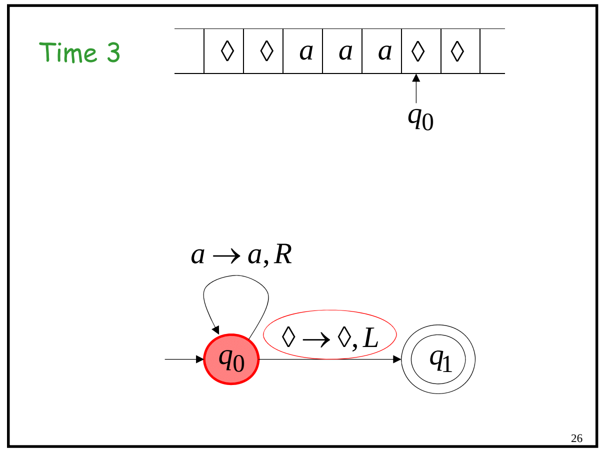
Time 1

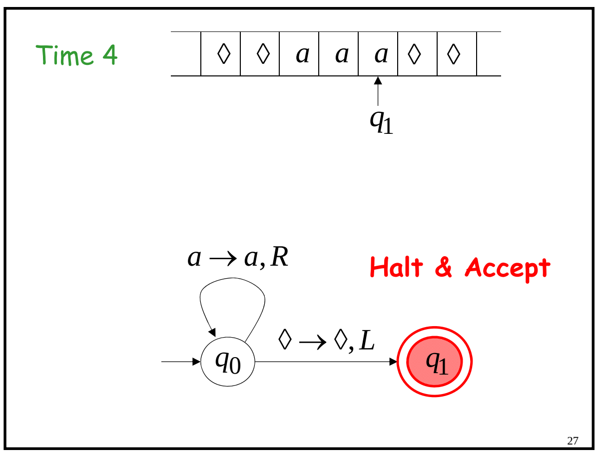


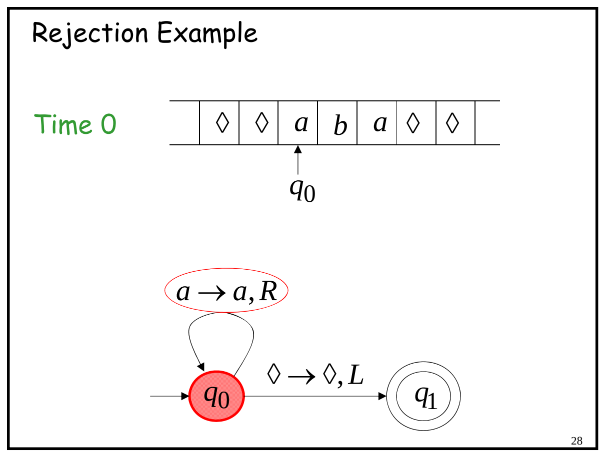


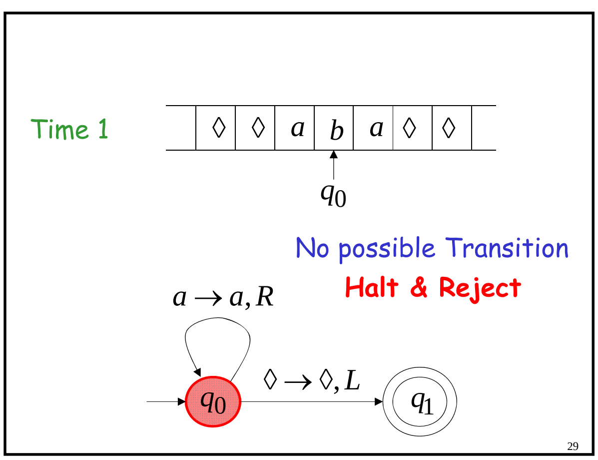
24

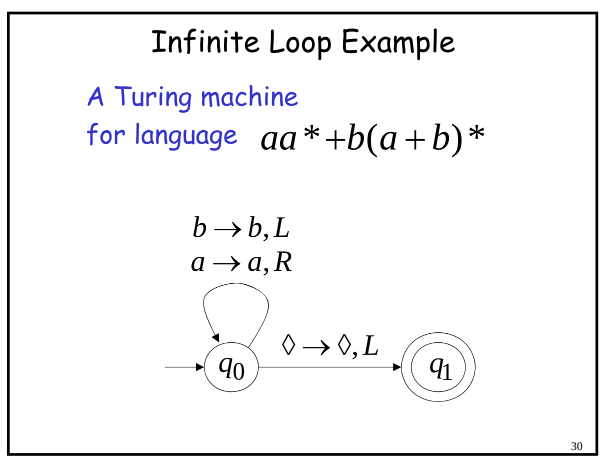

Time 2
◊ ◊ a a a ◊ ◊
q_0
$a \to a, R$
$◊ \to ◊, L$
q_0
q_1
25

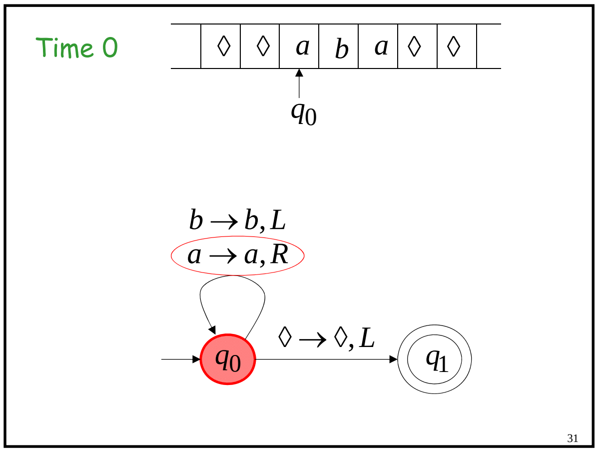

Time 3
◊ ◊ a a a ◊ ◊
q_0
$a \to a, R$
$◊ \to ◊, L$
q_0
q_1
26

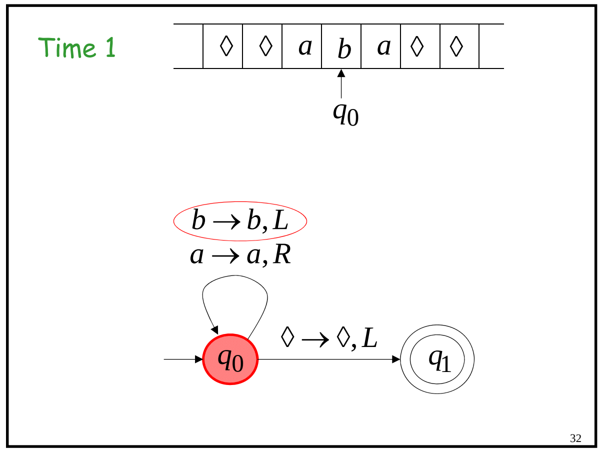

Time 4
◊ ◊ a a a ◊ ◊
q_1
$a \to a, R$
Halt & Accept
$◊ \to ◊, L$
q_0
q_1
27

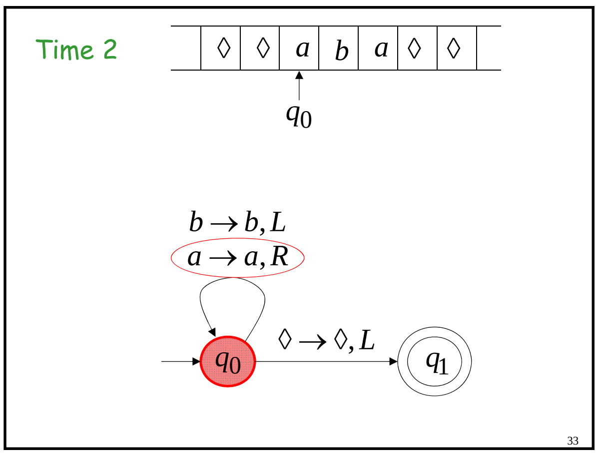

Rejection Example
Time 0
◊ ◊ a b a ◊ ◊
q_0
$a \to a, R$
$◊ \to ◊, L$
q_0
q_1
28

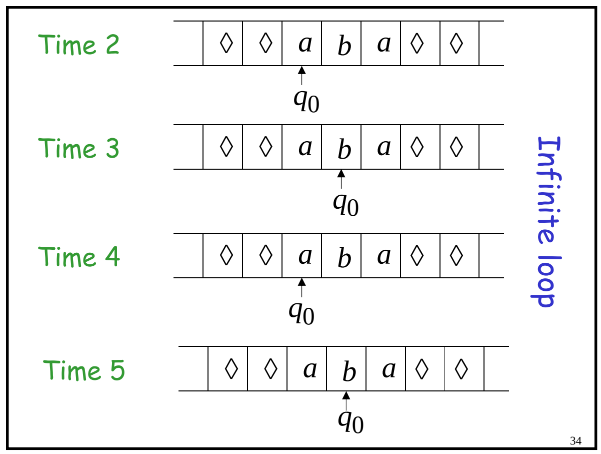

Time 1
◊ ◊ a b a ◊ ◊
q_0
No possible Transition
Halt & Reject
$a \to a, R$
$◊ \to ◊, L$
q_0
q_1
29


Infinite Loop Example
A Turing machine
for language $aa^* + b(a+b)^*$
$b \to b, L$
$a \to a, R$
$◊ \to ◊, L$
q_0
q_1
30









Because of the infinite loop:

- The final state cannot be reached
- The machine never halts
- The input is not accepted

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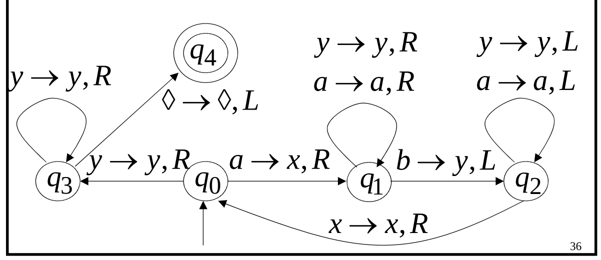
Another Turing Machine Example

Turing machine for the language $\{a^n b^n\}$



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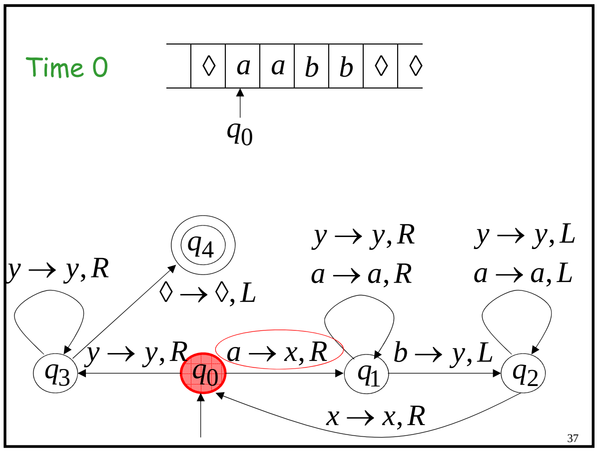
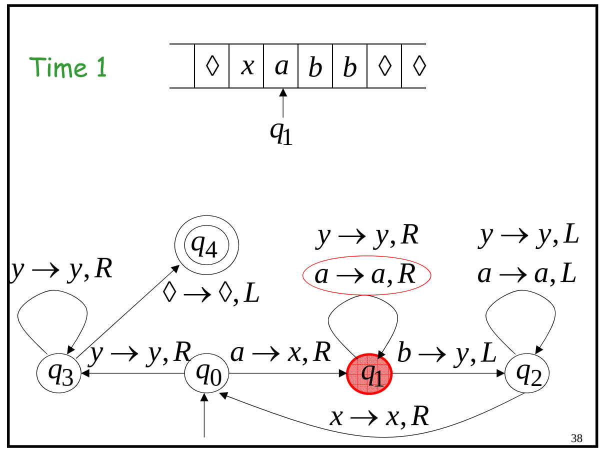
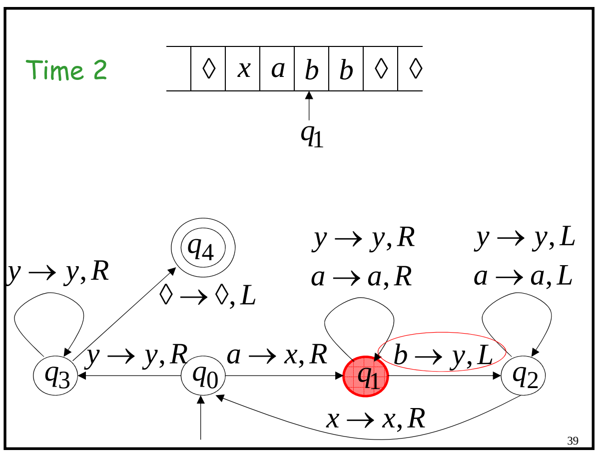
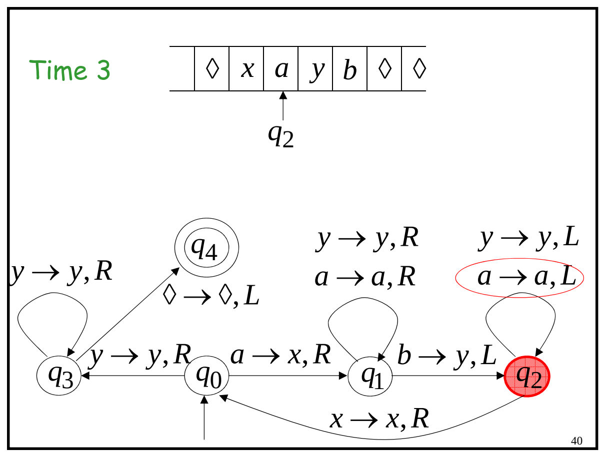
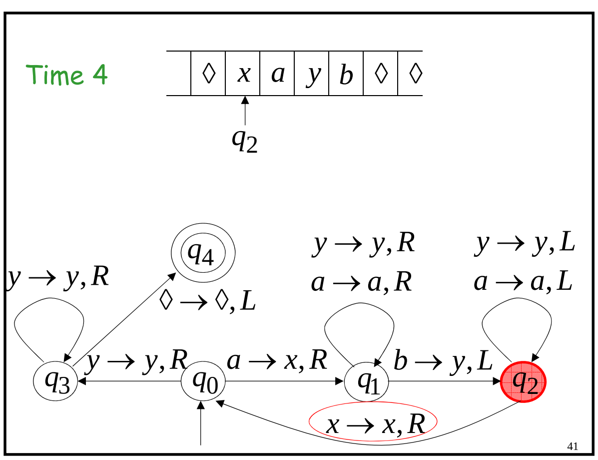
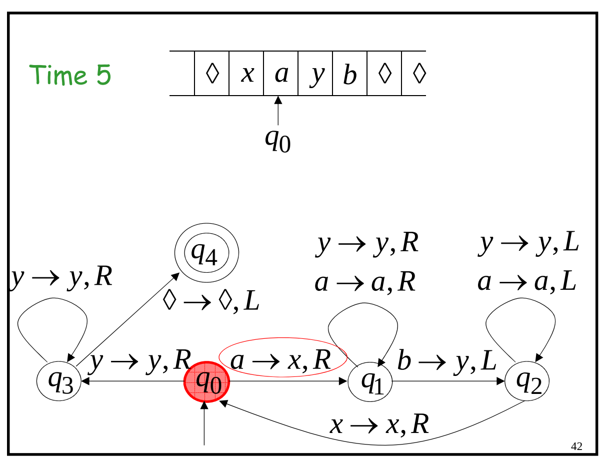

Time 0
◊ a a b b ◊ ◊
q0
q4
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
q3 q0 q1 q2
x → x, R
37

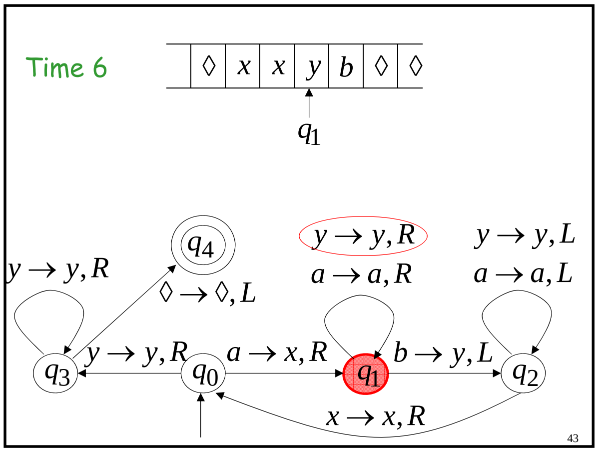

Time 1
◊ x a b b ◊ ◊
q1
q4
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
q3 q0 q1 q2
x → x, R
38

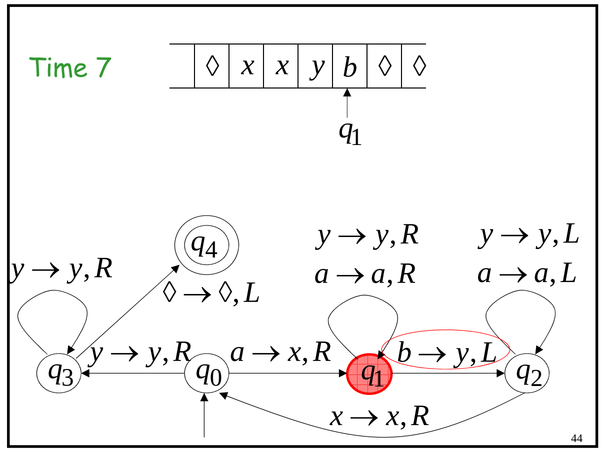

Time 2
◊ x a b b ◊ ◊
q1
q4
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
q3 q0 q1 q2
x → x, R
39

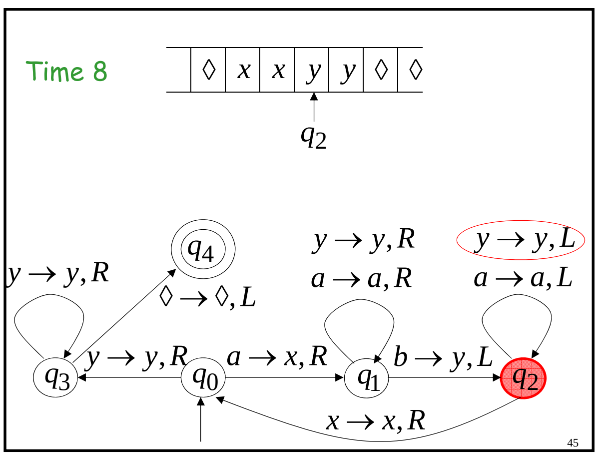

Time 3
◊ x a y b ◊ ◊
q2
q4
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
q3 q0 q1 q2
x → x, R
40

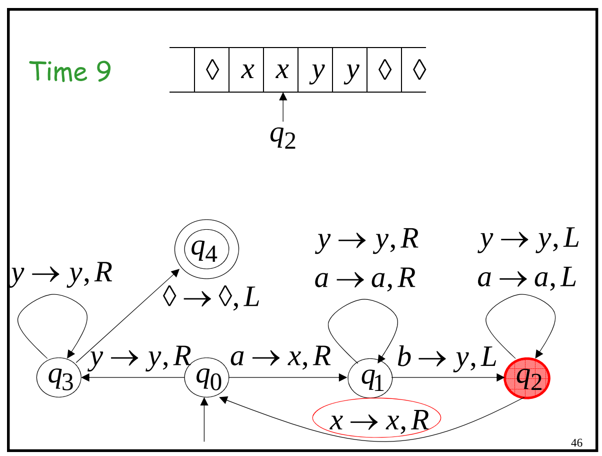

Time 4
◊ x a y b ◊ ◊
q2
q4
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
q3 q0 q1 q2
x → x, R
41

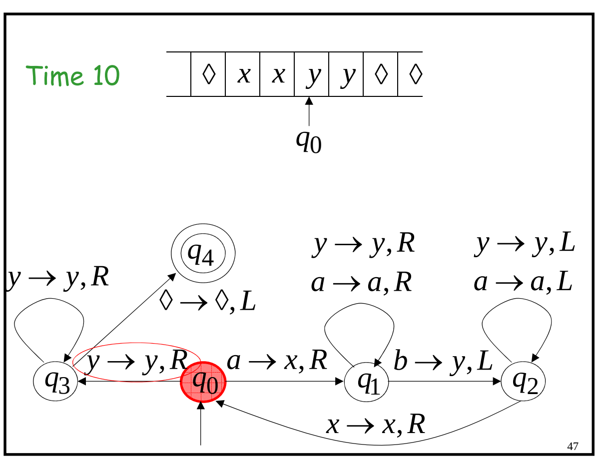

Time 5
◊ x a y b ◊ ◊
q0
q4
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
q3 q0 q1 q2
x → x, R
42

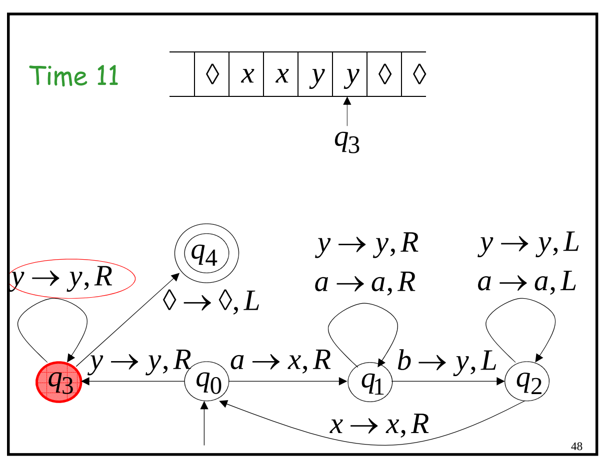

Time 6
◊ x x y b ◊ ◊
q_1
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
43


Time 7
◊ x x y b ◊ ◊
q_1
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
44

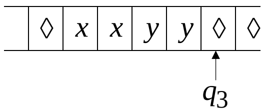

Time 8
◊ x x y y ◊ ◊
q_2
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
45

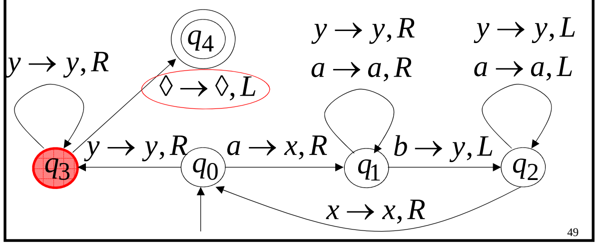

Time 9
◊ x x y y ◊ ◊
q_2
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
46


Time 10
◊ x x y y ◊ ◊
q_0
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
47

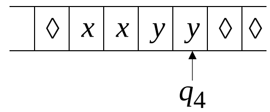

Time 11
◊ x x y y ◊ ◊
q_3
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
48

Time 12

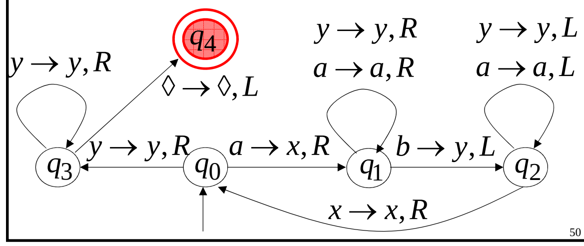




Time 13



Halt & Accept



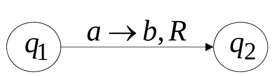
Observation:

If we modify the
machine for the language $\{a^n b^n\}$

we can easily construct
a machine for the language $\{a^n b^n c^n\}$

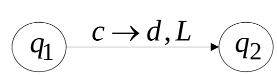
Formal Definitions for Turing Machines

Transition Function



$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function



$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

States, Input alphabet, Tape alphabet, Transition function, Initial state, blank, Final states

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Configuration

$\Diamond$	$\Diamond$	c	a	b	a	$\Diamond$	$\Diamond$
				$\uparrow q_1$			

Instantaneous description: $ca\; q_1\; ba$

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Time 4

$\Diamond$	x	a	y	b	$\Diamond$	$\Diamond$
	$\uparrow q_2$					

Time 5

$\Diamond$	x	a	y	b	$\Diamond$	$\Diamond$
		$\uparrow q_0$				

A Move: $q_2\; xayb \succ x\; q_0\; ayb$

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Time 4

$\Diamond$	x	a	y	b	$\Diamond$	$\Diamond$
	$\uparrow q_2$					

Time 5

$\Diamond$	x	a	y	b	$\Diamond$	$\Diamond$
		$\uparrow q_0$				

Time 6

$\Diamond$	x	x	y	b	$\Diamond$	$\Diamond$
			$\uparrow q_1$			

Time 7

$\Diamond$	x	x	y	b	$\Diamond$	$\Diamond$
				$\uparrow q_1$		

$$q_2\; xayb \succ x\; q_0\; ayb \succ xx\; q_1\; yb \succ xxy\; q_1\; b$$

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$$q_2\; xayb \succ x\; q_0\; ayb \succ xx\; q_1\; yb \succ xxy\; q_1\; b$$

Equivalent notation: $q_2\; xayb \overset{*}{\succ} xxy\; q_1\; b$

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Initial configuration: $q_0\; w$

Input string

w

$\Diamond$	a	a	b	b	$\Diamond$	$\Diamond$
	$\uparrow q_0$					

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The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \ q_0 \, w \overset{*}{\succ} x_1 \, q_f \, x_2\}$$

Initial state   Final state

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Standard Turing Machine

The machine we described is the standard:

- Deterministic
- Infinite tape in both directions
- Tape is the input/output file

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Computing Functions with Turing Machines

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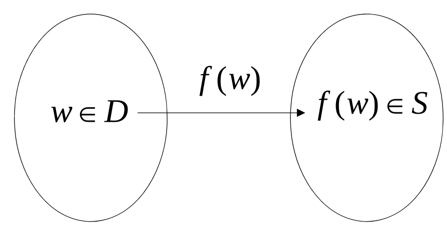
A function $f(w)$ has:

Domain: D Result Region: S



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A function may have many parameters:

Example: Addition function

$$f(x, y) = x + y$$

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Integer Domain

Decimal: 5

Binary: 101

Unary: 11111

We prefer **unary** representation:

easier to manipulate with Turing machines

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Definition:

A function f is computable if there is a Turing Machine M such that:

Initial configuration

$\Diamond$	w	$\Diamond$

q_0 initial state

Final configuration

$\Diamond$	$f(w)$	$\Diamond$

q_f final state

For all $w \in D$ Domain

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In other words:

A function f is computable if there is a Turing Machine M such that:

$$q_0\, w \overset{*}{\succ} q_f\, f(w)$$

Initial Configuration

Final Configuration

For all $w \in D$ Domain

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Example

The function $f(x, y) = x + y$ is computable

x, y are integers

Turing Machine:

Input string: $x0y$ unary

Output string: $xy0$ unary

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Start

$$\Diamond\ \overbrace{1\ 1\ \cdots\ 1}^{x}\ 0\ \overbrace{1\ \cdots\ 1}^{y}\ \Diamond$$

q_0

initial state

The 0 is the delimiter that separates the two numbers

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Start

$$\Diamond\ \overbrace{1\ 1\ \cdots\ 1}^{x}\ 0\ \overbrace{1\ \cdots\ 1}^{y}\ \Diamond$$

q_0 initial state

Finish

$$\Diamond\ \overbrace{1\ 1\ \cdots\ 1\ 1}^{x+y}\ 0\ \Diamond$$

q_f final state

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The 0 helps when we use the result for other operations

Finish

$$\Diamond\ \overbrace{1\ 1\ \cdots\ 1\ 1}^{x+y}\ 0\ \Diamond$$

q_f final state

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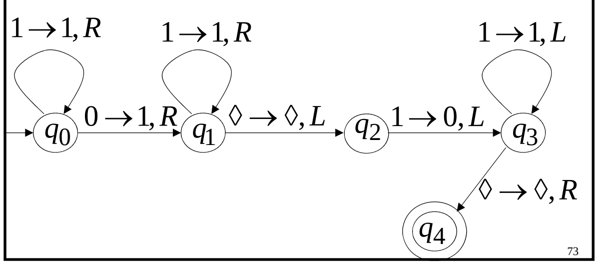
Turing machine for function $f(x,y)=x+y$



Execution Example:

$x = 11$ (2)

$y = 11$ (2)

Time 0

	x			y	
◊	1	1	0	1	1	◊

q_0

Final Result

$x+y$

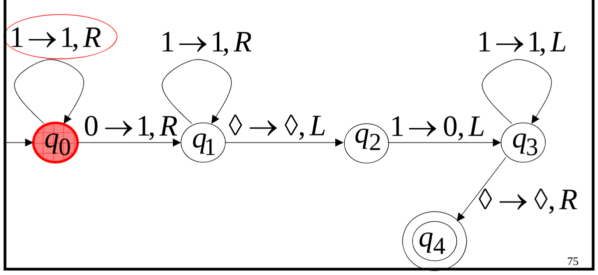
◊	1	1	1	1	0	◊

q_4

Time 0

◊	1	1	0	1	1	◊

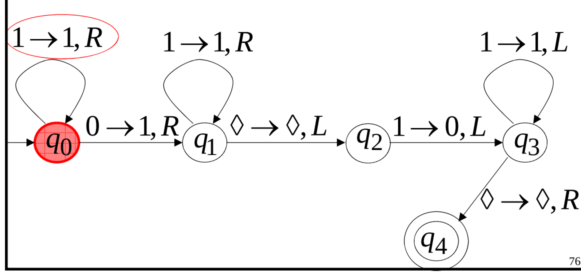
q_0



Time 1

◊	1	1	0	1	1	◊

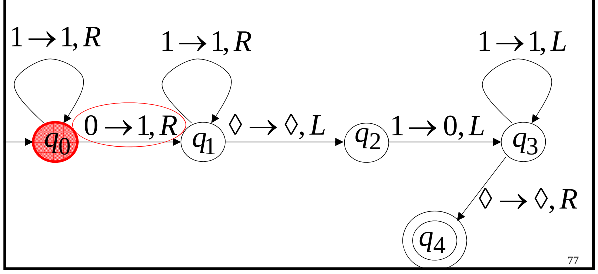
q_0



Time 2

◊	1	1	0	1	1	◊

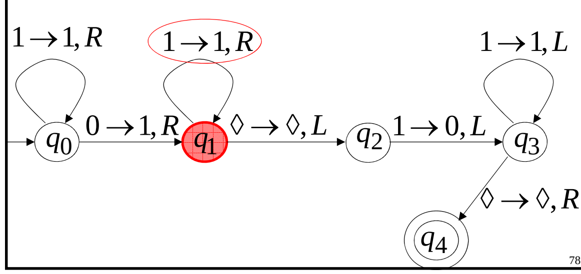
q_0

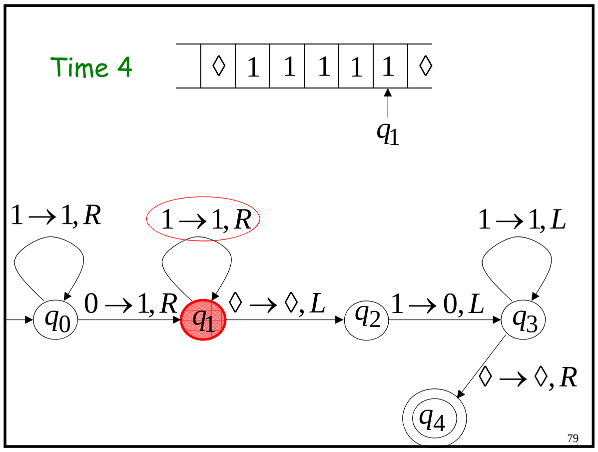


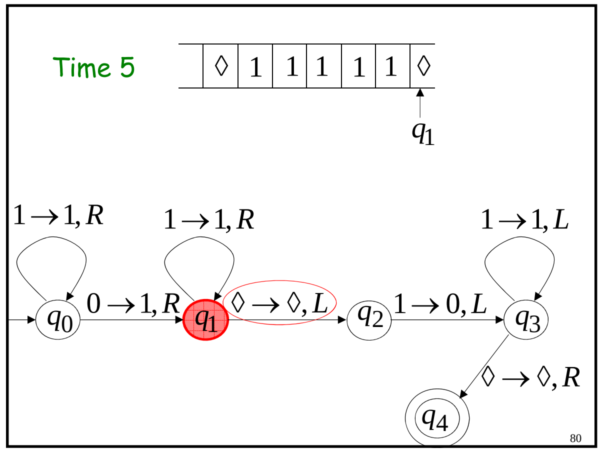
Time 3

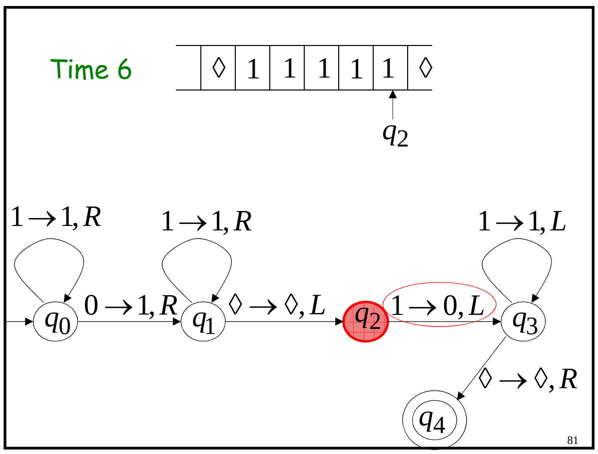
◊	1	1	1	1	1	◊

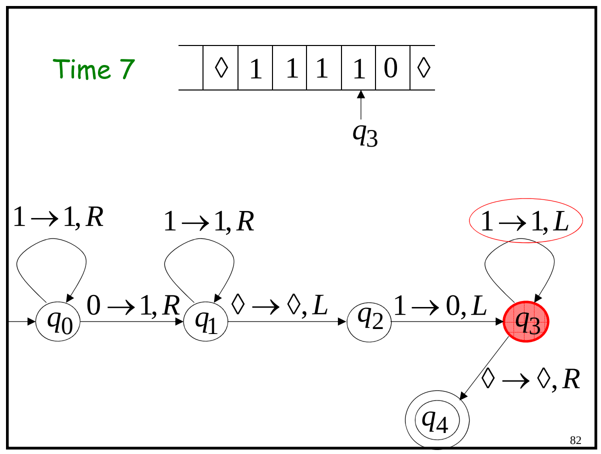
q_1

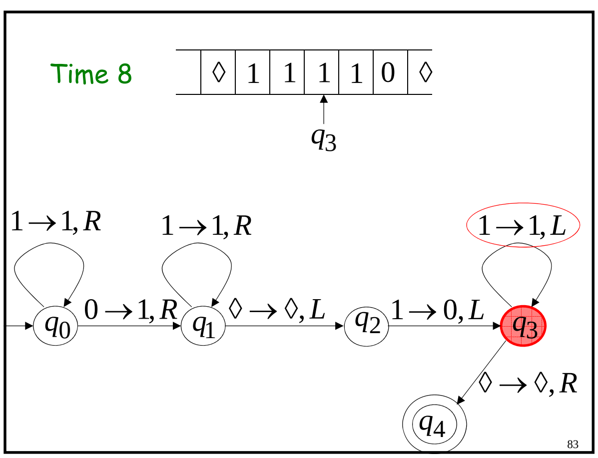


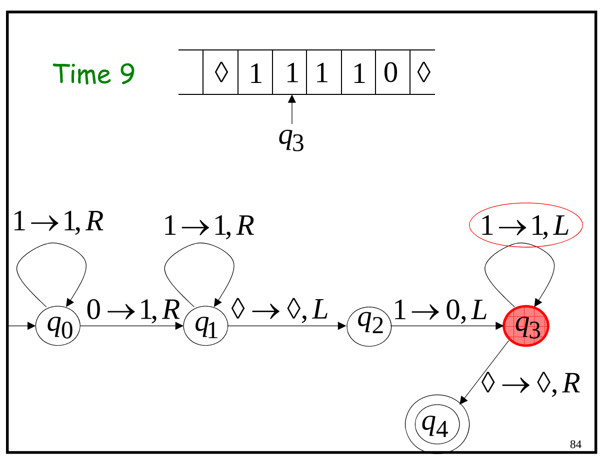

Time 4
◊ 1 1 1 1 1 ◊
q_1
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0 $0 \rightarrow 1, R$ q_1 $◊ \rightarrow ◊, L$ q_2 $1 \rightarrow 0, L$ q_3
$◊ \rightarrow ◊, R$
q_4
79


Time 5
◊ 1 1 1 1 1 ◊
q_1
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0 $0 \rightarrow 1, R$ q_1 $◊ \rightarrow ◊, L$ q_2 $1 \rightarrow 0, L$ q_3
$◊ \rightarrow ◊, R$
q_4
80

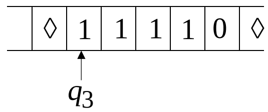

Time 6
◊ 1 1 1 1 1 ◊
q_2
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0 $0 \rightarrow 1, R$ q_1 $◊ \rightarrow ◊, L$ q_2 $1 \rightarrow 0, L$ q_3
$◊ \rightarrow ◊, R$
q_4
81

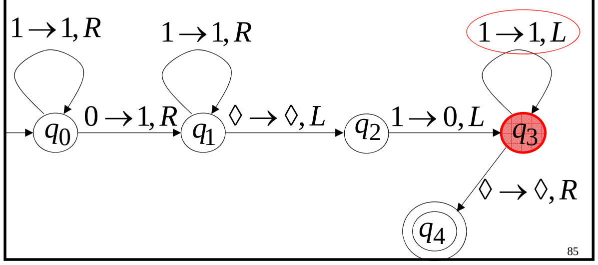

Time 7
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0 $0 \rightarrow 1, R$ q_1 $◊ \rightarrow ◊, L$ q_2 $1 \rightarrow 0, L$ q_3
$◊ \rightarrow ◊, R$
q_4
82


Time 8
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0 $0 \rightarrow 1, R$ q_1 $◊ \rightarrow ◊, L$ q_2 $1 \rightarrow 0, L$ q_3
$◊ \rightarrow ◊, R$
q_4
83


Time 9
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0 $0 \rightarrow 1, R$ q_1 $◊ \rightarrow ◊, L$ q_2 $1 \rightarrow 0, L$ q_3
$◊ \rightarrow ◊, R$
q_4
84

Time 10





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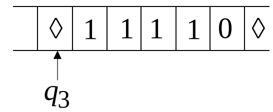
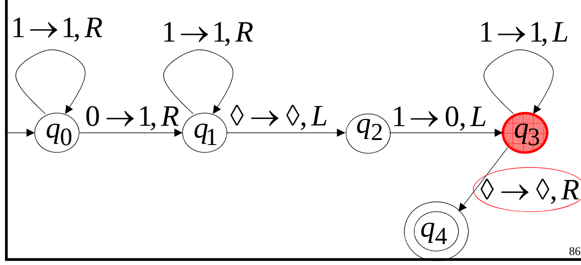
Time 11





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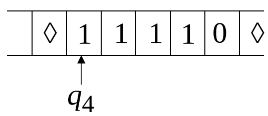
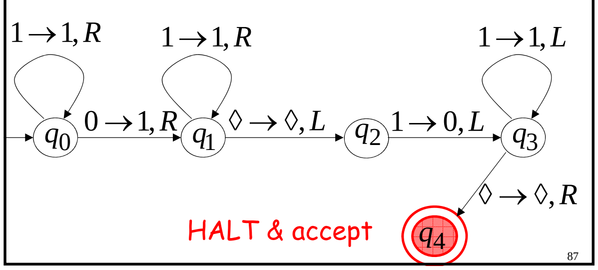
Time 12





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Another Example

The function $f(x) = 2x$ is computable

x is integer

Turing Machine:

Input string: x unary

Output string: xx unary

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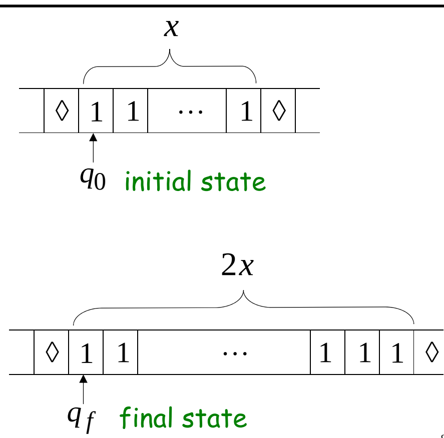
Start

Finish



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Turing Machine Pseudocode for $f(x) = 2x$

- Replace every 1 with $
- Repeat:
  - Find rightmost $, replace it with 1
  - Go to right end, insert 1

Until no more $ remain

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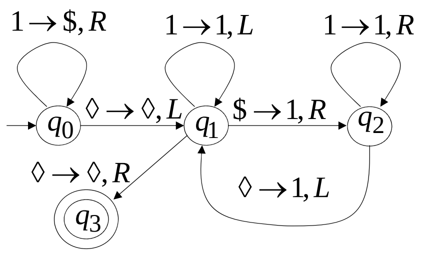
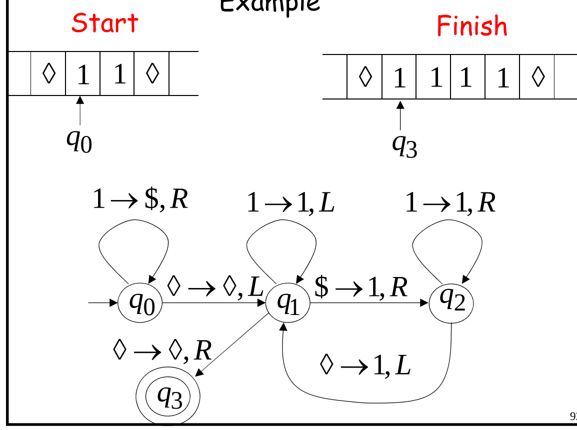
Turning Machine for $f(x) = 2x$



Example



Another Example

The function is computable

$$f(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Turing Machine for

$$f(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

- Repeat

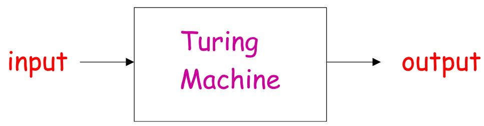
  Match a 1 from x with a 1 from y

  Until all of x or y is matched

- If a 1 from x is not matched
  erase tape, write 1 $(x > y)$
  else
  erase tape, write 0 $(x \leq y)$

Combining Turing Machines

Block Diagram



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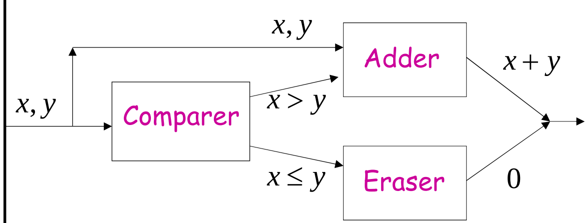
Example:

$$f(x,y) = \begin{cases} x+y & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$



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