Other Models of Computation

Models of computation:

- Turing Machines
- Recursive Functions
- Post Systems
- Rewriting Systems

Church's Thesis:

All models of computation are equivalent

Turing's Thesis:

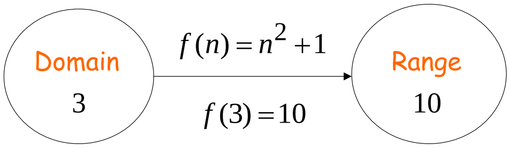
A computation is mechanical if and only if it can be performed by a Turing Machine

Church's and Turing's Thesis are similar:

Church-Turing Thesis

Recursive Functions

An example function:



We need a way to define functions

We need a set of basic functions

Basic Primitive Recursive Functions

Zero function: $z(x) = 0$

Successor function: $s(x) = x + 1$

Projection functions: $p_1(x_1, x_2) = x_1$

$p_2(x_1, x_2) = x_2$

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Building complicated functions:

Composition: $f(x, y) = h(g_1(x, y), g_2(x, y))$

Primitive Recursion:

$$f(x, 0) = g_1(x)$$

$$f(x, y+1) = h(g_2(x, y), f(x, y))$$

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Any function built from
the basic primitive recursive functions
is called:

Primitive Recursive Function

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A Primitive Recursive Function: $add(x, y)$

$add(x, 0) = x$ (projection)

$add(x, y+1) = s(add(x, y))$

(successor function)

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$$\begin{aligned} add(3,2) &= s(add(3,1)) \\ &= s(s(add(3,0))) \\ &= s(s(3)) \\ &= s(4) \\ &= 5 \end{aligned}$$

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Another Primitive Recursive Function:

$mult(x, y)$

$mult(x, 0) = 0$

$mult(x, y+1) = add(x, mult(x, y))$

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Theorem:

The set of primitive recursive functions is countable

Proof:

Each primitive recursive function can be encoded as a string

Enumerate all strings in proper order

Check if a string is a function

Theorem

there is a function that is not primitive recursive

Proof:

Enumerate the primitive recursive functions

$$f_1, f_2, f_3, \ldots$$

Define function $g(i) = f_i(i) + 1$

g differs from every f_i

g is not primitive recursive

END OF PROOF

A specific function that is <u>not</u> Primitive Recursive:

Ackermann's function:

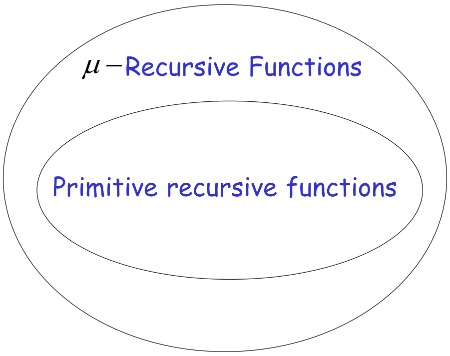
$$A(0, y) = y + 1$$
$$A(x, 0) = A(x - 1, 1)$$
$$A(x, y + 1) = A(x - 1, A(x, y))$$

Grows very fast, faster than any primitive recursive function

$\mu-$Recursive Functions

$$\mu y(g(x, y)) = \text{smallest } y \text{ such that } g(x, y) = 0$$

Accerman's function is a $\mu-$Recursive Function



Post Systems

- Have Axioms
- Have Productions

Very similar with unrestricted grammars

Example: Unary Addition

Axiom: $1+1=11$

Productions:

$$V_1+V_2=V_3 \rightarrow V_1 1+V_2=V_3 1$$
$$V_1+V_2=V_3 \rightarrow V_1+V_2 1=V_3 1$$

A production:

$$V_1+V_2=V_3 \rightarrow V_1 1+V_2=V_3 1$$

$$1+1=11 \Rightarrow 11+1=111 \Rightarrow 11+11=1111$$

$$V_1+V_2=V_3 \rightarrow V_1+V_2 1=V_3 1$$

Post systems are good for
proving mathematical statements
from a set of Axioms

Theorem:
A language is recursively enumerable
if and only if
a Post system generates it

Rewriting Systems

They convert one string to another

- Matrix Grammars
- Markov Algorithms
- Lindenmayer-Systems

Very similar to unrestricted grammars

Matrix Grammars

Example: $P_1: \; S \to S_1S_2$

$P_2: \; S_1 \to aS_1, \; S_2 \to bS_2c$

$P_3: \; S_1 \to \lambda, \; S_2 \to \lambda$

Derivation:

$$S \Rightarrow S_1S_2 \Rightarrow aS_1bS_2c \Rightarrow aaS_1bbS_2cc \Rightarrow aabbcc$$

A set of productions is applied simultaneously

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$P_1: \; S \to S_1S_2$

$P_2: \; S_1 \to aS_1, \; S_2 \to bS_2c$

$P_3: \; S_1 \to \lambda, \; S_2 \to \lambda$

$$L = \{a^n b^n c^n : n \ge 0\}$$

Theorem:

A language is recursively enumerable
if and only if
a Matrix grammar generates it

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Markov Algorithms

Grammars that produce λ

Example: $ab \to S$

$aSb \to S$

$S \to .\lambda$

Derivation:

$$aaabbb \Rightarrow aaSbb \Rightarrow aSb \Rightarrow S \Rightarrow \lambda$$

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$ab \to S$

$aSb \to S$

$S \to .\lambda$

$$L = \{a^n b^n : n \ge 0\}$$

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In general: $L = \{w : \; w \overset{*}{\Rightarrow} \lambda\}$

Theorem:

A language is recursively enumerable
if and only if
a Markov algorithm generates it

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Lindenmayer-Systems

They are parallel rewriting systems

Example: $a \to aa$

Derivation: $a \Rightarrow aa \Rightarrow aaaa \Rightarrow aaaaaaaa$

$$L = \{a^{2^n} : n \ge 0\}$$

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Lindenmayer-Systems are not general
As recursively enumerable languages

Extended Lindenmayer-Systems: $(x, a, y) \rightarrow u$

context

Theorem:
A language is recursively enumerable
if and only if an
Extended Lindenmayer-System generates it

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Computational Complexity

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Time Complexity: The number of steps during a computation

Space Complexity: Space used during a computation

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Time Complexity

- We use a multitape Turing machine
- We count the number of steps until a string is accepted
- We use the O(k) notation

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Example: $L = \{a^n b^n : n \geq 0\}$

Algorithm to accept a string w :

- Use a two-tape Turing machine
- Copy the a on the second tape
- Compare the a and b

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$L = \{a^n b^n : n \geq 0\}$

Time needed:

- Copy the a on the second tape $O(|w|)$
- Compare the a and b $O(|w|)$

Total time: $O(|w|)$

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$$L = \{a^n b^n : n \geq 0\}$$

For string of length n

time needed for acceptance: $O(n)$

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Language class: $DTIME(n)$

$DTIME(n)$

L_1 L_2 L_3

A Deterministic Turing Machine accepts each string of length n in time $O(n)$

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$DTIME(n)$

$\{a^n b^n : n \geq 0\}$

$\{ww\}$

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In a similar way we define the class

$$DTIME(T(n))$$

for any time function: $T(n)$

Examples: $DTIME(n^2), DTIME(n^3), \ldots$

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Example: The membership problem for context free languages

$$L = \{w : w \text{ is generated by grammar } G\}$$

$L \in DTIME(n^3)$ (CYK - algorithm)

Polynomial time

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Theorem: $DTIME(n^{k+1}) \subset DTIME(n^k)$

$DTIME(n^{k+1})$

$DTIME(n^k)$

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Polynomial time algorithms: $DTIME(n^k)$

Represent tractable algorithms:

For small k we can compute the result fast

The class P

$$P = \cup DTIME(n^k) \quad \text{for all } k$$

- Polynomial time
- All tractable problems

P

CYK-algorithm

$\{a^n b^n\}$

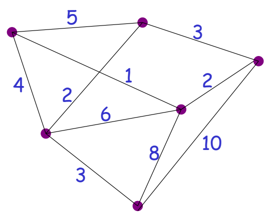
$\ldots$ $\{ww\}$

Exponential time algorithms: $DTIME(2^n)$

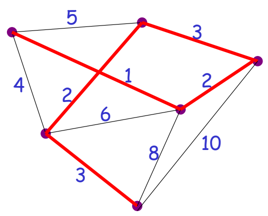
Represent intractable algorithms:

Some problem instances may take centuries to solve

Example: the Traveling Salesperson Problem



Question: what is the shortest route that connects all cities?



Question: what is the shortest route that connects all cities?

A solution: search exhuastively all hamiltonian paths

L = {shortest hamiltonian paths}

$$L \in DTIME(n!) \quad \approx \quad DTIME(2^n)$$

Exponential time

Intractable problem

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Example: The Satisfiability Problem

Boolean expressions in Conjunctive Normal Form:

$$t_1 \wedge t_2 \wedge t_3 \wedge \cdots \wedge t_k$$

$$t_i = x_1 \vee \bar{x}_2 \vee x_3 \vee \cdots \vee \bar{x}_p$$

Variables

Question: is expression satisfiable?

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Example: $(\bar{x}_1 \vee x_2) \wedge (x_1 \vee x_3)$

Satisfiable: $x_1 = 0,\ x_2 = 1,\ x_3 = 1$

$$(\bar{x}_1 \vee x_2) \wedge (x_1 \vee x_3) = 1$$

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Example: $(x_1 \vee x_2) \wedge \bar{x}_1 \wedge \bar{x}_2$

Not satisfiable

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$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

For n variables: $L \in DTIME(2^n)$

exponential

Algorithm:

search exhaustively all the possible binary values of the variables

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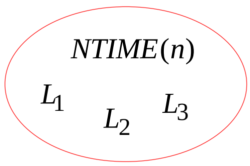
Non-Determinism

Language class: $NTIME(n)$



A Non-Deterministic Turing Machine accepts each string of length n in time $O(n)$

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Example: $L = \{ww\}$

Non-Deterministic Algorithm
to accept a string ww :

- Use a two-tape Turing machine
- Guess the middle of the string and copy w on the second tape
- Compare the two tapes

$L = \{ww\}$

Time needed:

- Use a two-tape Turing machine
- Guess the middle of the string and copy w on the second tape $O(|w|)$
- Compare the two tapes $O(|w|)$

Total time: $O(|w|)$

$NTIME(n)$

$L = \{ww\}$

In a similar way we define the class

$NTIME(T(n))$

for any time function: $T(n)$

Examples: $NTIME(n^2), NTIME(n^3), \ldots$

Non-Deterministic Polynomial time algorithms:

$L \in NTIME(n^k)$

The class NP

$P = \cup NTIME(n^k)$ for all k

Non-Deterministic Polynomial time

Example: The satisfiability problem

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

Non-Deterministic algorithm:

- Guess an assignment of the variables
- Check if this is a satisfying assignment

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$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

Time for n variables:

- Guess an assignment of the variables $O(n)$
- Check if this is a satisfying assignment $O(n)$

Total time: $O(n)$

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$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

$$L \in NP$$

The satisfiability problem is an NP - Problem

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Observation:

$$P \subseteq NP$$

Deterministic Polynomial

Non-Deterministic Polynomial

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Open Problem: $P = NP$?

WE DO NOT KNOW THE ANSWER

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Open Problem: $P = NP$?

Example: Does the Satisfiability problem have a polynomial time deterministic algorithm?

WE DO NOT KNOW THE ANSWER

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NP-Completeness

A problem is NP-complete if:

- It is in NP
- Every NP problem is reduced to it (in polynomial time)

Observation:

If we can solve any NP-complete problem in Deterministic Polynomial Time (P time) then we know:

$$P = NP$$

Observation:

If we prove that we cannot solve an NP-complete problem in Deterministic Polynomial Time (P time) then we know:

$$P \neq NP$$

Cook's Theorem:

The satisfiability problem is NP-complete

Proof:

Convert a Non-Deterministic Turing Machine to a Boolean expression in conjunctive normal form

Other NP-Complete Problems:

- The Traveling Salesperson Problem
- Vertex cover
- Hamiltonian Path

All the above are reduced to the satisfiability problem

Observations:

It is unlikely that NP-complete problems are in P

The NP-complete problems have exponential time algorithms

Approximations of these problems are in P