More Applications of The Pumping Lemma

class 14

The Pumping Lemma:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$ $\qquad$ $\{vv : v \in \{a,b\}\}$

Context-free languages

$\{a^n b^n : n \geq 0\}$ $\qquad$ $\{ww^R : w \in \{a,b\}^*\}$

Theorem: The language

$$L = \{vv : v \in \{a,b\}^*\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{vv : v \in \{a,b\}^*\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{vv : v \in \{a,b\}^*\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^m b^m a^m b^m \in L$

$$L = \{vv : v \in \{a,b\}^*\}$$

We can write: $a^m b^m a^m b^m = uvxyz$

with lengths $|vxy| \le m$ and $|vy| \ge 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \ge 0$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

We examine **<u>all</u>** the possible locations of string vxy in $a^m b^m a^m b^m$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 1: vxy is within the first a^m

$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \ge 1$

$$\overbrace{a\ldots\ldots a}^{m}\ \overbrace{b\ldots\ldots b}^{m}\ \overbrace{a\ldots\ldots a}^{m}\ \overbrace{b\ldots\ldots b}^{m}$$

$$u\ v\ x\ y \qquad\qquad z$$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 1: vxy is within the first a^m

$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \ge 1$

$$\overbrace{a\ldots\ldots\ldots\ldots\ldots a}^{m+k_1+k_2}\ \overbrace{b\ldots\ldots b}^{m}\ \overbrace{a\ldots\ldots a}^{m}\ \overbrace{b\ldots\ldots b}^{m}$$

$$u\ \ v^2\ \ x\ \ y^2 \qquad\qquad z$$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 1: vxy is within the first a^m

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2 xy^2 z \quad \notin L$$

$k_1 + k_2 \ge 1$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 1: vxy is within the first a^m

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2 xy^2 z \quad \notin L$$

However, from Pumping Lemma: $uv^2 xy^2 z \in L$

Contradiction!!!

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 2: v is in the first a^m
y is in the first b^m

$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \geq 1$

$$\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}\,\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}$$

$u \quad v \quad x \quad y \quad z$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 2: v is in the first a^m
y is in the first b^m

$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \geq 1$

$$\overbrace{a \ldots\ldots\ldots\ldots a}^{m+k_1}\,\overbrace{b \ldots\ldots\ldots\ldots b}^{m+k_2}\,\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}$$

$u \quad v^2 \quad x \quad y^2 \quad z$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 2: v is in the first a^m
y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

$k_1 + k_2 \geq 1$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 2: v is in the first a^m
y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \geq 1$

$$\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}\,\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}$$

$u \quad v \quad x \quad y \quad z$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \geq 1$

$$\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots b}^{k_2}\,\overbrace{a \ldots a}^{k_1}\,\overbrace{b \ldots\ldots\ldots b}^{m+k_3}\,\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}$$

$u \quad v^2 \quad x \quad y^2 \quad z$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{m+k_3} a^m b^m = uv^2xy^2z \quad \notin L$$

$k_1, k_2 \ge 1$

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{k_3} a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

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$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 4: v in the first a^m
y Overlaps the first $a^m b^m$

Analysis is similar to case 3

$$\overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}\ \overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}$$

$u \quad v \quad x \qquad y \qquad\qquad z$

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Other cases: vxy is within $a^m \boxed{b^m} a^m b^m$

or

$a^m b^m \boxed{a^m} b^m$

or

$a^m b^m a^m \boxed{b^m}$

Analysis is similar to case 1:

$\boxed{a^m} b^m a^m b^m$

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More cases: vxy overlaps $a^m \boxed{b^m a^m} b^m$

or

$a^m b^m \boxed{a^m b^m}$

Analysis is similar to cases 2,3,4:

$\boxed{a^m b^m} a^m b^m$

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There are no other cases to consider

Since $|vxy| \le m$, it is impossible
vxy to overlap:

$\boxed{a^m b^m a^m} b^m$

nor

$a^m \boxed{b^m a^m b^m}$

nor

$\boxed{a^m b^m a^m b^m}$

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In all cases we obtained a contradiction

Therefore: The original assumption that

$L = \{vv : v \in \{a,b\}^*\}$

is context-free must be wrong

Conclusion: L is not context-free

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Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$ $\{ww : w \in \{a,b\}\}$

$\{a^{n!} : n \geq 0\}$

Context-free languages

$\{a^n b^n : n \geq 0\}$ $\{ww^R : w \in \{a,b\}^*\}$

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Theorem: The language

$L = \{a^{n!} : n \geq 0\}$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

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$L = \{a^{n!} : n \geq 0\}$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

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$L = \{a^{n!} : n \geq 0\}$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^{m!} \in L$

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$L = \{a^{n!} : n \geq 0\}$

We can write: $a^{m!} = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

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$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **all** the possible locations of string vxy in $a^{m!}$

There is only one case to consider

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$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots a}^{m!}$$

$$u \quad v \quad x \quad y \quad z$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

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$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a}^{m! + k_1 + k_2}$$

$$u \quad v^2 \quad x \quad y^2 \quad z$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

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$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a}^{m! + k} \qquad k = k_1 + k_2$$

$$u \quad v^2 \quad x \quad y^2 \quad z$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k \leq m$$

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$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$a^{m!+k} = uv^2xy^2z$$

$$1 \leq k \leq m$$

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Since $1 \leq k \leq m$, for $m \geq 2$ we have:

$$\begin{aligned} m! + k &\leq m! + m \\ &< m! + m!m \\ &= m!(1+m) \\ &= (m+1)! \end{aligned}$$

$$\Downarrow$$

$$m! < m! + k < (m+1)!$$

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$$L = \{a^{n!} : n \geq 0\}$$

$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

$$m! < m! + k < (m+1)!$$

$$\Downarrow$$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

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$$L = \{a^{n!} : n \geq 0\}$$

$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

However, from Pumping Lemma: $uv^2xy^2z \in L$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

Contradiction!!!

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We obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n!} : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free

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Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$ $\qquad$ $\{ww : w \in \{a,b\}\}$

$\{a^{n^2} b^n : n \geq 0\}$ $\qquad$ $\{a^{n!} : n \geq 0\}$

Context-free languages

$\{a^n b^n : n \geq 0\}$ $\qquad$ $\{ww^R : w \in \{a,b\}^*\}$

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Theorem: The language

$$L = \{a^{n^2} b^n : n \geq 0\}$$

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Proof: Use the Pumping Lemma for context-free languages

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$$L = \{a^{n^2} b^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

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$$L = \{a^{n^2} b^n : n \geq 0\}$$

We can write: $a^{m^2} b^m = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

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$$L = \{a^{n^2} b^n : n \geq 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

We examine **all** the possible locations

of string vxy in $a^{m^2} b^m$

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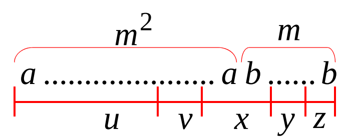
$$L = \{a^{n^2} b^n : n \geq 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Most complicated case: v is in a^m
y is in b^m



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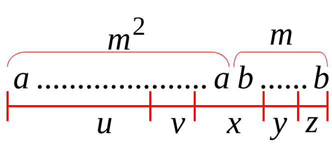
$$L = \{a^{n^2} b^n : n \geq 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

$v = a^{k_1}$ $\quad y = b^{k_2}$ $\quad 1 \leq k_1 + k_2 \leq m$



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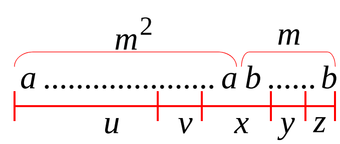
$$L = \{a^{n^2} b^n : n \geq 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$v = a^{k_1}$ $\quad y = b^{k_2}$ $\quad 1 \leq k_1 + k_2 \leq m$



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$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$\overbrace{a \ldots\ldots\ldots\ldots a}^{m^2 - k_1}\ \overbrace{b \ldots b}^{m - k_2}$$

$$\underbrace{\qquad\qquad}_{u}\ \underbrace{}_{v^0}\ \underbrace{\quad}_{x}\ \underbrace{}_{y^0}\ \underbrace{\quad}_{z}$$

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$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z$$

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$k_1 \neq 0$ and $k_2 \neq 0 \qquad 1 \leq k_1 + k_2 \leq m$

$$\Downarrow$$

$$(m - k_2)^2 \leq (m-1)^2$$
$$= m^2 - 2m + 1$$
$$< m^2 - k_1$$

$$\Downarrow$$

$$m^2 - k_1 \neq (m - k_2)^2$$

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$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$m^2 - k_1 \neq (m - k_2)^2$$

$$\Downarrow$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z \quad \notin L$$

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$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

However, from Pumping Lemma: $uv^0 xy^0 z \quad \in L$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z \quad \notin L$$

Contradiction!!!

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When we examine the rest of the cases
we also obtain a contradiction

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In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n^2} b^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free